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Research article

Uniform approximation for degenerate Bernstein polynomials

Murad Khalil[1], **Seongho Park**[1], **Si Hyeon Lee**[2] **and Kyo-Shin Hwang**[3,*]

[1] Department of Mathematics, Kwangwoon University, Seoul 01897, Republic of Korea
[2] School of Mathematics and Statistics, Shaanxi Normal University, Xi'an, Shaansi, 710119, China
[3] Graduate School of Education, Yeungnam University, Gyeongsan 38541, Republic of Korea

* **Correspondence:** Email: kshwang@yu.ac.kr.

Abstract: Let f be a real-valued continuous function defined on the interval $[0,1]$, and let $B_{n,\lambda}(f;\cdot)$ be the associated n-th degenerate Bernstein polynomial. We proved that, on any admissible interval, $|B_{n,\lambda}(f;\cdot) - f(\cdot)|$ converges uniformly to 0 as $n \to \infty$. We also established the uniform convergence of the first derivative of the degenerate Bernstein polynomial. To support the theoretical results, we tested the approximation theory on $f(x) = \sin(\pi x)$ and its first-order derivative $f'(x) = \pi\cos(\pi x)$. These showed that admissible pairs, and some formal pairs close to the admissible range, give better function approximation than the classical Bernstein case, and the first-derivative approximation convergence increases as $\lambda_n \to 0$.

Keywords: degenerate Bernstein polynomial; uniform convergence; modulus of continuity; forward difference operator
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1. Introduction

The Weierstrass approximation theorem is a fundamental result in mathematical analysis. It asserts that every continuous function defined on a closed interval can be uniformly approximated as closely as desired by a polynomial function. The theorem can be proved by a transformation with Bernstein polynomials [1]. By a linear substitution, the interval $[a,b]$ can be transformed into the unit interval $[0,1]$; therefore, the original statement of the theorem holds if and only if it holds for every continuous function f defined on the interval $[0,1]$.

The classical Bernstein polynomials, introduced by S. N. Bernstein [1,2], play a fundamental role in approximation theory by providing a constructive proof of the Weierstrass approximation theorem. Since then, they have become fundamental tools in diverse areas of mathematics and computer science. In particular, they provide a powerful method for approximating continuous functions on compact intervals [1]. For a continuous function $f \in C[0,1]$, the classical Bernstein polynomial approximation of $f(x)$ is defined by [1–3]

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\Big(\frac{k}{n}\Big).$$

It is well-known that $B_n(f;x) \to f(x)$ uniformly on $[0,1]$ as $n \to \infty$.

Theorem 1.1. *([2, 3]) If f is a function of $C[0,1]$ and $\epsilon > 0$ is arbitrary, then there exists an integer N such that*

$$|f(x) - B_n(f;x)| < \epsilon, \quad 0 \le x \le 1,$$

for all $n \ge N$.

The following theorem concerns the uniform convergence of the m-th derivative of classical Bernstein polynomials.

Theorem 1.2. *([2, 3]) If $f \in C^m[0,1]$, for some integer $m \ge 0$, then $B_n^{(m)}(f;x)$ converges uniformly to $f^{(m)}(x)$ on $[0,1]$.*

Several generalized versions of Bernstein-type operators have been studied, motivated by applications in combinatorics, special functions, and q-calculus. In the q-setting, Kim [4] studied identities on the q-integral representation of the product of several q-Bernstein-type polynomials. Among them, the notion of *degenerate* polynomials, introduced by Carlitz [5], has attracted considerable attention. Phillips [6] studied interpolation and approximation properties of Bernstein polynomials. Kim and Kim [7–9] introduced and studied properties of degenerate Bernstein polynomials, degenerate Euler polynomials, degenerate Stirling numbers of the second kind [10], degenerate Bell and Dowling polynomials [11], and heterogeneous Stirling numbers and heterogeneous Bell polynomials [12]. Wang, Ma, Kim and Kim [13] established probabilistic degenerate Bernstein polynomials. Qian, Riedel and Rosenberg [14] investigated uniform approximation and Bernstein polynomials with coefficients in the unit interval. Related contributions include the works of Cichon [15], Su, Mutlu, and Cekim [16], Ustaoglu [3], Paltanea [17], and Li, Ma, Kim and Kim [18].

In this paper, we study a degenerate analogue of the Bernstein polynomials constructed via the degenerate falling factorials $(x)_{n,\lambda}$. We investigate structural and approximation properties of the corresponding operators. Kim and Kim [9] introduced a degenerate Bernstein operator of order n. The primary objective of the present paper is to investigate whether the fundamental properties of the classical Bernstein polynomial approximation—such as positivity, preservation of constants (normalization), linearity, uniform approximation, and approximation of derivatives—remain valid in the degenerate setting when $\lambda \neq 0$. Motivated by this objective, we adopt a formulation that differs slightly from the degenerate Bernstein operator introduced in Kim and Kim [9]. This modification is designed to preserve the essential structural properties of the classical Bernstein operator while allowing us to establish approximation results analogous to those in the classical theory. In particular, the proposed normalization enables the basis functions to form a partition of unity and ensures the preservation of constant functions, which play a crucial role in the development of the approximation theory presented in this paper. In Section 2, we establish basic properties such as preservation of constants, positivity, and linearity, compute the first and second moments, and prove uniform convergence of the operator $B_{n,\lambda}$. We also derive a Korovkin-type theorem and establish Jensen's inequality. In Section 3, we analyze structural identities of the operators and derive a λ-Leibniz rule, connecting the theory with discrete forward difference techniques, which provide effective tools for approximation estimates. In Section 4, we obtain formulas for the m-th λ-difference of the basis functions and derive a representation formula for the m-th λ-difference of degenerate Bernstein polynomials. We further prove convergence of the first λ_n-difference to f' under a vanishing-step assumption. Finally, in Section 5, we present numerical results for the test function $f(x) = \sin(\pi x)$ and provide symbolic formulas together with graphical comparisons for several choices of n and λ.

2. Degenerate Bernstein polynomials

Let f be a real-valued continuous function on $[0,1]$. Let λ satisfy $0 \le \lambda < \frac{1}{2(n-1)}$ for integer $n \ge 2$, and let

$$I_{n,\lambda} = [(n-1)\lambda,\ 1-(n-1)\lambda].$$

We define the degenerate Bernstein operator $B_{n,\lambda} : C([0,1]) \to C(I_{n,\lambda})$ by

$$B_{n,\lambda}(f;x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(x)_{k,\lambda}(1-x)_{n-k,\lambda}}{(1)_{n,\lambda}} f\left(\frac{k}{n}\right), \qquad x \in I_{n,\lambda}. \tag{2.1}$$

Here $(x)_{k,\lambda}$ denotes the degenerate falling factorial defined by

$$(x)_{0,\lambda} = 1, \quad (x)_{k,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(k-1)\lambda), \quad k \ge 1.$$

We also define the associated basis polynomials

$$b_{n,k}^{(\lambda)}(x) = \binom{n}{k} \frac{(x)_{k,\lambda}(1-x)_{n-k,\lambda}}{(1)_{n,\lambda}}, \qquad k = 0,1,\ldots,n, \tag{2.2}$$

so that

$$B_{n,\lambda}(f;x) = \sum_{k=0}^{n} b_{n,k}^{(\lambda)}(x) f\left(\frac{k}{n}\right).$$

The operator $B_{n,\lambda}$ satisfies the following properties.

Proposition 2.1. *([9]) Let $f, g \in [0,1]$. Assume that λ satisfies $0 \le \lambda < \frac{1}{2(n-1)}$ for integer $n \ge 2$, and let $x \in I_{n,\lambda}$. Then we have*

(1) *Preservation of constants:*

$$B_{n,\lambda}(1;x) = 1.$$

(2) *Linearity: for all $\alpha, \beta \in \mathbb{R}$,*

$$B_{n,\lambda}(\alpha f + \beta g; x) = \alpha B_{n,\lambda}(f;x) + \beta B_{n,\lambda}(g;x).$$

Lemma 2.2. *Let λ satisfy $0 \le \lambda < \frac{1}{2(n-1)}$ for integer $n \ge 2$, and let $x \in I_{n,\lambda}$. If $f(t) \ge 0$ on $[0,1]$, then*

$$B_{n,\lambda}(f;x) \ge 0.$$

Proof. For $k = 0, 1, \ldots, n$, we have

$$(x)_{k,\lambda} \geq 0, \qquad (1-x)_{n-k,\lambda} \geq 0,$$

whenever $(n-1)\lambda \leq x \leq 1-(n-1)\lambda$. Moreover,

$$\binom{n}{k} \geq 0, \qquad (1)_{n,\lambda} > 0.$$

The basis functions

$$b_{n,k}^{(\lambda)}(x) = \binom{n}{k}\frac{(x)_{k,\lambda}(1-x)_{n-k,\lambda}}{(1)_{n,\lambda}}$$

satisfy

$$b_{n,k}^{(\lambda)}(x) \geq 0, \qquad k = 0, 1, \ldots, n.$$

If $f \geq 0$, then every summand in

$$B_{n,\lambda}(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k}^{(\lambda)}(x)$$

is non-negative, hence we have

$$B_{n,\lambda}(f; x) \geq 0.$$

This proves that $B_{n,\lambda}$ is a positive operator. □

Lemma 2.3. *Let λ satisfy $0 \leq \lambda < \frac{1}{2(n-1)}$ for integer $n \geq 2$, and let $x \in I_{n,\lambda}$. If there exist constants $c, d \in \mathbb{R}$ such that*

$$c \leq f(t) \leq d \qquad \textit{for all } t \in [0,1],$$

then

$$c \leq B_{n,\lambda}(f; x) \leq d.$$

Proof. Define constant functions

$$\varphi(t) = c, \qquad \psi(t) = d,$$

and then clearly

$$\varphi \leq f \leq \psi.$$

Since $B_{n,\lambda}$ is positive by Lemma 2.2, it preserves order, so we have

$$B_{n,\lambda}(\varphi; x) \leq B_{n,\lambda}(f; x) \leq B_{n,\lambda}(\psi; x).$$

Using the reproduction of constants

$$B_{n,\lambda}(\varphi; x) = c, \qquad B_{n,\lambda}(\psi; x) = d,$$

we have

$$c \leq B_{n,\lambda}(f; x) \leq d.$$

This completes the proof. □

We now compute the first and second moments.

Lemma 2.4. *Let λ satisfy $0 \le \lambda < \frac{1}{2(n-1)}$ for integer $n \ge 2$, and let $x \in I_{n,\lambda}$. Then we have*

$$B_{n,\lambda}(t;x) = x.$$

Proof. For $0 \le k \le n$, the degenerate Bernstein polynomial for $f(t) = t$ is given by

$$B_{n,\lambda}(f;x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(x)_{k,\lambda}(1-x)_{n-k,\lambda}}{(1)_{n,\lambda}} f\left(\frac{k}{n}\right),$$

so we have

$$B_{n,\lambda}(f;x) = \frac{1}{n} \sum_{k=0}^{n} \binom{n}{k} k \frac{(x)_{k,\lambda}(1-x)_{n-k,\lambda}}{(1)_{n,\lambda}}.$$

We get

$$\sum_{k=0}^{n} \binom{n}{k} k \frac{(x)_{k,\lambda}(1-x)_{n-k,\lambda}}{(1)_{n,\lambda}} = nx \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{(x-\lambda)_{l,\lambda}(1-x)_{n-1-l,\lambda}}{(1)_{n,\lambda}}$$

together with

$$k\binom{n}{k} = n\binom{n-1}{k-1} \quad \text{and} \quad (x)_{k,\lambda} = x(x-\lambda)_{k-1,\lambda}.$$

By the degenerate binomial identity

$$(x+y)_{n,\lambda} = \sum_{k=0}^{n} \binom{n}{k} (x)_{k,\lambda}(y)_{n-k,\lambda},$$

we have

$$\sum_{l=0}^{n-1} \binom{n-1}{l} (x-\lambda)_{l,\lambda}(1-x)_{n-1-l,\lambda} = (1-\lambda)_{n-1,\lambda} = (1)_{n,\lambda},$$

thus we obtain

$$B_{n,\lambda}(t;x) = x,$$

which completes the proof. □

Lemma 2.5. *Let λ satisfy $0 \le \lambda < \frac{1}{2(n-1)}$ for integer $n \ge 2$, and let $x \in I_{n,\lambda}$. Then we have*

$$B_{n,\lambda}(t^2;x) = \frac{x}{n} + \frac{n-1}{n} \frac{x(x-\lambda)}{1-\lambda}.$$

Proof. We have

$$B_{n,\lambda}(t^2;x) = \sum_{k=0}^{n} b_{n,k}^{(\lambda)}(x) \left(\frac{k}{n}\right)^2 = \frac{1}{n^2(1)_{n,\lambda}} \sum_{k=0}^{n} k^2 \binom{n}{k} (x)_{k,\lambda}(1-x)_{n-k,\lambda}.$$

Note that $k^2 = k(k-1) + k$, and we get

$$\begin{aligned}\sum_{k=0}^{n} k^2 \binom{n}{k} (x)_{k,\lambda}(1-x)_{n-k,\lambda} &= \sum_{k=0}^{n} k(k-1) \binom{n}{k} (x)_{k,\lambda}(1-x)_{n-k,\lambda} + \sum_{k=0}^{n} k \binom{n}{k} (x)_{k,\lambda}(1-x)_{n-k,\lambda} \\ &=: I_1 + I_2.\end{aligned}$$

By Lemma 2.4, we have

$$I_2 = \sum_{k=0}^{n} k\binom{n}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda} = n\sum_{k=0}^{n} \frac{k}{n}\binom{n}{k}\frac{(x)_{k,\lambda}(1-x)_{n-k,\lambda}}{(1)_{n,\lambda}}(1)_{n,\lambda} = nx(1)_{n,\lambda}. \tag{2.3}$$

For I_1, since

$$k(k-1)\binom{n}{k} = n(n-1)\binom{n-2}{k-2}, \qquad (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)_{n-2,\lambda}$$

and by using the degenerate binomial identity

$$\sum_{l=0}^{n-2}\binom{n-2}{l}(x-\lambda)_{l,\lambda}(1-x)_{n-2-l,\lambda} = (1-2\lambda)_{n-2,\lambda} = \frac{(1)_{n,\lambda}}{1-\lambda},$$

we have

$$\begin{aligned} I_1 &= n(n-1)x(x-\lambda)\sum_{l=0}^{n-2}\binom{n-2}{l}(x-2\lambda)_{l,\lambda}(1-x)_{n-2-l,\lambda} \\ &= n(n-1)x(x-\lambda)(1-2\lambda)_{n-2,\lambda} \\ &= n(n-1)x(x-\lambda)\frac{(1)_{n,\lambda}}{1-\lambda}. \end{aligned} \tag{2.4}$$

By (2.3) and (2.4), we get

$$\frac{1}{(1)_{n,\lambda}}(I_1+I_2) = nx + \frac{n(n-1)x(x-\lambda)}{1-\lambda},$$

hence we obtain

$$B_{n,\lambda}(t^2;x) = \tfrac{x}{n} + \tfrac{n-1}{n}\frac{x(x-\lambda)}{1-\lambda},$$

which is the desired conclusion. □

Remark 2.6. *Let $\lambda = 0$, and then the classical Bernstein polynomials are*

$$B_{n,0}(t;x) = x, \qquad B_{n,0}(t^2;x) = x^2 + \frac{x(1-x)}{n}.$$

Lemma 2.7. *Let λ satisfy $0 \le \lambda < \frac{1}{2(n-1)}$ for integer $n \ge 2$, and let $x \in I_{n,\lambda}$. Then we have*

$$B_{n,\lambda}((t-x)^2;x) = \frac{x(1-x)}{1-\lambda}\left(\tfrac{1}{n} - \lambda\right).$$

Proof. We begin with the identity $(t-x)^2 = t^2 - 2xt + x^2$. Applying the linearity of $B_{n,\lambda}$, we get

$$B_{n,\lambda}\big((t-x)^2;x\big) = B_{n,\lambda}(t^2;x) - 2x\,B_{n,\lambda}(t;x) + x^2 B_{n,\lambda}(1;x).$$

From Proposition 2.1(2) and Lemma 2.4, it follows that

$$B_{n,\lambda}(1;x) = 1, \qquad B_{n,\lambda}(t;x) = x,$$

and from Lemma 2.5,

$$B_{n,\lambda}(t^2;x) = \frac{x}{n} + \frac{n-1}{n}\frac{x(x-\lambda)}{1-\lambda},$$

thus we have

$$B_{n,\lambda}((t-x)^2;x) = \frac{x}{n} + \frac{n-1}{n}\frac{x(x-\lambda)}{1-\lambda} - x^2.$$

Write

$$-x^2 = -\frac{n-1}{n}x^2 - \frac{1}{n}x^2,$$

and then we have

$$\begin{aligned} B_{n,\lambda}((t-x)^2;x) &= \tfrac{x}{n} - \tfrac{x^2}{n} + \tfrac{n-1}{n}\left(\tfrac{x(x-\lambda)}{1-\lambda} - x^2\right) \\ &= \tfrac{x(1-x)}{n} - \tfrac{n-1}{n}\tfrac{\lambda x(1-x)}{1-\lambda} \\ &= x(1-x)\left(\tfrac{1}{n} - \tfrac{n-1}{n}\tfrac{\lambda}{1-\lambda}\right) \\ &= \tfrac{x(1-x)}{1-\lambda}\left(\tfrac{1}{n} - \lambda\right), \end{aligned}$$

which completes the proof. □

Remark 2.8. *By Lemma 2.2, let $f(t) = (t-x)^2$ for $t \in R$. Then $f(t) \geq 0$, and hence, $0 \leq B_{n,\lambda}(f;x)$. Also, set $g(x) = x(1-x)$ on R and then* $\max_R\{g(x)\} = 1/4$. *Therefore we have*

$$0 \leq B_{n,\lambda}((t-x)^2;x) \leq \tfrac{1}{4n(1-\lambda)}.$$

Also, by Jensen's inequality, we have

$$B_{n,\lambda}(|t-x|;x) \leq \sqrt{B_{n,\lambda}((t-x)^2;x)} \leq \sqrt{\tfrac{1}{4n(1-\lambda)}}. \tag{2.5}$$

Definition 2.9. *([19]) Let $f : [a,b] \to \mathbb{R}$ be continuous. The modulus of continuity of f is defined by*

$$\omega(f;h) = \sup\{|f(u) - f(v)| : |u-v| \leq h\}, \quad h \geq 0.$$

Proposition 2.10. *([19]) Let $f : [a,b] \to \mathbb{R}$ be continuous. Then*
(1) *$\omega(f;\cdot)$ is increasing,*
(2) $\lim_{h\to 0} \omega(f;h) = 0$*,*
(3) *$\omega(f;h_1 + h_2) \leq \omega(f;h_1) + \omega(f;h_2)$,*
(4) *$\omega(f;\lambda h) \leq (\lambda+1)\omega(f;h)$ for all $h, \lambda \geq 0$.*

We now prove the uniform convergence for $B_{n,\lambda}$.

Theorem 2.11. *Let $f \in C[0,1]$. Assume that $\{\lambda_n\}_{n\geq 2}$ is a sequence such that*

$$0 \leq \lambda_n < \tfrac{1}{2(n-1)} \quad \text{and} \quad \lambda_n(n-1) \to 0 \quad \text{as } n \to \infty.$$

Then the degenerate Bernstein operator $B_{n,\lambda}$ satisfies

$$\lim_{n\to\infty} \sup_{0<x<1} \left|B_{n,\lambda_n}(f;x) - f(x)\right| = 0.$$

Proof. Note that if $x \in (0,1)$, then $x \in [(n-1)\lambda_n, 1-(n-1)\lambda_n] =: I_{n,\lambda_n}$ for all sufficiently large n. By the positivity and linearity of the operator $B_{n,\lambda}$, it follows that, for any $x \in I_{n,\lambda_n}$ with $n \geq 2$,

$$\begin{aligned} |B_{n,\lambda_n}(f;x) - f(x)| &= \Big|\sum_{k=0}^{n} \binom{n}{k} f\Big(\tfrac{k}{n}\Big) b_{n,k}^{(\lambda_n)}(x) - f(x)\Big| \\ &\leq \sum_{k=0}^{n} \binom{n}{k} \Big| f\Big(\tfrac{k}{n}\Big) - f(x)\Big| b_{n,k}^{(\lambda_n)}(x) \\ &= B_{n,\lambda_n}(|f(t)-f(x)|;x). \end{aligned} \tag{2.6}$$

On the other hand, by Proposition 2.10(4), we have

$$|f(t) - f(x)| \leq \omega(f;\delta)\Big(1 + \tfrac{|t-x|}{\delta}\Big). \tag{2.7}$$

Therefore, by (2.6) and (2.7),

$$\begin{aligned} |B_{n,\lambda_n}(f;x) - f(x)| &\leq B_{n,\lambda_n}\Big(\omega(f;\delta)\Big(1 + \tfrac{|t-x|}{\delta}\Big);x\Big) \\ &\leq \omega(f;\delta) B_{n,\lambda_n}\Big(1 + \tfrac{|t-x|}{\delta};x\Big) \\ &\leq \omega(f;\delta)\Big(1 + \tfrac{1}{\delta} B_{n,\lambda_n}(|t-x|;x)\Big). \end{aligned}$$

By (2.5) in Remark 2.8, we have

$$B_{n,\lambda_n}(|t-x|;x) \leq \sqrt{B_{n,\lambda_n}((t-x)^2;x)}$$

and

$$B_{n,\lambda_n}((t-x)^2;x) \leq \tfrac{C}{n}.$$

For some constant $C > 0$ independent of x, it follows that

$$|B_{n,\lambda_n}(f;x) - f(x)| \leq \omega(f;\delta)\Big(1 + \tfrac{1}{\delta}\sqrt{\tfrac{C}{n}}\Big).$$

Choosing $\delta = n^{-1/2}$, we obtain

$$|B_{n,\lambda_n}(f;x) - f(x)| \leq 2\,\omega\Big(f;\sqrt{\tfrac{C}{n}}\Big).$$

Taking the supremum over I_{n,λ_n} gives

$$\sup_{x \in I_{n,\lambda_n}} |B_{n,\lambda}(f;x) - f(x)| \leq 2\,\omega\Big(f;\sqrt{\tfrac{C}{n}}\Big) \longrightarrow 0 \quad (n \to \infty).$$

Together with Proposition 2.10(2), the proof is complete. □

Remark 2.12. *In Theorem 2.11, $B_{n,\lambda}(f;x)$ converges uniformly to f on the admissible interval.*

Corollary 2.13 (Jensen inequality)**.** *Let λ satisfy $0 \leq \lambda < \frac{1}{2(n-1)}$ for integer $n \geq 2$, and let $x \in I_{n,\lambda}$. If f is convex on $[0,1]$, then*

$$f(x) \leq B_{n,\lambda}(f;x).$$

Proof. By assumptions, $b_{n,k}^{(\lambda)}(x) \geq 0$, and by (2) of Proposition 2.1,

$$\sum_{k=0}^{n} b_{n,k}^{(\lambda)}(x) = 1.$$

Moreover, by Lemma 2.4,

$$B_{n,\lambda}(t;x) = \sum_{k=0}^{n} b_{n,k}^{(\lambda)}(x)\frac{k}{n} = x.$$

Since f is convex, it follows that

$$f(x) = f\Big(\sum_{k=0}^{n} b_{n,k}^{(\lambda)}(x)\frac{k}{n}\Big) \leq \sum_{k=0}^{n} b_{n,k}^{(\lambda)}(x) f\Big(\frac{k}{n}\Big) = B_{n,\lambda}(f;x),$$

which proves

$$f(x) \leq B_{n,\lambda}(f;x).$$

□

3. Forward difference representation

Let D_λ denote the degenerate forward difference operator, E_λ the shift operator, and I the identity operator. For $f \in C[0,1]$, they are defined by

$$(E_\lambda f)(x) := f(x+\lambda), \qquad (If)(x) := f(x),$$

and the degenerate forward difference operator is given by

$$D_\lambda f := \frac{E_\lambda f - If}{\lambda}. \tag{3.1}$$

Accordingly, for any function f, we have

$$(D_\lambda f)(x) = \Big(\frac{E_\lambda f - If}{\lambda}\Big)(x) = \frac{(E_\lambda f)(x) - (If)(x)}{\lambda} = \frac{f(x+\lambda) - f(x)}{\lambda}.$$

The composition of operators is denoted by "$\circ$". In particular,

$$D_\lambda^2 = D_\lambda \circ D_\lambda, \quad E_\lambda^2 = E_\lambda \circ E_\lambda, \quad I^2 = I \circ I, \quad D_\lambda \circ E_\lambda = D_\lambda E_\lambda, \quad E_\lambda \circ D_\lambda = E_\lambda D_\lambda.$$

More generally, for $n \in \mathbb{N}$, we define

$$D_\lambda^n := \underbrace{D_\lambda \circ \cdots \circ D_\lambda}_{n \text{ times}}, \qquad E_\lambda^n := \underbrace{E_\lambda \circ \cdots \circ E_\lambda}_{n \text{ times}}, \qquad D_\lambda^0 = E_\lambda^0 = I.$$

For $m \geq 0$, we have

$$(E_\lambda^m f)(x) = (E_\lambda^{m-1} \circ E_\lambda f)(x) = (E_\lambda^{m-1} f)(x+\lambda) = (E_\lambda^{m-2} f)(x+2\lambda) = \cdots = f(x+m\lambda).$$

Moreover, the identity operator satisfies the relations

$$E_\lambda \circ I = E_\lambda = I \circ E_\lambda.$$

The following properties hold for the operators.

Theorem 3.1. *For $n \in \mathbb{N}$, we have*

$$\lambda^n D_\lambda^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} E_\lambda^k$$

and

$$E_\lambda^n = \sum_{k=0}^{n} \binom{n}{k} \lambda^k D_\lambda^k = \sum_{k=0}^{n} \binom{n}{k} \lambda^{n-k} D_\lambda^{n-k}.$$

Proof. From (3.1), we have

$$\lambda D_\lambda = E_\lambda - I.$$

Taking the n-th power on both sides, we obtain

$$\lambda^n D_\lambda^n = (E_\lambda - I)^n.$$

By the binomial theorem,

$$\lambda^n D_\lambda^n = (E_\lambda - I)^n = \sum_{k=0}^{n} \binom{n}{k} (-I)^{n-k} E_\lambda^k = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} E_\lambda^k.$$

Also, since

$$E_\lambda = I + \lambda D_\lambda,$$

we get

$$E_\lambda^n = (I + \lambda D_\lambda)^n.$$

Again by the binomial theorem,

$$E_\lambda^n = \sum_{k=0}^{n} \binom{n}{k} \lambda^k D_\lambda^k.$$

Replacing k by $n-k$, we also have

$$E_\lambda^n = \sum_{k=0}^{n} \binom{n}{k} \lambda^{n-k} D_\lambda^{n-k}.$$

□

Remark 3.2. *For any function $h \in C[0,1]$ and integer $n \geq 0$, we have*

$$h(x + n\lambda) = (E_\lambda^n h)(x) = \sum_{k=0}^{n} \binom{n}{k} \lambda^k (D_\lambda^k h)(x), \tag{3.2}$$

and

$$(D_\lambda^n h)(x) = \frac{1}{\lambda^n} \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} h(x + j\lambda).$$

Theorem 3.3. *For $m, n = 0, 1, 2, \cdots$, we have*

$$D_\lambda^m \circ E_\lambda^n = E_\lambda^n \circ D_\lambda^m.$$

Proof. We first show that D_λ and E_λ commute. Indeed,

$$D_\lambda \circ E_\lambda = \left(\frac{E_\lambda - I}{\lambda}\right) \circ E_\lambda = \frac{E_\lambda^2 - E_\lambda}{\lambda} = E_\lambda \circ \left(\frac{E_\lambda - I}{\lambda}\right) = E_\lambda \circ D_\lambda.$$

Since commuting operators preserve commutativity under powers, it follows that

$$D_\lambda^m \circ E_\lambda^n = E_\lambda^n \circ D_\lambda^m, \quad m, n \geq 0.$$

This completes the proof. □

Theorem 3.4. (The λ–Leibniz rule) *For $n \in \mathbb{N}$ and for $f, g \in C[0, 1]$, we have*

$$(D_\lambda^n(fg))(x) = \sum_{k=0}^{n} \binom{n}{k} (D_\lambda^k f)(x)(D_\lambda^{n-k} g)(x + k\lambda),$$

or equivalently,

$$(D_\lambda^n(fg))(x) = \sum_{k=0}^{n} \binom{n}{k} (D_\lambda^{n-k} f)(x)(D_\lambda^k g)(x + (n-k)\lambda).$$

Proof. We prove the theorem by mathematical induction on n. For the case $n = 1$, we have

$$(D_\lambda(fg))(x) = \frac{f(x+\lambda)g(x+\lambda) - f(x)g(x)}{\lambda}.$$

Adding and subtracting $f(x)g(x+\lambda)$, we obtain

$$\begin{aligned} (D_\lambda(fg))(x) &= \frac{f(x)g(x+\lambda) - f(x)g(x)}{\lambda} + \frac{f(x+\lambda)g(x+\lambda) - f(x)g(x+\lambda)}{\lambda} \\ &= f(x)(D_\lambda g)(x) + (D_\lambda f)(x) \cdot g(x+\lambda). \end{aligned} \tag{3.3}$$

Since

$$(D_\lambda^0 f)(x) = f(x), \qquad (D_\lambda^0 g)(x) = g(x),$$

we may rewrite the above equality as

$$(D_\lambda(fg))(x) = \binom{1}{0}(D_\lambda^0 f)(x)(D_\lambda^1 g)(x) + \binom{1}{1}(D_\lambda^1 f)(x)(D_\lambda^0 g)(x+\lambda).$$

Hence the statement is true for $n = 1$. Assume now that the formula holds for some fixed integer $n \geq 1$; namely,

$$(D_\lambda^n(fg))(x) = \sum_{k=0}^{n} \binom{n}{k} (D_\lambda^k f)(x)(D_\lambda^{n-k} g)(x + k\lambda). \tag{3.4}$$

We shall prove that the identity also holds for $n+1$. Applying the operator D_λ to both sides of (3.4), we obtain

$$(D_\lambda^{n+1}(fg))(x) = \sum_{k=0}^{n} \binom{n}{k} D_\lambda \Big((D_\lambda^k f)(x)(D_\lambda^{n-k} g)(x + k\lambda) \Big).$$

Using the product rule for D_λ with

$$F(x) = (D_\lambda^k f)(x), \qquad G(x) = (D_\lambda^{n-k} g)(x + k\lambda),$$

we have by using (3.3)

$$\begin{aligned}
&D_\lambda\Big((D_\lambda^k f)(x)(D_\lambda^{n-k} g)(x + k\lambda)\Big)\\
&= D_\lambda\Big(FG\Big)(x) = F(x)(D_\lambda G)(x) + (D_\lambda F)(x) \cdot G(x + \lambda)\\
&= (D_\lambda^{k+1} f)(x)(D_\lambda^{n-k} g)(x + (k + 1)\lambda) + (D_\lambda^k f)(x)(D_\lambda^{n-k+1} g)(x + k\lambda).
\end{aligned}$$

Therefore,

$$(D_\lambda^{n+1}(fg))(x) = \sum_{k=0}^{n} \binom{n}{k} (D_\lambda^{k+1} f)(x)(D_\lambda^{n-k} g)(x + (k + 1)\lambda) + \sum_{k=0}^{n} \binom{n}{k} (D_\lambda^k f)(x)(D_\lambda^{n-k+1} g)(x + k\lambda).$$

Next, we reindex the first summation by setting $j = k + 1$. Then, as k runs from 0 to n, the index j runs from 1 to $n + 1$. Hence,

$$\sum_{k=0}^{n} \binom{n}{k} (D_\lambda^{k+1} f)(x)(D_\lambda^{n-k} g)(x + (k + 1)\lambda) = \sum_{j=1}^{n+1} \binom{n}{j-1} (D_\lambda^j f)(x)(D_\lambda^{n+1-j} g)(x + j\lambda).$$

Substituting this into the previous expression yields

$$(D_\lambda^{n+1}(fg))(x) = \sum_{j=1}^{n+1} \binom{n}{j-1} (D_\lambda^j f)(x)(D_\lambda^{n+1-j} g)(x + j\lambda) + \sum_{k=0}^{n} \binom{n}{k} (D_\lambda^k f)(x)(D_\lambda^{n+1-k} g)(x + k\lambda).$$

Since the dummy indices are arbitrary, we rename j by k in the first sum:

$$(D_\lambda^{n+1}(fg))(x) = \sum_{k=1}^{n+1} \binom{n}{k-1} (D_\lambda^k f)(x)(D_\lambda^{n+1-k} g)(x + k\lambda) + \sum_{k=0}^{n} \binom{n}{k} (D_\lambda^k f)(x)(D_\lambda^{n+1-k} g)(x + k\lambda).$$

Combining the two sums term-by-term, we obtain

$$\begin{aligned}
(D_\lambda^{n+1}(fg))(x) = D_\lambda^0 f(x)(D_\lambda^{n+1} g)(x) + \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k}\right) (D_\lambda^k f)(x)(D_\lambda^{n+1-k} g)(x + k\lambda)\\
+ (D_\lambda^{n+1} f)(x) g(x + (n + 1)\lambda).
\end{aligned}$$

Applying Pascal's identity

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k},$$

we deduce that

$$(D_\lambda^{n+1}(fg))(x) = \sum_{k=0}^{n+1} \binom{n+1}{k} (D_\lambda^k f)(x)(D_\lambda^{n+1-k} g)(x + k\lambda).$$

Thus the statement holds for $n + 1$. Therefore, by the principle of mathematical induction, the formula is valid for all integers $n \in \mathbb{N}$. This completes the proof. □

Corollary 3.5. *([20]) For $n \in \mathbb{N}$ and for $f, g \in C[0,1]$, we have*

$$
\begin{aligned}
(D_\lambda^n(gf))(x) &= \sum_{k=0}^{n} \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} \lambda^l (D_\lambda^k f)(x)(D_\lambda^{n-k+l} g)(x) \\
&= \sum_{k=0}^{n} \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} \lambda^{k-l} (D_\lambda^k f)(x)(D_\lambda^{n-l} g)(x).
\end{aligned}
$$

Proof. From (3.2) in Remark 3.2, we have

$$
g(x+k\lambda) = \sum_{l=0}^{k} \binom{k}{l} \lambda^l (D_\lambda^l g)(x)
$$

and

$$
(D_\lambda^{n-k} g)(x+k\lambda) = D_\lambda^{n-k} \sum_{l=0}^{k} \binom{k}{l} \lambda^l (D_\lambda^l g)(x) = \sum_{l=0}^{k} \binom{k}{l} \lambda^l (D_\lambda^{n-k+l} g)(x),
$$

hence

$$
\begin{aligned}
D_\lambda^n(gf)(x) &= \sum_{k=0}^{n} \binom{n}{k} (D_\lambda^k f)(x)(D_\lambda^{n-k} g)(x+k\lambda) \\
&= \sum_{k=0}^{n} \binom{n}{k} (D_\lambda^k f)(x) \sum_{l=0}^{k} \binom{k}{l} \lambda^l (D_\lambda^{n-k+l} g)(x) \\
&= \sum_{k=0}^{n} \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} \lambda^l (D_\lambda^k f)(x)(D_\lambda^{n-k+l} g)(x).
\end{aligned}
$$

We also have

$$
D_\lambda^n(gf)(x) = \sum_{k=0}^{n} \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} \lambda^{k-l} (D_\lambda^k f)(x)(D_\lambda^{n-l} g)(x),
$$

together with

$$
g(x+k\lambda) = \sum_{l=0}^{k} \binom{k}{l} \lambda^{k-l} (D_\lambda^{k-l} g)(x),
$$

which is the desired conclusion. □

4. λ-difference representation of degenerate Bernstein polynomials

The following lemma presents a formula for the first derivative of the basis polynomials.

Lemma 4.1. *Let λ satisfy $0 \le \lambda < \frac{1}{2(n-1)}$ for integer $n \ge 2$, and let $x \in I_{n,\lambda}$. Set*

$$
(b_{n,k}^{(\lambda)}(\cdot))(x) = b_{n,k}^{(\lambda)}(x) = \begin{cases} \binom{n}{k} \frac{(x)_{k,\lambda}(1-x)_{n-k,\lambda}}{(1)_{n,\lambda}}, & 0 \le k \le n; \\ 0, & \textit{otherwise.} \end{cases}
$$

Then, for $0 \le k \le n+1$,

$$
(D_\lambda b_{n+1,k}^{(\lambda)}(\cdot))(x) = \frac{n+1}{1-n\lambda} \Big(b_{n,k-1}^{(\lambda)}(x) - b_{n,k}^{(\lambda)}(x+\lambda) \Big). \tag{4.1}
$$

Proof. From the above definition and (2.2) in Section 2, it follows that

$$b_{n+1,k}^{(\lambda)}(x) = \binom{n+1}{k}\frac{(x)_{k,\lambda}(1-x)_{n+1-k,\lambda}}{(1)_{n+1,\lambda}}.$$

By Theorem 3.4, the λ–Leibniz rule yields

$$(D_\lambda(uv))(x) = (D_\lambda u)(x)\,v(x+\lambda) + u(x)\,(D_\lambda v)(x).$$

Viewing $(x)_{k,\lambda}$ and $(1-x)_{m,\lambda}$ as the functions

$$x \mapsto (\cdot)_{k,\lambda}(x) \quad \text{and} \quad x \mapsto (1-\cdot)_{m,\lambda}(x),$$

respectively, we have

$$(D_\lambda(\cdot)_{k,\lambda})(x) = k(x)_{k-1,\lambda}, \qquad (D_\lambda(1-\cdot)_{m,\lambda})(x) = -m(1-x-\lambda)_{m-1,\lambda},$$

and we obtain

$$\Big(D_\lambda\Big((\cdot)_{k,\lambda}(1-\cdot)_{n+1-k,\lambda}\Big)\Big)(x) = k(x)_{k-1,\lambda}(1-x-\lambda)_{n+1-k,\lambda} - (x)_{k,\lambda}(n+1-k)(1-x-\lambda)_{n-k,\lambda}. \tag{4.2}$$

From the factorization property $(a)_{m+1,\lambda} = (a)_{m,\lambda}(a-m\lambda)$, we have

$$(1-x-\lambda)_{n+1-k,\lambda} = (1-x-(n+1-k)\lambda)(1-x-\lambda)_{n-k,\lambda}.$$

Substituting this into (4.2) gives

$$\begin{aligned}\Big(D_\lambda\Big((\cdot)_{k,\lambda}(1-\cdot)_{n+1-k,\lambda}\Big)\Big)(x) &= k(x)_{k-1,\lambda}\,(1-x-(n+1-k)\lambda)\,(1-x-\lambda)_{n-k,\lambda} \\ &\quad -(n+1-k)(x)_{k,\lambda}\,(1-x-\lambda)_{n-k,\lambda},\end{aligned} \tag{4.3}$$

and hence, by (4.3)

$$\begin{aligned}(D_\lambda b_{n+1,k}^{(\lambda)}(\cdot))(x) &= \binom{n+1}{k}\frac{1}{(1)_{n+1,\lambda}} \\ &\quad\times\Big[k(x)_{k-1,\lambda}(1-x-(n+1-k)\lambda)(1-x-\lambda)_{n-k,\lambda} \\ &\qquad -(n+1-k)(x)_{k,\lambda}(1-x-\lambda)_{n-k,\lambda}\Big].\end{aligned} \tag{4.4}$$

Next, use the combinatorial identities

$$\binom{n+1}{k}k = (n+1)\binom{n}{k-1}, \qquad \binom{n+1}{k}(n+1-k) = (n+1)\binom{n}{k},$$

and also the normalization relation

$$(1)_{n+1,\lambda} = (1-n\lambda)(1)_{n,\lambda}.$$

Then (4.4) becomes

$$\begin{aligned}&(D_\lambda b_{n+1,k}^{(\lambda)}(\cdot))(x) \\ &= \frac{n+1}{1-n\lambda}\frac{(1-x-\lambda)_{n-k,\lambda}}{(1)_{n,\lambda}}\Big[\binom{n}{k-1}(x)_{k-1,\lambda}(1-x-(n+1-k)\lambda) - \binom{n}{k}(x)_{k,\lambda}\Big].\end{aligned} \tag{4.5}$$

For the first term of (4.5), since

$$(1-x)_{n-(k-1),\lambda} = (1-x)_{n+1-k,\lambda} = (1-x)(1-x-\lambda)_{n-k,\lambda},$$

we have

$$\begin{aligned}
&\binom{n}{k-1}(x)_{k-1,\lambda}(1-x-(n+1-k)\lambda)(1-x-\lambda)_{n-k,\lambda}\\
&=\binom{n}{k-1}(x)_{k-1,\lambda}(1-x)(1-x-\lambda)_{n-k,\lambda}-\binom{n}{k-1}(x)_{k-1,\lambda}(n+1-k)\lambda(1-x-\lambda)_{n-k,\lambda}\\
&=\binom{n}{k-1}(x)_{k-1,\lambda}(1-x)_{n-(k-1)}-\binom{n}{k}k\lambda(x)_{k-1,\lambda}(1-x-\lambda)_{n-k,\lambda}\\
&=(1)_{n,\lambda}\,b^{(\lambda)}_{n,k-1}(x)-\binom{n}{k}k\lambda(x)_{k-1,\lambda}(1-x-\lambda)_{n-k,\lambda},
\end{aligned} \tag{4.6}$$

together with

$$\binom{n}{k-1}(n+1-k) = \binom{n}{k}k.$$

It follows that the first part of (4.5) equals

$$(1)_{n,\lambda}\,b^{(\lambda)}_{n,k-1}(x)-\binom{n}{k}k\lambda(x)_{k-1,\lambda}(1-x-\lambda)_{n-k,\lambda}.$$

For the second term of (4.5), we use the shift identity

$$(x+\lambda)_{k,\lambda} = (x)_{k,\lambda} + k\lambda(x)_{k-1,\lambda},$$

which gives

$$\binom{n}{k}\big((x)_{k,\lambda}+k\lambda(x)_{k-1,\lambda}\big)(1-x-\lambda)_{n-k,\lambda} = \binom{n}{k}(x+\lambda)_{k,\lambda}(1-x-\lambda)_{n-k,\lambda} = (1)_{n,\lambda}\,b^{(\lambda)}_{n,k}(x+\lambda). \tag{4.7}$$

Combining (4.6) and (4.7) in (4.5) yields

$$(D_\lambda b^{(\lambda)}_{n+1,k}(\cdot))(x) = \frac{n+1}{1-n\lambda}\Big(b^{(\lambda)}_{n,k-1}(x)-b^{(\lambda)}_{n,k}(x+\lambda)\Big),$$

which is exactly (4.1). □

The following lemma gives the first derivative representation of the degenerate Bernstein polynomial.

Lemma 4.2. *Let λ satisfy $0 \le \lambda < \frac{1}{2(n-1)}$ for integer $n \ge 2$, and let $x \in I_{n,\lambda}$. Then we have*

$$(D_\lambda B_{n+1,\lambda}(f;\cdot))(x) = \tfrac{n+1}{1-n\lambda}\sum_{k=0}^{n}\Delta f\Big(\tfrac{k}{n+1}\Big)b^{(\lambda)}_{n,k}(x) - \tfrac{\lambda(n+1)}{1-n\lambda}\sum_{k=0}^{n} f\Big(\tfrac{k}{n+1}\Big)(D_\lambda b^{(\lambda)}_{n,k}(\cdot))(x). \tag{4.8}$$

Proof. It follows from (2.1) of Section 2 that

$$B_{n+1,\lambda}(f;x) = \sum_{k=0}^{n+1} b^{(\lambda)}_{n+1,k}(x)\,f\Big(\tfrac{k}{n+1}\Big) = \sum_{k=0}^{n+1}\binom{n+1}{k}\tfrac{(x)_{k,\lambda}(1-x)_{n+1-k,\lambda}}{(1)_{n+1,\lambda}}f\Big(\tfrac{k}{n+1}\Big).$$

By the linearity of $B_{n+1,\lambda}$, we obtain

$$(D_\lambda B_{n+1,\lambda}(f;\cdot))(x) = \sum_{k=0}^{n+1} f\left(\tfrac{k}{n+1}\right)(D_\lambda b_{n+1,k}^{(\lambda)}(\cdot))(x).$$

It follows from Lemma 4.1 that

$$\begin{aligned}
(D_\lambda B_{n+1,\lambda}(f;\cdot))(x) &= \tfrac{n+1}{1-n\lambda}\sum_{k=0}^{n+1} f\left(\tfrac{k}{n+1}\right)\left(b_{n,k-1}^{(\lambda)}(x)-b_{n,k}^{(\lambda)}(x+\lambda)\right)\\
&= \tfrac{n+1}{1-n\lambda}\sum_{k=0}^{n}\left(b_{n,k}^{(\lambda)}(x)\,f\left(\tfrac{k+1}{n+1}\right)-b_{n,k}^{(\lambda)}(x+\lambda)\,f\left(\tfrac{k}{n+1}\right)\right)\\
&= \tfrac{n+1}{1-n\lambda}\sum_{k=0}^{n}\left(\left(f\left(\tfrac{k+1}{n+1}\right)-f\left(\tfrac{k}{n+1}\right)\right)b_{n,k}^{(\lambda)}(x)+\left(b_{n,k}^{(\lambda)}(x)-b_{n,k}^{(\lambda)}(x+\lambda)\right)f\left(\tfrac{k}{n+1}\right)\right)\\
&= \tfrac{n+1}{1-n\lambda}\sum_{k=0}^{n}\Delta f\left(\tfrac{k}{n+1}\right)b_{n,k}^{(\lambda)}(x)+\tfrac{n+1}{1-n\lambda}\sum_{k=0}^{n} f\left(\tfrac{k}{n+1}\right)\left(b_{n,k}^{(\lambda)}(x)-b_{n,k}^{(\lambda)}(x+\lambda)\right)\\
&= \tfrac{n+1}{1-n\lambda}\sum_{k=0}^{n}\Delta f\left(\tfrac{k}{n+1}\right)b_{n,k}^{(\lambda)}(x)-\tfrac{\lambda(n+1)}{1-n\lambda}\sum_{k=0}^{n} f\left(\tfrac{k}{n+1}\right)(D_\lambda b_{n,k}^{(\lambda)}(\cdot))(x),
\end{aligned}$$

together with

$$b_{n,k}^{(\lambda)}(x)-b_{n,k}^{(\lambda)}(x+\lambda) = -\lambda(D_\lambda b_{n,k}^{(\lambda)}(\cdot))(x),$$

which establishes the formula. □

The following lemma provides a formula for the second derivative of the basis polynomials.

Lemma 4.3. *Let λ satisfy $0 \le \lambda < \frac{1}{2(n-1)}$ for integer $n \ge 2$, and let $x \in I_{n,\lambda}$. Then we have*

$$(D_\lambda^2 b_{n+1,k}^{(\lambda)}(\cdot))(x) = \frac{(n+1)n}{(1-n\lambda)(1-(n-1)\lambda)}\left(b_{n-1,k-2}^{(\lambda)}(x)-2b_{n-1,k-1}^{(\lambda)}(x+\lambda)+b_{n-1,k}^{(\lambda)}(x+2\lambda)\right).$$

Proof. From Lemma 4.1 and the linearity of D_λ, we have

$$(D_\lambda^2 b_{n+1,k}^{(\lambda)}(\cdot))(x) = \frac{n+1}{1-n\lambda}\left((D_\lambda b_{n,k-1}^{(\lambda)}(\cdot))(x)-(D_\lambda b_{n,k}^{(\lambda)}(\cdot))(x+\lambda)\right).$$

We show that for $1 \le k \le n$,

$$(D_\lambda b_{n,k-1}^{(\lambda)}(\cdot))(x) = \frac{n}{1-(n-1)\lambda}\left(b_{n-1,k-2}^{(\lambda)}(x)-b_{n-1,k-1}^{(\lambda)}(x+\lambda)\right), \tag{4.9}$$

$$(D_\lambda b_{n,k}^{(\lambda)}(\cdot))(x+\lambda) = \frac{n}{1-(n-1)\lambda}\left(b_{n-1,k-1}^{(\lambda)}(x+\lambda)-b_{n-1,k}^{(\lambda)}(x+2\lambda)\right). \tag{4.10}$$

For (4.9), it follows from the definition that

$$b_{n,k-1}^{(\lambda)}(x) = \binom{n}{k-1}\frac{(x)_{k-1,\lambda}(1-x)_{n-k+1,\lambda}}{(1)_{n,\lambda}},$$

and we obtain

$$\begin{aligned}(D_\lambda b^{(\lambda)}_{n,k-1}(\cdot))(x) &= \binom{n}{k-1}\frac{1}{(1)_{n,\lambda}}\Big(D_\lambda\Big((\cdot)_{k-1,\lambda}(1-\cdot)_{n-k+1,\lambda}\Big)\Big)(x)\\ &= \binom{n}{k-1}\frac{1}{(1)_{n,\lambda}}\Big[(k-1)(x)_{k-2,\lambda}(1-x)_{n-k+1,\lambda}\\ &\qquad -(n-k+1)(x+\lambda)_{k-1,\lambda}(1-x-\lambda)_{n-k,\lambda}\Big],\end{aligned}$$

together with

$$(D_\lambda(fg))(x) = (D_\lambda f)(x)\cdot g(x) + f(x+\lambda)\,(D_\lambda g)(x).$$

Using $(1)_{n,\lambda} = (1-(n-1)\lambda)\,(1)_{n-1,\lambda}$ and the binomial coefficient identities

$$(k-1)\binom{n}{k-1} = n\binom{n-1}{k-2}, \qquad (n-k+1)\binom{n}{k-1} = n\binom{n-1}{k-1},$$

we rewrite the two terms as

$$\begin{aligned}&\frac{n}{1-(n-1)\lambda}\binom{n-1}{k-2}\frac{(x)_{k-2,\lambda}(1-x)_{n-k+1,\lambda}}{(1)_{n-1,\lambda}}\\ &-\frac{n}{1-(n-1)\lambda}\binom{n-1}{k-1}\frac{(x+\lambda)_{k-1,\lambda}(1-x-\lambda)_{n-k,\lambda}}{(1)_{n-1,\lambda}}.\end{aligned}$$

Recognizing these as $b^{(\lambda)}_{n-1,k-2}(x)$ and $b^{(\lambda)}_{n-1,k-1}(x+\lambda)$, we obtain (4.9).
To prove (4.10), we apply (4.9) with the index shift $k \mapsto k+1$, which gives the general identity

$$(D_\lambda b^{(\lambda)}_{n,k}(\cdot))(x) = \frac{n}{1-(n-1)\lambda}\Big(b^{(\lambda)}_{n-1,k-1}(x) - b^{(\lambda)}_{n-1,k}(x+\lambda)\Big).$$

Replacing x by $x+\lambda$ yields

$$(D_\lambda b^{(\lambda)}_{n,k}(\cdot))(x+\lambda) = \frac{n}{1-(n-1)\lambda}\Big(b^{(\lambda)}_{n-1,k-1}(x+\lambda) - b^{(\lambda)}_{n-1,k}(x+2\lambda)\Big),$$

which is (4.10). Combining (4.9) and (4.10), we obtain

$$(D^2_\lambda b^{(\lambda)}_{n+1,k}(\cdot))(x) \;=\; \frac{(n+1)n}{(1-n\lambda)(1-(n-1)\lambda)}\Big(b^{(\lambda)}_{n-1,k-2}(x) - 2b^{(\lambda)}_{n-1,k-1}(x+\lambda) + b^{(\lambda)}_{n-1,k}(x+2\lambda)\Big),$$

which completes the proof. □

The following theorem gives the second derivative representation of the degenerate Bernstein polynomial.

Lemma 4.4. *Let λ satisfy $0 \le \lambda < \frac{1}{2(n-1)}$ for integer $n \ge 2$, and let $x \in I_{n,\lambda}$. Then we have*

$$\begin{aligned}(D^2_\lambda B_{n+1,\lambda}(f;\cdot))(x) &= c_n c_{n-1}\sum_{k=0}^{n-1}\Big(\Delta f\Big(\tfrac{k+1}{n+1}\Big)b^{(\lambda)}_{n-1,k}(x) - \lambda f\Big(\tfrac{k+1}{n+1}\Big)(D_\lambda b^{(\lambda)}_{n-1,k}(\cdot))(x)\Big)\\ &\quad - c_n c_{n-1}\sum_{k=0}^{n-1}\Big(\Delta f\Big(\tfrac{k}{n+1}\Big)b^{(\lambda)}_{n-1,k}(x+\lambda) - \lambda f\Big(\tfrac{k}{n+1}\Big)(D_\lambda b^{(\lambda)}_{n-1,k}(\cdot))(x+\lambda)\Big),\end{aligned}$$

where

$$c_n = \frac{n+1}{1-n\lambda} \quad \textit{and} \quad c_{n-1} = \frac{n}{1-(n-1)\lambda}.$$

Proof. From Lemma 4.2, we already know that

$$(D_\lambda B_{n+1,\lambda}(f;\cdot))(x) = c_n \sum_{k=0}^{n} \left(b_{n,k}^{(\lambda)}(x) f\left(\tfrac{k+1}{n+1}\right) - b_{n,k}^{(\lambda)}(x+\lambda) f\left(\tfrac{k}{n+1}\right)\right). \tag{4.11}$$

Applying D_λ once more to (4.11), we get

$$(D_\lambda^2 B_{n+1,\lambda}(f;\cdot))(x) = c_n \sum_{k=0}^{n} \left(f\left(\tfrac{k+1}{n+1}\right)(D_\lambda b_{n,k}^{(\lambda)}(\cdot))(x) - f\left(\tfrac{k}{n+1}\right)(D_\lambda b_{n,k}^{(\lambda)}(\cdot))(x+\lambda)\right). \tag{4.12}$$

Similar to that in the proofs of (4.9) and (4.10), for $1 \le k \le n$,

$$\begin{aligned}(D_\lambda b_{n,k}^{(\lambda)}(\cdot))(x) &= c_{n-1}\left(b_{n-1,k-1}^{(\lambda)}(x) - b_{n-1,k}^{(\lambda)}(x+\lambda)\right),\\ (D_\lambda b_{n,k}^{(\lambda)}(\cdot))(x+\lambda) &= c_{n-1}\left(b_{n-1,k-1}^{(\lambda)}(x+\lambda) - b_{n-1,k}^{(\lambda)}(x+2\lambda)\right).\end{aligned}$$

Substituting these into (4.12), we get

$$\begin{aligned}&(D_\lambda^2 B_{n+1,\lambda}(f;\cdot))(x)\\ &= c_n c_{n-1} \sum_{k=0}^{n} \left(f\left(\tfrac{k+1}{n+1}\right)\left(b_{n-1,k-1}^{(\lambda)}(x) - b_{n-1,k}^{(\lambda)}(x+\lambda)\right) - f\left(\tfrac{k}{n+1}\right)\left(b_{n-1,k-1}^{(\lambda)}(x+\lambda) - b_{n-1,k}^{(\lambda)}(x+2\lambda)\right)\right)\\ &= c_n c_{n-1} \sum_{k=0}^{n-1} \left(f\left(\tfrac{k+2}{n+1}\right) b_{n-1,k}^{(\lambda)}(x) - 2f\left(\tfrac{k+1}{n+1}\right) b_{n-1,k}^{(\lambda)}(x+\lambda) + f\left(\frac{k}{n+1}\right) b_{n-1,k}^{(\lambda)}(x+2\lambda)\right),\end{aligned} \tag{4.13}$$

together with the boundary conditions. Note that we have

$$\begin{aligned}&f\left(\tfrac{k+2}{n+1}\right) b_{n-1,k}^{(\lambda)}(x) - 2f\left(\tfrac{k+1}{n+1}\right) b_{n-1,k}^{(\lambda)}(x+\lambda) + f\left(\tfrac{k}{n+1}\right) b_{n-1,k}^{(\lambda)}(x+2\lambda)\\ &= \left(f\left(\tfrac{k+2}{n+1}\right) - f\left(\tfrac{k+1}{n+1}\right)\right) b_{n-1,k}^{(\lambda)}(x) + f\left(\tfrac{k+1}{n+1}\right)\left(b_{n-1,k}^{(\lambda)}(x) - b_{n-1,k}^{(\lambda)}(x+\lambda)\right)\\ &\qquad - \left(f\left(\tfrac{k+1}{n+1}\right) - f\left(\tfrac{k}{n+1}\right)\right) b_{n-1,k}^{(\lambda)}(x+\lambda) - f\left(\tfrac{k}{n+1}\right)\left(b_{n-1,k}^{(\lambda)}(x+\lambda) - b_{n-1,k}^{(\lambda)}(x+2\lambda)\right)\\ &= \Delta f\left(\tfrac{k+1}{n+1}\right) b_{n-1,k}^{(\lambda)}(x) - \lambda f\left(\tfrac{k+1}{n+1}\right)(D_\lambda b_{n-1,k}^{(\lambda)}(\cdot))(x)\\ &\qquad - \Delta f\left(\tfrac{k}{n+1}\right) b_{n-1,k}^{(\lambda)}(x+\lambda) + \lambda f\left(\tfrac{k}{n+1}\right)(D_\lambda b_{n-1,k}^{(\lambda)}(\cdot))(x+\lambda).\end{aligned} \tag{4.14}$$

Substituting (4.13) into (4.14), we obtain

$$\begin{aligned}(D_\lambda^2 B_{n+1,\lambda}(f;\cdot))(x) = c_n c_{n-1} \sum_{k=0}^{n-1} &\left(\Delta f\left(\tfrac{k+1}{n+1}\right) b_{n-1,k}^{(\lambda)}(x) - \lambda f\left(\tfrac{k+1}{n+1}\right)(D_\lambda b_{n-1,k}^{(\lambda)}(\cdot))(x)\right.\\ &\left. - \Delta f\left(\tfrac{k}{n+1}\right) b_{n-1,k}^{(\lambda)}(x+\lambda) + \lambda f\left(\tfrac{k}{n+1}\right)(D_\lambda b_{n-1,k}^{(\lambda)}(\cdot))(x+\lambda)\right),\end{aligned}$$

which proves the theorem. □

The following lemma provides a formula for the m-th derivative of the basis polynomials.

Lemma 4.5. *Let $n \geq 2$ be an integer and suppose that $0 \leq \lambda < \frac{1}{2(n-1)}$. Let $x \in I_{n,\lambda}$. Then, for every integer $m \geq 1$ with $n \geq m$, we have*

$$(D_\lambda^m b_{n+1,k}^{(\lambda)}(\cdot))(x) = A_{n,m}(\lambda) \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} b_{n+1-m,k-j}^{(\lambda)}(x + (m-j)\lambda),$$

where

$$A_{n,m}(\lambda) = c_n c_{n-1} \cdots c_{n-m-1} = \prod_{j=0}^{m-1} \frac{n+1-j}{1-(n-j)\lambda}$$

and

$$c_n = \frac{n+1}{1-n\lambda},\ c_{n-1} = \frac{n}{1-(n-1)\lambda}, \cdots,\ c_{n-(m-1)} = \frac{n-m+2}{1-(n-(m-1))\lambda}.$$

Proof. Apply D_λ repeatedly to the identity

$$(D_\lambda b_{n+1,k}^{(\lambda)}(\cdot))(x) = c_{n-1}(b_{n,k-1}^{(\lambda)}(x) - b_{n,k}^{(\lambda)}(x+\lambda)).$$

As in Lemma 4.3, we obtain

$$(D_\lambda^2 b_{n+1,k}^{(\lambda)}(\cdot))(x) = c_n c_{n-1} \left(b_{n-1,k-2}^{(\lambda)}(x) - 2b_{n-1,k-1}^{(\lambda)}(x+\lambda) + b_{n-1,k}^{(\lambda)}(x+2\lambda) \right).$$

Proceeding by induction on m, we observe that each application of D_λ contributes a factor c_{n-r} and increases the alternating binomial sum by one. □

The following theorem gives the m-th derivative representation of the degenerate Bernstein polynomial.

Theorem 4.6. *Let $n \geq 2$ be an integer and suppose that $0 \leq \lambda < \frac{1}{2(n-1)}$. Let $m \in \mathbb{N}$ with $n \geq m$, and let $x \in I_{n,\lambda}$. Then, for every $f \in C[0,1]$, we have*

$$\begin{aligned}(D_\lambda^m B_{n+1,\lambda}(f;\cdot))(x) = B_{n+1,\lambda}^{(m)}(f;x) &= A_{n,m}(\lambda) \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} \\ &\times \sum_{k=0}^{n+1-m} \Big(\Delta f\Big(\tfrac{k+((m-1)-i)}{n+1}\Big) b_{n+1-m,k}^{(\lambda)}(x+i\lambda) \\ &\qquad - \lambda f\Big(\tfrac{k+((m-1)-i)}{n+1}\Big) (D_\lambda b_{n+1-m,k}^{(\lambda)}(\cdot))(x+i\lambda) \Big),\end{aligned}$$

where $A_{n,m}(\lambda)$ is in Lemma 4.5.

Proof. From Lemmas 4.2–4.4, we have

$$(D_\lambda B_{n+1,\lambda}(f;\cdot))(x) = c_n \sum_{k=0}^{n} \Delta f\Big(\tfrac{k}{n+1}\Big) b_{n,k}^{(\lambda)}(x) - c_n \sum_{k=0}^{n} f\Big(\tfrac{k}{n+1}\Big) (D_\lambda b_{n,k}^{(\lambda)}(\cdot))(x) \tag{4.15}$$

and

$$\begin{aligned}(D_\lambda^2 B_{n+1,\lambda}(f;\cdot))(x) = c_n c_{n-1} &\sum_{k=0}^{n-1} \Big(\Delta f\Big(\tfrac{k+1}{n+1}\Big) b_{n-1,k}^{(\lambda)}(x) - \lambda f\Big(\tfrac{k+1}{n+1}\Big) (D_\lambda b_{n-1,k}^{(\lambda)}(\cdot))(x) \Big) \\ - c_n c_{n-1} &\sum_{k=0}^{n-1} \Big(\Delta f\Big(\tfrac{k}{n+1}\Big) b_{n-1,k}^{(\lambda)}(x+\lambda) - \lambda f\Big(\tfrac{k}{n+1}\Big) (D_\lambda b_{n-1,k}^{(\lambda)}(\cdot))(x+\lambda) \Big),\end{aligned} \tag{4.16}$$

where c_n and c_{n-1} are in Lemma 4.5. By the same method as in Lemmas 4.1 and 4.3, we have

$$\begin{aligned}(D_\lambda^3 b_{n+1,k}^{(\lambda)}(\cdot))(x) = c_n c_{n-1} c_{n-2}\Big[&b_{n-2,k-3}^{(\lambda)}(x) - 3\, b_{n-2,k-2}^{(\lambda)}(x+\lambda)\\ &+ 3\, b_{n-2,k-1}^{(\lambda)}(x+2\lambda) - b_{n-2,k}^{(\lambda)}(x+3\lambda)\Big],\end{aligned} \tag{4.17}$$

and hence, in the same manner as in Lemmas 4.2 and 4.4, we can see that

$$\begin{aligned}(D_\lambda^3 B_{n+1,\lambda}(f;\cdot))(x) = c_n c_{n-1} c_{n-2} \sum_{i=0}^{2} (-1)^{2-i}\binom{2}{i} \sum_{k=0}^{n-1} \Big(&\Delta f\Big(\tfrac{k+(2-i)}{n+1}\Big) b_{n-2,k}^{(\lambda)}(x+i\lambda)\\ &- \lambda f\Big(\tfrac{k+(2-i)}{n+1}\Big)(D_\lambda b_{n-2,k}^{(\lambda)}(\cdot))(x+i\lambda)\Big).\end{aligned} \tag{4.18}$$

By the same argument as in Lemmas 4.1–4.3 and (4.17), we have, for $m \in \mathbb{N}$ and $n \geq m$,

$$(D_\lambda^m b_{n+1,k}^{(\lambda)}(\cdot))(x) = c_n c_{n-1} \cdots c_{n-(m-1)} \sum_{j=0}^{m} (-1)^{m-j}\binom{m}{j} b_{n+1-m,k-j}^{(\lambda)}(x+(m-j)\lambda).$$

From (4.15), (4.16) and (4.18), we argue by induction on $m \geq 1$:

$$\begin{aligned}(D_\lambda^m B_{n+1,\lambda}(f;\cdot))(x) &= c_n c_{n-1} c_{n-2} \cdots c_{n-(m-1)} \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i}\\ &\sum_{k=0}^{n+1-m} \Big(\Delta f\Big(\tfrac{k+((m-1)-i)}{n+1}\Big) b_{n+1-m,k}^{(\lambda)}(x+i\lambda) - \lambda f\Big(\tfrac{k+((m-1)-i)}{n+1}\Big)(D_\lambda b_{n+1-m,k}^{(\lambda)}(\cdot))(x+i\lambda)\Big),\end{aligned}$$

which is our claim. □

The following theorem establishes the uniform convergence of the first derivative of degenerate Bernstein polynomials.

Theorem 4.7. *Let $f \in C^1[0,1]$. Assume that $\{\lambda_n\}_{n\geq 2}$ is a sequence such that*

$$0 \leq \lambda_n < \frac{1}{2(n-1)} \quad \text{and} \quad \lambda_n(n-1) \to 0 \quad \text{as } n \to \infty.$$

Then

$$\lim_{n\to\infty} \sup_{0<x<1} \left| B_{n,\lambda_n}^{(1)}(f;x) - f^{(1)}(x) \right| = 0.$$

Proof. Note that if $x \in (0,1)$, then $x \in I_{n,\lambda_n}$ for all sufficiently large n. Set $g := f^{(1)} \in C[0,1]$. It follows from Lemma 4.2 that, for $x \in I_{n,\lambda_n}$,

$$B_{n,\lambda_n}^{(1)}(f;x) = c_{n-1} \sum_{k=0}^{n-1} \Delta f\Big(\tfrac{k}{n}\Big) b_{n-1,k}^{(\lambda_n)}(x) - \lambda_n c_{n-1} \sum_{k=0}^{n-1} f\Big(\tfrac{k}{n}\Big)(D_{\lambda_n} b_{n-1,k}^{(\lambda_n)}(\cdot))(x), \tag{4.19}$$

where

$$\Delta f\Big(\tfrac{k}{n}\Big) = f\Big(\tfrac{k+1}{n}\Big) - f\Big(\tfrac{k}{n}\Big).$$

By the mean-value theorem, for each k, there exists $\xi_{k,n} \in [\frac{k}{n}, \frac{k+1}{n}]$ such that

$$\Delta f\Big(\frac{k}{n}\Big) = \frac{1}{n}\, g(\xi_{k,n}).$$

Hence the first term in (4.19) becomes

$$\begin{aligned} c_{n-1}\sum_{k=0}^{n-1}\Delta f\Big(\frac{k}{n}\Big)\, b^{(\lambda_n)}_{n-1,k}(x) &= \frac{c_{n-1}}{n}\sum_{k=0}^{n-1} g(\xi_{k,n})\, b^{(\lambda_n)}_{n-1,k}(x) \\ &= \frac{c_{n-1}}{n}\sum_{k=0}^{n-1} g\Big(\frac{k}{n}\Big) b^{(\lambda_n)}_{n-1,k}(x) + \frac{c_{n-1}}{n}\sum_{k=0}^{n-1}\Big(g(\xi_{k,n}) - g\Big(\frac{k}{n}\Big)\Big) b^{(\lambda_n)}_{n-1,k}(x) \\ &= \frac{c_{n-1}}{n}\widetilde{B}_{n-1,\lambda_n}(g;x) + \frac{c_{n-1}}{n}\sum_{k=0}^{n-1}\Big(g(\xi_{k,n}) - g\Big(\frac{k}{n}\Big)\Big) b^{(\lambda_n)}_{n-1,k}(x), \end{aligned}$$

where

$$\widetilde{B}_{n-1,\lambda_n}(g;x) = \sum_{k=0}^{n-1} g\Big(\frac{k}{n}\Big) b^{(\lambda_n)}_{n-1,k}(x).$$

Since g is uniformly continuous on $[0,1]$, by Definition 2.9 and Proposition 2.10(1), we have

$$|g(\xi_{k,n}) - g(\tfrac{k}{n})| \le \omega(g; |\xi_{k,n} - \tfrac{k}{n}|) \le \omega(g; \tfrac{1}{n}),$$

together with

$$\Big|\xi_{k,n} - \frac{k}{n}\Big| \le \frac{1}{n} \quad \text{for all} \;\; \xi_{k,n} \in \Big[\frac{k}{n}, \frac{(k+1)}{n}\Big],$$

which gives, by using positivity and the fact that $\sum b^{(\lambda_n)}_{n-1,k}(x) = 1$,

$$\left|\sum_{k=0}^{n-1}\Big(g(\xi_{k,n}) - g\Big(\frac{k}{n}\Big)\Big) b^{(\lambda_n)}_{n-1,k}(x)\right| \le \sum_{k=0}^{n-1}\Big|g(\xi_{k,n}) - g\Big(\frac{k}{n}\Big)\Big| b^{(\lambda_n)}_{n-1,k}(x) \le \omega\Big(g; \frac{1}{n}\Big).$$

Therefore,

$$c_{n-1}\sum_{k=0}^{n-1}\Delta f\Big(\frac{k}{n}\Big)\, b^{(\lambda_n)}_{n-1,k}(x) = \frac{c_{n-1}}{n}\,\widetilde{B}_{n-1,\lambda_n}(g;x) + O\Big(\frac{c_{n-1}}{n}\,\omega\Big(g;\frac{1}{n}\Big)\Big), \tag{4.20}$$

uniformly in $x \in I_{n,\lambda_n}$.
We estimate the second sum in (4.19). By the linearity of D_λ, we have

$$\sum_{k=0}^{n-1} f\Big(\frac{k}{n}\Big)(D_{\lambda_n} b^{(\lambda_n)}_{n-1,k}(\cdot))(x) = D_{\lambda_n}\Big(\sum_{k=0}^{n-1} f\Big(\frac{k}{n}\Big) b^{(\lambda_n)}_{n-1,k}(\cdot)\Big)(x) = (D_{\lambda_n}\widetilde{B}_{n-1,\lambda_n}(f;\cdot))(x),$$

and hence

$$\begin{aligned} \left|\lambda_n c_{n-1}\sum_{k=0}^{n-1} f\Big(\frac{k}{n}\Big)(D_{\lambda_n} b^{(\lambda_n)}_{n-1,k}(\cdot))(x)\right| &= \left|\lambda_n c_{n-1}(D_{\lambda_n}\widetilde{B}_{n-1,\lambda_n}(f;\cdot))(x)\right| \\ &= \left|c_{n-1}\Big(\widetilde{B}_{n-1,\lambda_n}(f;x+\lambda_n) - \widetilde{B}_{n-1,\lambda_n}(f;x)\Big)\right|. \end{aligned}$$

Since $\widetilde{B}_{n-1,\lambda_n}(f;\cdot)$ is continuous on $[0,1]$ and f is fixed, it is uniformly continuous, that is,

$$\sup_x |\widetilde{B}_{n-1,\lambda_n}(f;x+\lambda_n)-\widetilde{B}_{n-1,\lambda_n}(f;x)| \to 0 \quad \text{as} \quad n\to\infty,$$

and therefore

$$\sup_{x\in I_{n,\lambda_n}} \left|\lambda_n c_{n-1}\sum_{k=0}^{n-1} f\left(\tfrac{k}{n}\right)(D_{\lambda_n} b_{n-1,k}^{(\lambda_n)}(\cdot))(x)\right| \longrightarrow 0. \tag{4.21}$$

Combining (4.19)–(4.21), we obtain

$$\sup_{x\in I_{n,\lambda_n}} \left|B_{n,\lambda_n}^{(1)}(f;x)-\tfrac{c_{n-1}}{n}\widetilde{B}_{n-1,\lambda_n}(g;x)\right| \le O\left(\tfrac{c_{n-1}}{n}\,\omega\left(g;\tfrac{1}{n}\right)\right)+o(1)\longrightarrow 0 \quad \text{as} \quad n\to\infty, \tag{4.22}$$

together with

$$\tfrac{c_{n-1}}{n} = \tfrac{1}{1-(n-1)\lambda_n} \longrightarrow 1 \quad \text{as} \quad n\to\infty, \tag{4.23}$$

under the assumptions $0\le\lambda_n<\frac{1}{2(n-1)}$ and $(n-1)\lambda_n\to 0$ as $n\to\infty$, and the fact from Proposition 2.10(2) that $\omega(g;1/n)\to 0$ as $n\to\infty$. Using the triangle inequality, we write

$$\begin{aligned}
\sup_{x\in I_{n,\lambda_n}} \left|B_{n,\lambda_n}^{(1)}(f;x)-g(x)\right| &\le \sup_{x\in I_{n,\lambda_n}} \left|B_{n,\lambda_n}^{(1)}(f;x)-\frac{c_{n-1}}{n}\widetilde{B}_{n-1,\lambda_n}(g;x)\right| \\
&\quad + \sup_{x\in I_{n,\lambda_n}} \left|\left(\tfrac{c_{n-1}}{n}-1\right)\widetilde{B}_{n-1,\lambda_n}(g;x)\right| + \sup_{x\in I_{n,\lambda_n}} \left|\widetilde{B}_{n-1,\lambda_n}(g;x)-g(x)\right| \\
&=: H_1(n)+H_2(n)+H_3(n).
\end{aligned} \tag{4.24}$$

For $H_1(n)$, it follows immediately from (4.22) that $H_1(n)\to 0$ as $n\to\infty$.
For $H_2(n)$, since g is continuous on $[0,1]$, we have $|g(x)|\le\|g\|_\infty$ for all $x\in[0,1]$. By the positivity of $\widetilde{B}_{n-1,\lambda_n}$ and the identity $\sum_{k=0}^{n-1} b_{n-1,k}^{(\lambda_n)}(x)=1$, we obtain

$$|\widetilde{B}_{n-1,\lambda_n}(g;x)| \le \widetilde{B}_{n-1,\lambda_n}(|g|;x) \le \widetilde{B}_{n-1,\lambda_n}(\|g\|_\infty;x) = \|g\|_\infty.$$

Hence, by (4.23),

$$\sup_{x\in I_{n,\lambda_n}} \left|\left(\tfrac{c_{n-1}}{n}-1\right)\widetilde{B}_{n-1,\lambda_n}(g;x)\right| \le \left|\tfrac{c_{n-1}}{n}-1\right|\|g\|_\infty \longrightarrow 0 \quad \text{as } n\to\infty.$$

For $H_3(n)$, by Theorem 2.11, the operator $\widetilde{B}_{n-1,\lambda}$ approximates the identity uniformly on I_{n,λ_n}; that is,

$$\sup_{x\in I_{n,\lambda_n}} \left|\widetilde{B}_{n-1,\lambda_n}(g;x)-g(x)\right| \longrightarrow 0 \quad \text{as } n\to\infty.$$

Combining the above three estimates in (4.24), we conclude that

$$\sup_{x\in I_{n,\lambda_n}} \left|B_{n,\lambda_n}^{(1)}(f;x)-g(x)\right| \longrightarrow 0 \quad \text{as } n\to\infty,$$

that is,

$$\lim_{n\to\infty}\sup_{0<x<1} \left|B_{n,\lambda_n}^{(1)}(f;x)-f^{(1)}(x)\right| = 0.$$

This completes the proof. □

Remark 4.8. *Let*

$$c_n = \frac{n+1}{1-n\lambda_n}, \qquad A_{n,m}(\lambda_n) = \prod_{j=0}^{m-1} c_{n-j},$$

where m is a fixed positive integer, $\{\lambda_n\}$ satisfies the admissibility condition $0 \le (n-1)\lambda_n < 1/2$ and $\lambda_n(n-1) \to 0 \quad as \quad n \to \infty$. The coefficients $A_{n,m}(\lambda_n)$ describe the leading growth of the m-th λ-difference structure in the derivative of the degenerate Bernstein polynomials. Their asymptotic behavior is

$$\lim_{n\to\infty} \frac{A_{n,m}(\lambda_n)}{n} = \begin{cases} 1, & m = 1, \\ +\infty, & m \ge 2. \end{cases}$$

Hence, let $f \in C^{(m)}[0,1]$, and then the uniform convergence

$$\lim_{n\to\infty} \sup_{x \in I_{n,\lambda_n}} \left| B^{(m)}_{n,\lambda_n}(f;x) - f^{(m)}(x) \right| = 0$$

holds only for the case $m = 1$ (Theorem 4.7). Consequently, the degenerate Bernstein operator preserves first-order differentiability in the limit, but higher-order differentiability is not automatically guaranteed without further normalization or structural conditions.

5. Numerical results

In this section, we present numerical examples for the degenerate Bernstein operator and its first λ-difference. The purpose of this section is to show both numerically and graphically that the degenerate Bernstein operator shows better convergence for approximation as compared to the classical Bernstein operator as explained in earlier sections. To show the convergence results, we take a test function $f(x)$ and its first derivative $f'(x)$ defined in (5.1).

Example 5.1.

$$f(x) = \sin(\pi x), \qquad f'(x) = \pi \cos(\pi x), \tag{5.1}$$

and we will use the following values of the scale levels n and λ,

$$n = 3, 5, 7, \text{ and } 9 \qquad \lambda = 0, 0.1, 0.2, 0.3, \text{ and } 0.4. \tag{5.2}$$

Case $\lambda = 0$ in (5.2) *corresponds to the classical Bernstein polynomials.*

In this example, we will use the results defined in (2.1), (2.2), and Lemma 4.2. Before explaining and showing the results, we first explain some terminologies that we will use in discussions.

We say that a pair (n, λ) is *admissible* when

$$0 \le \lambda < \frac{1}{2(n-1)},$$

so that the interval $I_{n,\lambda}$ is nonempty and the operator in (2.1) and the first λ-difference in (4.19) are considered on their natural domain.

We say that a pair (n, λ) is *formal* when the algebraic expression is still defined but the admissibility conditions fail.

We call a pair (n,λ) *undefined* when

$$(1)_{n,\lambda}=0 \quad \text{or} \quad 1-(n-1)\lambda=0,$$

so the operator or its first λ-difference cannot be evaluated.

In both numerical and graphical interpretations, we will present both admissible and formal cases to test their impact on the results. The pairs $(7,0.2)$ and $(9,0.2)$ are undefined because

$$(1)_{7,0.2}=0, \qquad (1)_{9,0.2}=0.$$

Therefore, we will not do the numerical calculations and graphical representations for these two pairs.

Now we first calculate the numerical expressions for the test function $f(x)$, defined in (5.1). All the cases of the values of the scale levels n and λ are considered.

For $n=3$. The degenerate Bernstein basis polynomials, defined in (2.2), can be written as

$$b_{3,k}^{(\lambda)}(x)=\binom{3}{k}\frac{(x)_{k,\lambda}(1-x)_{3-k,\lambda}}{(1-\lambda)(1-2\lambda)}, \qquad k=0,1,2,3. \tag{5.3}$$

The degenerate Bernstein operator $B_{n,\lambda}(f;x)$ (defined in (2.1)) for the test function $f(x)=\sin(\pi x)$ and $n=3$ is given by

$$B_{3,\lambda}(\sin(\pi x);x)=\frac{1}{(1-\lambda)(1-2\lambda)}\Big[3(x)_{1,\lambda}(1-x)_{2,\lambda}\frac{\sqrt{3}}{2}+3(x)_{2,\lambda}(1-x)_{1,\lambda}\frac{\sqrt{3}}{2}\Big]. \tag{5.4}$$

We evaluate (5.4) for all λ values given in (5.2) as

$$\begin{aligned}
B_{3,0.1}(\sin(\pi x);x)&=\frac{1}{(0.9)(0.8)}\Big[3(x)_{1,0.1}(1-x)_{2,0.1}\frac{\sqrt{3}}{2}+3(x)_{2,0.1}(1-x)_{1,0.1}\frac{\sqrt{3}}{2}\Big],\\
B_{3,0.2}(\sin(\pi x);x)&=\frac{1}{(0.8)(0.6)}\Big[3(x)_{1,0.2}(1-x)_{2,0.2}\frac{\sqrt{3}}{2}+3(x)_{2,0.2}(1-x)_{1,0.2}\frac{\sqrt{3}}{2}\Big],\\
B_{3,0.3}(\sin(\pi x);x)&=\frac{1}{(0.7)(0.4)}\Big[3(x)_{1,0.3}(1-x)_{2,0.3}\frac{\sqrt{3}}{2}+3(x)_{2,0.3}(1-x)_{1,0.3}\frac{\sqrt{3}}{2}\Big],\\
B_{3,0.4}(\sin(\pi x);x)&=\frac{1}{(0.6)(0.2)}\Big[3(x)_{1,0.4}(1-x)_{2,0.4}\frac{\sqrt{3}}{2}+3(x)_{2,0.4}(1-x)_{1,0.4}\frac{\sqrt{3}}{2}\Big].
\end{aligned} \tag{5.5}$$

The graphical representation of all the numerical calculations in (5.5) is shown in Figure 1(a).

For $n=5$, the degenerate Bernstein basis polynomials, defined in (2.2), for $n=5$ can be written as

$$b_{5,k}^{(\lambda)}(x)=\binom{5}{k}\frac{(x)_{k,\lambda}(1-x)_{5-k,\lambda}}{(1-\lambda)(1-2\lambda)(1-3\lambda)(1-4\lambda)}, \qquad k=0,1,2,3,4,5. \tag{5.6}$$

The degenerate Bernstein operator $B_{n,\lambda}(f;x)$ for $n=5$ is given by

$$\begin{aligned}
B_{5,\lambda}(\sin(\pi x);x)=&\frac{1}{(1-\lambda)(1-2\lambda)(1-3\lambda)(1-4\lambda)}\Big[5(x)_{1,\lambda}(1-x)_{4,\lambda}(0.5878)\\
&+10(x)_{2,\lambda}(1-x)_{3,\lambda}(0.9511)+10(x)_{3,\lambda}(1-x)_{2,\lambda}(0.9511)+5(x)_{4,\lambda}(1-x)_{1,\lambda}(0.5878)\Big].
\end{aligned} \tag{5.7}$$

We evaluate (5.7) for all λ values given in (5.2) as

$$\begin{aligned}
B_{5,0.1}(\sin(\pi x);x) &= \frac{1}{(0.9)(0.8)(0.7)(0.6)}\Big[5(x)_{1,0.1}(1-x)_{4,0.1}(0.5878)\\
&\quad +10(x)_{2,0.1}(1-x)_{3,0.1}(0.9511)+10(x)_{3,0.1}(1-x)_{2,0.1}(0.9511)\\
&\quad +5(x)_{4,0.1}(1-x)_{1,0.1}(0.5878)\Big],\\
B_{5,0.2}(\sin(\pi x);x) &= \frac{1}{(0.8)(0.6)(0.4)(0.2)}\Big[5(x)_{1,0.2}(1-x)_{4,0.2}(0.5878)\\
&\quad +10(x)_{2,0.2}(1-x)_{3,0.2}(0.9511)+10(x)_{3,0.2}(1-x)_{2,0.2}(0.9511)\\
&\quad +5(x)_{4,0.2}(1-x)_{1,0.2}(0.5878)\Big],\\
B_{5,0.3}(\sin(\pi x);x) &= \frac{1}{(0.7)(0.4)(0.1)(-0.2)}\Big[5(x)_{1,0.3}(1-x)_{4,0.3}(0.5878)\\
&\quad +10(x)_{2,0.3}(1-x)_{3,0.3}(0.9511)+10(x)_{3,0.3}(1-x)_{2,0.3}(0.9511)\\
&\quad +5(x)_{4,0.3}(1-x)_{1,0.3}(0.5878)\Big],\\
B_{5,0.4}(\sin(\pi x);x) &= \frac{1}{(0.6)(0.2)(-0.2)(-0.6)}\Big[5(x)_{1,0.4}(1-x)_{4,0.4}(0.5878)\\
&\quad +10(x)_{2,0.4}(1-x)_{3,0.4}(0.9511)+10(x)_{3,0.4}(1-x)_{2,0.4}(0.9511)\\
&\quad +5(x)_{4,0.4}(1-x)_{1,0.4}(0.5878)\Big].
\end{aligned} \tag{5.8}$$

The graphical representation of all the numerical calculations in (5.8) is shown in Figure 1(b).

Similarly, the test function $f(x)$ can be represented in terms of degenerate Bernstein polynomials for $n = 7, 9$ and $\lambda = 0, 0.1, 0.2, 0.3, 0.4$. However, for brevity, we are showing just the graphical interpretations for cases $n = 7$ and $n = 9$ in Figure 1(c) and Figure 1(d), respectively.

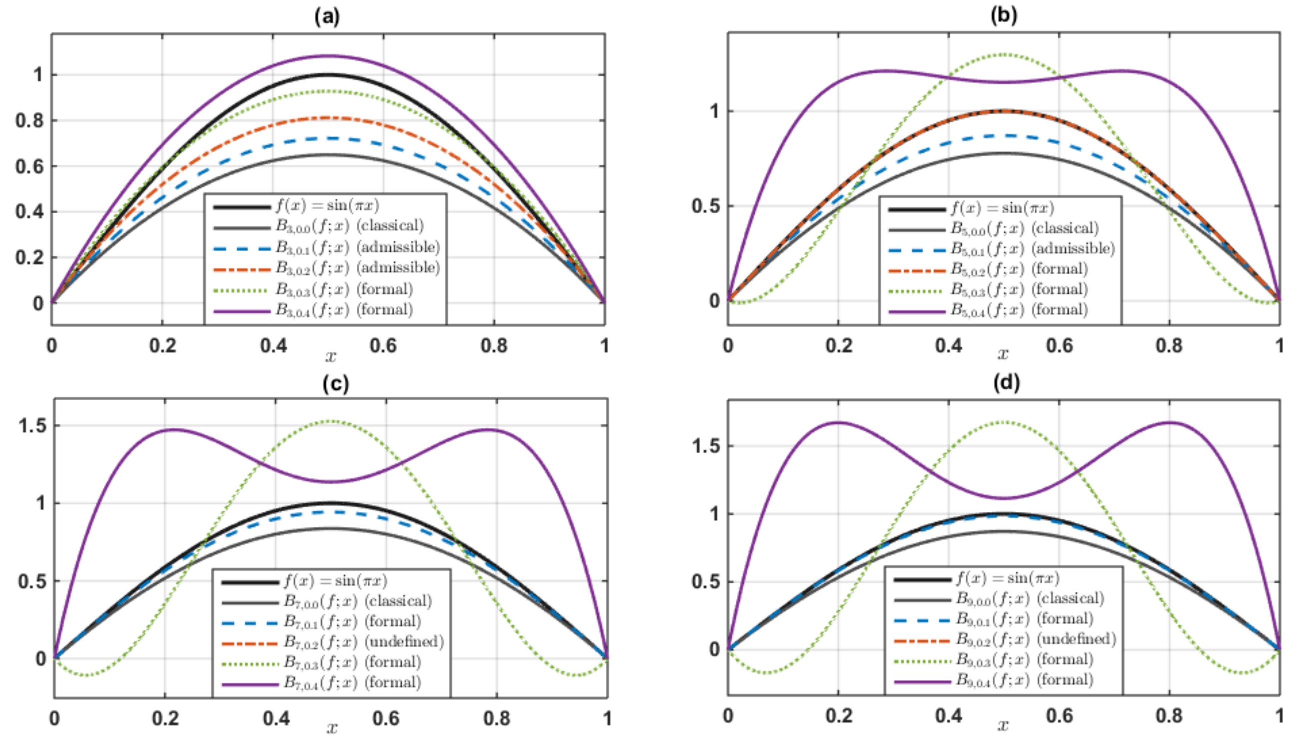

Figure 1. The test function and its approximation for $n = 3, 5, 7, 9$ and $\lambda = 0, 0.1, 0.2, 0.3, 0.4$, where $\lambda = 0$ denotes the classical Bernstein polynomials.

We now express the first-order derivative $f'(x) = \pi\cos(\pi x)$ of the test function in terms of degenerate Bernstein polynomials for different cases of scale level n and λ values. For this, we will use the result defined in Lemma 4.2.

Case $n = 3$. For $n = 3$, we can write (4.8) as

$$B^{(1)}_{3,\lambda}(\sin(\pi x); x) = \frac{3}{1-2\lambda}\Big[\frac{\sqrt{3}}{2}b^{(\lambda)}_{2,0}(x) - \frac{\sqrt{3}}{2}b^{(\lambda)}_{2,2}(x)\Big] - \frac{3\lambda}{1-2\lambda}\Big[\frac{\sqrt{3}}{2}D_\lambda b^{(\lambda)}_{2,1}(x) + \frac{\sqrt{3}}{2}D_\lambda b^{(\lambda)}_{2,2}(x)\Big]. \tag{5.9}$$

We evaluate (5.9) for all λ values given in (5.2) as

$$\begin{aligned}
B^{(1)}_{3,0.1}(\sin(\pi x); x) &= \frac{3}{0.8}\Big[\frac{\sqrt{3}}{2}b^{(0.1)}_{2,0}(x) - \frac{\sqrt{3}}{2}b^{(0.1)}_{2,2}(x)\Big] - \frac{3(0.1)}{0.8}\Big[\frac{\sqrt{3}}{2}D_{0.1}b^{(0.1)}_{2,1}(x) + \frac{\sqrt{3}}{2}D_{0.1}b^{(0.1)}_{2,2}(x)\Big],\\
B^{(1)}_{3,0.2}(\sin(\pi x); x) &= \frac{3}{0.6}\Big[\frac{\sqrt{3}}{2}b^{(0.2)}_{2,0}(x) - \frac{\sqrt{3}}{2}b^{(0.2)}_{2,2}(x)\Big] - \frac{3(0.2)}{0.6}\Big[\frac{\sqrt{3}}{2}D_{0.2}b^{(0.2)}_{2,1}(x) + \frac{\sqrt{3}}{2}D_{0.2}b^{(0.2)}_{2,2}(x)\Big],\\
B^{(1)}_{3,0.3}(\sin(\pi x); x) &= \frac{3}{0.4}\Big[\frac{\sqrt{3}}{2}b^{(0.3)}_{2,0}(x) - \frac{\sqrt{3}}{2}b^{(0.3)}_{2,2}(x)\Big] - \frac{3(0.3)}{0.4}\Big[\frac{\sqrt{3}}{2}D_{0.3}b^{(0.3)}_{2,1}(x) + \frac{\sqrt{3}}{2}D_{0.3}b^{(0.3)}_{2,2}(x)\Big],\\
B^{(1)}_{3,0.4}(\sin(\pi x); x) &= \frac{3}{0.2}\Big[\frac{\sqrt{3}}{2}b^{(0.4)}_{2,0}(x) - \frac{\sqrt{3}}{2}b^{(0.4)}_{2,2}(x)\Big] - \frac{3(0.4)}{0.2}\Big[\frac{\sqrt{3}}{2}D_{0.4}b^{(0.4)}_{2,1}(x) + \frac{\sqrt{3}}{2}D_{0.4}b^{(0.4)}_{2,2}(x)\Big].
\end{aligned} \tag{5.10}$$

The graphical representation of all the numerical calculations in (5.10) is shown in Figure 2(a).

Case $n = 5$. For $n = 5$, we can write (4.8) as

$$\begin{aligned}B^{(1)}_{5,\lambda}(\sin(\pi x); x) = \frac{5}{1-4\lambda}\Big[&0.5878 b^{(\lambda)}_{4,0}(x) + \big(0.9511 - 0.5878\big) b^{(\lambda)}_{4,1}(x) - \big(0.9511 - 0.5878\big) b^{(\lambda)}_{4,3}(x)\\ &- 0.5878 b^{(\lambda)}_{4,4}(x)\Big] - \frac{5\lambda}{1-4\lambda}\Big[0.5878 D_\lambda b^{(\lambda)}_{4,1}(x)\\ &+ 0.9511 D_\lambda b^{(\lambda)}_{4,2}(x) + 0.9511 D_\lambda b^{(\lambda)}_{4,3}(x) + 0.5878 D_\lambda b^{(\lambda)}_{4,4}(x)\Big].\end{aligned} \tag{5.11}$$

We evaluate (5.11) for all λ values given in (5.2) as

$$\begin{aligned}B^{(1)}_{5,0.1}(\sin(\pi x); x) &= \frac{5}{0.6}\Big[0.5878 b^{(0.1)}_{4,0}(x) + \big(0.9511 - 0.5878\big) b^{(0.1)}_{4,1}(x) - \big(0.9511 - 0.5878\big) b^{(0.1)}_{4,3}(x)\\ &\quad - 0.5878 b^{(0.1)}_{4,4}(x)\Big] - \frac{5(0.1)}{0.6}\Big[0.5878 D_{0.1} b^{(0.1)}_{4,1}(x)\\ &\quad + 0.9511 D_{0.1} b^{(0.1)}_{4,2}(x) + 0.9511 D_{0.1} b^{(0.1)}_{4,3}(x) + 0.5878 D_{0.1} b^{(0.1)}_{4,4}(x)\Big],\\[2ex]
B^{(1)}_{5,0.2}(\sin(\pi x); x) &= \frac{5}{0.2}\Big[0.5878 b^{(0.2)}_{4,0}(x) + \big(0.9511 - 0.5878\big) b^{(0.2)}_{4,1}(x) - \big(0.9511 - 0.5878\big) b^{(0.2)}_{4,3}(x)\\ &\quad - 0.5878 b^{(0.2)}_{4,4}(x)\Big] - \frac{5(0.2)}{0.2}\Big[0.5878 D_{0.2} b^{(0.2)}_{4,1}(x)\\ &\quad + 0.9511 D_{0.2} b^{(0.2)}_{4,2}(x) + 0.9511 D_{0.2} b^{(0.2)}_{4,3}(x) + 0.5878 D_{0.2} b^{(0.2)}_{4,4}(x)\Big],\\[2ex]
B^{(1)}_{5,0.3}(\sin(\pi x); x) &= \frac{5}{-0.2}\Big[0.5878 b^{(0.3)}_{4,0}(x) + \big(0.9511 - 0.5878\big) b^{(0.3)}_{4,1}(x) - \big(0.9511 - 0.5878\big) b^{(0.3)}_{4,3}(x)\\ &\quad - 0.5878 b^{(0.3)}_{4,4}(x)\Big] - \frac{5(0.3)}{-0.2}\Big[0.5878 D_{0.3} b^{(0.3)}_{4,1}(x)\\ &\quad + 0.9511 D_{0.3} b^{(0.3)}_{4,2}(x) + 0.9511 D_{0.3} b^{(0.3)}_{4,3}(x) + 0.5878 D_{0.3} b^{(0.3)}_{4,4}(x)\Big],\\[2ex]
B^{(1)}_{5,0.4}(\sin(\pi x); x) &= \frac{5}{-0.6}\Big[0.5878 b^{(0.4)}_{4,0}(x) + \big(0.9511 - 0.5878\big) b^{(0.4)}_{4,1}(x) - \big(0.9511 - 0.5878\big) b^{(0.4)}_{4,3}(x)\\ &\quad - 0.5878 b^{(0.4)}_{4,4}(x)\Big] - \frac{5(0.4)}{-0.6}\Big[0.5878 D_{0.4} b^{(0.4)}_{4,1}(x)\\ &\quad + 0.9511 D_{0.4} b^{(0.4)}_{4,2}(x) + 0.9511 D_{0.4} b^{(0.4)}_{4,3}(x) + 0.5878 D_{0.4} b^{(0.4)}_{4,4}(x)\Big].\end{aligned} \tag{5.12}$$

The graphical representation of all the numerical calculations in (5.12) is shown in Figure 2(b).

Similarly, the function $f'(x)$ can be represented in terms of degenerate Bernstein polynomials for $n = 7, 9$ and $\lambda = 0, 0.1, 0.2, 0.3, 0.4$. However, we are showing just the graphical interpretations for cases $n = 7$ and $n = 9$ in Figure 2(c) and Figure 2(d), respectively.

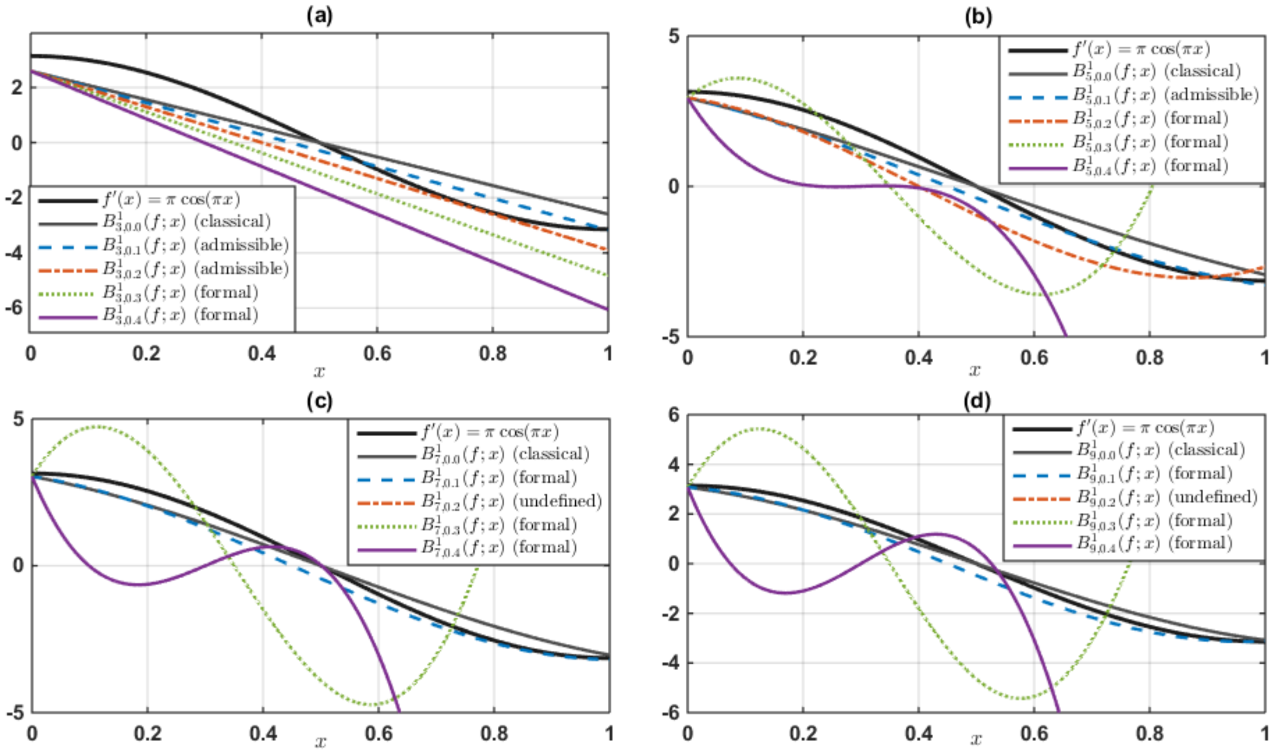

Figure 2. Graphical interpretation of the first order derivative $f'(x) = \pi\cos(\pi x)$ and its approximations for $n = 3, 5, 7, 9$ and $\lambda = 0, 0.1, 0.2, 0.3, 0.4$, where $\lambda = 0$ denotes the classical Bernstein polynomials.

Discussion

In Figures 1 and 2, test function and its first derivative, defined in (5.1), and their approximations in terms of degenerate Bernstein polynomials for scale levels $n = 3, 5, 7, 9$ and $\lambda = 0, 0.1, 0.2, 0.3, 0.4$ are shown. In Figure 1, the solid black curve shows the exact test function $f(x)$, the curve for the pair $(n, 0)$ corresponds to its approximation with classical Bernstein polynomials, and all other curves show its approximations with degenerate Bernstein polynomials at selected scale level n and λ values. Similarly, in Figure 2, the solid black curve shows the exact first derivative $f'(x)$ of the test function, where the curve for the pair $(n, 0)$ corresponds to its approximation with classical Bernstein polynomials, and all other curves show its approximations with degenerate Bernstein polynomials.

In Figure 1(a), the approximation results for the test function $f(x)$ are shown at $n = 3$ and for selected λ values. For $n = 3$, the pairs $(3, 0.1)$, $(3, 0.2)$ are *admissible*, the pairs $(3, 0.3)$, $(3, 0.4)$ are *formal*, and the pair $(3, 0)$ corresponds to approximation with classical Bernstein polynomials. We can observe that the approximation for the classical case ($\lambda = 0$) is weak compared to admissible and formal pairs, which shows the effectiveness of degenerate Bernstein polynomials. It is also shown that the approximation results for the classical case $\lambda = 0$ are even weaker than the formal cases.

In Figure 1(b), the approximation results are shown for scale level $n = 5$, and all the considered λ values. In this case, only the pair $(5, 0.1)$ is *admissible*, and the remaining cases are *formal*. We see the different pattern in this case: The approximation results are good for the admissible pair and the formal pair where $\lambda = 0.2$, which is closer to the admissible value. The formal pairs $(5, 0.3)$ and $(5, 0.4)$ do not show good approximation for $n = 5$ because in these pairs, the λ values are significant larger than admissible λ values. We can also see that the approximation with classical Bernstein polynomials for $n = 5$ is better than the scale level ($n = 3$).

In Figure 1(c), the approximation results are shown for scale level $n = 7$, and all the selected λ values. In this case, there is no admissible pair, three pairs are formal, and the pair $(7, 0.2)$ is undefined. No curve is shown for the undefined case. The formal pair $(7, 0.1)$ shows better results than all other pairs including the classical one.

In Figure 1(d), the approximation results are shown for scale level $n = 9$, and all the selected λ values. In this case, there is also no admissible pair, three pairs are formal, and the pair $(9, 0.2)$ is undefined. The formal pair $(9, 0.1)$ shows better results than all other pairs including the classical one.

From this result, we conclude that the admissible pairs and those formal pairs in which the λ value is closer to the admissible case show better approximation results than classical Bernstein polynomials. The test function approximation results are consistent with Theorem 2.11, which shows that $B_{n,\lambda_n}(f; x)$ converges uniformly to $f(x)$ under the admissibility assumptions.

In Figure 2(a), we have the approximation results for the first-order derivative $f'(x)$ at $n = 3$ for selected λ values. For $n = 3$, the pairs $(3, 0.1)$, $(3, 0.2)$ are admissible, the pairs $(3, 0.3)$, $(3, 0.4)$ are formal, and the pair $(3, 0)$ corresponds to approximation with classical Bernstein polynomials. We can see that the results are consistent with Theorem 4.7, which gives the corresponding convergence for the first derivative when $(n - 1)\lambda_n \to 0$. Therefore, we can observe that the approximation of the first-order derivative is more accurate than all other cases. The reason is that as λ increases, according to the convergence condition $(3 - 1)\lambda$, the convergence becomes weaker. The same pattern can be observed for all other cases for $n = 5, 7, 9$ as shown in Figures 2(b)–(d). All these results show that the first-order derivative approximation with classical Bernstein polynomials is better than with degenerate Bernstein polynomials.

6. Conclusions

In this paper, we studied the approximation behavior of degenerate Bernstein polynomials on the admissible interval. We established the basic properties of the operator, including positivity, linearity, and moment formulas, and then proved the uniform convergence of $B_{n,\lambda}(f; x)$ to $f(x)$. We also developed forward-difference and λ-difference representations for the basis functions and the corresponding degenerate Bernstein polynomials. Uniform convergence of the first derivative under the condition that $\lambda_n \to 0$ with the admissibility assumptions was also proved.

The numerical examples for $f(x) = \sin(\pi x)$ and $f'(x) = \pi \cos(\pi x)$ support the theoretical results. The function approximation shows that admissible pairs, and some formal pairs close to the admissible range, can provide better results than the classical Bernstein case. On the other hand, for the first derivative, the degenerate approximation becomes weaker as λ increases.

Author contributions

All authors contributed equally to this work. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest.

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