Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(7): 19808–19823.
DOI: 10.3934/math.2026804
Received: 22 March 2026
Revised: 11 June 2026
Accepted: 22 June 2026
Published: 07 July 2026

Research article

On convergence of some normalized extreme order statistics

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Abstract: We investigate normalized extreme order statistics for distributions supported on a finite interval. Under local endpoint assumptions of the form $f(x) \sim C(b-x)^{r-1}$ $(x \to b-)$, and its left-endpoint analogue, we prove that the normalized sample maximum and minimum are uniformly integrable in every positive power. As a consequence, weak convergence upgrades to convergence of all positive moments. We then establish the joint weak convergence of the normalized lower and upper extremes, and prove that they are asymptotically independent. This leads to the limit law of their difference and an explicit asymptotic variance formula. These results are applied to the sample midrange in symmetric bounded location families. In particular, when the endpoint exponent satisfies $1 \leq r < 2$, the sample midrange has asymptotic variance of order $n^{-2/r}$ and is therefore asymptotically more efficient than the sample mean and interior sample quantiles, whose variances are typically of order n^{-1}. We also obtain the asymptotic law of the sample range as a natural companion result. The main statistical application remains the sample midrange as an estimator of the center parameter. Semicircular and geometric localization examples are discussed, and a simulation study confirms the asymptotic results.

Keywords: extreme order statistics; moment convergence; sample midrange; sample range; uniform integrability; Weibull domain of attraction
Mathematics Subject Classification: 62F10, 62G30

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with common distribution function F supported on a finite interval $[a, b]$, where $-\infty < a < b < \infty$. Denote the associated order statistics by

$$X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}.$$

Throughout the paper, in addition to using $\bar{X}_n$ and M_n to denote the conventional sample mean and sample median, we also use

$$R_n = \frac{X_{1:n} + X_{n:n}}{2},$$

to represent the sample midrange (see [9]).

Extreme order statistics on bounded supports belong naturally to the Weibull domain of attraction. A standard theorem states that if the right endpoint

$$b_F := \sup\{x : F(x) < 1\}$$

is finite and if there exists $r > 0$ such that

$$\lim_{t\to\infty} \frac{1 - F(b_F - x/t)}{1 - F(b_F - 1/t)} = x^r, \qquad x > 0,$$

then an appropriately normalized sample maximum converges in distribution to a Weibull-type limit; see, for instance, Arnold, Balakrishnan and Nagaraja [2]. The general extreme-value and regular-variation background is classical; see Resnick [10] and de Haan and Ferreira [7]. Moment convergence and uniform integrability for normalized maxima have also been studied, for example, by Leadbetter, Lindgren and Rootzén [8] and DasGupta [6]. Midrange-type statistics have a separate history; see Rider [11], Broffitt [5], Akhundov, Balakrishnan and Nevzorov [1], and the closely related asymptotic results of Barakat and Nigm [3] and Barakat, Nigm and Elsawah [4].

For bounded models, such limits provide the correct first-order description of extremes. However, weak convergence alone does not imply convergence of moments, which is often the quantity needed in statistical applications. For the special case of positive endpoint densities, Wang, Deng, Li and Zhou [14] obtained asymptotic formulas for variances and covariances of certain extreme order statistics. The present paper extends that perspective in two directions. First, we allow general endpoint exponents $r, r^* \geq 1$, thereby covering a broader class of bounded densities with polynomial endpoint behavior. Second, instead of focusing only on second moments, we prove uniform integrability of the normalized extremes in every positive power and consequently obtain convergence of all positive moments. Compared with classical domain-of-attraction results that typically focus on one-sided limits (Resnick [10], de Haan & Ferreira [7]), our contribution is a two-sided bounded-support formulation derived directly from local endpoint-density conditions—exactly the structure needed for the joint analysis of minimum, maximum, midrange and range. Relative to the uniform-integrability treatments in DasGupta [6] and Leadbetter, Lindgren and Rootzén [8], we give a self-contained argument tailored to this local density setting and use it to obtain convergence of all positive moments for both endpoint gaps simultaneously. In contrast to the broad catalogue of extremal functionals in Barakat and Nigm [3] and Barakat, Nigm and Elsawah [4], our emphasis is on a unified endpoint condition that yields joint asymptotic independence and explicit variance consequences for the sample midrange and, as a by-product, the sample range. Unlike earlier studies that treat only the maximum or minimum under Weibull-type conditions (see, e.g., [2, 10]), the present work simultaneously handles both endpoints of the support in symmetric bounded models. This two-ended formulation is the specific novelty of the paper.

To state the limit laws, let $X_{e,1}, X_{e,2}, Y_e$ be mutually independent random variables such that $X_{e,1}$ and $X_{e,2}$ have cumulative distribution function

$$F_e(x) = (1 - e^{-x^r})\mathbf{1}_{[0,\infty)}(x),$$

while Y_e has cumulative distribution function

$$F_e^*(y) = (1 - e^{-y^{r^*}})\mathbf{1}_{[0,\infty)}(y).$$

These are Weibull-type limit variables supported on $[0, \infty)$.

Our main contributions are as follows:

(i) Under endpoint density conditions, we prove that the normalized sample maximum and minimum converge in distribution and in every positive moment.
(ii) We establish joint convergence of the normalized lower and upper extremes, prove their asymptotic independence, and derive the limiting distribution and asymptotic variance of their difference.
(iii) For symmetric bounded location families, we obtain the asymptotic law and variance of the sample midrange. In particular, if the density near the boundary behaves like $(b - x)^{r-1}$ with $1 \leq r < 2$, then the sample midrange is asymptotically more efficient than the sample mean and the sample median.
(iv) As a by-product, we derive the corresponding first-order limit theorem and variance asymptotics for the sample range.

The restriction $r \geq 1$ is used because the present proof assumes a continuous density on the closed support. If $0 < r < 1$, the density has an integrable singularity at the endpoint and is no longer continuous at the boundary. We expect analogous conclusions under modified local assumptions, but such singular endpoint models require a different technical treatment and are left for future work.

The paper is organized as follows. Section 2 contains the main results. Auxiliary lemmas are collected in Section 3. Proofs are presented in Section 4. Section 5 provides a simulation study that corroborates the asymptotic theory. Applications and illustrations are discussed in Section 6, followed by a brief conclusion in Section 7.

2. Main results

We start with the right endpoint.

Theorem 2.1. *Let X have support $[a, b]$, where $-\infty < a < b < \infty$, and suppose that X has a continuous density f on $[a, b]$. Assume that there exist constants $C > 0$ and $r \geq 1$ such that*

$$\lim_{x \to b-} \frac{f(x)}{(b - x)^{r-1}} = C. \tag{2.1}$$

Then, for every $w > 0$,

$$(b - X_{n:n})\left(\frac{nC}{r}\right)^{1/r} \xrightarrow{d} X_{e,1}, \tag{2.2}$$

and

$$\lim_{n\to\infty} E\left[\left\{(b - X_{n:n})(nC/r)^{1/r}\right\}^w\right] = E(X_{e,1}^w) = \Gamma\left(1 + \frac{w}{r}\right). \tag{2.3}$$

Corollary 2.2. *Let X have support $[a, b]$ and a continuous density f on $[a, b]$. Assume that there exist constants $C^* > 0$ and $r^* \geq 1$ such that*

$$\lim_{x \to a+} \frac{f(x)}{(x - a)^{r^*-1}} = C^*. \tag{2.4}$$

Then, for every $w > 0$,

$$(X_{1:n} - a)\left(\frac{nC^*}{r^*}\right)^{1/r^*} \xrightarrow{d} Y_e, \tag{2.5}$$

and

$$\lim_{n\to\infty} E\left[\left\{(X_{1:n} - a)(nC^*/r^*)^{1/r^*}\right\}^w\right] = E(Y_e^w) = \Gamma\left(1 + \frac{w}{r^*}\right). \tag{2.6}$$

Theorem 2.3. *Let X have support $[a, b]$ and a continuous density f on $[a, b]$. Assume that* (2.1) *and* (2.4) *hold for some constants $C, C^* > 0$ and exponents $r, r^* \geq 1$. Then*

$$(X_{1:n} - a)\left(\frac{nC^*}{r^*}\right)^{1/r^*} - (b - X_{n:n})\left(\frac{nC}{r}\right)^{1/r} \xrightarrow{d} Y_e - X_{e,1}, \tag{2.7}$$

and

$$\lim_{n\to\infty} \mathrm{Var}\left((X_{1:n} - a)(nC^*/r^*)^{1/r^*} - (b - X_{n:n})(nC/r)^{1/r}\right) \tag{2.8}$$

$$= \mathrm{Var}(Y_e - X_{e,1}) = \Gamma\left(1 + \frac{2}{r^*}\right) + \Gamma\left(1 + \frac{2}{r}\right) - \Gamma\left(1 + \frac{1}{r^*}\right)^2 - \Gamma\left(1 + \frac{1}{r}\right)^2. \tag{2.9}$$

Remark 2.4. *The separate convergences in* (2.2) *and Corollary 2.2 do not automatically imply* (2.7)*. A joint convergence argument is needed. In the proof we shall also show that the two limiting variables are independent.*

We now turn to the sample midrange. Let f_b be a continuous even density supported on $[-b, b]$, and consider the location family

$$\mathcal{F}_b = \{f(x, \theta) : f(x, \theta) = f_b(x - \theta),\ \theta \in \mathbb{R}\}.$$

Corollary 2.5. *Suppose that $X \sim f(x, \theta) \in \mathcal{F}_b$ and that*

$$\lim_{x\to b-} \frac{f_b(x)}{(b - x)^{r-1}} = C \tag{2.10}$$

for some $C > 0$ and $r \geq 1$. Then the sample midrange $R_n = (X_{1:n} + X_{n:n})/2$ satisfies

$$(R_n - \theta)(nC)^{1/r} \xrightarrow{d} \frac{r^{1/r}}{2}(X_{e,1} - X_{e,2}), \tag{2.11}$$

where $X_{e,1}$ and $X_{e,2}$ are independent and identically distributed with cdf F_e. Moreover,

$$\mathrm{Var}(R_n) \sim \frac{r^{2/r}}{2(nC)^{2/r}}\left[\Gamma\left(1 + \frac{2}{r}\right) - \Gamma\left(1 + \frac{1}{r}\right)^2\right]. \tag{2.12}$$

In particular, $\mathrm{Var}(R_n) = O(n^{-2/r})$.

Remark 2.6. *For symmetric bounded distributions, both $\bar{X}_n$ and R_n are unbiased for the center θ; the same is true for M_n whenever the median is uniquely determined by symmetry. Since interior sample quantiles and the sample mean typically have variances of order n^{-1}, Corollary 2.5 implies that when $1 \leq r < 2$, the sample midrange is asymptotically more efficient than the sample mean and the sample median.*

Corollary 2.7. *Under the assumptions of Corollary 2.5, if in addition*

$$\lim_{x\to b-} f_b(x) = C > 0,$$

then $r = 1$ and

$$\mathrm{Var}(R_n) \sim \frac{1}{2(nC)^2}.$$

For the uniform model $X \sim U[\theta - b, \theta + b]$, where $C = 1/(2b)$, this reduces to

$$\mathrm{Var}(R_n) \sim \frac{2b^2}{n^2}.$$

Corollary 2.8 (Sample range)**.** *Under the assumptions of Corollary 2.5, let $W_n = X_{n:n} - X_{1:n}$ be the sample range. Then*

$$(2b - W_n)\left(\frac{nC}{r}\right)^{1/r} \xrightarrow{d} X_{e,1} + X_{e,2}, \tag{2.13}$$

where $X_{e,1}$ and $X_{e,2}$ are independent copies with cdf F_e. Moreover, by the moment convergence already established for both endpoint gaps,

$$\mathrm{Var}(2b - W_n) \sim 2\left(\frac{r}{nC}\right)^{2/r}\left[\Gamma\left(1 + \frac{2}{r}\right) - \Gamma\left(1 + \frac{1}{r}\right)^2\right]. \tag{2.14}$$

This result is included to address the natural range functional generated by the same two extreme order statistics. The statistical focus of the paper remains the sample midrange, because R_n estimates the location center θ, whereas W_n estimates the support width $2b$.

3. Auxiliary lemmas

Lemma 3.1 (Lévy–Cramér continuity theorem in dimension two)**.** *Let (ξ_n, η_n), $n \geq 1$, and (ξ, η) be random vectors in $\mathbb{R}^2$. If $(\xi_n, \eta_n) \xrightarrow{d} (\xi, \eta)$, then $\xi_n \xrightarrow{d} \xi$, $\eta_n \xrightarrow{d} \eta$, and $\xi_n - \eta_n \xrightarrow{d} \xi - \eta$. If, in addition,*

$$\lim_{n\to\infty} P(\xi_n \leq x, \eta_n \leq y) = \lim_{n\to\infty} P(\xi_n \leq x) \lim_{n\to\infty} P(\eta_n \leq y)$$

for every continuity point (x, y) of the limiting distribution, then ξ and η are independent.

Lemma 3.2 (Moment convergence criterion [12])**.** *Let $\xi_n \xrightarrow{d} \xi$, and let $w > 0$. Then*

$$\lim_{n\to\infty} E|\xi_n|^w = E|\xi|^w < \infty,$$

if and only if the family $\{|\xi_n|^w : n \geq 1\}$ is uniformly integrable.

Lemma 3.3 (Correlation inequality for order statistics, see [13])**.** *Let $X_1, \ldots, X_n$ be independent and identically distributed. If $i \leq j$ and both $X_{i:n}$ and $X_{j:n}$ have finite variances, then*

$$0 \leq \mathrm{Corr}(X_{i:n}, X_{j:n}) \leq \left(\frac{i(n+1-j)}{j(n+1-i)}\right)^{1/2}.$$

In particular,

$$\mathrm{Corr}(X_{1:n}, X_{n:n}) \leq \frac{1}{n}.$$

4. Proofs

4.1. Proof of Theorem 2.1

Intuitive explanation: The condition $f(x) \sim C(b-x)^{r-1}$ as $x \to b^-$ implies that the tail probability near b behaves like $Cr^{-1}(b-x)^r$. Heuristically, this means $b - X_{n:n}$ should be on the order of $n^{-1/r}$, so the normalized maximum converges to a non-degenerate limit. Now we proceed with the formal proof. Let $b_n > 0$ be determined by $1 - F(b - b_n) = 1/n$, $n \geq 1$. Since f is continuous and satisfies (2.1), L'Hospital's rule gives

$$\lim_{x \to b-} \frac{1-F(x)}{(b-x)^r} = \lim_{x \to b-} \frac{f(x)}{r(b-x)^{r-1}} = \frac{C}{r}. \tag{4.1}$$

Hence

$$1 - F(x) \sim \frac{C}{r}(b-x)^r, \qquad x \to b-. \tag{4.2}$$

Substituting $x = b - b_n$ into (4.2) yields

$$b_n^r \sim \frac{r}{nC}, \qquad b_n \sim \left(\frac{r}{nC}\right)^{1/r}. \tag{4.3}$$

For fixed $x > 0$, (4.2) implies

$$\frac{1 - F(b - b_n x)}{1 - F(b - b_n)} \to x^r,$$

so

$$1 - F(b - b_n x) = \frac{x^r + o(1)}{n}.$$

Therefore,

$$P\left(\frac{b - X_{n:n}}{b_n} > x\right) = F(b - b_n x)^n \to e^{-x^r},$$

which shows that

$$V_n := \frac{b - X_{n:n}}{b_n} \xrightarrow{d} X_{e,1}. \tag{4.4}$$

Combining (4.4) with (4.3) proves (2.2).

It remains to establish moment convergence. We now give a complete proof of uniform integrability. The proof is based on a uniform Weibull-type upper tail bound for the normalized endpoint gap. From (4.3), there exist constants $0 < c_1 < c_2 < \infty$ such that

$$c_1 n^{-1/r} \leq b_n \leq c_2 n^{-1/r}, \qquad n \geq 1. \tag{4.5}$$

By (4.2), there are constants $\delta > 0$ and $A_1, A_2 > 0$ such that, for $0 < t \leq \delta$,

$$A_1 t^r \leq 1 - F(b - t) \leq A_2 t^r. \tag{4.6}$$

Since $1 - F(b - b_n) = 1/n$, (4.6) also gives another direct proof of (4.5). We claim that there exist constants $K, c > 0$, independent of n and x, such that

$$P(V_n > x) \leq K \exp(-cx^r), \qquad x \geq 0, \ n \geq 1, \tag{4.7}$$

where $V_n = (b - X_{n:n})/b_n$. To prove this, first suppose $b_n x \le \delta$. Then, using $1 - u \le e^{-u}$ and (4.6),

$$P(V_n > x) = F(b - b_n x)^n \le \exp\{-nA_1 b_n^r x^r\} \le \exp(-c_0 x^r),$$

for some $c_0 > 0$, because nb_n^r is bounded away from zero by (4.5). Next suppose $b_n x > \delta$. Put $\rho = F(b - \delta) < 1$. Then $F(b - b_n x) \le \rho$, and hence $P(V_n > x) \le \rho^n$. At the same time, $x \le (b - a)/b_n$, since $V_n \le (b - a)/b_n$ almost surely; therefore $x^r \le C_0 n$ for a constant $C_0 > 0$. Choosing $c_1' > 0$ sufficiently small gives $\rho^n \le \exp(-c_1' x^r)$ throughout this second region. Combining the two regions proves (4.7), after increasing K to cover bounded values of n and x.

Now let $T > 0$. By the tail-integral formula for nonnegative random variables,

$$\begin{aligned} E\{V_n^w \mathbf{1}(V_n > T)\} &= T^w P(V_n > T) + \int_T^\infty w x^{w-1} P(V_n > x)\, dx \\ &\le K T^w e^{-cT^r} + Kw \int_T^\infty x^{w-1} e^{-cx^r}\, dx. \end{aligned} \tag{4.8}$$

The right-hand side is independent of n, and because $T^w e^{-cT^r} \to 0$ as $T \to \infty$ and the integral $\int_T^\infty x^{w-1} e^{-cx^r} dx$ also tends to 0, it can be made arbitrarily small by choosing T large enough. Hence

$$\sup_n E\{V_n^w \mathbf{1}(V_n > T)\} \to 0 \quad \text{as } T \to \infty,$$

which means that $\{V_n^w : n \ge 1\}$ is uniformly integrable. This argument is self-contained; it is also consistent with the uniform-integrability criteria for normalized extremes discussed by DasGupta [6] and Leadbetter, Lindgren and Rootzén [8].

Finally, (4.4) and the moment convergence criterion imply

$$E(V_n^w) \to E(X_{e,1}^w) = \Gamma\left(1 + \frac{w}{r}\right).$$

Replacing b_n by its equivalent in (4.3) gives (2.3). This completes the proof.

4.2. *Proof of Corollary 2.2*

Apply Theorem 2.1 to the reflected variable $-X$, whose right endpoint equals $-a$. This yields both assertions.

4.3. *Proof of Theorem 2.3*

Intuitively, because the left and right tails are controlled by independent Poisson-type point processes, the normalized gaps to the endpoints become asymptotically independent. The joint convergence is then obtained by a direct computation of the probability that all observations fall between two moving boundaries.

Let b_n and $b_n^* > 0$ be defined by

$$1 - F(b - b_n) = \frac{1}{n}, \qquad F(a + b_n^*) = \frac{1}{n}.$$

By the right- and left-endpoint expansions,

$$F(b - b_n x) = 1 - \frac{x^r + \varepsilon_n(x)}{n}, \tag{4.9}$$

$$F(a+b_n^* y) = \frac{y^{r^*} + \delta_n(y)}{n}, \tag{4.10}$$

where $\varepsilon_n(x) \to 0$ and $\delta_n(y) \to 0$ for each fixed $x, y > 0$. For fixed compact subsets of $(0, \infty)$, the convergence is locally uniform by the regular variation implied by the endpoint density assumptions. In the present fixed-x, y calculation, pointwise convergence is sufficient.

For fixed $x, y > 0$ and large n, $a + b_n^* y < b - b_n x$. Hence

$$\begin{aligned} P\left(\frac{b - X_{n:n}}{b_n} \le x, \frac{X_{1:n} - a}{b_n^*} \le y\right) &= 1 - P(X_{n:n} < b - b_n x) - P(X_{1:n} > a + b_n^* y) \\ &\quad + P(a + b_n^* y < X_i < b - b_n x \text{ for all } i) \\ &= 1 - F(b - b_n x)^n - (1 - F(a + b_n^* y))^n + \left(F(b - b_n x) - F(a + b_n^* y)\right)^n. \end{aligned}$$

Using (4.9) and (4.10) and the elementary limit $(1 - z_n/n)^n \to e^{-z}$ whenever $z_n \to z$, the three powers converge respectively to e^{-x^r}, $e^{-y^{r^*}}$, and $e^{-x^r - y^{r^*}}$. Therefore

$$P\left(\frac{b - X_{n:n}}{b_n} \le x, \frac{X_{1:n} - a}{b_n^*} \le y\right) \to (1 - e^{-x^r})(1 - e^{-y^{r^*}}).$$

The limit is the product of the marginal Weibull distribution functions. Thus

$$\left(\frac{b - X_{n:n}}{b_n}, \frac{X_{1:n} - a}{b_n^*}\right) \xrightarrow{d} (X_{e,1}, Y_e),$$

where the two components are independent. Using $b_n \sim (r/nC)^{1/r}$ and $b_n^* \sim (r^*/nC^*)^{1/r^*}$ gives the joint convergence in the normalization of the theorem, and (2.7) follows by the continuous mapping theorem.

Finally, we note that (2.7) also holds trivially when $x \le 0$ or $y \le 0$, since in those cases one of the events $\{(b - X_{n:n})/b_n \le x\}$ or $\{(X_{1:n} - a)/b_n^* \le y\}$ is either always false or always true for all n, matching the limit. Thus the claimed convergence in (2.7) is valid for all x, y.

For the variance, define

$$U_n = (X_{1:n} - a)(nC^*/r^*)^{1/r^*}, \qquad V_n = (b - X_{n:n})(nC/r)^{1/r}.$$

Theorem 2.1 and Corollary 2.2 with $w = 1, 2$ give

$$\sup_n \operatorname{Var}(U_n) < \infty, \qquad \sup_n \operatorname{Var}(V_n) < \infty,$$

and moreover $\operatorname{Var}(U_n) \to \operatorname{Var}(Y_e)$ and $\operatorname{Var}(V_n) \to \operatorname{Var}(X_{e,1})$. Lemma 3.3 yields

$$0 \le \operatorname{Corr}(X_{1:n}, X_{n:n}) \le \frac{1}{n}.$$

Since U_n is increasing in $X_{1:n}$ while V_n is decreasing in $X_{n:n}$,

$$-\frac{1}{n} \le \operatorname{Corr}(U_n, V_n) \le 0.$$

Consequently,

$$|\operatorname{Cov}(U_n, V_n)| \le \frac{1}{n} \sqrt{\operatorname{Var}(U_n) \operatorname{Var}(V_n)} \to 0,$$

which gives the desired rigorous covariance control. Therefore

$$\mathrm{Var}(U_n - V_n) = \mathrm{Var}(U_n) + \mathrm{Var}(V_n) - 2\,\mathrm{Cov}(U_n, V_n) \to \mathrm{Var}(Y_e) + \mathrm{Var}(X_{e,1}).$$

The explicit expression follows from

$$E(X_{e,1}^m) = \Gamma\left(1 + \frac{m}{r}\right), \qquad E(Y_e^m) = \Gamma\left(1 + \frac{m}{r^*}\right), \qquad m > 0.$$

4.4. *Proof of Corollary 2.5*

Because f_b is even on $[-b, b]$, the left-endpoint condition holds with the same exponent r and constant C. Hence,

$$(X_{1:n} - (\theta - b))\left(\frac{nC}{r}\right)^{1/r} \xrightarrow{d} X_{e,2}, \qquad ((\theta + b) - X_{n:n})\left(\frac{nC}{r}\right)^{1/r} \xrightarrow{d} X_{e,1},$$

where $X_{e,1}$ and $X_{e,2}$ are independent copies. Since

$$R_n - \theta = \frac{1}{2}\left[(X_{1:n} - (\theta - b)) - ((\theta + b) - X_{n:n})\right],$$

we obtain

$$(R_n - \theta)\left(\frac{nC}{r}\right)^{1/r} \xrightarrow{d} \frac{X_{e,2} - X_{e,1}}{2}.$$

Multiplying both sides by $r^{1/r}$ gives

$$(R_n - \theta)(nC)^{1/r} \xrightarrow{d} \frac{r^{1/r}}{2}(X_{e,2} - X_{e,1}).$$

Since $X_{e,1}$ and $X_{e,2}$ are i.i.d., the random variables $X_{e,2} - X_{e,1}$ and $X_{e,1} - X_{e,2}$ have the same distribution. Thus the convergence can equivalently be written in the stated form

$$(R_n - \theta)(nC)^{1/r} \xrightarrow{d} \frac{r^{1/r}}{2}(X_{e,1} - X_{e,2}).$$

For the variance,

$$\mathrm{Var}\left(\frac{X_{e,2} - X_{e,1}}{2}\right) = \frac{1}{2}\,\mathrm{Var}(X_{e,1}) = \frac{1}{2}\left[\Gamma\left(1 + \frac{2}{r}\right) - \Gamma\left(1 + \frac{1}{r}\right)^2\right],$$

which implies the stated variance formula.

4.5. *Proof of Corollary 2.7*

Since $\lim_{x\to b-} f_b(x) = C > 0$, we must have $r = 1$ by definition of the exponent in (2.10). Applying Corollary 2.5 with $r = 1$ yields

$$(R_n - \theta)(nC) \xrightarrow{d} \frac{1}{2}(X_{e,1} - X_{e,2}),$$

and the variance formula follows directly from the general variance expression. □

4.6. Proof of Corollary 2.8

For the symmetric location family,

$$2b - W_n = ((\theta + b) - X_{n:n}) + (X_{1:n} - (\theta - b)).$$

The joint convergence of the two normalized endpoint gaps, together with equality of the endpoint constants and exponents, gives (2.13). Uniform integrability of every positive power, proved in Theorem 2.1 and Corollary 2.2, gives convergence of second moments. It remains to verify that the covariance term vanishes asymptotically. Define

$$\widetilde{V}_n = ((\theta + b) - X_{n:n})\left(\frac{nC}{r}\right)^{1/r}, \qquad \widetilde{U}_n = (X_{1:n} - (\theta - b))\left(\frac{nC}{r}\right)^{1/r}.$$

By Lemma 3.3, $0 \le \mathrm{Corr}(X_{1:n}, X_{n:n}) \le 1/n$. Since $\widetilde{V}_n$ is decreasing in $X_{n:n}$ and $\widetilde{U}_n$ is increasing in $X_{1:n}$, we have

$$-\frac{1}{n} \le \mathrm{Corr}(\widetilde{U}_n, \widetilde{V}_n) \le 0,$$

and consequently

$$|\,\mathrm{Cov}(\widetilde{U}_n, \widetilde{V}_n)| \le \frac{1}{n}\sqrt{\mathrm{Var}(\widetilde{U}_n)\,\mathrm{Var}(\widetilde{V}_n)} = o\Big((nC/r)^{-2/r}\Big).$$

Thus the covariance term is negligible, and the variance of $2b - W_n$ is asymptotically the sum of the two individual variances. Since the two limiting endpoint gaps are independent,

$$\mathrm{Var}(X_{e,1} + X_{e,2}) = 2\,\mathrm{Var}(X_{e,1}),$$

which yields (2.14).

5. Numerical illustrations

To complement the asymptotic theory, we conduct a Monte Carlo study using three bounded symmetric distributions on the interval $[-1, 1]$. For each distribution we explicitly state the density, the right-endpoint exponent r, and the normalizing constant C (or C^*) appearing in (2.1) (or (2.4)).

- **Uniform**$[-1, 1]$**:** $f(x) = \frac{1}{2}\mathbf{1}_{[-1,1]}(x)$. Here $r = 1$ and $C = \frac{1}{2}$ (since $f(x)$ is constant near the endpoints).
- **Wigner semicircle:** $f(x) = \frac{2}{\pi}\sqrt{1 - x^2}\,\mathbf{1}_{[-1,1]}(x)$. Expanding near $x = 1$ gives $f(x) \sim \frac{2\sqrt{2}}{\pi}(1 - x)^{1/2}$, so $r = \frac{3}{2}$ and $C = \frac{2\sqrt{2}}{\pi}$.
- **Symmetric polynomial family:** $f(x) = \frac{r}{2}(1 - |x|)^{r-1}\,\mathbf{1}_{[-1,1]}(x)$, with $r = 2.5$. Near $x = 1$, $f(x) = \frac{r}{2}(1 - x)^{r-1}$, hence $C = \frac{r}{2} = 1.25$. This case serves as a control where $r > 2$ and the midrange is no longer guaranteed to outperform the sample mean or median in variance.

All three distributions are symmetric about $\theta = 0$, so the center parameter coincides with the symmetry point.

For each model we generated independent samples of size $n \in \{20, 50, 100, 200, 500, 1000, 2000\}$, with $B = 5000$ replications per combination. We recorded the normalized upper extreme

$$Z_n^{\max} = (1 - X_{n:n})(nC/r)^{1/r},$$

and compared its empirical mean and variance with the theoretical Weibull limits $E[Z_n^{\max}] \to \Gamma(1+1/r)$ and $\mathrm{Var}(Z_n^{\max}) \to \Gamma(1+2/r) - \Gamma(1+1/r)^2$ given by Theorem 2.1. We also evaluated the sample midrange $R_n = (X_{1:n} + X_{n:n})/2$, the sample mean $\bar{X}_n$, and the sample median M_n as estimators of the center $\theta = 0$.

5.1. Moment convergence of normalized extremes

Table 1 reports the empirical moments of $Z_n^{\max}$ for selected sample sizes. For all three distributions, both the mean and the variance approach the theoretical limits as n grows, confirming the uniform integrability and moment convergence proved in Theorem 2.1. In the semicircle case, $E[Z_n^{\max}]$ moves from 0.910 at $n = 20$ to 0.906 at $n = 1000$, matching the limit 0.9027 within sampling error; the variance stabilizes around the theoretical 0.3757. The convergence is slightly slower for $r = 2.5$, as expected from the heavier endpoint tail.

Table 1. Empirical moments of $Z_n^{\max}$. Theoretical limits are shown in the last row.

	$r = 1$ (Uniform)		$r = 1.5$ (Semicircle)		$r = 2.5$ (Polynomial)	
n	$E[Z_n^{\max}]$	$\mathrm{Var}(Z_n^{\max})$	$E[Z_n^{\max}]$	$\mathrm{Var}(Z_n^{\max})$	$E[Z_n^{\max}]$	$\mathrm{Var}(Z_n^{\max})$
20	0.927	0.776	0.910	0.374	0.879	0.134
50	0.989	0.917	0.888	0.369	0.889	0.136
100	0.990	0.895	0.896	0.376	0.890	0.140
200	1.001	0.990	0.896	0.372	0.883	0.140
500	1.002	0.994	0.913	0.374	0.886	0.143
1000	1.008	1.032	0.906	0.377	0.878	0.146
2000	0.995	0.989	0.901	0.375	0.882	0.142
Theory	1.000	1.000	0.903	0.376	0.887	0.144

Reproducibility note. All simulations were carried out in R 4.3.2, using random seed 42. The R code used to generate Table 1 and Figure 1 is not submitted together with this manuscript; the script (e.g. `simulate_extremes.R`) is available from the corresponding author upon request. To quantify simulation uncertainty, Appendix A reports Monte Carlo standard errors for the empirical moments in Table 1 and the empirical coverage of the asymptotic 90% interval in Section 6.

5.2. Variance comparison of center estimators

Figure 1 plots log(Var) against $\log n$ for R_n, $\bar{X}_n$, and M_n. For the uniform case ($r = 1$), the midrange slope is approximately -2 (since $2/r = 2$), while the mean and median have slope near -1, so R_n is vastly more efficient. The same pattern holds for the semicircle ($r = 1.5$), where the midrange variance decreases as $n^{-4/3} \approx n^{-1.33}$, still faster than the n^{-1} rate of the other estimators. In contrast, for $r = 2.5$, all three estimators decay roughly at the n^{-1} rate; the midrange no longer enjoys a clear advantage, and indeed becomes slightly less efficient than the mean at larger n. This confirms the theoretical boundary $r < 2$ for the midrange to be asymptotically superior, as identified in Corollary 2.5.

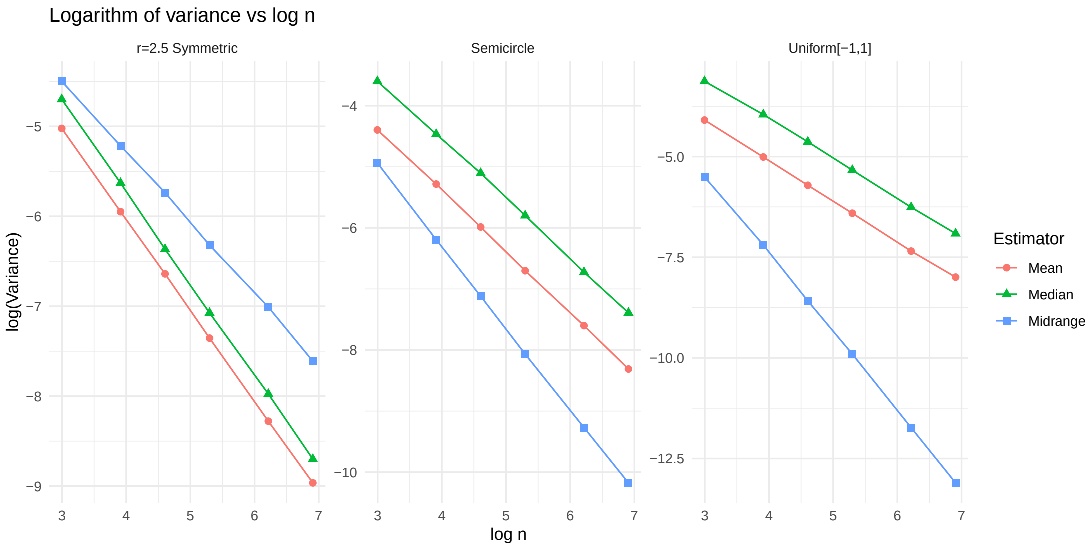

Figure 1. Logarithm of sample variance versus $\log n$ for the midrange, mean, and median. Fitted slopes for the midrange are -1.948, -1.339, and -0.792 in the three cases, respectively.

The Monte Carlo results are fully consistent with the theoretical findings of Section 6. In particular, for $r = 1$ and $r = 3/2$, the midrange exhibits significantly smaller mean squared error (MSE) than the sample mean and median even at moderate sample sizes, supporting the practical relevance of the asymptotic statements.

6. Applications and illustrations

6.1. Semicircular distribution

Consider the semicircular location family

$$f(x,\theta) = \frac{2}{\pi}\sqrt{1-(x-\theta)^2}\,\mathbf{1}_{[\theta-1,\theta+1]}(x), \qquad \theta \in \mathbb{R}.$$

The core density is

$$f_b(x) = \frac{2}{\pi}\sqrt{1-x^2}\,\mathbf{1}_{[-1,1]}(x),$$

and satisfies

$$\lim_{x\to 1-} \frac{f_b(x)}{(1-x)^{1/2}} = \frac{2\sqrt{2}}{\pi} =: C.$$

Thus (2.10) holds with $r = 3/2$. By Corollary 2.5,

$$(R_n - \theta)\left(\frac{2\sqrt{2}}{\pi}n\right)^{2/3} \xrightarrow{d} \frac{(3/2)^{2/3}}{2}(X_{e,1} - X_{e,2}).$$

Consequently, an approximate 90% confidence interval for θ can be obtained from the limiting distribution. Let $q_{0.95}$ satisfy

$$P\left(\left|\frac{(3/2)^{2/3}}{2}(X_{e,1} - X_{e,2})\right| \le q_{0.95}\right) = 0.90.$$

Then

$$\left(R_n - q_{0.95}\left(\frac{2\sqrt{2}}{\pi}n\right)^{-2/3}, R_n + q_{0.95}\left(\frac{2\sqrt{2}}{\pi}n\right)^{-2/3}\right)$$

is an approximate 90% confidence interval. The length of this interval is of order $n^{-2/3}$, whereas the variance of the semicircular distribution equals $1/4$, so a confidence interval based on the sample mean has length $O(n^{-1/2})$. Hence the midrange-based interval is asymptotically shorter, illustrating the efficiency gain when $r < 2$.

The empirical coverage of this interval for the semicircle model is reported in Appendix A.

6.2. *A simple localization model*

A second illustration is a stylized geometric localization problem. Suppose that observed debris locations arise from points distributed approximately on a sphere centered at $(\theta_1, \theta_2, \theta_3)$ with radius ρ. If one observes planar coordinates (X_i, Y_i), a natural idealized model is that (X_i, Y_i, Z_i) are i.i.d. uniformly distributed on the sphere

$$(x - \theta_1)^2 + (y - \theta_2)^2 + (z - \theta_3)^2 = \rho^2.$$

For a point uniformly distributed on this sphere, the marginals of X and Y are uniform on $[\theta_1 - \rho, \theta_1 + \rho]$ and $[\theta_2 - \rho, \theta_2 + \rho]$, respectively. Hence the positive-endpoint-density case applies coordinatewise, and the estimator

$$(R_x, R_y) = \left(\frac{X_{1:n} + X_{n:n}}{2}, \frac{Y_{1:n} + Y_{n:n}}{2}\right)$$

has asymptotic variance of order n^{-2} in each coordinate, outperforming the coordinatewise sample mean and sample median, whose variances are of order n^{-1}. The same reasoning applies to other radially symmetric bounded models. For example, if points are uniformly distributed on a disc $\{(x, y) : x^2 + y^2 \leq \rho^2\}$, the marginal distribution of X has density $f_X(x) = 2\sqrt{\rho^2 - x^2}/(\pi\rho^2)$, which near the endpoints behaves like a semicircle with $r = 3/2$. Thus the midrange again yields a variance of order $n^{-4/3}$, still faster than the n^{-1} rate of the sample mean. These examples show that the efficiency crossing at $r = 2$ provides a simple, practical criterion to decide whether extreme-order-based estimators are worth using in a given bounded-support problem.

7. Conclusions

We proved that normalized extreme order statistics on bounded supports satisfy not only weak convergence but also convergence of all positive moments under natural endpoint density assumptions. The key technical step is uniform integrability of the normalized extremes. We then obtained the joint limit law of the normalized minimum and maximum, together with an explicit asymptotic variance formula for their difference. As an application, we derived the asymptotic distribution and variance of the sample midrange for symmetric bounded location families and showed that, when $1 \leq r < 2$, the sample midrange can be asymptotically more efficient than the sample mean and the sample median.

A closely related statistic is the sample range $W_n = X_{n:n} - X_{1:n}$. Corollary 2.8 gives a first-order limit theorem and variance asymptotics for W_n under symmetric endpoint assumptions. The present paper nevertheless keeps its main statistical focus on estimation of the center parameter by the sample midrange. A broader treatment of the sample range—for example, unequal endpoint exponents, higher-order expansions, and inference for the support width—is a natural topic for future work.

Two natural directions for future work are worth mentioning. First, it would be interesting to treat endpoint behavior involving slowly varying factors, singular endpoint densities with $0 < r < 1$, or weaker smoothness assumptions. Second, sharper non-asymptotic comparisons between the sample midrange and competing location estimators for bounded families may provide additional practical insight.

Author contributions

H. Qiu conceived the research idea, developed the main theoretical results, and wrote the initial manuscript; Y. Zhang contributed to the literature review, refined the proofs, and helped prepare the numerical illustrations and simulation study. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the Doctoral Scientific Research Project of Changsha Social Work College (2025JB38).

Supplementary materials

`simulate_extremes.R`: R script for the Monte Carlo simulations (not submitted with the manuscript, available from the corresponding author upon request).

Conflict of interest

The authors declare that they have no conflict of interest.

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A. Appendix

Table 2 reports Monte Carlo standard errors for the empirical moments in Table 1. The standard error of the mean is computed as the replication sample standard deviation divided by $\sqrt{B}$; for the variance it is estimated via the empirical fourth moment across replications.

Table 2. Monte Carlo standard errors for the empirical moments of $Z_n^{\max}$. Values are multiplied by 10^3 for readability.

	$r = 1$ (Uniform)		$r = 1.5$ (Semicircle)		$r = 2.5$ (Polynomial)	
n	MCSE(Mean)	MCSE(Var)	MCSE(Mean)	MCSE(Var)	MCSE(Mean)	MCSE(Var)
20	12.46	26.91	8.64	9.44	5.17	2.45
50	13.55	35.77	8.59	9.75	5.21	2.65
100	13.38	32.26	8.67	9.74	5.29	2.64
200	14.07	42.26	8.63	9.20	5.29	2.66
500	14.10	38.27	8.64	8.95	5.34	2.80
1000	14.36	42.10	8.69	9.48	5.40	2.79
2000	10.10	29.80	6.12	6.70	3.80	1.97

Figure 2 shows the empirical coverage of the asymptotic 90% confidence interval for θ derived in Section 6. The coverage is evaluated for the semicircle model and approaches the nominal level as n increases.

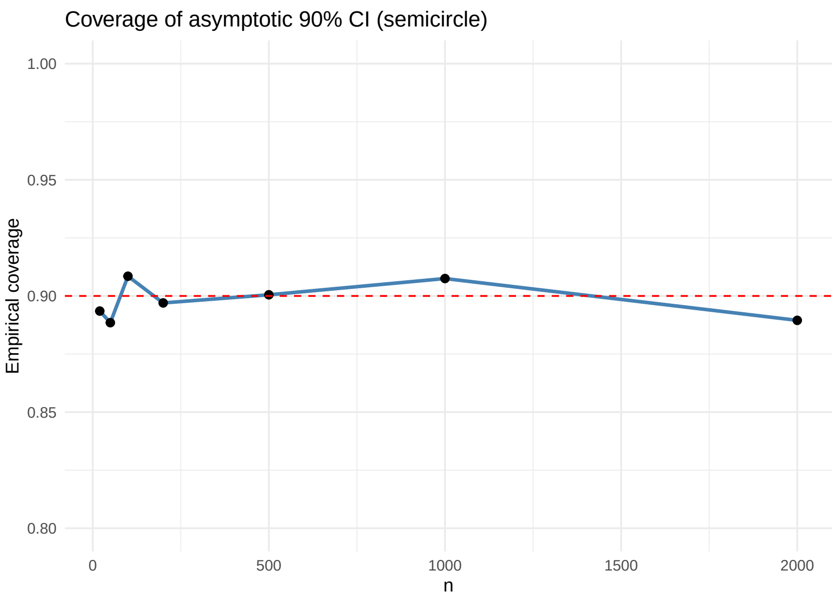

Figure 2. Empirical coverage of the asymptotic 90% confidence interval for θ in the semicircle model. The horizontal line marks the nominal 0.90 level.

The empirical coverage ranges from 0.8885 to 0.9085 across all sample sizes, confirming that the asymptotic interval achieves the nominal level within Monte Carlo uncertainty.

Table 3 compares the empirical variances of the midrange, mean, and median for the three distributions.

Table 3. Empirical variance ratios for the midrange relative to the sample mean and median.

	Uniform		Semicircle		Polynomial	
n	$R_n/\bar{X}_n$	R_n/M_n	$R_n/\bar{X}_n$	R_n/M_n	$R_n/\bar{X}_n$	R_n/M_n
20	0.2435	0.0929	0.5827	0.2627	1.6888	1.2221
50	0.1126	0.0392	0.4027	0.1778	2.0781	1.5105
100	0.0566	0.0192	0.3221	0.1334	2.4690	1.8759
200	0.0302	0.0104	0.2555	0.1034	2.8112	2.1235
500	0.0125	0.0042	0.1884	0.0785	3.5559	2.6266
1000	0.0060	0.0020	0.1550	0.0617	3.8683	2.9702
2000	0.0031	0.0010	0.1102	0.0441	4.2015	3.1987