Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(7): 19772–19783.
DOI: 10.3934/math.2026802
Received: 18 May 2026
Revised: 17 June 2026
Accepted: 01 July 2026
Published: 07 July 2026

Research article

Forcing and domination in graphs

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Abstract: Let G be a connected graph. We investigated structural relationships between the zero forcing number $Z(G)$ and connected domination. Our main result proved that the complement of every minimal connected dominating set formed a zero forcing set, establishing the converse direction to a known result of Amos et al. and yielding the inequality $Z(G) + \gamma_c(G) \leq |V(G)|$. We further studied graphs attaining equality and showed that several natural graph families, including complete multipartite graphs, satisfied it. Motivated by the duality between forcing and domination, we introduced the dom-forcing number $F_d(G)$, the minimum size of a set that was both dominating and zero forcing. We established sharp general bounds, determined its exact values for paths, stars, complete graphs, and complete bipartite graphs, and characterized when it equaled the zero forcing number or the domination number. Finally, we discussed several open problems related to forcing-domination dualities.

Keywords: zero forcing set; connected domination; dom-forcing set
Mathematics Subject Classification: 05C10, 05C82

1. Introduction

The zero forcing process was introduced independently in the context of minimum rank problems for graphs [1] and has since become a fundamental tool in linear algebra, quantum control, and graph searching. A set of initially colored vertices is called a *zero forcing set* if, by repeatedly applying the rule that a colored vertex with exactly one uncolored neighbor forces that neighbor to become colored, every vertex of the graph eventually becomes colored. The minimum size of such a set is the *zero forcing number* $Z(G)$. This parameter provides an upper bound on the maximum nullity of symmetric matrices whose nonzero off-diagonal entries correspond to the edges of G, and it has been extensively studied in relation to cut-vertex reductions [2] and structural characterizations [3]. Nolan et al. [4] determined upper and lower bounds on the zero forcing number of 2-connected outerplanar graphs in terms of the structure of the weak dual. Kheyridoost et al. [5] studied the zero forcing number and propagation time of some graphs. Several variants of zero forcing have since been introduced, including k-forcing [6], connected forcing, and loop zero forcing, reflecting the rich structure of propagation processes on graphs.

Domination is another classical theme in graph theory. A set $D \subseteq V(G)$ is *dominating* if every vertex outside D has at least one neighbor in D, and it is *connected dominating* if, in addition, the induced subgraph $G[D]$ is connected. The connected domination number $\gamma_c(G)$, introduced by Sampathkumar and Walikar [7], arises naturally in network design and routing problems [8–10]. The question of enumerating all inclusion-wise minimal connected dominating sets in a graph of order n in time significantly less than 2^n is an open problem that has been asked in many places. Abu-Khzam et al. [11] answered this question affirmatively by providing an enumeration algorithm that runs in time $O(1.9896^n)$ using only polynomial space. More generally, k-dominating sets and connected k-dominating sets have been investigated in the literature. The concept of forcing in domination settings has also been explored; for instance, the forcing domination number measures the minimum number of vertices whose forced inclusion in a minimum dominating set uniquely determines the set. Connected domination is naturally linked to zero forcing through the observation that minimal connected dominating sets often determine structural partitions of the vertex set that are compatible with propagation processes.

The interaction between forcing parameters and domination parameters has attracted growing attention in recent years. Amos et al. [6] proved that if S is a minimum k-forcing set of a k-connected graph such that the induced subgraph on $V \setminus S$ is connected, then $V \setminus S$ is a connected k-dominating set. For $k = 1$, this gives a sufficient condition for the complement of a minimum zero forcing set to be a connected dominating set. Davila et al. [3] established lower bounds for the zero forcing number in terms of minimum degree for graphs with large girth, and Fürst and Rautenbach [12] later gave a short proof of a related lower bound. The notion of power domination, which combines domination with a propagation rule, further highlights the natural interplay between these two areas.

Despite these developments, a systematic structural connection between *minimal* connected dominating sets and zero forcing sets has not been fully established. In particular, while the result of Amos et al. [6] establishes a sufficient condition for the complement of a zero forcing set to be a connected dominating set, the converse direction—whether the complement of every minimal connected dominating set is necessarily zero forcing—has not been rigorously treated in the literature. Our Theorem 3.2 fills this gap by establishing the reverse implication: from a minimal connected dominating set to a zero forcing set. This provides a structural counterpart to the result of Amos et al. and yields the inequality $Z(G) + \gamma_c(G) \leq n$, which, to the best of our knowledge, has not appeared explicitly in this general form.

The main contributions of this paper are threefold.

First, we prove that for every connected graph G, if Q is a *minimal* connected dominating set, then $V(G) \setminus Q$ is a zero forcing set. This yields the general inequality

$$Z(G) + \gamma_c(G) \leq n,$$

which unifies and extends several previously known instances of equality. While the result of Amos et al. [6] establishes a sufficient condition for the complement of a zero forcing set to be a connected dominating set, our Theorem 3.2 provides the converse structural connection: every minimal connected dominating set, without any additional connectivity assumptions, yields a zero forcing set as its complement. This completes the duality between the two parameters in a way that was previously not recognized. Our proof uses the block–cut-vertex structure of the induced subgraph $G[Q]$ and provides a self-contained argument.

Second, we study the equality case $Z(G) + \gamma_c(G) = n$. We show that the equality class includes complete multipartite graphs with all parts of size at least two, in addition to complete graphs and cycles. The question of a full characterization of this equality class remains open; we discuss this in detail in Problem 6.1.

Third, motivated by the duality between forcing and domination, we introduce the *dom-forcing number* $F_d(G)$, defined as the minimum size of a set that is both dominating and zero forcing. We establish sharp bounds

$$\max\{Z(G), \gamma(G)\} \leq F_d(G) \leq Z(G) + \gamma(G),$$

and discuss extremal examples and open problems concerning its computational complexity and structural characterizations.

The remainder of the paper is organized as follows. Section 2 recalls the necessary definitions and known results. Section 3 presents the main structural theorem and its proof. Section 4 investigates graphs attaining equality in the bound. Section 5 introduces the dom-forcing number and derives its basic properties. Section 6 concludes with several open problems.

Throughout the paper all graphs are finite, simple, undirected, and connected.

2. Preliminaries

For a graph $G = (V, E)$, let $N(v)$ denote the open neighborhood of v, and let $N[v] = N(v) \cup \{v\}$. The order of G is denoted by $n = |V(G)|$. For a subset $S \subseteq V(G)$, we denote by $G[S]$ the subgraph induced by S.

Definition 2.1 (Zero forcing set)**.** *The following definition of zero forcing was introduced in [1].*

Let $S \subseteq V(G)$ be initially colored. If a colored vertex has exactly one uncolored neighbor, then it forces that neighbor to become colored.

A set S is a zero forcing set if repeated applications of this rule color all vertices. The minimum cardinality of a zero forcing set is the zero forcing number $Z(G)$.

Definition 2.2 (Dominating set)**.** *A set $D \subseteq V(G)$ is dominating if every vertex in $V(G) \setminus D$ has at least one neighbor in D. The domination number is denoted by $\gamma(G)$.*

Definition 2.3 (Connected dominating set)**.** *Connected domination was introduced by Sampathkumar and Walikar [7].*

A dominating set D is connected if $G[D]$ is connected. The minimum cardinality of a connected dominating set is called the connected domination number and is denoted by $\gamma_c(G)$.

Definition 2.4 (Dom-forcing set)**.** *Motivated by the interaction between domination and forcing, we introduce the following parameter.*

A set $D \subseteq V(G)$ is a dom-forcing set if:

(1) D is dominating;

(2) *D is zero forcing.*

The minimum cardinality of such a set is called the dom-forcing number and is denoted by $F_d(G)$.

For a graph H, a *block* is a maximal connected subgraph of H that has no cut-vertex; equivalently, it is either a maximal 2-connected subgraph, a bridge, or an isolated vertex. A block is called a *leaf block* if it contains at most one cut-vertex of H. The *block-cut tree* of H is a bipartite graph whose vertices correspond to the blocks and cut-vertices of H, with an edge between a block-vertex and a cut-vertex if the cut-vertex belongs to the block. It is well known that the block-cut tree of a connected graph is a tree.

3. Zero forcing and connected domination

Lemma 3.1. *Let H be a connected graph and let $X \subseteq V(H)$ be a nonempty set of vertices such that every vertex of X is a cut-vertex of H. Then, there exists a leaf block B of H such that B contains exactly one vertex of X.*

Proof. Consider the block-cut tree $\mathcal{T}$ of H. Since H is connected, $\mathcal{T}$ is a tree. The vertices of X correspond to a subset of the cut-vertex nodes in $\mathcal{T}$. Take a leaf block B of H that is at maximum distance from the set X in $\mathcal{T}$. Suppose, for contradiction, that B contains two distinct vertices $x_1, x_2 \in X$. Then, B would be incident to at least two cut-vertex nodes in $\mathcal{T}$, contradicting the definition of a leaf block, which has degree one in $\mathcal{T}$. Hence, B contains at most one vertex of X. Since X is nonempty and B is a leaf block, the unique cut-vertex of B (if any) belongs to X; otherwise, if B contained no vertex of X, then moving to the neighboring block in the unique path from B toward X would yield a leaf block at greater distance from X, contradicting the maximality of B. Therefore, B contains exactly one vertex of X. □

We begin with a basic relation between connected domination and zero forcing.

Theorem 3.2. *Let Q be a minimal connected dominating set of a connected graph G. Then*

$$V(G) \setminus Q$$

is a zero forcing set.

Proof. Let $S = V(G) \setminus Q$. Start the zero forcing process with all vertices of S colored.

Suppose, to the contrary, that the process stops before all vertices become colored. Let $U \subseteq Q$ be the set of vertices that remain uncolored at the end of the process. Since the process has stopped, no colored vertex has exactly one uncolored neighbor. Equivalently,

$$\forall x \in S, \quad |N(x) \cap U| \neq 1. \tag{1}$$

We first establish a standard structural property of minimal connected dominating sets.

Claim 1. *For every vertex $q \in Q$, at least one of the following holds:*

(1) *q is a cut-vertex of $G[Q]$;*
(2) *there exists a vertex $x \in V(G) \setminus Q$ such that $N(x) \cap Q = \{q\}$.*

Proof of Claim. Suppose neither condition holds for some $q \in Q$. Then q is not a cut-vertex of $G[Q]$, so $Q \setminus \{q\}$ induces a connected subgraph. Moreover, every vertex outside Q still has a neighbor in $Q \setminus \{q\}$, since condition (ii) fails. Hence $Q \setminus \{q\}$ remains a connected dominating set, contradicting the minimality of Q. □

Now consider the induced subgraph $G[U]$. Since $U \subseteq Q$, every vertex of U is a cut-vertex of $G[Q]$ by Claim 1; otherwise, if some $u \in U$ were not a cut-vertex of $G[Q]$, then either u has a private neighbor outside Q (which would be a colored vertex with exactly one uncolored neighbor, contradicting (1)) or $Q \setminus \{u\}$ remains connected dominating, contradicting minimality. More formally, applying Claim 1 to each $u \in U$, if u is not a cut-vertex of $G[Q]$, then there exists $x \in S$ with $N(x) \cap Q = \{u\}$; but then x has exactly one uncolored neighbor, namely, u, contradicting (1). Therefore, every vertex of U is a cut-vertex of $G[Q]$.

By Lemma 3.1 applied to $H = G[Q]$ and $X = U$, there exists a leaf block B of $G[U]$ such that B contains exactly one vertex of U. Let u be the unique vertex of U in B, and let $x \in S$ be a colored vertex adjacent to u.

By Claim 1, since $u \in Q$, either u is a cut-vertex of $G[Q]$ or there exists $x \in S$ such that $N(x) \cap Q = \{u\}$. We claim the first alternative cannot occur. If u were a cut-vertex of $G[Q]$, then $G[Q] - u$ would have at least two components. Since B is a leaf block of $G[U]$ containing u and no other vertex of U, the set $B \setminus \{u\}$ lies entirely in one component of $G[U] - u$. In the block-cut tree of $G[U]$, the block B corresponds to a leaf node; its only cut-vertex is u. Thus, B cannot separate distinct components of U. This contradicts the fact that u is a cut-vertex of $G[Q]$ separating vertices of U. Therefore, the first alternative is impossible.

Hence, there exists a colored vertex $x \in S$ such that

$$N(x) \cap Q = \{u\}.$$

Since every uncolored vertex lies in Q, the vertex u is the unique uncolored neighbor of x. Thus, x forces u, contradicting (1).

Therefore, $U = \emptyset$. Hence, all vertices are eventually colored, and $V(G) \setminus Q$ is a zero forcing set. □

Corollary 3.3. *For every connected graph G,*

$$Z(G) + \gamma_c(G) \leq n.$$

Proof. Let Q be a minimum connected dominating set. By Theorem 3.2,

$$V(G) \setminus Q$$

is a zero forcing set. Hence

$$Z(G) \leq n - \gamma_c(G).$$

Rearranging yields

$$Z(G) + \gamma_c(G) \leq n.$$

□

Example 3.4. *Consider the path P_4 with vertices v_1, v_2, v_3, v_4 in order. Let $Q = \{v_2, v_3\}$. Then, $G[Q]$ is connected and every vertex outside Q (namely, v_1 and v_4) has at least one neighbor in Q, so Q is a connected dominating set. It is minimal: removing v_2 would leave v_1 undominated, and removing v_3 would leave v_4 undominated. By Theorem 3.2, the complement $S = V \setminus Q = \{v_1, v_4\}$ should be zero forcing. Indeed, v_1 has unique uncolored neighbor v_2, so it forces v_2; then v_2 has unique uncolored neighbor v_3, so it forces v_3. Thus, all vertices become colored.*

Inequalities relating zero forcing and domination parameters have been studied previously in various settings; see, for example, [1,2].

4. Extremal graphs

We now study graphs attaining equality in Corollary 3.3.

The equality class is substantially larger than previously expected. In particular, complete multipartite graphs provide infinite additional families.

Proposition 4.1. *For every complete graph K_n,*

$$Z(K_n) + \gamma_c(K_n) = n.$$

Proof. Since every vertex is adjacent to all others, $\gamma_c(K_n) = 1$. Also, $Z(K_n) = n - 1$. Hence, $Z(K_n) + \gamma_c(K_n) = n$. □

Proposition 4.2. *For every cycle C_n with $n \geq 3$,*

$$Z(C_n) + \gamma_c(C_n) = n.$$

Proof. It is well known that

$$Z(C_n) = 2$$

for every cycle C_n with $n \geq 3$; see [1].

Also,

$$\gamma_c(C_n) = n - 2.$$

Hence,

$$Z(C_n) + \gamma_c(C_n) = n.$$

□

Proposition 4.3. *Let*

$$G = K_{n_1, n_2, \ldots, n_r}, \qquad r \geq 2, \qquad n_i \geq 2.$$

Then,

$$Z(G) + \gamma_c(G) = n.$$

Proof. Let $V_1, V_2, \ldots, V_r$ be the partite sets of G, where $|V_i| = n_i$ for each i. Choose vertices u, v from two different parts. Then, every vertex outside these two parts is adjacent to both u and v. Hence,

$$\gamma_c(G) = 2.$$

We next show

$$Z(G) = n - 2.$$

Choose vertices $a \in V_1,\ b \in V_2$, and let $S = V(G) \setminus \{a, b\}$.

Every colored vertex in $V_1 \setminus \{a\}$ has exactly one uncolored neighbor, namely b, and therefore forces b. Similar vertices in $V_2 \setminus \{b\}$ force a. Hence

$$Z(G) \leq n - 2.$$

Now suppose some sets T of size $n - 3$ were zero forcing. Then, at least three vertices are initially uncolored.

If these uncolored vertices lie in at least two different parts, then every colored vertex adjacent to one uncolored vertex is also adjacent to another uncolored vertex, so no colored vertex has exactly one uncolored neighbor. Hence, no forcing move is possible at the first step.

If all three uncolored vertices lie in the same part, say V_i, then every colored vertex outside V_i has at least two uncolored neighbors (since it is adjacent to all vertices in V_i), and every colored vertex in V_i has no uncolored neighbor at all (since all uncolored vertices are in V_i). Again no colored vertex has exactly one uncolored neighbor, so no forcing move is possible.

Thus, in either case, no forcing move can be made, contradicting the assumption that T is zero forcing. Hence, no set of size $n - 3$ is zero forcing. Therefore,

$$Z(G) \geq n - 2.$$

Consequently,

$$Z(G) = n - 2,$$

and thus

$$Z(G) + \gamma_c(G) = n.$$

□

Remark 4.4. *The graphs considered in Propositions 4.1–4.3—complete graphs, cycles, and complete multipartite graphs with all parts of size at least two—all satisfy $Z(G) + \gamma_c(G) = n$. These examples suggest that the equality class may be significantly larger than previously expected. In particular, complete multipartite graphs provide infinite families of examples beyond the classical cases noted in [6].*

It is natural to ask whether additional families of graphs can be characterized. We propose the following open problem (see also Problem 6.1).

5. Dom-forcing number

We now study the dom-forcing number.

Theorem 5.1. *For every connected graph G,*

$$\max\{Z(G), \gamma(G)\} \leq F_d(G) \leq Z(G) + \gamma(G).$$

Both bounds are sharp.

Proof. Every dom-forcing set is simultaneously dominating and zero forcing. Hence

$$F_d(G) \geq Z(G), \quad F_d(G) \geq \gamma(G).$$

Therefore

$$F_d(G) \geq \max\{Z(G), \gamma(G)\}.$$

Now let S be a minimum zero forcing set and D a minimum dominating set. Since every set containing a zero forcing set is again zero forcing, $S \cup D$ is zero forcing. It is also dominating because it contains D. Hence

$$F_d(G) \leq |S \cup D| \leq |S| + |D| = Z(G) + \gamma(G).$$

This proves the bounds. Sharpness is demonstrated by the examples below. □

We now determine exact values of $F_d(G)$ for several fundamental graph classes.

Proposition 5.2. *For the complete graph K_n with $n \geq 1$,*

$$F_d(K_n) = n - 1.$$

Proof. Every set of $n-1$ vertices in K_n is dominating, since the single vertex outside the set is adjacent to all vertices in the set. It is also zero forcing, as the unique uncolored vertex has at least one colored neighbor. Hence, $F_d(K_n) \leq n - 1$.

Now let S be any set of size at most $n - 2$. Then, at least two vertices remain uncolored. In a complete graph, every colored vertex is adjacent to all uncolored vertices, so every colored vertex has at least two uncolored neighbors; hence, no forcing move is possible. Thus, S is not zero forcing. Therefore, $F_d(K_n) \geq n - 1$.

Combining the two inequalities gives $F_d(K_n) = n - 1$. □

Proposition 5.3. *For the complete bipartite graph $K_{p,q}$ with $2 \leq p \leq q$,*

$$F_d(K_{p,q}) = p + q - 2.$$

Proof. Let the partite sets be A and B with $|A| = p$ and $|B| = q$. Choose vertices $a \in A$ and $b \in B$, and set $S = V(K_{p,q}) \setminus \{a, b\}$. Then, $|S| = p + q - 2$.

We first show that S is dominating. The vertex a is adjacent to every vertex in $B \setminus \{b\}$, which is nonempty since $q \geq 2$; hence, a has at least one neighbor in S. Similarly, b is adjacent to every vertex in $A \setminus \{a\}$, which is nonempty since $p \geq 2$. Thus, every vertex outside S is dominated by S.

We next show that S is zero forcing. Every vertex in $A \setminus \{a\}$ is adjacent to all vertices in B; among these, its only uncolored neighbor is b (since all vertices in $B \setminus \{b\}$ are in S and hence colored). Thus each vertex in $A \setminus \{a\}$ has exactly one uncolored neighbor, namely, b, and can force b. Similarly, each vertex in $B \setminus \{b\}$ has exactly one uncolored neighbor, namely, a, and can force a. Hence all vertices become colored. Therefore, S is zero forcing, so $F_d(K_{p,q}) \leq p + q - 2$.

It is known that $Z(K_{p,q}) = p + q - 2$ for $p, q \geq 2$ (see [1]). Since $F_d(G) \geq Z(G)$ for every graph G, we have $F_d(K_{p,q}) \geq p + q - 2$. Therefore, equality holds. □

Proposition 5.4. *For the path P_n with $n \geq 2$,*

$$F_d(P_n) = \left\lceil \frac{n}{3} \right\rceil.$$

Proof. It is well known that the domination number of the path P_n is $\gamma(P_n) = \lceil n/3 \rceil$. Since every dom-forcing set is a dominating set, we have

$$F_d(P_n) \geq \gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil.$$

It remains to show that there exists a minimum dominating set that is also zero forcing. Let the vertices of P_n be $v_1, v_2, \ldots, v_n$ in order. We construct a dominating set D of size $\lceil n/3 \rceil$ that contains v_1. If $n = 3t$, take

$$D = \{v_1, v_4, v_7, \ldots, v_{3t-2}\};$$

if $n = 3t + 1$, take

$$D = \{v_1, v_4, v_7, \ldots, v_{3t-2}, v_{3t+1}\};$$

if $n = 3t + 2$, take

$$D = \{v_1, v_4, v_7, \ldots, v_{3t-2}, v_{3t+2}\}.$$

In each case, $|D| = \lceil n/3 \rceil$, $v_1 \in D$, and every vertex not in D is within distance at most one from some vertex in D. Hence, D is a minimum dominating set.

Since $v_1 \in D$ and D is initially colored, the vertex v_1 has exactly one uncolored neighbor v_2 (if $v_2 \notin D$); it forces v_2. Once v_2 is colored, it forces v_3 if uncolored; the process continues along the path. Thus, every vertex of P_n eventually becomes colored. Therefore, D is zero forcing. Consequently,

$$F_d(P_n) \leq |D| = \left\lceil \frac{n}{3} \right\rceil.$$

Combining the two inequalities gives the result. □

Proposition 5.5. *For the star graph $K_{1,m}$ with $m \geq 2$,*

$$F_d(K_{1,m}) = m.$$

Proof. Let c be the center and let L be the set of m leaves. We first show that $F_d(K_{1,m}) \leq m$. The set L of all leaves is dominating, since c is adjacent to every vertex of L, and every leaf belongs to L. It is also zero forcing: initially all leaves are colored and c is the unique uncolored vertex; any leaf has c as its unique uncolored neighbor and thus forces c. Hence, L is a dom-forcing set, so $F_d(K_{1,m}) \leq m$.

We now prove the lower bound. Let $S \subseteq V(K_{1,m})$ with $|S| \leq m - 1$. We show that S is not a dom-forcing set.

If $c \notin S$, then $S \subseteq L$ and $|S| \leq m - 1$, so there exists a leaf $l \in L \setminus S$. The only neighbor of l is c, and $c \notin S$. Hence, l is not dominated by S, so S is not dominating.

If $c \in S$, then at least two leaves, say l_1, l_2, are not in S (since $|S| \leq m - 1$ and $c \in S$ accounts for one vertex). Initially l_1 and l_2 are uncolored. The center c is colored but has two uncolored neighbors, so it cannot perform a forcing move. Any colored leaf has c as its unique uncolored neighbor; however, forcing c would not reduce the set of uncolored vertices since c is already colored. Moreover, the two uncolored leaves l_1, l_2 have no neighbors other than c, so no vertex can force either of them. Thus l_1 and l_2 remain uncolored forever, and S is not zero forcing.

Therefore, no set of size at most $m - 1$ is both dominating and zero forcing. Hence $F_d(K_{1,m}) \geq m$. Combining with the upper bound gives equality. □

The following two observations follow directly from the definition of $F_d(G)$.

Proposition 5.6. *The equality*

$$F_d(G) = Z(G)$$

holds if, and only if, some minimum zero forcing set is dominating.

Proof. If some minimum zero forcing set S is dominating, then S is a dom-forcing set, so $F_d(G) \leq |S| = Z(G)$. The reverse inequality $F_d(G) \geq Z(G)$ holds for every graph. Hence,the equality holds.

Conversely, if $F_d(G) = Z(G)$, let D be a minimum dom-forcing set of size $Z(G)$. Then, D is zero forcing by definition and has size $Z(G)$, so it is a minimum zero forcing set; it is also dominating by definition. Thus, some minimum zero forcing set is dominating. □

Proposition 5.7. *The equality*

$$F_d(G) = \gamma(G)$$

holds if, and only if, some minimum dominating set is zero forcing.

Proof. The proof is analogous. If some minimum dominating set is zero forcing, it is a dom-forcing set, so $F_d(G) \leq \gamma(G)$. The reverse inequality $F_d(G) \geq \gamma(G)$ holds for every graph. Conversely, if $F_d(G) = \gamma(G)$, a minimum dom-forcing set of size $\gamma(G)$ is a minimum dominating set that is also zero forcing. □

Remark 5.8. *From Theorem 5.1 and Corollary 3.3, we obtain*

$$F_d(G) \leq \min\{Z(G) + \gamma(G),\ n - \gamma_c(G) + \gamma(G)\}.$$

The second bound follows by taking Q to be a minimum connected dominating set; by Theorem 3.2, $V(G) \setminus Q$ is zero forcing, and adding a minimum dominating set D yields a dom-forcing set of size at most $n - \gamma_c(G) + \gamma(G)$.

The lower bound $F_d(G) = Z(G)$ is attained by complete graphs and complete bipartite graphs with both parts of size at least two (Propositions 5.2 and 5.3). The lower bound $F_d(G) = \gamma(G)$ is attained by paths (Proposition 5.4). The upper bound $F_d(G) = Z(G) + \gamma(G)$ is attained by star graphs (Proposition 5.5), since $Z(K_{1,m}) = m - 1$, $\gamma(K_{1,m}) = 1$, and $F_d(K_{1,m}) = m$.

6. Open problems

The results of this paper suggest several natural directions for future work.

Problem 6.1. *Characterize all connected graphs satisfying*

$$Z(G) + \gamma_c(G) = n.$$

Complete graphs, cycles, and complete multipartite graphs with all parts of size at least two are known to satisfy the equality (Propositions 4.1–4.3). However, the full characterization remains open. In particular, it would be interesting to determine:

(i) whether the equality class is closed under certain graph operations, such as joins, Cartesian products, or lexicographic products;

(ii) whether there exist graphs satisfying the equality that are not complete multipartite or cycle-like;

(iii) whether the equality can be characterized by a simple condition on the block-cut tree or by a forbidden structure.

Computational experiments on graphs of small order suggest that the family may be broader than the examples given in this paper; a systematic computational study is left for future work.

Problem 6.2. *Determine the computational complexity of the dom-forcing number $F_d(G)$.*

Problem 6.3. *Find structural characterizations of graphs satisfying $F_d(G) = Z(G)$ or $F_d(G) = \gamma(G)$.*

Problem 6.4. *Investigate forcing-domination dualities for:*

- *connected forcing;*
- *total domination;*
- *k-forcing;*
- *directed graphs.*

Author contributions

Haining Jiang: Methodology, Validation, Writing–original draft, Writing–review and editing; Haiyun Wan: Writing–original draft, Writing–review and editing, Conceptualization. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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