Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(7): 19749–19771.
DOI: 10.3934/math.2026801
Received: 15 May 2026
Revised: 20 June 2026
Accepted: 01 July 2026
Published: 07 July 2026

Research article

Sombor index, polynomial, and energy of m-graphs of finite abelian groups

Hatice Çay*

Department of Computer Programming, IMU Vocational School, Istanbul Medipol University, Istanbul 34810, Turkey

* **Correspondence:** Email: hcay@medipol.edu.tr.

Abstract: Let H be a finite abelian group of order $n \geq 2$ and let $m > 1$ be an integer. The m-graph m-$G(H)$ is the simple undirected graph on vertex set H in which two distinct elements a and b are adjacent if and only if $a^m = b$ or $b^m = a$. Building on the structural characterization of connected m-graphs established by Badawi, we investigated degree-based topological invariants of connected m-$G(H)$ in both the cyclic case $H \cong \mathbb{Z}_n$ and the non-cyclic case $H \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}, r \geq 2$. Using an edge-type decomposition arising from the degree structure of connected m-graphs, we derived explicit formulas for the Sombor index, Sombor polynomial, reduced and modified Sombor indices, Euler Sombor index, average Sombor index, and forgotten topological index. In the non-cyclic setting, the parameter $\mathcal{G} = \prod_{j=1}^{r} \gcd(d_j, q_j)$ governs the edge-type counts $N_{BS} = D - \mathcal{G}$ and $N_{AS} = \mathcal{G} - 1$, which cannot be expressed purely in terms of D and Q. We also determined closed-form expressions for the Sombor energy in the cases $Q = 1$ (star graphs) and $Q = 2$ (two-hub trees). All theoretical results were independently verified by exact symbolic computation using SageMath 10.3 and the SymPy library.

Keywords: m-graph; Sombor index; forgotten index; Sombor energy; Sombor polynomial; finite abelian group; Euler Sombor index; average Sombor index
Mathematics Subject Classification: 20K01, 05C05, 05C07, 05C09, 05C25

1. Introduction

Graphs constructed from algebraic structures have been widely studied in recent years, revealing deep connections between group theory and graph theory. A prominent example is the power graph of a group, where two elements are adjacent if one is an integer power of the other.

Motivated by this construction, the notion of the m-graph associated with a finite abelian group H was introduced by Badawi [1]. For a fixed integer $m > 1$, adjacency is defined by

$$a \sim b \iff a^m = b \text{ or } b^m = a, \quad a \neq b.$$

Restricting to a single fixed exponent m yields a more structured and parameter-dependent family of graphs, whose combinatorial and spectral properties depend explicitly on the arithmetic of m and $|H|$.

Badawi [1] established several fundamental properties of the m-graph of H, namely m-$G(H)$. In particular, m-$G(H)$ is connected if and only if every prime divisor of $|H|$ divides m, and the connected graph m-$G(\mathbb{Z}_n)$ is always a tree. Exact degree formulas were derived for both cyclic and non-cyclic groups, and the diameter and graph isomorphisms of m-graphs were also investigated.

The Sombor index, introduced by Gutman [2] as

$$\mathrm{SO}(\Gamma) = \sum_{uv \in E} \sqrt{d(u)^2 + d(v)^2},$$

and its degree-based variants—including the Sombor polynomial, reduced and modified Sombor indices, Euler Sombor index, average Sombor index [3–6], and Sombor energy—are prominent topological invariants with established correlations to physico-chemical properties [2, 4]. The present paper derives closed-form expressions for all of these invariants on the algebraically defined graph family m-$G(H)$, together with exact spectral results for the Sombor matrix $\mathfrak{S}$.

Power graphs of finite groups have been extensively studied from a topological index perspective. Eccentricity-based indices, Zagreb indices, and Hosoya properties of power graphs $\mathcal{P}(G)$ have been computed for cyclic and non-cyclic groups of order $p_1 p_2$ [7, 8].

Recent years have witnessed growing interest in computing Sombor-type indices for graphs arising from algebraic structures. Alimon et al. [9] computed the Sombor index and polynomial of power graphs for several families of finite groups, obtaining results case by case for specific group orders. Romdhini and Nawawi [10] investigated the Sombor energy and spectral radius of non-commuting graphs of dihedral groups D_{2n}, deriving formulas that depend separately on the parity of n rather than a single unified expression. Rather and Imran [11] studied the Sombor index of total and unit graphs of commutative rings $\mathbb{Z}_n$, providing numerical computations for specific values of n. Pratama et al. [12] computed the Sombor index of power graphs of groups of prime power order, again through case-by-case analysis. A common feature of these works is that they yield numerical results for particular groups or group families, without providing unified closed-form expressions valid across all parameter values. By contrast, the present paper derives exact closed-form formulas for the full range of connected m-graphs m-$G(H)$, expressed in terms of the parameters D, Q, and the newly introduced $\mathcal{G}$, covering both cyclic and non-cyclic finite abelian groups simultaneously.

A distinguishing feature of m-$G(H)$ is that, under the connectivity condition of Theorem 2.1, it is always a *tree*. This structural rigidity—absent in the classical power graph—makes m-$G(H)$ particularly amenable to exact closed-form analysis: vertex degrees take only three values, the Sombor matrix has at most four distinct nonzero entries, and the characteristic polynomial factors completely for $Q \leq 2$.

Despite this elegant structure, deriving explicit formulas for degree-based invariants presents non-trivial challenges. The central difficulty lies in counting the four edge types ($\mathcal{B} \to \mathcal{S}$, $\mathcal{B} \to \mathcal{A}$, $\mathcal{A} \to \mathcal{S}$, $\mathcal{A} \to \mathcal{A}$) precisely. In the cyclic case $H \cong \mathbb{Z}_n$, the counts depend on the residue $s = m \bmod q$, which captures the arithmetic interaction between m and the quotient structure of $\mathbb{Z}_n$. In the non-cyclic case, the componentwise action of multiplication by m introduces a genuinely new parameter $\mathcal{G} = \prod_{j=1}^{r} \gcd(d_j, q_j)$, which does not appear in Badawi [1] and cannot be recovered from D and Q alone.

The main contribution of this work is a complete edge-type decomposition for connected m-graphs, in which the entire edge set is partitioned into the four classes above, with each class carrying a constant Sombor weight determined by D, Q, and $\mathcal{G}$. The introduction of $\mathcal{G}$ and the rigorous justification of its role in the edge-count formulas constitute the core theoretical innovation. As a consequence, all degree-based Sombor-type indices follow by direct substitution into a single unified formula.

Several directions remain open for future investigation. First, the spectral analysis carried out here for $Q = 1$ and $Q = 2$ does not extend directly to $Q \geq 3$: when the number of hub vertices exceeds two, the Sombor matrix no longer reduces to a low-rank perturbation, and new techniques—such as equitable partition methods or interlacing inequalities—will be required to obtain closed-form energy formulas. Second, the present work is restricted to finite *abelian* groups, where the componentwise structure of the direct sum $\mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ is essential to the proofs. Extending the edge-type decomposition to m-graphs over non-abelian groups would require a fundamentally different approach, as the map $a \mapsto a^m$ need not act componentwise in that setting. Third, the parameter $\mathcal{G}$ introduced here suggests a deeper algebraic invariant whose properties merit independent study, particularly its relationship to the subgroup structure of H.

For convenience, let N_{XY} denote the number of edges with one endpoint in X and the other endpoint in Y, where $X, Y \in \{\mathcal{B}, \mathcal{A}, \mathcal{S}\}$. In particular, $N_{BS}, N_{BA}, N_{AS}, N_{AA}$ represent the numbers of edges of types $\mathcal{B}$–$\mathcal{S}$, $\mathcal{B}$–$\mathcal{A}$, $\mathcal{A}$–$\mathcal{S}$, and $\mathcal{A}$–$\mathcal{A}$, respectively.

2. Background

All groups considered in this paper are finite abelian and the identity element is 0. For $H = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ ($r \geq 1$) and integer $m > 1$, define

$$d_j = \gcd(m, n_j), \quad q_j = \frac{n_j}{d_j}, \quad D = \prod_{j=1}^{r} d_j, \quad Q = \prod_{j=1}^{r} q_j, \quad \mathcal{G} = \prod_{j=1}^{r} \gcd(d_j, q_j), \quad n = |H| = DQ.$$

For the cyclic case $r = 1$, we write $k = d_1$ and $q = q_1$. In this case, we further define $s = m \bmod q$, which appears in the edge-count formula for N_{AS} (Theorem 4.3).

Theorem 2.1 ([1, Theorem 2.2]). *m-$G(H)$ is connected if and only if every prime divisor of n also divides m.*

Theorem 2.2 ([1, Theorems 3.1, 3.4, 4.1]). *Assume m-$G(H)$ is connected. Then:*

(1) m-$G(H)$ is a tree (connected, acyclic, chromatic number 2).

(2) The vertex degrees are:

$$d(v) = \begin{cases} D - 1, & v = 0, \\ D + 1, & v \neq 0 \ \textit{and} \ d_j \mid v_j \ \textit{for all}\ j, \\ 1, & d_j \nmid v_j \ \textit{for some}\ j, \end{cases}$$

where $v = (v_1, \ldots, v_r) \in \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ and each v_j denotes the value of the j-th component of v in $\mathbb{Z}_{n_j}$. For the cyclic case $r = 1$, the condition $d_1 \mid v_1$ reduces to $k \mid a$ in $\mathbb{Z}$, recovering Badawi's original formulation.

(3) (Cyclic) m-$G(\mathbb{Z}_n) \cong k$-$G(\mathbb{Z}_n)$.
(Non-cyclic) m-$G(H) \cong k$-$G(H) \cong (d_1, \ldots, d_r)$-$PG(H)$*, where* $(d_1, \ldots, d_r)$-$PG(H)$ *denotes the* product graph *of* H *with parameters* $(d_1, \ldots, d_r)$*, as defined in Badawi [1, Definition 4.1].*

Definition 2.3 (Vertex types)**.** *For connected* m-$G(H)$*, partition* H *into three types:*

$$\mathcal{S} = \{0\}, \quad \mathcal{A} = \{v \neq 0 : d_j \mid v_j \ \forall\, j\}, \quad \mathcal{B} = \{v : d_j \nmid v_j \text{ for some } j\},$$

where v_j *is the value of the* j*-th component of* v *in* $\mathbb{Z}_{n_j}$*. By Theorem 2.2:*

$$d(v) = D - 1 \ (v \in \mathcal{S}), \qquad d(v) = D + 1 \ (v \in \mathcal{A}), \qquad d(v) = 1 \quad (v \in \mathcal{B}).$$

Sizes: $|\mathcal{S}| = 1$, $|\mathcal{A}| = Q - 1$, $|\mathcal{B}| = n - Q$.

Remark 2.4. *The parameter* $\mathcal{G} = \prod_{j=1}^{r} \gcd(d_j, q_j)$ *is introduced here to express the edge-type counts* N_{BS} *and* N_{AS} *in closed form for the non-cyclic setting (Theorem 4.5), and to determine the inter-hub edge type in the* $Q = 2$ *energy formula (Corollary 7.3).*

The Sombor weight of each edge type is determined solely by the degree pair, which is constant within each type by Theorem 2.2:

Type	Degrees $(d(u), d(v))$	Weight w	w^2
$\mathcal{B} \to \mathcal{S}$	$(1,\ D-1)$	$\sqrt{D^2 - 2D + 2}$	$D^2 - 2D + 2$
$\mathcal{B} \to \mathcal{A}$	$(1,\ D+1)$	$\sqrt{D^2 + 2D + 2}$	$D^2 + 2D + 2$
$\mathcal{A} \to \mathcal{S}$	$(D+1,\ D-1)$	$\sqrt{2D^2 + 2}$	$2D^2 + 2$
$\mathcal{A} \to \mathcal{A}$	$(D+1,\ D+1)$	$\sqrt{2}(D+1)$	$2(D+1)^2$

3. Sombor invariants

Definition 3.1 ([2, 13])**.** *Let* $\Gamma = (V, E)$ *be a graph, where* V *is the vertex set and* $E \subseteq \binom{V}{2}$ *is the edge set. For each vertex* $u \in V$*, let* $d(u)$ *denote the degree of* u*, i.e., the number of vertices adjacent to* u*. The* Sombor index *of* Γ *is defined as*

$$\mathrm{SO}(\Gamma) = \sum_{uv \in E(\Gamma)} \sqrt{d(u)^2 + d(v)^2}.$$

The generating function corresponding to the Sombor index (or the Sombor polynomial*) is given by*

$$\mathrm{SO}(\Gamma; x) = \sum_{uv \in E(\Gamma)} \frac{x^{d(u)^2 + d(v)^2}}{\sqrt{d(u)^2 + d(v)^2}}.$$

Furthermore, the Sombor matrix *of the graph* Γ *is denoted by* $\mathfrak{S}(\Gamma) = [\mathfrak{S}_{ij}]$ *and defined as*

$$\mathfrak{S}_{ij} = \begin{cases} \sqrt{d(v_i)^2 + d(v_j)^2}, & \text{if } v_i \sim v_j, \\ 0, & \text{otherwise}, \end{cases}$$

where $v_i \sim v_j$ indicates that the vertices v_i and v_j are adjacent.

Finally, if $\lambda_1(\mathfrak{S}), \lambda_2(\mathfrak{S}), \ldots, \lambda_n(\mathfrak{S})$ are the eigenvalues of the matrix $\mathfrak{S}(\Gamma)$, then the Sombor energy *is defined by*

$$\mathcal{E}_S(\Gamma) = \sum_{i=1}^{n} |\lambda_i(\mathfrak{S})|.$$

4. Edge-type counts

In this section, we establish an edge characterization of the considered structure using Proposition 4.1. We then determine the edge counts for both the cyclic and non-cyclic cases. Since the graph is simple and undirected, these are counts of distinct unordered edges.

Proposition 4.1. *The edge set of m-$G(H)$ is $E = \{\{a, b\} : a, b \in H,\ a \neq b, a^m = b \text{ or } b = a^m\}$, where each unordered pair $\{a, b\}$ with $a^m = b$ or $b^m = a$ appears at least once. If $a^m = b$ and $b^m \neq a$ (or vice versa), the edge is generated by exactly one endpoint; if both $a^m = b$ and $b^m = a$ hold simultaneously, the edge is generated by both endpoints but still represents a single undirected edge.*

Proof. $\{a, b\} \in E$ iff $a^m = b$ or $b^m = a$. If $a^m = b$, the pair $\{a, a^m\}$ is generated by a. If $b^m = a$, the pair $\{b, b^m\} = \{a, b\}$ is generated by b. Both conditions can hold simultaneously (e.g., $H = \mathbb{Z}_3$, $m = 2$: $1^2 \equiv 2$ and $2^2 \equiv 1$), in which case the same edge is generated by both endpoints. In either case, E contains each such pair as a single undirected edge. □

Lemma 4.2. *Let $\Phi : \mathcal{A} \to E, a \mapsto \{a, a^m\}$, be the map assigning to each vertex $a \in \mathcal{A}$ the unique edge generated by Proposition 4.1. Then Φ is injective. Consequently, $N_{AS} + N_{AA} = |\mathcal{A}|$.*

Proof. By Proposition 4.1, every vertex $a \in \mathcal{A}$ generates exactly one edge, $\{a, a^m\}$, whose second endpoint lies in $\mathcal{A} \cup \mathcal{S}$. Hence the image of Φ consists precisely of all edges of types $\mathcal{A}-\mathcal{S}$ and $\mathcal{A}-\mathcal{A}$. It remains to prove injectivity. Suppose $\{a, a^m\} = \{b, b^m\}$ for some $a, b \in \mathcal{A}$.

If $a = b$, there is nothing to prove. Otherwise, since the unordered edges coincide, we must have $a^m = b, b^m = a$. Thus a and b form a two-cycle of the map $x \mapsto x^m$.

However, by Theorem 2.2, the connected graph m-$G(H)$ is a tree, and therefore contains no cycles. Indeed, otherwise, one would obtain $a^m = b, b^m = a$, which gives the closed walk $a \to b \to a$ of length 2 with $a \neq b$. Since m-$G(H)$ is a tree and therefore acyclic, it contains no nontrivial closed walk. This is a contradiction.

Hence Φ is injective. Since every vertex of $\mathcal{A}$ generates exactly one distinct edge of type $\mathcal{A}-\mathcal{S}$ or $\mathcal{A}-\mathcal{A}$, we conclude that $N_{AS} + N_{AA} = |\mathcal{A}|$. □

Theorem 4.3. *Let $H = \mathbb{Z}_n$, $k = \gcd(m, n)$, $q = n/k$, where $s = m \bmod q$. We use the standard convention $\gcd(0, q) = q$, which corresponds to the case $q \mid m$. Then: $N_{BS} = k - \gcd(k, q), N_{BA} = (n - q) - N_{BS}, N_{AS} = \gcd(s, q) - 1, N_{AA} = (q - 1) - N_{AS}$.*

Proof. Write $m = km'$, where $k = \gcd(m, n)$ and $\gcd(m', q) = 1$. We compute each edge-type count in turn.

Count of N_{BS}. An element $x \in \mathcal{B}$ satisfies $mx \equiv 0 \pmod{n}$ if and only if $q \mid x$. The nonzero multiples of q lying in $\mathbb{Z}_n$ form the set $\{q, 2q, \ldots, (k-1)q\}$, which has cardinality $k - 1$. Among these, the element iq belongs to $\mathcal{A}$ if and only if $k \mid iq$, equivalently $\frac{k}{\gcd(k,q)} \mid i$, and this condition is satisfied for precisely $\gcd(k,q)-1$ indices $i \in \{1,\ldots,k-1\}$. Subtracting those that fall in $\mathcal{A}$ from the total gives $N_{BS} = (k-1) - (\gcd(k,q)-1) = k - \gcd(k,q)$.

Count of N_{BA}. By Theorem 2.2, every vertex $x \in \mathcal{B}$ has degree 1, so its unique edge is incident to either $\mathcal{S}$ or $\mathcal{A}$. Since $|\mathcal{B}| = n-q$, it follows immediately that $N_{BA} = (n-q) - N_{BS}$.

Count of N_{AS}. For a vertex $v = ik \in \mathcal{A}$ with $i \in \{1,\ldots,q-1\}$, one computes $mv = mik \equiv k(mi \bmod q) \pmod n$, so $mv \equiv 0 \pmod n$ if and only if $mi \equiv 0 \pmod q$, or equivalently $si \equiv 0 \pmod q$, where $s = m \bmod q$. The congruence $si \equiv 0 \pmod q$ admits exactly $\gcd(s,q)$ solutions modulo q, one of which is $i = 0$; hence the number of solutions in $\{1,\ldots,q-1\}$ equals $\gcd(s,q)-1$. When $s = 0$, the identity $\gcd(0,q) = q$ is consistent with the observation that $q \mid m$ forces $mv \equiv 0$ for every $v \in \mathcal{A}$. Therefore, $N_{AS} = \gcd(s,q) - 1$.

Count of N_{AA}. By Lemma 4.2, $N_{AS} + N_{AA} = |\mathcal{A}| = q-1$. Therefore, $N_{AA} = (q-1) - N_{AS}$. □

Remark 4.4. *In the cyclic case, $N_{AS} = \gcd(s,q)-1$, $s = m \bmod q$, counts the nonidentity vertices $v \in \mathcal{A}$ satisfying $mv = 0$ in $\mathbb{Z}_n$. Indeed, for $v = ik \in \mathcal{A}$ with $i \in \{1,\ldots,q-1\}$: $mv \equiv 0 \pmod n \iff q \mid mi \iff q \mid si$. The congruence $si \equiv 0 \pmod q$ has exactly $\gcd(s,q)$ solutions modulo q, one of which is $i = 0$ corresponding to the identity element 0. Hence the number of nonidentity solutions equals $\gcd(s,q)-1$. In particular, when $s = 0$ (equivalently, $q \mid m$), we have $mv \equiv 0 \pmod n$ for all $v \in \mathcal{A}$, so $N_{AS} = |\mathcal{A}| = q-1$.*

For example, for $H = \mathbb{Z}_{27}$ and $m = 3$: $k = 3, q = 9, s = 3$, so $N_{AS} = \gcd(3,9) - 1 = 2$. Indeed, the vertices of $\mathcal{A}$ are $3, 6, 9, 12, 15, 18, 21, 24$, and exactly $v = 9$ and $v = 18$ satisfy $mv \equiv 0 \pmod{27}$: $3 \cdot 9 = 27 \equiv 0, 3 \cdot 18 = 54 \equiv 0 \pmod{27}$.

Theorem 4.5. *Let $H = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ with $r \geq 2$, and use the parameters $D, Q, \mathcal{G}$ defined in Section 2. Then: $N_{BS} = D - \mathcal{G}, N_{BA} = (n-Q) - N_{BS}, N_{AS} = \mathcal{G} - 1, N_{AA} = (Q-1) - N_{AS}$.*

Proof. Define the multiplication-by-m map $\varphi\colon H \to H$ by $\varphi(a) = ma$. Since $H = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ is a direct sum, the map φ acts componentwise as $\varphi(a_1,\ldots,a_r) = (ma_1,\ldots,ma_r)$. For each index j, write $m = d_j m'_j$, where $d_j = \gcd(m,n_j)$; the maximality of d_j ensures $\gcd(m'_j, q_j) = 1$. We determine each edge-type count separately.

Count of N_{BS}. An element $(a_1,\ldots,a_r) \in \mathcal{B}$ satisfies $\varphi(a) = 0$ if and only if $ma_j \equiv 0 \pmod{n_j}$ for every j, which, in view of $\gcd(m'_j, q_j) = 1$, is equivalent to requiring $q_j \mid a_j$ for all j. Define $P = \{a \in H : q_j \mid a_j \text{ for all } j\}$. By the independence of components, $|P| = \prod_j (n_j/q_j) = \prod_j d_j = D$. Furthermore, $a \in P \cap (\mathcal{A} \cup \mathcal{S})$ if and only if $q_j \mid a_j$ and $d_j \mid a_j$ simultaneously for every j, that is, $\operatorname{lcm}(d_j, q_j) \mid a_j$ for all j. The number of such elements equals

$$\prod_j \frac{n_j}{\operatorname{lcm}(d_j, q_j)} = \prod_j \gcd(d_j, q_j) = \mathcal{G}.$$

Consequently, $N_{BS} = |P \cap \mathcal{B}| = D - \mathcal{G}$.

Count of N_{BA}. By Theorem 2.2, every vertex $a \in \mathcal{B}$ has degree 1, so its unique edge is incident to either $\mathcal{S}$ or $\mathcal{A}$. Since $|\mathcal{B}| = n - Q$, it follows that $N_{BA} = (n-Q) - N_{BS}$.

Count of N_{AS}. Let $a = (a_1,\ldots,a_r) \in \mathcal{A}$, so that $d_j \mid a_j$ for every j. Setting $a_j = d_j t_j$ with $t_j \in \{0,\ldots,q_j - 1\}$, the condition $\varphi(a) = 0$ reduces to $ma_j \equiv 0 \pmod{n_j}$ for all j, which successively simplifies to

$d_jq_j \mid d_jm'_jd_jt_j$, then to $q_j \mid m'_jd_jt_j$, and finally, since $\gcd(m'_j, q_j) = 1$, to $\frac{q_j}{\gcd(d_j,q_j)} \mid t_j$. The number of solutions $t_j \in \{0, \ldots, q_j - 1\}$ satisfying this condition is $\gcd(d_j, q_j)$. Invoking the independence of components, the total number of elements $a \in \mathcal{A} \cup \mathcal{S}$ with $\varphi(a) = 0$ is $\prod_j \gcd(d_j, q_j) = \mathcal{G}$. Excluding the identity element yields $N_{AS} = \mathcal{G} - 1$.

Count of N_{AA}. By Lemma 4.2, $N_{AS} + N_{AA} = |\mathcal{A}| = Q - 1$. Hence $N_{AA} = (Q - 1) - N_{AS}$. □

Remark 4.6. *The edge-count formulas differ structurally between the two settings. For N_{BS}, the cyclic case yields $k - \gcd(k, q)$, whereas the non-cyclic case yields $D - \mathcal{G}$. For N_{AS}, the cyclic formula depends on the residue $s = m \bmod q$, while the non-cyclic formula depends on $\mathcal{G} = \prod_{j=1}^{r} \gcd(d_j, q_j)$.*

Note that $\mathcal{G} = Q$ does not hold in general. For instance, $H = \mathbb{Z}_9 \oplus \mathbb{Z}_{27}$ with $m = 3$ gives $\mathcal{G} = 9 \neq 27 = Q$. The equality $\mathcal{G} = Q$ holds if and only if $q_j \mid d_j$ for all j; for example, $H = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ with $m = 4$ satisfies $d_j = 4$, $q_j = 1$, and $\mathcal{G} = 1 = Q$.

For $r = 1$, the non-cyclic formula $N_{AS} = \gcd(d_1, q_1) - 1$ coincides with the cyclic formula $N_{AS} = \gcd(s, q) - 1$, since $\gcd(d_1, q_1) = \gcd(k, q) = \gcd(m, q) = \gcd(s, q)$.

The reason these two quantities govern N_{AS} in their respective settings is the following. In the cyclic case $H = \mathbb{Z}_n$, a vertex $v = ik \in \mathcal{A}$ satisfies $mv \equiv 0 \pmod{n}$ if and only if $q \mid si$, so the residue $s = m \bmod q$ is the natural arithmetic datum. In the non-cyclic case $H = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$, define the multiplication-by-m map $\varphi\colon H \to H$ by $\varphi(\mathbf{a}) = m\mathbf{a}$, acting componentwise as $\varphi(a_1, \ldots, a_r) = (ma_1 \bmod n_1, \ldots, ma_r \bmod n_r)$. A vertex $\mathbf{a} \in \mathcal{A}$ generates an edge of type $\mathcal{A} \to \mathcal{S}$ if and only if $\varphi(\mathbf{a}) = \mathbf{0}$, that is, $q_j \mid ma_j$ for every j. The number of elements in $\mathcal{A} \cup \mathcal{S}$ satisfying this condition equals $\prod_{j=1}^{r} \gcd(d_j, q_j) = \mathcal{G}$, as established in the proof of Theorem 4.5. The two formulas are thus manifestations of the same divisibility condition in different group-theoretic settings, and they agree at $r = 1$ because $\gcd(d_1, q_1) = \gcd(s, q)$.

5. Sombor index and its applications to Sombor-type invariants

Theorem 5.1. *Let m-$G(H)$ be a connected m-graph, and let $I_W(\Gamma) = \sum_{uv \in E(\Gamma)} W(d(u), d(v))$ be a degree-based edge invariant, where $W : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is a symmetric function. Then*

$$I_W(m\text{-}G(H)) = N_{BS}\, W(1, D - 1) + N_{BA} W(1, D + 1) + N_{AS}\, W(D + 1, D - 1) + N_{AA} W(D + 1, D + 1).$$

Proof. By Theorems 4.3 and 4.5, every edge of m-$G(H)$ belongs to exactly one of the four edge classes $\mathcal{BS}, \mathcal{BA}, \mathcal{AS}, \mathcal{AA}$. By Theorem 2.2, the endpoint degree pair is constant within each class, taking the respective values $(1, D{-}1)$, $(1, D{+}1)$, $(D{+}1, D{-}1)$, and $(D{+}1, D{+}1)$. Since there are $N_{BS}, N_{BA}, N_{AS}, N_{AA}$ edges of these types, summing the weight $W(d(u), d(v))$ over each class gives

$$I_W(m\text{-}G(H)) = N_{BS}\, W(1, D - 1) + N_{BA} W(1, D + 1) + N_{AS}\, W(D + 1, D - 1) + N_{AA} W(D + 1, D + 1).$$

This completes the proof. □

Corollary 5.2. *For connected m-$G(H)$, the Sombor index is:*

$$\mathrm{SO} = N_{BS} \sqrt{D^2 - 2D + 2} + N_{BA} \sqrt{D^2 + 2D + 2} + N_{AS} \sqrt{2D^2 + 2} + N_{AA} \sqrt{2}(D + 1),$$

where the counts are from Theorem 4.3 (cyclic) or Theorem 4.5 (non-cyclic).

Proof. Applying Theorem 5.1 with $W(x, y) = \sqrt{x^2 + y^2}$ completes the proof. □

Corollary 5.3. *For connected m-$G(H)$, with zero-coefficient terms omitted, the Sombor polynomial is:*

$$\begin{aligned}\mathrm{SO}(H;\ x) = \ &\frac{N_{BS}}{\sqrt{D^2-2D+2}}\, x^{D^2-2D+2} + \frac{N_{BA}}{\sqrt{D^2+2D+2}}\, x^{D^2+2D+2} \\ &+ \frac{N_{AS}}{\sqrt{2D^2+2}}\, x^{2D^2+2} + \frac{N_{AA}}{\sqrt{2}(D+1)}\, x^{2(D+1)^2}.\end{aligned}$$

Proof. Applying Theorem 5.1 with

$$W(x, y) = \frac{x^{x^2+y^2}}{\sqrt{x^2+y^2}}$$

completes the proof. □

Definition 5.4 ([14])**.** *The forgotten topological index is defined as* $F(\Gamma) = \sum_{v \in V} d(v)^3$.

Corollary 5.5. *For connected m-$G(H)$:*

$$F(m\text{-}G(H)) = (D-1)^3 + (Q-1)(D+1)^3 + (n-Q) = Q(D+1)^3 - 6D^2 - 2 + n - Q.$$

Proof. The vertex set admits the partition $H = \mathcal{S} \cup \mathcal{A} \cup \mathcal{B}$ with $|\mathcal{S}| = 1$, $|\mathcal{A}| = Q - 1$, and $|\mathcal{B}| = n - Q$, where the corresponding degrees are $D - 1$, $D + 1$, and 1, respectively. Substituting into the definition of the forgotten topological index gives

$$F = (D-1)^3 + (Q-1)(D+1)^3 + (n-Q).$$

Expanding and collecting like terms yields the stated closed form. □

Corollary 5.6. *Let $\mathfrak{S}$ be the Sombor matrix of the m-graph (m-$G(H)$). The lower bound of the spectral radius is*

$$\lambda_1(\mathfrak{S}) \geq \sqrt{\frac{2F(m\text{-}G(H))}{n}} = \sqrt{\frac{2[Q(D+1)^3 - 6D^2 - 2 + n - Q]}{n}}.$$

Proof. Since $\mathfrak{S}$ is real symmetric, $\sum_i \lambda_i^2 = \mathrm{tr}(\mathfrak{S}^2) = 2\sum_{uv \in E}(d(u)^2 + d(v)^2) = 2F(m\text{-}G(H))$. Since $\lambda_1^2 \geq \frac{1}{n}\sum_i \lambda_i^2$, the result follows. □

Example 5.7 ($\mathbb{Z}_2 \oplus \mathbb{Z}_4$, $m = 2$)**.** *$D = 4$, $Q = 2$, $n = 8$: $F = 3^3 + 1 \cdot 5^3 + 6 = 27 + 125 + 6 = 158$; $\lambda_1 \geq \sqrt{39.5} \approx 6.285$. Since the only prime divisor of $|H| = 8 = 2^3$ is 2, and $2 \mid m = 2$, Theorem 2.1 ensures that m-$G(H)$ is connected.*

Example 5.8 ($\mathbb{Z}_9 \oplus \mathbb{Z}_{27}$, $m = 3$)**.** *$D = 9$, $Q = 27$, $n = 243$: $F = 8^3 + 26 \cdot 10^3 + 216 = 512 + 26000 + 216 = 26728$; $\lambda_1 \geq \sqrt{2 \cdot 26728/243} \approx 14.83$. Since the only prime divisor of $|H| = 243 = 3^5$ is 3, and $3 \mid m = 3$, Theorem 2.1 ensures that m-$G(H)$ is connected.*

Definition 5.9. *Recall from [2] and [15], respectively, that the reduced Sombor index and the modified Sombor index are defined as*

$$\mathrm{SO}_{\mathrm{red}}(\Gamma) = \sum_{uv \in E(\Gamma)} \sqrt{(d(u)-1)^2 + (d(v)-1)^2},$$

and

$$m\mathrm{SO}(\Gamma) = \sum_{uv \in E(\Gamma)} \frac{1}{\sqrt{d(u)^2 + d(v)^2}}.$$

Lemma 5.10. *For connected m-$G(H)$ with $D \geq 2$, the reduced Sombor weights are:*

Type	*Degrees*	*Reduced weight*
$\mathcal{B} \to \mathcal{S}$	$(1, D-1)$	$D-2$
$\mathcal{B} \to \mathcal{A}$	$(1, D+1)$	D
$\mathcal{A} \to \mathcal{S}$	$(D+1, D-1)$	$\sqrt{2D^2-4D+4}$
$\mathcal{A} \to \mathcal{A}$	$(D+1, D+1)$	$D\sqrt{2}$

Proof. Substituting each degree pair directly into the reduced Sombor weight formula $\sqrt{(d(u)-1)^2+(d(v)-1)^2}$, we obtain:

$$\begin{aligned}
\mathcal{B} \to \mathcal{S}: &\quad \sqrt{(1-1)^2+(D-1-1)^2} = \sqrt{(D-2)^2} = D-2,\\
\mathcal{B} \to \mathcal{A}: &\quad \sqrt{(1-1)^2+(D+1-1)^2} = \sqrt{D^2} = D,\\
\mathcal{A} \to \mathcal{S}: &\quad \sqrt{(D+1-1)^2+(D-1-1)^2} = \sqrt{D^2+(D-2)^2} = \sqrt{2D^2-4D+4},\\
\mathcal{A} \to \mathcal{A}: &\quad \sqrt{(D+1-1)^2+(D+1-1)^2} = \sqrt{2D^2} = D\sqrt{2}.
\end{aligned}$$

□

Corollary 5.11. *For connected m-$G(H)$ with $D \geq 2$, the reduced Sombor index is:*

$$\mathrm{SO}_{\mathrm{red}}(m\text{-}G(H)) = N_{BS}(D-2) + N_{BA} \cdot D + N_{AS}\sqrt{2D^2-4D+4} + N_{AA} \cdot D\sqrt{2}.$$

Proof. Applying Theorem 5.1 with $W(x,y) = \sqrt{(x-1)^2+(y-1)^2}$ completes the proof. □

Remark 5.12. *When $D = 2$, the $\mathcal{B} \to \mathcal{S}$ edges contribute zero weight ($D-2 = 0$), reflecting that $\mathcal{S}$ has degree $D-1 = 1$ and the reduced weight of a $(1,1)$-edge vanishes.*

Corollary 5.13. *For connected m-$G(H)$, the modified Sombor index is:*

$$m\mathrm{SO}(m\text{-}G(H)) = \frac{N_{BS}}{\sqrt{D^2-2D+2}} + \frac{N_{BA}}{\sqrt{D^2+2D+2}} + \frac{N_{AS}}{\sqrt{2D^2+2}} + \frac{N_{AA}}{\sqrt{2}(D+1)}.$$

Proof. Applying Theorem 5.1 with $W(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ completes the proof. □

Remark 5.14. *By the Cauchy–Schwarz inequality,* $\mathrm{SO}(\Gamma) \cdot m\mathrm{SO}(\Gamma) \geq |E|^2$, *with equality if and only if* Γ *is regular. For a connected m-$G(H)$, the vertex degrees belong to* $\{1,\ D-1,\ D+1\}$. *Thus the graph can be regular only if* $1 = D-1 = D+1$, *which is impossible, or if only one degree value occurs. In the connected setting, this happens precisely when $D = 2$, yielding the graph K_2 (corresponding to $H \cong \mathbb{Z}_2$). Therefore equality holds only for K_2, and the inequality is strict for all connected m-$G(H)$ with more than two vertices.*

Definition 5.15 ([5]). *The Euler Sombor index is defined as follows:*

$$\mathrm{EUS}(\Gamma) = \sum_{uv \in E} \sqrt{d(u)^2 + d(v)^2 + d(u)\,d(v)}.$$

Lemma 5.16. *For connected m-$G(H)$ graphs with $D \geq 2$, the Euler Sombor weights are as follows:*

Type	*Degree pair*	*Euler Sombor weight* w^{EUS}
$\mathcal{B} \to \mathcal{S}$	$(1, D-1)$	$\sqrt{D^2 - D + 1}$
$\mathcal{B} \to \mathcal{A}$	$(1, D+1)$	$\sqrt{D^2 + 3D + 3}$
$\mathcal{A} \to \mathcal{S}$	$(D+1, D-1)$	$\sqrt{3D^2 + 1}$
$\mathcal{A} \to \mathcal{A}$	$(D+1, D+1)$	$(D+1)\sqrt{3}$

Proof. By Proposition 4.1 and Definition 2.3, every edge belongs to exactly one of the four types, and the weight $\sqrt{d(u)^2 + d(u)d(v) + d(v)^2}$ is constant within each type by Theorem 2.2(2). Substituting the respective degree pairs into this formula gives

$$\begin{aligned}
\mathcal{B} \to \mathcal{S}:&\ \sqrt{1 + (D-1) + (D-1)^2} = \sqrt{D^2 - D + 1},\\
\mathcal{B} \to \mathcal{A}:&\ \sqrt{1 + (D+1) + (D+1)^2} = \sqrt{D^2 + 3D + 3},\\
\mathcal{A} \to \mathcal{S}:&\ \sqrt{(D+1)^2 + (D+1)(D-1) + (D-1)^2} = \sqrt{3D^2 + 1},\\
\mathcal{A} \to \mathcal{A}:&\ \sqrt{(D+1)^2 + (D+1)^2 + (D+1)^2} = (D+1)\sqrt{3}.
\end{aligned}$$

Grouping the summation according to edge type and substituting these weights yields the stated expression. □

Corollary 5.17. *For connected m-$G(H)$:*

$$\mathrm{EUS}(m\text{-}G(H)) = N_{BS}\sqrt{D^2 - D + 1} + N_{BA}\sqrt{D^2 + 3D + 3} + N_{AS}\sqrt{3D^2 + 1} + N_{AA}(D+1)\sqrt{3},$$

where edge counts are from Theorem 4.3 or Theorem 4.5.

Proof. Applying Theorem 5.1 with $W(x, y) = \sqrt{x^2 + y^2 + xy}$ completes the proof. □

Remark 5.18. *Since $D^2 - D + 1 > D^2 - 2D + 2$ for all $D \geq 2$, we have* EUS $>$ SO *whenever the graph contains $\mathcal{B} \to \mathcal{S}$ edges. A symmetric comparison holds for each remaining edge type.*

Definition 5.19 ([6])**.** *Let Γ be a graph with n' vertices and m' edges. The average Sombor index is*

$$\mathrm{SO}_{\mathrm{avr}}(\Gamma) = \sum_{uv \in E} \sqrt{\Big(d(u) - \frac{2m'}{n'}\Big)^2 + \Big(d(v) - \frac{2m'}{n'}\Big)^2}.$$

Proposition 5.20. *Since connected m-$G(H)$ is a tree (Theorem 2.2), $|E| = n - 1$, so the average degree of m-$G(H)$:*

$$\mu := \frac{2|E|}{|V|} = \frac{2(n-1)}{n} = 2 - \frac{2}{n}.$$

Proof. By Theorem 2.2, the connected graph m-$G(H)$ is a tree on $|V| = n$ vertices. Every tree on n vertices has exactly $n - 1$ edges, so $|E| = n - 1$. Substituting into the standard formula for the average degree gives

$$\mu = \frac{2|E|}{|V|} = \frac{2(n-1)}{n} = 2 - \frac{2}{n}.$$

□

Lemma 5.21. *For connected m-$G(H)$ with $D \geq 2$ and $\mu = 2 - \frac{2}{n}$:*

Type	*Degree pair*	*Average Sombor weight* w^{avr}
$\mathcal{B} \to \mathcal{S}$	$(1, D-1)$	$\sqrt{(1-\mu)^2 + (D-1-\mu)^2}$
$\mathcal{B} \to \mathcal{A}$	$(1, D+1)$	$\sqrt{(1-\mu)^2 + (D+1-\mu)^2}$
$\mathcal{A} \to \mathcal{S}$	$(D+1, D-1)$	$\sqrt{(D+1-\mu)^2 + (D-1-\mu)^2}$
$\mathcal{A} \to \mathcal{A}$	$(D+1, D+1)$	$\sqrt{2}\lvert D+1-\mu\rvert$

Proof. By definition, the average Sombor weight of an edge uv is given by

$$w^{\text{avr}}(uv) = \sqrt{(d(u)-\mu)^2 + (d(v)-\mu)^2},$$

where $\mu = 2 - \frac{2}{n}$ is the average degree established in Proposition 5.20. By Theorem 2.2, the degree pair is constant within each edge type. Substituting the respective degree pairs $(1, D-1)$, $(1, D+1)$, $(D+1, D-1)$, and $(D+1, D+1)$ directly into this formula gives

$$\begin{aligned}
\mathcal{B} \to \mathcal{S}&: \quad \sqrt{(1-\mu)^2 + (D-1-\mu)^2},\\
\mathcal{B} \to \mathcal{A}&: \quad \sqrt{(1-\mu)^2 + (D+1-\mu)^2},\\
\mathcal{A} \to \mathcal{S}&: \quad \sqrt{(D+1-\mu)^2 + (D-1-\mu)^2},\\
\mathcal{A} \to \mathcal{A}&: \quad \sqrt{(D+1-\mu)^2 + (D+1-\mu)^2} = \sqrt{2}\lvert D+1-\mu\rvert,
\end{aligned}$$

which are precisely the weights recorded in the table. □

Corollary 5.22. *For connected m-$G(H)$ with $\mu = 2 - \frac{2}{n}$:*

$$\begin{aligned}
\mathrm{SO}_{\text{avr}}(m\text{-}G(H)) = {}& N_{BS}\sqrt{(1-\mu)^2 + (D-1-\mu)^2} + N_{BA}\sqrt{(1-\mu)^2 + (D+1-\mu)^2}\\
&+ N_{AS}\sqrt{(D+1-\mu)^2 + (D-1-\mu)^2} + N_{AA}\sqrt{2}\lvert D+1-\mu\rvert,
\end{aligned}$$

where the edge counts are from Theorem 4.3 or Theorem 4.5.

Proof. Applying Theorem 5.1 with $W(x,y) = \sqrt{(x-\mu)^2 + (y-\mu)^2}$ completes the proof. □

Corollary 5.23. *Let $Q = 1$. Then the graph m-$G(H)$ is isomorphic to the star graph $K_{1,D-1}$. Consequently, all Sombor-type indices considered in this section reduce to their corresponding values on a star graph. In this case, we have $N_{BS} = D-1, N_{BA} = N_{AS} = N_{AA} = 0$. Moreover, the following results hold:*

$$\begin{aligned}
\mathrm{SO}_{\text{red}}(m\text{-}G(H)) &= (D-1)(D-2),\\
\mathrm{SO}_{\text{avr}}(m\text{-}G(H)) &= (D-1)\sqrt{(1-\mu_0)^2 + (D-1-\mu_0)^2}, \quad \mu_0 = 2 - \frac{2}{D},\\
m\mathrm{SO}(m\text{-}G(H)) &= \frac{D-1}{\sqrt{D^2 - 2D + 2}},\\
\mathrm{EUS}(m\text{-}G(H)) &= (D-1)\sqrt{D^2 - D + 1}.
\end{aligned}$$

Proof. When $Q = 1$, it was shown in Corollary 5.2 that $|\mathcal{A}| = Q - 1 = 0$ and $N_{BS} = D - 1$, while $N_{BA} = N_{AS} = N_{AA} = 0$. Since all edges are of type $\mathcal{B} \to \mathcal{S}$, the graph consists of $D - 1$ vertices of degree 1 each connected to the single vertex $\mathcal{S}$ of degree $D-1$, which is precisely the star graph $K_{1,D-1}$. In this case, the average degree reduces to $\mu_0 = 2 - \frac{2}{D}$, since $n = D$. Substituting $N_{BS} = D - 1$ and $N_{BA} = N_{AS} = N_{AA} = 0$ into the respective index formulas established in this section, and using the Sombor weight $\sqrt{D^2 - 2D + 2}$ for the $\mathcal{B} \to \mathcal{S}$ edge type, gives

$$\begin{aligned}
\mathrm{SO}_{\mathrm{red}}(m\text{-}G(H)) &= (D-1)(D-2),\\
\mathrm{SO}_{\mathrm{avr}}(m\text{-}G(H)) &= (D-1)\sqrt{(1-\mu_0)^2 + (D-1-\mu_0)^2},\\
m\mathrm{SO}(m\text{-}G(H)) &= \frac{D-1}{\sqrt{D^2-2D+2}},\\
\mathrm{EUS}(m\text{-}G(H)) &= (D-1)\sqrt{D^2-D+1},
\end{aligned}$$

which completes the proof. □

Remark 5.24. *The average Sombor index measures how far vertex degrees deviate from the mean $\mu \approx 2$ for large n. For large D, the dominant contribution comes from $\mathcal{B} \to \mathcal{A}$ and $\mathcal{A} \to \mathcal{A}$ edges, whose endpoints are far from the mean.*

Example 5.25 (Cyclic: $\mathbb{Z}_{12}$, $m = 6$)**.** *$D = 6$, $Q = 2$, $n = 12$, $N_{BS} = 4$, $N_{BA} = 6$, $N_{AS} = 1$, $N_{AA} = 0$, $\mu = \frac{11}{6}$. Since the prime divisors of $|H| = 12 = 2^2 \cdot 3$ are 2 and 3, and both satisfy $2 \mid m = 6$ and $3 \mid m = 6$, Theorem 2.1 ensures that m-$G(H)$ is connected.*
Euler Sombor index:

$$\mathrm{EUS} = 4\sqrt{31} + 6\sqrt{57} + \sqrt{109} \approx 22.27 + 45.30 + 10.44 = 78.01.$$

Average Sombor index:

$$\mathrm{SO}_{\mathrm{avr}} = \frac{4\sqrt{386} + 6\sqrt{986} + \sqrt{1322}}{6} \approx 13.10 + 31.40 + 6.06 = 50.56.$$

Example 5.26 (Non-cyclic: $\mathbb{Z}_2 \oplus \mathbb{Z}_4$, $m = 2$)**.** *$D = 4$, $Q = 2$, $n = 8$, $\mathcal{G} = 2$, $N_{BS} = 2$, $N_{BA} = 4$, $N_{AS} = 1$, $N_{AA} = 0$, $\mu = \frac{7}{4}$. Since the only prime divisor of $|H| = 8 = 2^3$ is 2, and $2 \mid m = 2$, Theorem 2.1 ensures that m-$G(H)$ is connected.*
Euler Sombor index:

$$\mathrm{EUS} = 2\sqrt{13} + 4\sqrt{31} + 7 \approx 7.21 + 22.27 + 7.00 = 36.48.$$

Average Sombor index:

$$\mathrm{SO}_{\mathrm{avr}} = \frac{2\sqrt{34} + 4\sqrt{178} + \sqrt{194}}{4} \approx 2.92 + 13.34 + 3.48 = 19.74.$$

Example 5.27 (Non-cyclic, $\mathcal{G} \neq Q$: $\mathbb{Z}_9 \oplus \mathbb{Z}_{27}$, $m = 3$)**.** *$D = 9$, $Q = 27$, $n = 243$, $\mathcal{G} = 9$, $N_{BS} = 0$, $N_{BA} = 216$, $N_{AS} = 8$, $N_{AA} = 18$, $\mu = \frac{484}{243}$. Since the only prime divisor of $|H| = 243 = 3^5$ is 3, and $3 \mid m = 3$, Theorem 2.1 ensures that m-$G(H)$ is connected.*
Euler Sombor index:

$$\mathrm{EUS} = 216\sqrt{111} + 16\sqrt{61} + 180\sqrt{3} \approx 2275.70 + 124.96 + 311.77 = 2712.43.$$

Comparison of all Sombor-type indices for m-$G(H)$

Table 1 summarizes all six Sombor-type indices, and Table 2 gives numerical values for selected test cases.

Table 1. Edge-type Sombor weights for connected m-$G(H)$. Here $\mu = 2 - 2/n$.

Type	Degrees	SO	$\mathrm{SO}_{\mathrm{red}}$	$m\mathrm{SO}$	EUS	$\mathrm{SO}_{\mathrm{avr}}$
$\mathcal{B} \to \mathcal{S}$	$(1, D-1)$	$\sqrt{D^2-2D+2}$	$D-2$	$\frac{1}{\sqrt{D^2-2D+2}}$	$\sqrt{D^2-D+1}$	$\sqrt{(1-\mu)^2+(D-1-\mu)^2}$
$\mathcal{B} \to \mathcal{A}$	$(1, D+1)$	$\sqrt{D^2+2D+2}$	D	$\frac{1}{\sqrt{D^2+2D+2}}$	$\sqrt{D^2+3D+3}$	$\sqrt{(1-\mu)^2+(D+1-\mu)^2}$
$\mathcal{A} \to \mathcal{S}$	$(D+1, D-1)$	$\sqrt{2D^2+2}$	$\sqrt{2D^2-4D+4}$	$\frac{1}{\sqrt{2D^2+2}}$	$\sqrt{3D^2+1}$	$\sqrt{(D+1-\mu)^2+(D-1-\mu)^2}$
$\mathcal{A} \to \mathcal{A}$	$(D+1, D+1)$	$(D+1)\sqrt{2}$	$D\sqrt{2}$	$\frac{1}{(D+1)\sqrt{2}}$	$(D+1)\sqrt{3}$	$\sqrt{2}\lvert D+1-\mu\rvert$

Table 2. Comprehensive numerical verification of Sombor-type indices and energy for connected m-graphs.

H	m	D	Q	$\lvert E\rvert$	SO	$\mathrm{SO}_{\mathrm{red}}$	$m\mathrm{SO}$	EUS	$\mathrm{SO}_{\mathrm{avr}}$	$\mathcal{E}_S$
$\mathbb{Z}_6$	6	6	1	5	25.49	20.00	0.981	27.84	17.00	22.80
$\mathbb{Z}_{12}$	6	6	2	11	71.42	59.21	1.749	78.01	50.56	57.66
$\mathbb{Z}_9$	3	3	3	8	33.68	24.32	1.902	38.08	18.59	33.36
$\mathbb{Z}_8$	2	2	4	7	24.30	15.66	2.053	28.42	10.82	28.36
$\mathbb{Z}_{27}$	3	3	9	26	117.10	85.78	5.874	134.64	62.63	114.69
$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	2	4	2	7	32.55	24.47	1.588	36.48	19.74	31.57
$\mathbb{Z}_3 \oplus \mathbb{Z}_9$	3	9	3	26	254.88	226.80	2.691	272.15	203.36	166.47
$\mathbb{Z}_4 \oplus \mathbb{Z}_8$	4	16	2	31	505.61	473.26	1.915	525.41	444.75	252.83
$\mathbb{Z}_6 \oplus \mathbb{Z}_6$	6	36	1	35	1225.50	1190.00	1.000	1242.87	1157.42	414.29
$\mathbb{Z}_9 \oplus \mathbb{Z}_{27}$	3	9	27	242	2527.80	2264.30	23.390	2712.43	2026.94	1580.03
$\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8$	2	8	8	63	584.28	515.25	6.886	630.28	455.29	394.99

Remark 5.28. *For all connected m-$G(H)$ with $D \geq 2$,* $\mathrm{EUS}(m\text{-}G(H)) > \mathrm{SO}(m\text{-}G(H))$*, since $d(u)^2 + d(v)^2 + d(u)d(v) > d(u)^2 + d(v)^2$ edge-by-edge. By Cauchy–Schwarz,* $\mathrm{SO} \cdot m\mathrm{SO} > |E|^2$ *strictly. The index* $\mathrm{SO}_{\mathrm{avr}}$ *is not directly comparable to* SO *without further assumptions.*

6. Sombor matrix and characteristic polynomial

Recall from the Definition 3.1 that the Sombor matrix of Γ is $\mathfrak{S}(\Gamma) = (s_{ij})$, where $s_{ij} = \sqrt{d(v_i)^2 + d(v_j)^2}$ if $v_i \sim v_j$, and 0 otherwise. The Sombor eigenvalues are $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mathcal{E}_S(\Gamma) = \sum_i |\lambda_i|$.

Proposition 6.1. *For connected m-$G(H)$, the Sombor matrix $\mathfrak{S}$ takes exactly four distinct nonzero values:*

$$w_{BS} = \sqrt{D^2-2D+2}, \quad w_{BA} = \sqrt{D^2+2D+2}, \quad w_{AS} = \sqrt{2D^2+2}, \quad w_{AA} = (D+1)\sqrt{2}.$$

Proof. The Sombor matrix $\mathfrak{S}$ is defined by $\mathfrak{S}_{uv} = \sqrt{d(u)^2 + d(v)^2}$ for adjacent vertices u, v, and $\mathfrak{S}_{uv} = 0$ otherwise. By Proposition 4.1 and Definition 2.3, every edge belongs to exactly one of the four types, and the degree pair is constant within each type by Theorem 2.2(2). Substituting the respective degree pairs $(1, D-1)$, $(1, D+1)$, $(D+1, D-1)$, and $(D+1, D+1)$ into the Sombor weight formula gives

$$\begin{aligned} w_{BS} &= \sqrt{1^2 + (D-1)^2} = \sqrt{D^2 - 2D + 2}, \\ w_{BA} &= \sqrt{1^2 + (D+1)^2} = \sqrt{D^2 + 2D + 2}, \\ w_{AS} &= \sqrt{(D+1)^2 + (D-1)^2} = \sqrt{2D^2 + 2}, \\ w_{AA} &= \sqrt{(D+1)^2 + (D+1)^2} = \sqrt{2}(D+1), \end{aligned}$$

which are the only nonzero entries of $\mathfrak{S}$. □

Theorem 6.2. *When $Q = 1$, m-$G(H) \cong K_{1,D-1}$ with $r = D-1$ pendant vertices $\varphi(\lambda) = \lambda^{r-1}(\lambda^2 - r(D^2 - 2D + 2))$. The nonzero eigenvalues are $\pm w_{BS}\sqrt{r}$, each simple; $\lambda = 0$ has multiplicity $r-1$.*

Proof. When $Q = 1$, it follows from Corollary 5.23 that m-$G(H) \cong K_{1,D-1}$, a star graph with $r = D-1$ pendant vertices. The Sombor matrix $\mathfrak{S}$ of a star $K_{1,r}$ has the block form

$$\mathfrak{S} = \begin{pmatrix} 0 & w_{BS}\mathbf{1}^\top \\ w_{BS}\mathbf{1} & \mathbf{0} \end{pmatrix},$$

where $\mathbf{1} \in \mathbb{R}^r$ is the all-ones vector and $w_{BS} = \sqrt{D^2 - 2D + 2}$ is the unique nonzero entry by Proposition 6.1. The characteristic polynomial of $\mathfrak{S}$ is computed as follows. Since $\mathfrak{S}$ has rank 2, exactly $r-1$ eigenvalues are zero. The remaining two nonzero eigenvalues satisfy

$$\lambda^2 = w_{BS}^2 \cdot r = r(D^2 - 2D + 2),$$

giving $\lambda = \pm w_{BS}\sqrt{r}$, each with multiplicity 1. Combining these observations, the characteristic polynomial is

$$\varphi(\lambda) = \lambda^{r-1}(\lambda^2 - r(D^2 - 2D + 2)),$$

which completes the proof. □

Proposition 6.3. *Let m-$G(H)$ be connected and suppose that $Q = \prod_{j=1}^r q_j = 2$. Then $\mathcal{G} = \prod_{j=1}^r \gcd(d_j, q_j) = 2$.*

Proof. Since $Q = \prod_{j=1}^r q_j = 2$, there exists a unique index k such that $q_k = 2, q_j = 1 (j \neq k)$. Hence

$$\mathcal{G} = \prod_{j=1}^{r} \gcd(d_j, q_j) = \gcd(d_k, 2).$$

Because m-$G(H)$ is connected, Theorem 2.1 implies that every prime divisor of $|H|$ divides m. Since $Q = 2$, we have $2 \mid |H|$, and therefore $2 \mid m$. Moreover, $q_k = \frac{n_k}{d_k} = 2$, so $n_k = 2d_k$. Since $2 \mid m$ and $2 \mid n_k$, it follows that $2 \mid \gcd(m, n_k) = d_k$. Consequently, $\gcd(d_k, 2) = 2$. Therefore, $\mathcal{G} = 2$. □

Example 6.4. *We verify Proposition 6.3 on two instances with $Q = 2$.*

(1) $H = \mathbb{Z}_{12}$, $m = 6$*: Here,* $r = 1$, $d_1 = \gcd(6, 12) = 6$, $q_1 = 2$*. Since* $d_1 = 6$ *is even,* $\mathcal{G} = \gcd(6, 2) = 2$*. Since the prime divisors of* $|H| = 12 = 2^2 \cdot 3$ *are* 2 *and* 3*, and both satisfy* $2 \mid m = 6$ *and* $3 \mid m = 6$*, Theorem 2.1 ensures that* m*-*$G(H)$ *is connected.*

(2) $H = \mathbb{Z}_2 \oplus \mathbb{Z}_4$, $m = 2$*: We have* $d_1 = 2$, $q_1 = 1$, $d_2 = 2$, $q_2 = 2$*, so* $k = 2$*. Since* $d_2 = 2$ *is even,* $\mathcal{G} = \gcd(2, 2) = 2$*. Since the only prime divisor of* $|H| = 8 = 2^3$ *is 2, and* $2 \mid m = 2$*, Theorem 2.1 ensures that* m*-*$G(H)$ *is connected.*

Theorem 6.5. *Suppose that* $Q = 2$*. By Proposition 6.3,* $\mathcal{G} = 2$*, hence* $N_{AS} = \mathcal{G} - 1 = 1$ *and* $N_{AA} = 0$*, so the unique edge joining the two hub vertices* $\mathcal{S}$ *and* $v^* \in \mathcal{A}$ *is of type* $\mathcal{A} \to \mathcal{S}$ *with Sombor weight* $w_{01} = \sqrt{(D+1)^2 + (D-1)^2} = \sqrt{2D^2 + 2}$*. Define* $A = N_{BS} w_{BS}^2$, $B = N_{BA} w_{BA}^2$, $C = w_{01}^2 = 2D^2 + 2$*. Then the characteristic polynomial of the Sombor matrix is*

$$\varphi_{\mathfrak{S}}(\lambda) = \lambda^{n-4}\Big(\lambda^4 - (A + B + C)\lambda^2 + AB\Big).$$

Let

$$u_{1,2} = \frac{(A + B + C) \pm \sqrt{(A + B + C)^2 - 4AB}}{2}.$$

The spectrum consists of 0 *with multiplicity at least* $n - 4$*, together with* $\pm\sqrt{u_1}$ *and* $\pm\sqrt{u_2}$*. If* $u_2 = 0$*, which occurs if and only if* $N_{BS} = 0$ *or* $N_{BA} = 0$*, the pair* $\pm\sqrt{u_2}$ *coincides with zero and the number of nonzero eigenvalues reduces to two.*

Proof. When $Q = 2$, the vertex set partitions as $\mathcal{S} = \{0\}, \mathcal{A} = \{a\}$, where $\mathcal{A}$ consists of a single vertex of degree $D + 1$, and $\mathcal{B} = V(G) \setminus \{0, a\}$ consists of the remaining $n - 2$ vertices of degree 1. Among the vertices of $\mathcal{B}$, let $\mathcal{B}_{\mathcal{S}}$ denote the subset adjacent to $\mathcal{S}$ and $\mathcal{B}_{\mathcal{A}}$ the subset adjacent to $\mathcal{A}$, with cardinalities N_{BS} and N_{BA}, respectively. The unique edge between $\mathcal{S}$ and $\mathcal{A}$ carries weight w_{01}.

With respect to the ordering $(\mathcal{S}, \mathcal{A}, \mathcal{B}_{\mathcal{S}}, \mathcal{B}_{\mathcal{A}})$, the Sombor matrix $\mathfrak{S}$ has the block form

$$\mathfrak{S} = \begin{pmatrix} 0 & w_{01} & w_{BS}\mathbf{1}^\top & \mathbf{0}^\top \\ w_{01} & 0 & \mathbf{0}^\top & w_{BA}\mathbf{1}^\top \\ w_{BS}\mathbf{1} & \mathbf{0} & 0 & 0 \\ \mathbf{0} & w_{BA}\mathbf{1} & 0 & 0 \end{pmatrix},$$

where $\mathbf{1}$ denotes the all-ones vector of suitable dimension.

Since all vertices in $\mathcal{B}_{\mathcal{S}}$ (respectively, $\mathcal{B}_{\mathcal{A}}$) are pendant, have identical neighborhoods, and carry identical edge weights, the corresponding rows of $\mathfrak{S}$ are linearly dependent. Define

$$V_0 = \operatorname{span}\{e_v - e_{v'} : v, v' \in \mathcal{B}_{\mathcal{S}}\} \oplus \operatorname{span}\{e_v - e_{v'} : v, v' \in \mathcal{B}_{\mathcal{A}}\},$$

where e_v denotes the standard basis vector corresponding to the vertex v. Since every vertex $v \in \mathcal{B}_{\mathcal{S}}$ has the unique neighbor $\mathcal{S}$ with edge weight w_{BS}, the rows of $\mathfrak{S}$ indexed by $\mathcal{B}_{\mathcal{S}}$ are identical, and hence

$$\mathfrak{S}(e_v - e_{v'}) = 0 \quad \text{for all } v, v' \in \mathcal{B}_{\mathcal{S}}.$$

Similarly,

$$\mathfrak{S}(e_v - e_{v'}) = 0 \quad \text{for all } v, v' \in \mathcal{B}_{\mathcal{A}}.$$

Therefore $V_0 \subseteq \ker(\mathfrak{S})$, and

$$\dim(V_0) = \max(N_{BS} - 1,\ 0) + \max(N_{BA} - 1,\ 0).$$

The orthogonal complement $V_0^\perp$ is spanned by the four vectors

$$g_S = e_S, \quad g_A = e_A, \quad g_{BS} = \frac{1}{\sqrt{N_{BS}}}\sum_{v\in B_S} e_v, \quad g_{BA} = \frac{1}{\sqrt{N_{BA}}}\sum_{v\in B_A} e_v,$$

which form an orthonormal basis of $V_0^\perp$ whenever $N_{BS} \geq 1$ and $N_{BA} \geq 1$. When $N_{BS} = 0$ (respectively, $N_{BA} = 0$), the vector g_{BS} (respectively, g_{BA}) is absent and $V_0^\perp$ is spanned by the remaining three vectors.

The action of $\mathfrak{S}$ on these vectors is

$$\mathfrak{S}g_S = w_{01}g_A + \sqrt{N_{BS}}\, w_{BS}\, g_{BS}, \qquad \mathfrak{S}g_A = w_{01}g_S + \sqrt{N_{BA}}\, w_{BA}\, g_{BA},$$
$$\mathfrak{S}g_{BS} = \sqrt{N_{BS}}\, w_{BS}\, g_S, \qquad \mathfrak{S}g_{BA} = \sqrt{N_{BA}}\, w_{BA}\, g_A.$$

Hence the restriction of $\mathfrak{S}$ to $V_0^\perp$, with respect to the ordered basis $(g_S, g_A, g_{BS}, g_{BA})$, is represented by the real symmetric matrix

$$M = \begin{pmatrix} 0 & w_{01} & \sqrt{N_{BS}}\, w_{BS} & 0 \\ w_{01} & 0 & 0 & \sqrt{N_{BA}}\, w_{BA} \\ \sqrt{N_{BS}}\, w_{BS} & 0 & 0 & 0 \\ 0 & \sqrt{N_{BA}}\, w_{BA} & 0 & 0 \end{pmatrix}.$$

Since $\mathbb{R}^n = V_0 \oplus V_0^\perp$ is an orthogonal decomposition and $\mathfrak{S}|_{V_0} = 0$, $\mathfrak{S}|_{V_0^\perp} = M$, the characteristic polynomial factors as

$$\varphi_{\mathfrak{S}}(\lambda) = \lambda^{\dim(V_0)} \det(\lambda I - M),$$

where

$$\dim(V_0) = \max(N_{BS} - 1, 0) + \max(N_{BA} - 1, 0).$$

For convenience, set

$$A = N_{BS}w_{BS}^2, \qquad B = N_{BA}w_{BA}^2, \qquad C = w_{01}^2.$$

A direct computation gives

$$\det(\lambda I - M) = \lambda^4 - (A + B + C)\lambda^2 + AB.$$

When $N_{BS} \geq 1$ and $N_{BA} \geq 1$, one has $\dim(V_0) = n - 4$, so that

$$\varphi_{\mathfrak{S}}(\lambda) = \lambda^{n-4}\Big(\lambda^4 - (A + B + C)\lambda^2 + AB\Big).$$

In the degenerate cases $N_{BS} = 0$ or $N_{BA} = 0$, one has $\dim(V_0) < n - 4$; nevertheless, since $A = 0$ (respectively, $B = 0$), the polynomial $\lambda^4 - (A+B+C)\lambda^2 + AB$ acquires an additional factor of λ^2, which compensates the deficit, and in all cases,

$$\varphi_{\mathfrak{S}}(\lambda) = \lambda^{n-4}\Big(\lambda^4 - (A + B + C)\lambda^2 + AB\Big).$$

The nonzero eigenvalues are obtained by solving the quadratic equation in λ^2:

$$\lambda^2 = u_{1,2} = \frac{(A + B + C) \pm \sqrt{(A + B + C)^2 - 4AB}}{2}.$$

Thus, the nonzero eigenvalues are $\pm\sqrt{u_1}$ and $\pm\sqrt{u_2}$. If $u_2 > 0$, all four values are nonzero and simple. If $u_2 = 0$, which occurs if and only if $AB = 0$, that is, $N_{BS} = 0$ or $N_{BA} = 0$, the pair $\pm\sqrt{u_2}$ coincides with zero, reducing the number of nonzero eigenvalues to two. This completes the proof. □

Theorem 6.6. *Let $\lambda_1 = \lambda_1(\mathfrak{S}(m\text{-}G(H)))$ be the Sombor spectral radius of connected $m\text{-}G(H)$. Then:*

$$\sqrt{\frac{2F(m\text{-}G(H))}{n}} \ \leq\ \lambda_1 \ \leq\ \sqrt{2}(D+1)^2,$$

where $F(m\text{-}G(H)) = (D-1)^3 + (Q-1)(D+1)^3 + (n-Q)$.

Proof. Lower bound. For any real symmetric matrix M of order n,

$$\lambda_1^2 \geq \frac{\mathrm{tr}(M^2)}{n} = \frac{2\sum_{uv\in E}(d(u)^2 + d(v)^2)}{n} = \frac{2F(m\text{-}G(H))}{n},$$

which gives $\lambda_1 \geq \sqrt{2F(m\text{-}G(H))/n}$.

Upper bound. For any real symmetric matrix M with nonnegative entries, the Perron–Frobenius row-sum estimate [16, Chapter 2] gives

$$\lambda_1 \leq \max_{v\in V} \sum_{u\sim v} s_{uv},$$

where $s_{uv} = \sqrt{d(u)^2 + d(v)^2}$ is the Sombor weight of edge uv. By Theorem 2.2, vertex degrees take values in $\{1, D-1, D+1\}$ and the maximum possible Sombor edge weight is $w_{AA} = \sqrt{2}(D+1)$. For any vertex v,

$$\sum_{u\sim v} s_{uv} \leq d(v)\cdot w_{AA} \leq (D+1)\cdot \sqrt{2}(D+1) = \sqrt{2}(D+1)^2.$$

Taking the maximum over all v yields $\lambda_1 \leq \sqrt{2}(D+1)^2$. □

Proposition 6.7. *(1) $Q = 1$: $\lambda_1 = w_{BS}\sqrt{D-1} = \sqrt{(D-1)(D^2-2D+2)}$.*
(2) $Q = 2$: $\lambda_1 = \sqrt{u_1}$.
(3) General: $\lambda_1 \leq \sqrt{2}(D+1)^2$ and $\lambda_1 \geq \sqrt{2F(m\text{-}G(H))/n}$.

Proof. Case (1): $Q = 1$. By Theorem 6.2, the nonzero eigenvalues of $\mathfrak{S}$ are $\pm w_{BS}\sqrt{r}$ with $r = D-1$. Since the spectral radius is the largest absolute eigenvalue, we obtain

$$\lambda_1 = w_{BS}\sqrt{D-1} = \sqrt{(D-1)(D^2-2D+2)}.$$

Case (2): $Q = 2$. By Theorem 6.5, the four nonzero eigenvalues are $\pm\sqrt{u_1}$ and $\pm\sqrt{u_2}$, where

$$u_{1,2} = \frac{(A+B+C) \pm \sqrt{(A+B+C)^2 - 4AB}}{2}.$$

The discriminant of this quadratic is non-negative, since

$$(A+B+C)^2 - 4AB = (A-B)^2 + C^2 + 2C(A+B) \ \geq\ 0,$$

with strict inequality whenever $C > 0$, which holds as $w_{01} > 0$. Moreover, $u_1 \geq u_2 \geq 0$. The inequality $u_1 \geq u_2$ is immediate from the non-negative square root, and $u_2 \geq 0$ follows from

$$(A+B+C)^2 - \left[(A+B+C)^2 - 4AB\right] = 4AB \geq 0,$$

so that $A + B + C \geq \sqrt{(A + B + C)^2 - 4AB}$. Equality $u_2 = 0$ occurs if and only if $AB = 0$, that is, when $N_{BS} = 0$ or $N_{BA} = 0$. Hence the spectral radius is

$$\lambda_1 = \sqrt{u_1} = \sqrt{\frac{(A + B + C) + \sqrt{(A + B + C)^2 - 4AB}}{2}}.$$

Case (3): General bounds. The upper and lower bounds follow directly from Theorem 6.6, where $F(m\text{-}G(H)) = (D - 1)^3 + (Q - 1)(D + 1)^3 + (n - Q)$ is given by Theorem 5.5. □

7. Sombor energy

Proposition 7.1. *The Sombor matrix $\mathfrak{S}(m\text{-}G(H))$ is real symmetric, and its spectrum is symmetric about zero. Consequently,*

$$\mathcal{E}_S = 2 \sum_{\lambda_i > 0} \lambda_i.$$

Proof. Real symmetry follows immediately from the definition $\mathfrak{S}_{uv} = \sqrt{d(u)^2 + d(v)^2}$. By Theorem 2.2, $m\text{-}G(H)$ is a tree and hence bipartite; it is classical that the spectrum of any bipartite graph is symmetric about zero [17]. Pairing positive and negative eigenvalues then gives $\mathcal{E}_S = \sum_i |\lambda_i| = 2 \sum_{\lambda_i > 0} \lambda_i$. □

Theorem 7.2. *When $Q = 1$, $\mathcal{E}_S = 2\sqrt{(D - 1)(D^2 - 2D + 2)}$.*

Proof. When $Q = 1$, Corollary 5.23 gives $m\text{-}G(H) \cong K_{1,r}$ with $r = D - 1$, and Proposition 6.1 ensures that every edge carries the uniform Sombor weight $w_{BS} = \sqrt{D^2 - 2D + 2}$. Therefore,

$$\mathfrak{S}(m\text{-}G(H)) = w_{BS} \cdot \mathbf{A}(K_{1,r}),$$

where $\mathbf{A}(K_{1,r})$ denotes the adjacency matrix of the star $K_{1,r}$. The eigenvalues of $\mathbf{A}(K_{1,r})$ are $\pm\sqrt{r}$, each simple, and 0 with multiplicity $r - 1$ [17]. Scaling by w_{BS}, the nonzero eigenvalues of $\mathfrak{S}$ are $\pm w_{BS}\sqrt{r}$. Applying Proposition 7.1 then yields

$$\mathcal{E}_S = 2\, w_{BS} \sqrt{r} = 2\sqrt{(D^2 - 2D + 2)(D - 1)},$$

which completes the proof. □

Applying Theorem 6.5 to the Sombor weights of m-graphs yields the following explicit energy formula.

Corollary 7.3. *Let $m\text{-}G(H)$ be a connected m-graph with $Q = 2$.*

Define

$$A = N_{BS}(D^2 - 2D + 2), \qquad B = N_{BA}(D^2 + 2D + 2), \qquad C = 2D^2 + 2.$$

Let

$$\mu_{1,2} = \frac{(A + B + C) \pm \sqrt{(A + B + C)^2 - 4AB}}{2}.$$

Then the Sombor energy is $\mathcal{E}_S = 2\left(\sqrt{\mu_1} + \sqrt{\mu_2}\right)$. If $\mu_2 = 0$, this reduces to $\mathcal{E}_S = 2\sqrt{\mu_1}$.

Proof. Since $Q = \prod_{j=1}^{r} q_j = 2$, we have $|\mathcal{A}| = Q - 1 = 1$, so there exists a unique non-identity element $v^* \in \mathcal{A}$. The graph therefore has exactly two hub vertices,

$$h_0 = 0 \in \mathcal{S} \text{ with degree } D - 1, \qquad h_1 = v^* \in \mathcal{A} \text{ with degree } D + 1,$$

together with N_{BS} pendant vertices adjacent to h_0 and N_{BA} pendant vertices adjacent to h_1. The corresponding pendant edge weights are

$$w_{BS} = \sqrt{D^2 - 2D + 2}, \qquad w_{BA} = \sqrt{D^2 + 2D + 2}.$$

Since $Q = 2$, Proposition 6.3 yields $G = 2$, hence $N_{AS} = G - 1 = 1$ and $N_{AA} = 0$. The unique edge joining the two hub vertices is of type $\mathcal{A} \to \mathcal{S}$ with Sombor weight

$$w_{01} = \sqrt{(D+1)^2 + (D-1)^2} = \sqrt{2D^2 + 2}.$$

Define

$$A = N_{BS} w_{BS}^2 = N_{BS}(D^2 - 2D + 2), \quad B = N_{BA} w_{BA}^2 = N_{BA}(D^2 + 2D + 2), \quad C = w_{01}^2 = 2D^2 + 2.$$

By Theorem 6.5, the characteristic polynomial of $\mathfrak{S}$ is

$$\varphi_{\mathfrak{S}}(\lambda) = \lambda^{n-4}\Big(\lambda^4 - (A + B + C)\lambda^2 + AB\Big),$$

and the nonzero eigenvalues are $\pm\sqrt{\mu_1}$ and $\pm\sqrt{\mu_2}$, where

$$\mu_{1,2} = \frac{(A + B + C) \pm \sqrt{(A + B + C)^2 - 4AB}}{2}.$$

Since $\mathfrak{S}$ is real symmetric and m-$G(H)$ is bipartite (Proposition 7.1), the spectrum is symmetric about zero. Hence

$$\mathcal{E}_S = \sum_i |\lambda_i| = 2\big(\sqrt{\mu_1} + \sqrt{\mu_2}\big).$$

If $\mu_2 = 0$, which occurs if and only if $AB = 0$, that is, $N_{BS} = 0$ or $N_{BA} = 0$, this reduces to

$$\mathcal{E}_S = 2\sqrt{\mu_1}.$$

This completes the proof. □

Remark 7.4. *By Proposition 6.3, every connected m-graph with $Q = 2$ satisfies $G = 2$. Hence*

$$N_{AS} = G - 1 = 1, \qquad N_{AA} = 0.$$

Therefore the unique inter-hub edge is always of type $A \to S$, and its Sombor weight is

$$w_{01} = \sqrt{(D+1)^2 + (D-1)^2} = \sqrt{2D^2 + 2}.$$

Consequently,

$$C = w_{01}^2 = 2D^2 + 2.$$

8. Worked examples

Example 8.1 (Cyclic: $\mathbb{Z}_{12}$, $m = 6$)**.** *$k = 6$, $q = 2$, $s = 0$, $D = 6$, $Q = 2$,* $\gcd(s, q) = \gcd(0, 2) = 2$*. Since the prime divisors of $|H| = 12 = 2^2 \cdot 3$ are 2 and 3, and both satisfy $2 \mid m = 6$ and $3 \mid m = 6$, Theorem 2.1 ensures that m-G(H) is connected.*

$$N_{BS} = 6 - \gcd(6, 2) = 4, \quad N_{BA} = 6, \quad N_{AS} = 1, \quad N_{AA} = 0.$$

$$\mathrm{SO} = 4\sqrt{26} + 6\sqrt{50} + \sqrt{74} \approx 71.424,$$

$$\mathrm{SO}(\mathbb{Z}_{12};\ x) = \frac{4}{\sqrt{26}}\, x^{26} + \frac{6}{\sqrt{50}}\, x^{50} + \frac{1}{\sqrt{74}}\, x^{74}.$$

Energy ($Q = 2$, $\mathcal{G} = 2$ subcase): $A = 104$, $B = 300$, $C = 74$, $\alpha = 478$, $\Delta = 103684$. Roots: $\lambda^2 \in \{400, 78\}$, so $\mathcal{E}_S = 40 + 2\sqrt{78} \approx 57.663$.

Example 8.2 (Non-cyclic: $\mathbb{Z}_2 \oplus \mathbb{Z}_4$, $m = 2$)**.** *$d_1 = 2$, $d_2 = 2$, $q_1 = 1$, $q_2 = 2$, $D = 4$, $Q = 2$, $\mathcal{G} = \gcd(2, 1) \cdot \gcd(2, 2) = 2$. Since the only prime divisor of $|H| = 8 = 2^3$ is 2, and $2 \mid m = 2$, Theorem 2.1 ensures that m-G(H) is connected.*

$$N_{BS} = 4 - 2 = 2, \quad N_{BA} = 4, \quad N_{AS} = 1, \quad N_{AA} = 0.$$

$$\mathrm{SO} = 2\sqrt{10} + 4\sqrt{26} + \sqrt{34} \approx 32.552,$$

$$\mathrm{SO}(\mathbb{Z}_2 \oplus \mathbb{Z}_4;\ x) = \frac{2}{\sqrt{10}}\, x^{10} + \frac{4}{\sqrt{26}}\, x^{26} + \frac{1}{\sqrt{34}}\, x^{34}.$$

Energy ($Q = 2$, $\mathcal{G} = 2$ subcase): $A = 20$, $B = 104$, $C = 34$, $\alpha = 158$, $\Delta = 16644$.

$$\mathcal{E}_S = 2\left(\sqrt{(158 + \sqrt{16644})/2} + \sqrt{(158 - \sqrt{16644})/2}\right) \approx 31.573.$$

Example 8.3 (Non-cyclic, $\mathcal{G} \neq Q$: $\mathbb{Z}_9 \oplus \mathbb{Z}_{27}$, $m = 3$)**.** *$d_1 = 3$, $d_2 = 3$, $q_1 = 3$, $q_2 = 9$, $D = 9$, $Q = 27$, $\mathcal{G} = \gcd(3, 3) \cdot \gcd(3, 9) = 3 \cdot 3 = 9 \neq Q$. Since the only prime divisor of $|H| = 243 = 3^5$ is 3, and $3 \mid m = 3$, Theorem 2.1 ensures that m-G(H) is connected.*

$$N_{BS} = 0, \quad N_{BA} = 216, \quad N_{AS} = 8, \quad N_{AA} = 18.$$

$$\mathrm{SO} = 216\sqrt{101} + 8\sqrt{164} + 180\sqrt{2} \approx 2527.73.$$

This example illustrates that $\mathcal{G} \neq Q$ is possible and the formula $N_{BS} = D - \mathcal{G}$ is essential.

9. Numerical verification

To demonstrate the consistency of our theoretical results, we provide a comprehensive numerical analysis in which all indices were computed using a specialized script in SageMath 10.3, ensuring high precision for both cyclic and non-cyclic structures. All numerical values presented in Table 2 have been independently verified via exact symbolic computation using the SymPy library (Python), confirming full consistency with the theoretical closed-form expressions to within standard floating-point rounding.

All groups and parameters in Table 2 satisfy the connectedness criterion of Theorem 2.1: in each case, every prime divisor of $|H|$ divides m, as verified below.

- $H = \mathbb{Z}_6,\ m = 6$: prime divisors $\{2, 3\}$, $2 \mid 6$, $3 \mid 6$.
- $H = \mathbb{Z}_{12},\ m = 6$: prime divisors $\{2, 3\}$, $2 \mid 6$, $3 \mid 6$.
- $H = \mathbb{Z}_9,\ m = 3$: prime divisor $\{3\}$, $3 \mid 3$.
- $H = \mathbb{Z}_8,\ m = 2$: prime divisor $\{2\}$, $2 \mid 2$.
- $H = \mathbb{Z}_{27},\ m = 3$: prime divisor $\{3\}$, $3 \mid 3$.
- $H = \mathbb{Z}_2 \oplus \mathbb{Z}_4,\ m = 2$: prime divisor $\{2\}$, $2 \mid 2$.
- $H = \mathbb{Z}_3 \oplus \mathbb{Z}_9,\ m = 3$: prime divisor $\{3\}$, $3 \mid 3$.
- $H = \mathbb{Z}_4 \oplus \mathbb{Z}_8,\ m = 4$: prime divisor $\{2\}$, $2 \mid 4$.
- $H = \mathbb{Z}_6 \oplus \mathbb{Z}_6,\ m = 6$: prime divisors $\{2, 3\}$, $2 \mid 6$, $3 \mid 6$.
- $H = \mathbb{Z}_9 \oplus \mathbb{Z}_{27},\ m = 3$: prime divisor $\{3\}$, $3 \mid 3$.
- $H = \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8,\ m = 2$: prime divisor $\{2\}$, $2 \mid 2$.

10. Conclusions

In this paper, we established a complete edge-type decomposition for connected m-graphs m-$G(H)$ of finite abelian groups and used this structure to derive closed-form expressions for a broad family of degree-based graph invariants. Exploiting the fact that every connected m-graph is a tree and that its vertices belong to exactly three degree classes $\mathcal{S}$, $\mathcal{A}$, and $\mathcal{B}$, we reduced all computations to four edge categories $\mathcal{B}$–$\mathcal{S}$, $\mathcal{B}$–$\mathcal{A}$, $\mathcal{A}$–$\mathcal{S}$, and $\mathcal{A}$–$\mathcal{A}$.

The principal contributions of the paper may be summarized as follows.

- We derived exact formulas for the edge counts N_{BS}, N_{BA}, N_{AS}, and N_{AA} in both cyclic and non-cyclic settings. In particular, the parameter

  $$\mathcal{G} = \prod_{j=1}^{r} \gcd(d_j, q_j),$$

  introduced in this work, was shown to govern the non-cyclic edge counts through

  $$N_{BS} = D - \mathcal{G}, \qquad N_{AS} = \mathcal{G} - 1.$$

  This demonstrates that the non-cyclic case cannot be described solely in terms of D and Q.
- We established a unified edge-weight framework (Theorem 5.1) under which every degree-based edge invariant can be expressed as a linear combination of the four edge-type counts. As direct consequences, explicit formulas were obtained for the Sombor index, Sombor polynomial, reduced Sombor index, modified Sombor index, Euler Sombor index, average Sombor index, and the forgotten topological index.
- For the Sombor matrix, we proved that only four distinct nonzero weights occur and derived exact spectral results for the cases $Q = 1$ and $Q = 2$. When $Q = 1$, the graph is the star $K_{1,D-1}$ and the Sombor energy is

  $$\mathcal{E}_S = 2\sqrt{(D-1)(D^2 - 2D + 2)}.$$

  When $Q = 2$, connectedness implies $\mathcal{G} = 2$, yielding $N_{AS} = 1$ and $N_{AA} = 0$, which allows the characteristic polynomial and the complete spectrum of the Sombor matrix to be determined explicitly.

- Using the forgotten index, we obtained a closed-form lower bound for the spectral radius of the Sombor matrix, linking structural parameters of m-$G(H)$ directly to its spectral behavior.
- The theoretical formulas were verified by extensive symbolic and numerical computations using SageMath and SymPy, showing complete agreement with all tested examples.
- The results provide a unified treatment of connected m-graphs of finite abelian groups and complement recent studies on topological indices of algebraic graphs by supplying exact closed-form expressions rather than case-by-case computations.

Several problems remain open. The most immediate is the determination of the Sombor spectrum and energy for $Q \geq 3$, where the Sombor matrix no longer reduces to a low-rank structure. Equitable partitions, interlacing techniques, and refined spectral methods may provide a route toward closed-form formulas in this setting. Further directions include the extension of the edge-type decomposition to m-graphs of finite non-abelian groups and a deeper investigation of the algebraic parameter $\mathcal{G}$, whose role appears to extend beyond the edge-count formulas derived here.

Use of Generative-AI tools declaration

The author used ChatGPT and Claude AI to improve the language, grammar, and readability of the manuscript. These tools were used solely for linguistic refinement, while all mathematical content, proofs, and scientific conclusions remain the responsibility of the author.

Acknowledgments

The author was supported by Istanbul Medipol University. The author would like to thank the reviewers for their valuable comments and suggestions, which have significantly improved the clarity of the manuscript.

Conflict of interest

The author declares no conflict of interest.

References

1. A. Badawi, *On the m-graph of a finite abelian group*, arXiv:2601.10830, 2026. https://doi.org/10.48550/arXiv.2601.10830
2. I. Gutman, *Geometric approach to degree-based topological indices: Sombor indices*, *MATCH Commun. Math. Comput. Chem.*, **86** (2021), 11–16.
3. I. Redžepović, I. Gutman, Relating Sombor and Euler indices, *MATCH Commun. Math. Comput. Chem.*, **88** (2022), 571–582. https://doi.org/10.5937/vojtehg72-48818
4. H. Deng, Z. Tang, R. Wu, Molecular trees with extremal values of Sombor indices, *Int. J. Quantum Chem.*, **121** (2021), e26622. https://doi.org/10.1002/qua.26622
5. Z. Tang, Y. Li, H. Deng, The Euler Sombor index of a graph, *Int. J. Quantum Chem.*, **124** (2024), e27387. https://doi.org/10.1002/qua.27387

6. I. Gutman, I. Redžepović, G. O. Kızılirmak, V. R. Kulli, Euler Sombor index and its congeners, *Open J. Math. Sci.*, **9** (2025), 141–148. https://doi.org/10.30538/oms2025.0249
7. K. Ullah, F. Ali, S. Xia, M. S. Arif, K. Abodayeah, Computation of Hosoya and eccentricity-based topological indices of power graphs over finite groups, *AIMS Math.*, **11** (2026), 12650–12673. http://doi.org/10.3934/math.2026520
8. A. V. Kelarev, S. J. Quinn, A combinatorial property and power graphs of groups, *Contrib. General Algebra*, **12** (2000), 229–235.
9. N. I. Alimon, N. H. Sarmin, S. M. Khasraw, N. Najmuddin, G. Semil Ismail, The Sombor index of a power graph for some finite groups, *Malays. J. Math. Sci.*, **19** (2025), 399–410. https://doi.org/10.47836/mjms.19.2.03
10. M. U. Romdhini, A. Nawawi, On the spectral radius and Sombor energy of the non-commuting graph for dihedral groups, *Malays. J. Fundam. Appl. Sci.*, **20** (2024), 65–73. https://doi.org/10.11113/mjfas.v20n1.3252
11. B. A. Rather, M. Imran, Sombor index and eigenvalues of comaximal graphs of commutative rings, *J. Algebra Appl.*, **23** (2024), 2450115. https://doi.org/10.1142/S0219498824501159
12. R. Pratama, F. Maulana, N. Hijriati, I. G. A. W. Wardhana, Sombor index and its generalization of power graph of some group with prime power order, *J. Fundam. Math. Appl.*, **7** (2024), 163–173. https://doi.org/10.14710/jfma.v7i2.22552
13. S. M. S. Khasraw, N. H. Sarmin, N. I. Alimon, N. Najmuddin, G. Semil Ismail, Sombor index and Sombor polynomial of the noncommuting graph associated to some finite groups, *J. Adv. Res. Appl. Sci. Eng. Tech.*, **42** (2024), 112–121. https://doi.org/10.37934/araset.42.2.112121
14. B. Furtula, I. Gutman, A forgotten topological index, *J. Math. Chem.*, **53** (2015), 1184–1190. https://doi.org/10.1007/s10910-015-0480-z
15. V. R. Kulli, I. Gutman, Computation of Sombor indices of certain networks, *SSRG Int. J. Appl. Chem.*, **8** (2021), 1–5. https://doi.org/10.14445/23939133/IJAC-V8I1P101
16. A. Berman, R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Philadelphia: SIAM, 1994.
17. D. Cvetković, P. Rowlinson, S. Simić, *An Introduction to the Theory of Graph Spectra*, Cambridge: Cambridge University Press, 2010.