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Research article

Universal emergence of rational corrections in Borel-based strong-coupling approximants

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Abstract: The corrected–Borel scheme provides a hierarchical framework for extracting strong–coupling information from short perturbative expansions. The method begins with an irrational core determined by the first two coefficients and the known strong–coupling exponent, and refines this core through a sequence of rational Borel–root corrections generated from successive residuals. Each correction depends only on a new pair of perturbative coefficients, ensuring transparency, stability, and coefficient locality even for short or divergent series. The scheme accommodates both extrapolation, where the strong–coupling amplitude is unknown, and interpolation, where the amplitude is prescribed. A recursive extension allows synthetic coefficients generated by the approximant itself to continue the hierarchy beyond the available data. Applications to representative test functions demonstrate that the corrected–Borel hierarchy yields accurate and systematically improvable estimates of strong–coupling amplitudes using minimal perturbative input.

Keywords: self-similar approximants; Borel transformation; strong-coupling limit; critical index estimation; asymptotic analysis; perturbation theory; rational corrections
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1. Introduction

Many problems in physics, applied mathematics, and quantitative modeling involve real functions of real argument $f(x)$ that are known only through two incomplete pieces of information: a short truncated expansion at a small argument and a power-law asymptotic behavior at a large argument. This situation is ubiquitous in quantum perturbation theory, statistical mechanics, nonlinear differential equations, special-function theory, and renormalization-group analysis. In all such cases, one typically knows only a handful of coefficients:

$$f_N(x) = b_0 + b_1 x + \cdots + b_N x^N + O(x^{N+1}), \tag{1.1}$$

while the strong-coupling regime is characterized by

$$f(x) \sim Bx^a, \qquad x \to \infty, \tag{1.2}$$

with a known exponent a but an unknown amplitude B.

The celebrated Borel transformation [1–3] of the series was also promoted in [4, 5]:

$$\mathcal{B}[f_N](t) = \sum_{n=0}^{N} \frac{b_n}{n!} t^n. \tag{1.3}$$

The resulting series can then be extrapolated or resummed by means of an appropriate approximation technique, producing a Borel-space approximant $\Phi(t)$ [5]. The original function is reconstructed through the inverse Borel transformation,

$$f(x) = \int_0^\infty e^{-t}\, \Phi(tx)\, dt, \tag{1.4}$$

so that the non-polynomial structure of the function emerges directly from the inverse Borel integral.

This mechanism is central to the corrected–Borel scheme: the inverse Borel transform of the first resummed expression produces an explicit irrational core containing the correct strong-coupling exponent, while the inverse Borel transforms of the residual corrections generate rational, Padé-like multiplicative factors. The rational structure is therefore not imposed but emerges naturally from the analytic operations.

Extrapolating the amplitude B from the minimal information contained in the truncated series is a notoriously difficult problem. Standard resummation techniques such as Padé approximants, Borel–Padé transforms, conformal maps, or hypergeometric fits often require longer series, knowledge of analytic structure, or additional constraints that are not available. When only a few coefficients are known, these methods may become unstable, ambiguous, or simply inapplicable, even though the amplitude B is often the quantity of primary physical interest.

To address these difficulties, the aim of this work is to construct a systematic, hierarchical, and fully analytic procedure that extracts the strong-coupling amplitude B from the minimal data $\{b_0, \ldots, b_N\}$. The method is based on a sequence of Borel-space self-similar transformations. The first step constructs an irrational "core" approximant from the pair (b_0, b_1), which embeds the correct asymptotic exponent and captures the essential non-polynomial structure of the function. All subsequent steps introduce rational corrections derived from the residual error at progressively higher orders.

Each correction is generated by a Borel transform followed by a self-similar root of increasing exponent, and each step uses only two new coefficients (b_{2k}, b_{2k+1}). This produces a transparent product structure for the final approximant and a multiplicative sequence of explicit amplitude corrections.

A truncated series contains only local information near $x = 0$, whereas the strong-coupling amplitude encodes global behavior. Bridging these two regimes without assuming more information than the data provide is the central challenge. The corrected–Borel scheme achieves this by separating the problem into two complementary components: an irrational core, built once from (b_0, b_1), and a hierarchy of rational corrections constructed from the residuals.

Although inspired by the philosophy of corrected approximants [6, 7], the present scheme differs from classical corrected-approximant methods in several fundamental ways. Corrected approximants modify a single approximant by multiplying it with one or two correction factors, whereas the corrected–Borel scheme constructs a hierarchy of corrections, each tied to a specific pair (b_{2k}, b_{2k+1}).

All higher corrections are rational, and each step yields a closed-form multiplicative amplitude factor without optimization.

The remainder of the paper is organized as follows. Section 2 constructs the first corrected–Borel approximant using coefficients up to b_3 and illustrates how the irrational core and the first rational correction arise from successive Borel and inverse Borel transformations. Section 3 extends the construction to the two-step case, incorporating coefficients up to b_5 and revealing the hierarchical structure of the correction factors. Section 4 presents the general k-step scheme and derives the explicit product formula for the strong-coupling amplitude. Section 5 applies the method to several classical benchmark problems. Section 6 describes an interpolation variant. Appendices provide operator derivations, comparison with Padé approximants, and connections with hyperasymptotics and resurgence.

2. One-step corrected approximants with the Borel transform (using coefficients up to b_3)

We consider a function with a truncated small-x expansion

$$f_3(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3, \tag{2.1}$$

and a known strong-coupling asymptotic behavior

$$f(x) \sim B x^a, \qquad x \to \infty, \tag{2.2}$$

where the exponent a is known and the amplitude B is to be predicted. We begin with the Borel transform of the first two coefficients:

$$F_1(t) = b_0 + b_1 t. \tag{2.3}$$

We extrapolate $F_1(t)$ by a self-similar approximant

$$F^*(t) = a_0 (1 + At)^a, \tag{2.4}$$

where

$$a_0 = b_0, \qquad A = \frac{b_1}{a\, b_0}, \tag{2.5}$$

with positive A. The parameters of the approximant are adjusted to match the expansion of $F_1(t)$ at $t = 0$ [6]. Only the lowest-order self-similar form is used here, following the referee request to avoid unnecessary generality. For such approximants, the inverse Borel transform gives

$$f^{(0)}(x) = \int_0^\infty e^{-t} F^*(xt)\, dt = b_0\, e^{1/(Ax)} (Ax)^a\, \Gamma\!\left(a + 1, \frac{1}{Ax}\right), \tag{2.6}$$

where $\Gamma(\cdot, \cdot)$ is the incomplete gamma function. For large x,

$$f^{(0)}(x) \sim B^{(0)} x^a, \qquad B^{(0)} = b_0\, \Gamma(a+1)\, A^a = \Gamma(a+1)\, \frac{b_1^a}{a^a}\, b_0^{1-a}, \tag{2.7}$$

expressed through the Γ-function. This quantity $B^{(0)}$ is the strong-coupling amplitude of the irrational core. The irrational core $f^{(0)}(x)$ is the minimal Borel-based approximant that matches (b_0, b_1), embeds the correct exponent a, and introduces no additional structure.

Define the residual factor

$$\Phi^{(1)}(x) = \frac{f_3(x)}{f^{(0)}(x)}. \tag{2.8}$$

Expanding at small x,

$$\Phi^{(1)}(x) = 1 + v_0 x^2 + v_1 x^3 + O(x^4), \tag{2.9}$$

where

$$v_0 = \frac{1}{2!}\,\frac{d^2}{dx^2}\left(\frac{f_3(x)}{f^{(0)}(x)}\right)\bigg|_{x=0}, \qquad v_1 = \frac{1}{3!}\,\frac{d^3}{dx^3}\left(\frac{f_3(x)}{f^{(0)}(x)}\right)\bigg|_{x=0}, \tag{2.10}$$

we isolate the linear part and then take its inverse:

$$g^{(1)}(x) = v_0 + v_1 x, \qquad h^{(1)}(x) = \frac{1}{g^{(1)}(x)} = c_0 + c_1 x + O(x^2). \tag{2.11}$$

The Borel transform of $h^{(1)}(x)$ is

$$H^{(1)}(t) = c_0 + c_1 t. \tag{2.12}$$

We approximate it by a second-order self-similar root form

$$H_*^{(1)}(t) = c_0(1 + Ct)^2, \qquad C = \frac{c_1}{2c_0}. \tag{2.13}$$

The inverse Borel transform gives

$$h_*^{(1)}(x) = c_0 + c_1 x + \frac{c_1^2}{2c_0}x^2. \tag{2.14}$$

Thus the corrected linear part is

$$g_*^{(1)}(x) = \frac{1}{h_*^{(1)}(x)}. \tag{2.15}$$

The complete corrected residual factor becomes

$$\Phi_*^{(1)}(x) = 1 + x^2 g_*^{(1)}(x) = 1 + \frac{x^2}{c_0 + c_1 x + \frac{c_1^2}{2c_0}x^2}. \tag{2.16}$$

As $x \to \infty$,

$$\Phi_*^{(1)}(x) \to 1 + \left(\frac{c_1^2}{2c_0}\right)^{-1} =: C_{\text{amp}}^{(1)}. \tag{2.17}$$

The full one-step corrected approximant is

$$f^{(1)}(x) = f^{(0)}(x)\,\Phi_*^{(1)}(x), \tag{2.18}$$

with strong-coupling amplitude

$$B^{(1)} = C_{\text{amp}}^{(1)}\,B^{(0)}. \tag{2.19}$$

This expression is fully analytic and depends only on the known coefficients b_0, b_1, b_2, b_3 and the exponent a.

3. Two-step corrected–Borel approximant (using coefficients up to b_5)

We consider a truncated expansion

$$f_5(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5, \tag{3.1}$$

with known strong-coupling asymptotic

$$f(x) \sim B x^a, \qquad x \to \infty. \tag{3.2}$$

This section extends the one-step construction by incorporating the next pair of coefficients (b_4, b_5), producing a second rational correction factor.

3.1. Step 1: Borel core and first correction (coefficients b_0–b_3)

Step 1 is completely analogous to the construction in Section 2. The irrational core $f^{(0)}(x)$ and the first rational correction $\Phi_*^{(1)}(x)$ are obtained exactly as before, yielding the one-step approximant

$$f^{(1)}(x) = f^{(0)}(x)\,\Phi_*^{(1)}(x), \tag{3.3}$$

with strong-coupling amplitude

$$B^{(1)} = C_{\mathrm{amp}}^{(1)}\,B^{(0)}. \tag{3.4}$$

We now proceed to the next correction, which uses only the new coefficients b_4 and b_5, consistent with the coefficient-local design of the hierarchy.

3.2. Step 2: Second correction (coefficients b_4, b_5)

Define the residual at the next order:

$$\Phi^{(2)}(x) = \frac{f_5(x)}{f^{(1)}(x)} = 1 + x^4(w_1 + w_2 x) + O(x^6), \tag{3.5}$$

and isolate the linear part:

$$g^{(2)}(x) = w_1 + w_2 x, \qquad h^{(2)}(x) = \frac{1}{g^{(2)}(x)} = d_0 + d_1 x + O(x^2). \tag{3.6}$$

The Borel transform is

$$H^{(2)}(t) = d_0 + d_1 t. \tag{3.7}$$

We approximate it by a fourth-order self-similar root:

$$H_*^{(2)}(t) = d_0(1 + Wt)^4, \qquad W = \frac{d_1}{4d_0}. \tag{3.8}$$

The inverse Borel transform yields

$$h_*^{(2)}(x) = d_0 + 4d_0 W x + 12 d_0 W^2 x^2 + 24 d_0 W^3 x^3 + 24 d_0 W^4 x^4. \tag{3.9}$$

Thus

$$g_*^{(2)}(x) = \frac{1}{h_*^{(2)}(x)}, \qquad \Phi_*^{(2)}(x) = 1 + x^4 g_*^{(2)}(x). \tag{3.10}$$

As $x \to \infty$,

$$\Phi_*^{(2)}(x) \to 1 + \frac{1}{24\, d_0 W^4} =: C_{\text{amp}}^{(2)}. \tag{3.11}$$

The full two-step approximant is

$$f^{(2)}(x) = f^{(0)}(x)\, \Phi_*^{(1)}(x)\, \Phi_*^{(2)}(x), \tag{3.12}$$

with strong-coupling amplitude

$$B^{(2)} = C_{\text{amp}}^{(2)}\, C_{\text{amp}}^{(1)}\, B^{(0)}. \tag{3.13}$$

This expression uses all coefficients up to b_5 and consists of one irrational factor (from the first Borel inversion) and two rational corrections. The resulting multiplicative structure is Padé-like, but it is not imposed: it emerges naturally from the Borel–root hierarchy.

4. General k-step corrected–Borel scheme

We consider a real function $f(x)$ known through a truncated perturbative expansion

$$f_N(x) = \sum_{n=0}^{N} b_n x^n, \tag{4.1}$$

and a strong-coupling asymptotic form

$$f(x) \sim B x^a, \qquad x \to \infty, \tag{4.2}$$

with known exponent a and unknown amplitude B. The corrected–Borel scheme constructs a hierarchical approximant that reproduces the truncated series to any prescribed order and yields an explicit analytic expression for the amplitude B.

The construction separates into two complementary components:

- a single irrational core, built once from (b_0, b_1), which embeds the strong-coupling exponent a;
- a sequence of rational Borel-root corrections, each depending only on the pair (b_{2k}, b_{2k+1}).

4.1. Step 0: Irrational core from (b_0, b_1)

The first-order Borel transform of the two-term truncation is

$$F_1(t) = b_0 + b_1 t. \tag{4.3}$$

We approximate it by the self-similar form

$$F_1^*(t) = b_0(1 + At)^a, \qquad A = \frac{b_1}{a\, b_0}, \tag{4.4}$$

which matches the Taylor expansion of $F_1(t)$ at $t = 0$.

The inverse Borel transform yields the irrational core

$$f^{(0)}(x) = b_0\, e^{1/(Ax)}(Ax)^a\, \Gamma\!\left(a+1, \frac{1}{Ax}\right), \tag{4.5}$$

where $\Gamma(\cdot,\cdot)$ is the incomplete gamma function. Its strong-coupling amplitude is

$$B^{(0)} = \Gamma(a+1)\, b_0^{1-a} b_1^a\, a^{-a}, \tag{4.6}$$

expressed through the Γ-function. This is the only irrational component in the entire hierarchy; all subsequent corrections are rational.

4.2. *Recursive step $k \geq 1$*

Assume that, after $(k-1)$ steps, we have an approximant

$$f^{(k-1)}(x) \sim B^{(k-1)} x^a, \qquad x \to \infty. \tag{4.7}$$

Define the residual

$$\Phi^{(k)}(x) = \frac{f_{2k+1}(x)}{f^{(k-1)}(x)}, \tag{4.8}$$

whose small-x expansion has the universal form

$$\Phi^{(k)}(x) = 1 + x^{2k}(w_1^{(k)} + w_2^{(k)} x) + O(x^{2k+2}), \tag{4.9}$$

where $w_1^{(k)}$ and $w_2^{(k)}$ are determined by matching the truncated series.

Extract the linear part:

$$g^{(k)}(x) = w_1^{(k)} + w_2^{(k)} x, \qquad h^{(k)}(x) = \frac{1}{g^{(k)}(x)} = d_0^{(k)} + d_1^{(k)} x + O(x^2). \tag{4.10}$$

The Borel transform is

$$H^{(k)}(t) = d_0^{(k)} + d_1^{(k)} t. \tag{4.11}$$

We approximate it by a $2k$-th self-similar root:

$$H_*^{(k)}(t) = d_0^{(k)}(1 + C_k t)^{2k}, \qquad C_k = \frac{d_1^{(k)}}{2k\, d_0^{(k)}}. \tag{4.12}$$

The inverse Borel transform gives the finite sum

$$h_*^{(k)}(x) = d_0^{(k)} \sum_{n=0}^{2k} \binom{2k}{n} (C_k x)^n n!. \tag{4.13}$$

Define the rational correction factor:

$$\Phi_*^{(k)}(x) = 1 + x^{2k} g_*^{(k)}(x), \qquad g_*^{(k)}(x) = \frac{1}{h_*^{(k)}(x)}. \tag{4.14}$$

As $x \to \infty$, the dominant term in the sum gives

$$h_*^{(k)}(x) \sim d_0^{(k)} (C_k x)^{2k} (2k)!, \tag{4.15}$$

so that

$$\Phi_*^{(k)}(x) \to 1 + \frac{1}{d_0^{(k)}(2k)!\, C_k^{2k}} =: C_{\mathrm{amp}}^{(k)}. \tag{4.16}$$

4.3. Updated approximant and amplitude

The k-step approximant is

$$f^{(k)}(x) = f^{(0)}(x) \prod_{j=1}^{k} \Phi_*^{(j)}(x), \tag{4.17}$$

with strong-coupling amplitude

$$B^{(k)} = B^{(0)} \prod_{j=1}^{k} C_{\text{amp}}^{(j)}. \tag{4.18}$$

Each pair (b_{2k}, b_{2k+1}) contributes exactly one multiplicative amplitude refinement. The exponent a remains fixed throughout the hierarchy, and the analytic structure is built incrementally from local information.

4.4. Analytic properties

The corrected–Borel hierarchy satisfies:

- **Coefficient locality:** $\Phi^{(k)}$ depends only on (b_{2k}, b_{2k+1}).
- **Exact matching:** $f^{(k)}(x)$ reproduces the truncated series up to x^{2k+1}.
- **Exponent preservation:** Only the core contains a; all corrections are rational.
- **Monotonic refinement:** For alternating-sign coefficients, $C_{\text{amp}}^{(k)} > 0$ and $B^{(k)}$ stabilizes rapidly.
- **Stability:** Perturbations in higher coefficients affect only their corresponding correction factors.

The corrected–Borel scheme thus provides a transparent, hierarchical, and analytically controlled bridge between weak- and strong-coupling regimes. Appendix A presents a compact operator formulation of the same hierarchy.

5. Examples

In this section, we illustrate the corrected–Borel scheme on several classical test problems where both a truncated perturbative expansion and a known strong-coupling asymptotic form are available. For comparison, we also compute the amplitudes obtained from the corrected approximants of Refs. [6, 7] and from the corrected Padé construction (5.9). To quantify performance, we additionally report the relative percentage errors:

$$\varepsilon = \frac{|B_{\text{approx}} - B_{\text{exact}}|}{B_{\text{exact}}} 100. \tag{5.1}$$

Following Ref. [7], we begin by defining the core approximation:

$$F^*(x) = b_0 \left(1 + \frac{b_1}{a b_0} x\right)^a, \tag{5.2}$$

and form the ratio

$$\frac{f_N(x)}{F^*(x)} = 1 + p_0 x^2 + p_1 x^3. \tag{5.3}$$

The first correction factor is

$$\text{cor}_1(x) = 1 + p_0 x^2 \left(1 - \frac{p_1}{2p_0} x\right)^{-2}, \tag{5.4}$$

leading to the amplitude estimate

$$F^{(1)} = F^{(0)}\left(1 + p_0\left(1 - \frac{p_1}{2p_0}\right)^{-2}\right), \qquad F^{(0)} = b_0\left(\frac{b_1}{ab_0}\right)^a. \tag{5.5}$$

Proceeding to the next order, we expand

$$\frac{f_N(x)}{F^*(x)\,\mathrm{cor}_1(x)} = 1 + q_0x^4 + q_1x^5, \tag{5.6}$$

and introduce the second correction

$$\mathrm{cor}_2(x) = 1 + q_0x^4\left(1 - \frac{q_1}{4q_0}x\right)^{-4}, \tag{5.7}$$

which yields

$$F^{(2)} = F^{(1)}\left(1 + q_0\left(1 - \frac{q_1}{4q_0}\right)^{-4}\right). \tag{5.8}$$

Finally, yet another technique of the corrected Padé method constructs

$$f(x) \approx F^*(x)\,P_{n/n}(x) \qquad (2n = N \text{ or } 2n = N - 1), \tag{5.9}$$

with $P_{n/n}(x)$ chosen to reproduce the truncated series, see also Appendix B.2. Although a direct application of the classical Padé approximants is not feasible as shown in Appendix B, some canonical modification is possible, which can account for power-law asymptotic behavior and reduce calculations to Padé approximants. First, the inverse power-transformation is applied to the original series. To the transformed series, the technique of Padé approximants is applied as it turns out that only approximants behaving as x^{-1} are relevant when the amplitude B is the quantity of interest. After yet another power-transformation to ensure correct behavior at small x, Bender and Boettcher arrived to the estimate for the amplitude [8]. We refer to calculations according to [8] as the BB-method.

The comparison formulas above are included or explained for completeness and to ensure that all benchmark methods are explicit within the manuscript.

5.1. Example 1: Massive Schwinger model

The massive Schwinger model in Hamiltonian lattice formulation [9, 10] provides a benchmark problem with rapidly growing perturbative coefficients. Consider the energy gap $\Delta(z)$ between the lowest and second excited states as a function of $z = (1/ga)^4$, where g is a coupling parameter and a is the lattice spacing. The small-z expansion is

$$\Delta(z) = \sum_{n=0}^{\infty} b_n z^n, \qquad z \to 0, \tag{5.10}$$

with coefficients

$$b_0 = 1, \quad b_1 = 6, \quad b_2 = -26, \quad b_3 = 190.66667,$$
$$b_4 = -1756.66667, \quad b_5 = 18048.33651.$$

In the continuum limit $z \to \infty$, the gap behaves as

$$\Delta(z) \sim 1.1284\, z^{1/4}, \tag{5.11}$$

so the exact amplitude is $B = 1.1284$. The corrected–Borel scheme yields

$$B^{(0)} = 1.0031, \qquad B^{(1)} = 1.05037, \qquad B^{(2)} = 1.05414.$$

The corrected-approximant method gives

$$F^{(1)} = 1.28738, \qquad F^{(2)} = 1.38822,$$

and the corrected Padé estimate is

$$B = F^{(0)} P_{2/2}(\infty) = 1.44337.$$

The BB-method of [8] gives $B = 1.50457$. The relative percentage errors incurred by all tested methods are summarized in Table 1.

Table 1. Relative percentage errors for Example 1.

Method	Error ε
Corrected–Borel $B^{(1)}$	6.91500
Corrected–Borel $B^{(2)}$	6.58092
Corrected approximant $F^{(1)}$	14.08900
Corrected approximant $F^{(2)}$	23.02510
Corrected Padé	27.91260
BB-method [8]	33.33650

The corrected–Borel scheme provides the smallest errors among all methods tested. Classical corrected approximants and the corrected Padé estimate deteriorate as higher orders are included, reflecting the difficulty of handling strongly divergent coefficients.

5.2. *Example 2: Debye–Hückel function*

The Debye–Hückel function

$$D(x) = \frac{2}{x} - \frac{2}{x^2}\left(1 - e^{-x}\right) \tag{5.12}$$

appears in the theory of strong electrolytes and Gaussian polymers [11, 12]. It is convenient to work with its inverse

$$f(x) = D(x)^{-1}, \tag{5.13}$$

whose small-x expansion is

$$f(x) = 1 + \frac{x}{3} + \frac{x^2}{36} - \frac{x^3}{540} - \frac{x^4}{6480} + \frac{x^5}{27216} + \cdots . \tag{5.14}$$

The large-x behavior is

$$f(x) \sim \frac{1}{2}x, \qquad x \to \infty. \tag{5.15}$$

The corrected–Borel scheme gives

$$B^{(0)} = 0.333333, \qquad B^{(1)} = 0.449074, \qquad B^{(2)} = 0.460493.$$

The corrected-approximant method yields

$$F^{(1)} = 0.564815, \qquad F^{(2)} = 0.580297,$$

and the corrected Padé estimate is

$$B = F^{(0)} P_{2/2}(\infty) = 0.62069.$$

The BB-method of [8] gives $B = 0.4$. The relative percentage errors incurred by all tested methods are summarized in Table 2.

Table 2. Relative percentage errors for Example 2.

Method	Error ε
Corrected–Borel $B^{(1)}$	10.18520
Corrected–Borel $B^{(2)}$	7.90131
Corrected approximant $F^{(1)}$	12.96300
Corrected approximant $F^{(2)}$	16.05940
Corrected Padé	24.13790
BB-method [8]	20.00000

Even in this structurally challenging example, the corrected–Borel scheme remains competitive and significantly outperforms the corrected Padé.

5.3. Example 3: Nonlinear Schrödinger equation

The ground-state energy $E(c)$ of Bose-condensed atoms in a spherically symmetric harmonic trap satisfies the short expansion

$$E(c) = \frac{3}{2} + \frac{1}{2}c - \frac{3}{16}c^2 + \frac{9}{64}c^3 - \frac{35}{256}c^4 + O(c^5), \qquad c \to 0, \tag{5.16}$$

and the strong-coupling behavior

$$E(c) \sim \frac{5}{4}c^{2/5}, \qquad c \to \infty. \tag{5.17}$$

Assume the next coefficient is $b_5 = 0$.

The corrected–Borel scheme gives

$$B^{(0)} = 1.23729, \qquad B^{(1)} = 1.24612, \qquad B^{(2)} = 1.24676.$$

The corrected-approximant method yields

$$F^{(1)} = 1.31355, \qquad F^{(2)} = 1.31307,$$

and the corrected Padé estimate is

$$B = F^{(0)} P_{2/2}(\infty) = 1.28211.$$

The BB-method of [8] gives $B = 1.31097$. The relative percentage errors incurred by all tested methods are summarized in Table 3.

Table 3. Relative percentage errors for Example 3.

Method	Error ε
Corrected–Borel $B^{(1)}$	0.310191
Corrected–Borel $B^{(2)}$	0.259576
Corrected approximant $F^{(1)}$	5.084120
Corrected approximant $F^{(2)}$	5.045840
Corrected Padé	2.568650
BB-method [8]	4.877700

This example highlights the strength of the corrected–Borel scheme: even with only a few coefficients, it produces an amplitude accurate to within 0.3%. Across all three test problems, the corrected–Borel scheme consistently produces stable and competitive amplitude estimates. In the Schwinger model and nonlinear Schrödinger examples, it clearly outperforms both corrected approximants and corrected Padé methods. The problem with the canonical BB-method of [8] is not cosmetic, as explained in [7].

6. Interpolation variant of the corrected–Borel scheme

The corrected–Borel scheme has so far been presented as an extrapolation method for predicting the strong-coupling amplitude B. In many physical problems, however, the situation is reversed: the amplitude is known more reliably than the higher perturbative coefficients. In such cases, the hierarchical structure of the scheme can be adapted to an interpolation setting.

Instead of taking all perturbative coefficients as fixed, one may treat the last coefficient entering the construction—or even an additional, artificial coefficient beyond the known series—as an adjustable quantity. This reflects the practical observation that the final terms of a divergent or rapidly growing series are often less trustworthy than global information such as the strong-coupling amplitude.

The adjustable coefficient is then chosen so that the resulting approximant reproduces the prescribed amplitude at infinity. This approach replaces one piece of local perturbative data with a global asymptotic constraint, while leaving the hierarchical structure of the corrected–Borel scheme entirely intact.

When the amplitude B is known in advance from independent theoretical or numerical considerations, the scheme can therefore be adapted to an interpolation procedure without altering its architecture. The key observation is that the amplitude produced by the K-step corrected–Borel approximant,

$$B^{(K)}(b_{2K+1}) = B^{(0)} \prod_{k=1}^{K} C_{\mathrm{amp}}^{(k)}(b_{2K+1}), \tag{6.1}$$

depends explicitly on the highest coefficient entering the K-th correction step, typically b_{2K+1}.

When the true strong-coupling amplitude B is known, this coefficient may be treated as an adjustable parameter determined from the interpolation condition

$$B^{(K)}(b_{2K+1}) = B. \tag{6.2}$$

This introduces only a single scalar degree of freedom and leaves all stages of the corrected–Borel hierarchy unchanged. Lower-order coefficients $\{b_0, \ldots, b_{2K}\}$ remain fixed by the perturbative expansion, while the final coefficient b_{2K+1} is selected so that the full K-step approximant reproduces the prescribed strong-coupling amplitude.

Thus the corrected–Borel framework naturally accommodates both extrapolation (unknown B) and interpolation (known B).

A recursive self-improvement extension of the scheme is described in Appendix D. In this extension, synthetic coefficients generated by the approximant itself allow the hierarchy to continue beyond the available perturbative data.

The interpolation variant therefore provides a flexible mechanism for incorporating global asymptotic information into the corrected–Borel hierarchy without modifying its analytic structure.

7. Conclusions and discussion

The corrected–Borel scheme offers a hierarchical and analytically transparent framework for extracting strong-coupling information from short perturbative expansions. Its central architectural feature is the clean separation between two complementary components: a single irrational core, constructed once from the initial pair (b_0, b_1), and a sequence of rational Borel-root corrections generated from progressively higher-order residuals. This division allows the method to embed the correct strong-coupling exponent at the outset while incorporating all additional information through explicit, coefficient-local refinements. Because each correction depends only on the new pair (b_{2k}, b_{2k+1}), the scheme remains scalable and robust even when only a few coefficients are available.

A key conceptual insight is that the rational structure of the correction factors is not an imposed ansatz but an emergent consequence of the method's internal operations. In the present context, "emergent" is used in a precise sense: the rational correction factors arise automatically from the fixed sequence of residual extraction, inversion, Borel transformation, application of a self-similar root, and inverse Borel transformation. Because the Borel transform of a finite Taylor polynomial is a polynomial, and the inverse of such a polynomial is rational, the Padé-like form of the corrections is a structural consequence of the procedure rather than an assumed form. This distinguishes the present approach from classical Padé theory, where rationality is postulated globally from the outset.

Although the corrected–Borel approximants reproduce the truncated small-x expansion to any prescribed order, the mechanism of matching differs fundamentally from that of Padé approximants. Instead of solving a new algebraic system at each order, the corrected–Borel scheme enforces matching iteratively: information is passed forward through the residuals, and each step corrects only what the previous step could not capture. This iterative structure eliminates the need for global reoptimization and provides a transparent view of how each pair of coefficients contributes to the final strong-coupling amplitude.

The method's strengths follow naturally from this architecture. The coefficient-local design ensures that each refinement is analytically explicit and easy to interpret. The separation between irrational and rational components keeps the non-polynomial structure confined to the core, while all higher corrections remain rational and structurally simple. The Borel-space formulation stabilizes the treatment of factorially growing coefficients and provides a natural setting for self-similar transformations. Together, these features yield a flexible and extensible analytic tool capable of handling divergent or rapidly growing series with minimal input.

At the same time, the scheme has clear boundaries. Its residual extraction assumes a two-term structure at each order, which may require modification for functions with more intricate small-x behavior. The strong-coupling exponent must be known or estimated separately, and inaccuracies in early steps may propagate multiplicatively through the hierarchy.

Although the corrected–Borel scheme is designed as an analytic extrapolation method rather than an exact reconstruction procedure, there exist special situations in which exact recovery of the target function is in fact possible from only a few perturbative coefficients. The simplest example is the irrational core itself: for functions whose Borel transform is exactly of the form $b_0(1 + At)^a$, the inverse Borel transform reproduces the full function from the pair (b_0, b_1) alone. In such cases, the corrected–Borel hierarchy collapses, and the core already yields the exact strong–coupling amplitude.

A second class of examples arises when the function is a product of the core with a finite rational factor. Because each rational correction in the hierarchy is generated from a linear residual and a finite Borel polynomial, the scheme can reproduce such functions exactly once the corresponding pair (b_{2k}, b_{2k+1}) is included. These cases illustrate that the emergence of rational corrections is not merely structural but can, in favorable circumstances, lead to exact recovery of the full function.

An even more striking situation occurs in zero–dimensional field theory, where low-order Borel-Leroy summation is known to reconstruct the exact partition function from only a handful of coefficients [4]. Because the corrected–Borel scheme builds its hierarchy around the same Borel machinery, it is natural to expect that the correction framework developed here can be applied to zero-dimensional models as well. A systematic study of this connection—and of the conditions under which exact reconstruction occurs—is reserved for future work.

While the present corrected–Borel scheme assumes the strong-coupling exponent (or critical index) a is known a priori, methods for its independent estimation have been extensively developed within related self-similar frameworks [13, 14]. Most importantly, the calculation of the critical index can effectively be reduced to the calculation of strong-coupling amplitudes as a special case. The systematic integration of these index-estimation techniques into the current hierarchical scheme represents an important and natural avenue for future work.

Overall, the corrected–Borel scheme provides a conceptually clear, hierarchical, and analytically controlled bridge between weak- and strong-coupling regimes. Its emergent rational structure, coefficient-local design, and stability in the short-series regime make it a promising tool for a wide range of problems in perturbation theory, statistical mechanics, and nonlinear analysis. Crucially, this methodology demonstrates that rational approximations can be systematically derived, rather than merely postulated as an a priori ansatz.

Use of Generative-AI tools declaration

The author used Microsoft Copilot to assist with language editing, structural polishing, and consistency checking. All AI-generated suggestions were reviewed, verified, and fully edited by the author, who takes complete responsibility for the final content of the manuscript.

Conflict of interest

Dr. Simon Gluzman is the Guest Editor of special issue "Unified Perspectives on Approximation Methods and Expansions" for AIMS Mathematics. Dr. Simon Gluzman was not involved in the editorial review and the decision to publish this article.

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Appendix

A. Operator structure of the corrected–Borel hierarchy

This appendix provides a formal, operator-theoretic representation of the corrected–Borel hierarchy. By decomposing the procedure into elementary operators acting on truncated series, the emergence of the rational corrections becomes structurally explicit.

Let $\mathcal{B}$ denote the Borel transform,

$$\mathcal{B}\left[\sum_{n=0}^{N} b_n x^n\right](t) = \sum_{n=0}^{N} \frac{b_n}{n!} t^n, \tag{A.1}$$

and let $\mathcal{B}^{-1}$ denote the inverse Borel transform,

$$\mathcal{B}^{-1}[\Phi](x) = \int_0^{\infty} e^{-t}\Phi(tx)\,dt. \tag{A.2}$$

Define the root operator $\mathcal{R}_m$ acting on a linear polynomial $p(t) = c_0 + c_1 t$ by

$$(\mathcal{R}_m p)(t) = c_0(1 + Ct)^m, \qquad C = \frac{c_1}{m\,c_0}. \tag{A.3}$$

This operator encapsulates the self-similar root transformation applied at each stage of the hierarchy. The exponent m increases with the step index, reflecting the progressively higher order of the residual.

Let $\mathcal{I}$ denote the inversion operator,

$$\mathcal{I}[g](x) = \frac{1}{g(x)}. \tag{A.4}$$

Let $\mathcal{T}_k$ denote the shifted truncation operator that extracts the linear part of the residual at order $2k$. For a residual of the form $\Phi(x) = 1 + x^{2k}(w_1 + w_2 x) + O(x^{2k+2})$, we define:

$$(\mathcal{T}_k\Phi)(x) = \left[x^{-2k}(\Phi(x) - 1)\right]_{\le 1} = w_1 + w_2 x. \tag{A.5}$$

The corrected–Borel hierarchy can now be written as a composition of these operators, making the analytic deduction of the method explicit.

A.1. Core operator

The irrational core is generated by the sequence:

$$f^{(0)}(x) = \mathcal{B}^{-1}[\mathcal{R}_a(\mathcal{B}[b_0 + b_1 x])](x). \tag{A.6}$$

A.2. Recursive correction operator

Define the k-th correction operator as the composition:

$$C_k = \mathcal{B}^{-1} \circ \mathcal{R}_{2k} \circ \mathcal{B} \circ \mathcal{I} \circ \mathcal{T}_k. \tag{A.7}$$

Applied to the residual

$$\Phi^{(k)}(x) = \frac{f_{2k+1}(x)}{f^{(k-1)}(x)}, \tag{A.8}$$

the operator produces the rational correction factor:

$$\Phi_*^{(k)}(x) = 1 + x^{2k}\, C_k[\Phi^{(k)}](x). \tag{A.9}$$

A.3. Full hierarchy

The full K-step corrected–Borel approximant is the product:

$$f^{(K)}(x) = f^{(0)}(x) \prod_{k=1}^{K} \Phi_*^{(k)}(x). \tag{A.10}$$

Its strong-coupling amplitude is given by:

$$B^{(K)} = B^{(0)} \prod_{k=1}^{K} C_{\mathrm{amp}}^{(k)}, \tag{A.11}$$

where each $C_{\mathrm{amp}}^{(k)}$ is obtained from the large-x limit of $\Phi_*^{(k)}(x)$.

This operator formulation highlights the modularity of the hierarchy: each step is generated by the same fixed sequence of operations, differing only in the truncation order and the corresponding pair of coefficients. The irrational structure is introduced exactly once, while all higher corrections emerge as rational and coefficient-local factors.

A.4. Formal lemma on the structure of corrections

Lemma A.1. *Let $H_*(t) = d_0(1 + Ct)^{2k}$ be the self-similar root form obtained from a linear residual, where d_0 and $C > 0$ are constants. Define the function $h(x)$ via the inverse Borel transform:*

$$h(x) = \mathcal{B}^{-1}[H_*(t)](x). \tag{A.12}$$

Then $h(x)$ is a polynomial of exact degree $2k$, and the resulting correction factor

$$\Phi(x) = 1 + \frac{x^{2k}}{h(x)} \tag{A.13}$$

is strictly rational. Moreover, its asymptotic behavior is:

$$\Phi(x) \to 1 + \frac{1}{d_0 (2k)!\, C^{2k}} \qquad \textit{as } x \to \infty. \tag{A.14}$$

Proof. Expanding $(1 + Ct)^{2k}$ binomially and utilizing the identity $\int_0^\infty e^{-t} t^n \, dt = n!$ yields:

$$h(x) = d_0 \sum_{n=0}^{2k} \binom{2k}{n} n! \, (Cx)^n, \tag{A.15}$$

which is a polynomial of exact degree $2k$. The asymptotic limit follows directly from isolating the leading term $n = 2k$, where $h(x) \sim d_0 (2k)! \, C^{2k} x^{2k}$. Substituting this asymptotic form into the definition of $\Phi(x)$ yields the stated limit. □

This lemma formalizes the mechanism by which each correction factor acquires a Padé-like rational form.

Remark A.1. The lemma demonstrates that the Padé-like structure of the correction factors is not an imposed ansatz but a necessary structural consequence of the operator chain. This justifies the terminology of "emergent rationality".

A.5. *Invariance of the strong-coupling exponent*

To rigorously establish that the recursive corrections modify only the amplitude and never distort the fundamental analytic archetype, we formalize the preservation of the critical index.

Lemma A.2. *Let $f^{(k-1)}(x)$ possess the strong-coupling asymptotic behavior $f^{(k-1)}(x) \sim B^{(k-1)} x^a$ as $x \to \infty$, where $B^{(k-1)}$ is a non-zero amplitude and a is the established critical exponent. Under the action of the k-th correction operator C_k, the updated approximant $f^{(k)}(x) = f^{(k-1)}(x) \Phi_*^{(k)}(x)$ strictly preserves the exponent a.*

Proof. By definition, the k-th correction factor is generated via the operator sequence as $\Phi_*^{(k)}(x) = 1 + x^{2k} C_k[\Phi^{(k)}](x)$. From Lemma A.1, $\Phi_*^{(k)}(x)$ is a strictly rational function that converges to a constant scalar limit as $x \to \infty$:

$$\lim_{x \to \infty} \Phi_*^{(k)}(x) = C_{\text{amp}}^{(k)} = 1 + \frac{1}{d_0^{(k)} (2k)! \, C_k^{2k}}. \tag{A.16}$$

Because this limit is a finite, non-zero constant, the asymptotic behavior of the multiplicatively updated approximant evaluates to:

$$f^{(k)}(x) \sim \left(B^{(k-1)} x^a \right) C_{\text{amp}}^{(k)} = \left(B^{(k-1)} C_{\text{amp}}^{(k)} \right) x^a = B^{(k)} x^a \qquad \text{as } x \to \infty. \tag{A.17}$$

Since the multiplication by the rational surrogate $\Phi_*^{(k)}(x)$ acts purely as a scalar transformation in the $x \to \infty$ limit, the strong-coupling exponent a remains strictly invariant across all hierarchy levels k. □

Remark A.2. This invariance is a direct consequence of the structural separation embedded in the corrected–Borel methodology. The irrational scaling law x^a is postulated exactly once by the core operator, while the hierarchical operators C_k are mathematically restricted to generating zero-degree asymptotic shifts, ensuring pure analytic deduction at each refinement stage.

Remark A.3 (Physical interpretation via hierarchical condensation). The multiplicative chain of the corrected–Borel hierarchy admits a compelling physical interpretation analogous to a nested or hierarchical Bose–Einstein condensation (i.e., a macroscopic "condensate on top of a condensate"). The irrational core $f^{(0)}(x)$ acts as the primary condensate, fixing the global quantum numbers and the fundamental non-polynomial archetype via the critical index a. The subsequent residuals represent the remaining fluctuational fields. Rather than treating these fluctuations via standard additive perturbations—which typically trigger asymptotic divergence—the recursive operator sequence C_k isolates and stabilizes them. Each rational factor $\Phi_*^{(k)}(x)$ thus signifies a successive, localized condensation of the residual fluctuations, systematically refining the macroscopic amplitude $B^{(K)}$ while leaving the underlying ground-state symmetry strictly invariant.

B. Comparison with Padé approximants

Padé approximants provide a classical method for extending the domain of a truncated power series. Given

$$f_N(x) = \sum_{n=0}^{N} b_n x^n, \tag{B.1}$$

a diagonal Padé approximant $P_{n/n}(x)$ is defined by

$$f_N(x) - P_{n/n}(x) = O(x^{2n+1}), \qquad 2n = N \text{ or } 2n = N - 1. \tag{B.2}$$

The Padé approximant is a rational function

$$P_{n/n}(x) = \frac{A_0 + A_1 x + \cdots + A_n x^n}{1 + C_1 x + \cdots + C_n x^n}, \tag{B.3}$$

whose coefficients are determined by solving a linear system that enforces the matching condition.

This global rational ansatz is structurally very different from the corrected–Borel hierarchy, where rationality appears only in the correction factors and only at the level required by the residual structure.

B.1. Structural differences

The key distinctions between Padé approximants and the corrected–Borel scheme are:

- **Global vs. local structure.** Padé imposes a global rational form from the outset. The corrected–Borel scheme introduces irrational structure only once (in the core) and rational structure incrementally through coefficient-local corrections.
- **Coefficient usage.** Padé uses all coefficients simultaneously to determine the rational function. The corrected–Borel scheme uses only (b_0, b_1) for the core and each pair (b_{2k}, b_{2k+1}) for the k-th correction.
- **Stability.** Padé approximants may develop spurious poles or converge to incorrect limits when the analytic continuation problem is indeterminate. The corrected–Borel scheme avoids global rational constraints and therefore does not suffer from pole proliferation.
- **Exponent handling.** Padé does not incorporate the strong-coupling exponent a unless it is built into the ansatz. The corrected–Borel scheme embeds a directly into the irrational core.

These differences explain why Padé approximants often deteriorate in the short-series regime, while the corrected–Borel hierarchy remains stable and systematically improvable.

B.2. Corrected Padé construction

For completeness, we recall the corrected Padé method used in the numerical comparisons of Section 5. One writes

$$f(x) \approx F^*(x)\, P_{n/n}(x), \tag{B.4}$$

where $F^*(x)$ is the core approximant

$$F^*(x) = b_0 \left(1 + \frac{b_1}{ab_0} x\right)^a, \tag{B.5}$$

and $P_{n/n}(x)$ is chosen to reproduce the truncated series.

For $N = 4$ or $N = 5$, one uses $n = 2$, giving

$$P_{2/2}(x) = \frac{1 + p_1 x + p_2 x^2}{1 + q_1 x + q_2 x^2}. \tag{B.6}$$

The amplitude estimate is then

$$B_{\text{Padé}} = F^{(0)}\, P_{2/2}(\infty), \tag{B.7}$$

where $F^{(0)} = b_0(\frac{b_1}{ab_0})^a$. This construction is included because it is often used in the literature, but it is not structurally aligned with the corrected–Borel hierarchy.

B.3. Summary

Padé approximants are powerful when long series are available or when the analytic continuation problem is well-posed. However, in the short-series regime considered here, their global rational structure can lead to instabilities, spurious poles, and inaccurate amplitude estimates.

The corrected–Borel scheme avoids these issues by building analytic structure hierarchically from local information, introducing irrationality only once, and generating rational corrections through a universal Borel–root mechanism.

C. Relation to hyperasymptotics and resurgence

This appendix clarifies the conceptual relationship between the corrected–Borel hierarchy and the broader frameworks of hyperasymptotics and resurgence. The goal is not to claim equivalence, but to explain how the structure of the corrected–Borel scheme fits naturally within the analytic landscape shaped by these theories.

Hyperasymptotic analysis, developed by Berry, Howls, Olde Daalhuis, and others [15–19], refines classical asymptotics by repeatedly re-expanding the remainder after optimal truncation of a divergent series. Each stage of the hyperasymptotic expansion extracts exponentially small contributions that are invisible to ordinary perturbation theory. The procedure is recursive: the remainder at one level becomes the input to the next.

Resurgence theory, initiated by Écalle, provides a deeper structural interpretation of this process. In the resurgent framework, the Borel transform of a divergent series is a multivalued analytic function whose singularities encode the entire transseries structure of the solution. The Stokes phenomenon, alien derivatives, and bridge equations describe how information flows between different asymptotic sectors.

The corrected–Borel hierarchy does not attempt to reconstruct the full resurgent structure of a function. Instead, it uses a simplified mechanism that mirrors one aspect of hyperasymptotics: the iterative extraction and resummation of residual information.

In particular:

- The Borel transform converts factorial divergence into polynomial structure, as in both hyperasymptotics and resurgence.
- Each residual in the corrected–Borel scheme plays a role analogous to a hyperasymptotic remainder: it captures the part of the function not accounted for at the previous stage.
- The self-similar root transformation acts as a simplified surrogate for the analytic continuation and saddle-point structure that appear in hyperasymptotics.
- The inverse Borel transform reintroduces non-polynomial structure in a controlled way, analogous to the reconstruction of exponentially small terms.

These parallels are structural rather than literal. The corrected–Borel scheme does not require knowledge of Borel-plane singularities, Stokes constants, or transseries sectors. Instead, it uses only the minimal local information encoded in the truncated perturbative coefficients.

From the perspective of resurgence, the corrected–Borel hierarchy may be viewed as a local resurgent procedure: it extracts information from the Borel transform near the origin and propagates it through a sequence of algebraic operations. This is in contrast to full resurgence, which requires global analytic control of the Borel plane.

The corrected–Borel scheme therefore occupies a middle ground: it is more structured than classical resummation methods such as Padé or Borel–Padé, but less demanding than full hyperasymptotic or resurgent analysis. Its strength lies in its ability to extract strong-coupling amplitudes from very short series without requiring global analytic information.

D. Recursive self-improvement extension

The goal below is to present a clean and operational description of how synthetic coefficients can be generated and used to extend the corrected–Borel hierarchy beyond the available perturbative data.

The corrected–Borel scheme, as presented in the main text, uses the perturbative coefficients only up to b_{2K+1} in constructing the K-step approximant. In many applications, however, one may wish to extend the hierarchy further even when no additional perturbative coefficients are available. This motivates a recursive self-improvement procedure in which the approximant itself is used to generate synthetic higher-order coefficients.

Let $f^{(K)}(x)$ denote the K-step corrected–Borel approximant. Its small-x expansion can be written as

$$f^{(K)}(x) = \sum_{n=0}^{\infty} \tilde{b}_n^{(K)} x^n, \tag{D.1}$$

where the coefficients $\tilde{b}_n^{(K)}$ are obtained by expanding the product

$$f^{(K)}(x) = f^{(0)}(x) \prod_{j=1}^{K} \Phi_*^{(j)}(x). \tag{D.2}$$

The key idea is to treat the next pair of coefficients $(\tilde{b}^{(K)}_{2K+2}, \tilde{b}^{(K)}_{2K+3})$ as synthetic data that can be fed back into the hierarchy to construct the $(K+1)$-st correction.

Define

$$b^{(\mathrm{syn})}_{2K+2} = \tilde{b}^{(K)}_{2K+2}, \qquad b^{(\mathrm{syn})}_{2K+3} = \tilde{b}^{(K)}_{2K+3}. \tag{D.3}$$

These synthetic coefficients are then used in place of the true (unknown) coefficients in the next residual:

$$\Phi^{(K+1)}(x) = \frac{b_0 + b_1 x + \cdots + b_{2K+1} x^{2K+1} + b^{(\mathrm{syn})}_{2K+2} x^{2K+2} + b^{(\mathrm{syn})}_{2K+3} x^{2K+3}}{f^{(K)}(x)}. \tag{D.4}$$

The same Borel-root procedure as before is then applied to construct the $(K+1)$-st correction factor $\Phi^{(K+1)}_*(x)$.

This recursive step effectively allows the hierarchy to continue beyond the available perturbative data, using the approximant itself to supply the missing coefficients. The procedure is analogous in spirit to the generation of synthetic terms in certain self-similar and renormalization-group approaches.

The resulting $(K+1)$-step approximant is

$$f^{(K+1)}(x) = f^{(K)}(x)\, \Phi^{(K+1)}_*(x), \tag{D.5}$$

with strong-coupling amplitude

$$B^{(K+1)} = B^{(K)}\, C^{(K+1)}_{\mathrm{amp}}. \tag{D.6}$$

Because the synthetic coefficients are generated from the approximant itself, the recursive extension is not guaranteed to converge in all cases. However, numerical experiments indicate that for many problems with alternating-sign or moderately growing coefficients, the self-improvement procedure stabilizes the amplitude and can significantly enhance accuracy. This recursive mechanism should be viewed as an analytic tool for exploring the structure of the corrected–Borel hierarchy beyond the strict limits of the available perturbative data.

In summary, the corrected–Borel hierarchy is not a substitute for hyperasymptotics or resurgence, but it is consistent with their analytic philosophy. It captures, in a simplified and coefficient-local form, the idea that residual information can be systematically extracted and resummed to improve asymptotic predictions. Resurgence theory posits that the perturbative sector “knows” about non-perturbative sectors through the global structure of Borel singularities. Our approach exploits a similar philosophy but through a more direct, algebraic mechanism. The two theories are compared in Table 4.

Table 4. Comparison between the recursive corrected-Borel bootstrap and resurgence theory, highlighting their respective local and global analytic structures.

Feature	Recursive corrected-Borel	Resurgence theory
Primary Input	Local Bootstrap: Short truncated series $\{b_n\}$.	Global Reconstruction: Borel plane structure and singularities.
Mechanism	Recursive rational corrections via residual inversion and Borel-root steps.	Alien calculus, bridge equations, and transseries expansions.
Core Logic	Self-similar root inversion and algebraic recursion.	Analytic continuation, Stokes phenomena, and monodromy.
Implementation	Fully explicit, algebraic, and computationally light.	Conceptually deep; requires complex analytic machinery.
Goal	Rapid stabilization of strong-coupling amplitudes.	Full reconstruction of perturbative and non-perturbative sectors.
Data Density	Effective in coefficient-poor settings.	Requires large-order growth of the coefficients or detailed singularity data.