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Research article

Bifurcation and meromorphic exact solutions of five nonlinear partial differential equations

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Abstract: Differential equations model how scientific systems change over time and space. They are essential tools across physics, engineering, biology, and economics. Searching for solutions to these equations is of great significance, and simultaneously seeking meromorphic exact solutions of multiple nonlinear partial differential equations (PDEs) would be innovative. In view of this, we transform five nonlinear PDEs into ordinary differential equations (ODEs) through appropriate transformations and summarize these ODEs into one equation, then explore the bifurcation of this equation with commonalities. Subsequently, we utilize the complex method and the $\exp(-\vartheta(z))$-expansion approach to derive meromorphic exact solutions for five nonlinear PDEs at once, including the Tzizeica equation, the Zhiber–Shabat (ZS) equation, the modified Kortweg de Vries (mKdV) equation, the Tzitzeica–Dodd–Bullough (TDB) equation, and the modified double sine-Gordon (mDSG) equation. To better understand the dynamic structure of the results, two-dimensional, three-dimensional, and contour graphs are given to illustrate the solutions. The conceptual framework developed here could be extended to analyze additional nonlinear physical equations in scientific research.

Keywords: differential equation; complex method; $\exp(-\vartheta(z))$-expansion method; bifurcation; exact solution; meromorphic function
Mathematics Subject Classification: 30D35, 34A05

1. Introduction

Nonlinear differential equations (NLDEs) serve as essential mathematical frameworks for modeling complex behaviors in diverse natural and engineered systems, where interactions produce outcomes beyond simple summation. Their applications span numerous disciplines, including solid-state physics, plasma physics, nonlinear optics, fluid dynamics, chemistry, and biology. To gain a thorough understanding of solution phenomena associated with NLDEs, scholars have conducted extensive research. Numerous methods to search explicit solutions of NLDEs have been established, for instance, the Hirota bilinear approach [1–3], the (G'/G)-expansion method [4, 5], the modified Kudryashov method [6–8], the generalized Riccati equation mapping approach [9], the sine-Gordon expansion method [10, 11], the three-wave method [12], the exp function method [13], the variational method [14], the dynamic systems method [15], the logistic method [16–18], the $\exp(-\vartheta(z))$-expansion method [19–21], and the complex method [22,23], among others [24,25]. Although these methods are effective, it is difficult to get the solution for differential equations in the complex plane because of the existence of singularity. Furthermore, the types of singularities are various, pole, logarithmic singularity, essential singularity, movable singularity, and so on. The behavior of solutions to a differential equation in the neighborhood of these singularities are extraordinarily complicated as well. Within the singularities mentioned above, the properties of the pole type are the best; therefore, studying the meromorphic solutions to NLDEs is very important. Meromorphic solutions of auxiliary differential equations are studied [29]. Some significant results have been developed, such as the growth order, classifications and characterizations of meromorphic solutions [26–28]. However, in the case where it is difficult to obtain solutions to NLDEs as described earlier, it would be innovative to attain meromorphic exact solutions to multiple nonlinear PDEs at once. Therefore, this article will obtain meromorphic exact solutions of five nonlinear PDEs, including the Tzitzeica equation [30, 31], the Zhiber–Shabat (ZS) equation [32–34], the modified Kortweb de Vries (mKdV) equation [35, 36], the Tzitzeica–Dodd–Bullough (TDB) equation [37, 38], and modified double sine-Gordon (mDSG) equation [39, 40], in one go. The specific expressions of these equations are shown in Table 1.

Table 1. Expressions of five nonlinear PDEs.

Name	Equation	Number
Tzitzeica equation	$u_{tt} - u_{xx} + e^{-2u} - e^{u} = 0$	(1.1)
ZS equation	$u_{xt} + \rho e^{u} + \sigma e^{-u} + \gamma e^{-2u} = 0$	(1.2)
mKdV equation	$4uu_{xxx} + uu_{xxt} - u_x u_{xt} - 4u^3 u_t - 4u_x u_{xx} - 16u^3 u_x = 0$	(1.3)
TDB equation	$u_{xt} = e^{-2u} + e^{-u}$	(1.4)
mDSG equation	$u_{xt} = c \sin u + b \sin 2u$	(1.5)

These five equations are all important integrable models in nonlinear mathematical physics, playing a central role in soliton theory, geometric surfaces, field theory, and integrable systems. The Tzitzeica equation originates from affine geometry, describing a class of surfaces with special scaling invariance, and its solutions also appear in discrete integrable systems and conformal field theory; the ZS equation is a more general sine-Gordon-type equation, serving as the bridge in studying integrable layers and conservation laws; the mKdV equation is an integrable deformation of the classical Kortweg de Vries (KdV) equation, featuring peaked soliton solutions; the TDB equation (sometimes referred to as the Dodd–Bullough equation) is a special form or alternative formulation of the Tzitzeica equation, commonly found in optics and relativistic field theory; and the mDSG equation is an integrable variant of the double sine-Gordon equation used to describe coupled bosonic fields or ladder models. All these equations possess rich integrable structures (such as Lax pairs and infinite conservation laws) and diverse analytical solutions (solitons, rational solutions, etc.) and are interconnected in the classification of modern integrable systems, forming a key set of examples in nonlinear waves and geometric physics.

In order to achieve meromorphic exact solutions of these five nonlinear partial differential equations (PDEs), we need to first transform them into ordinary differential equations (ODEs) through appropriate transformations [30–40].

1) If we let $u = lnw$, then Eq (1.1) becomes

$$ww_{tt} - ww_{xx} - w_t^2 + w_x^2 - w^3 + 1 = 0. \tag{1.6}$$

Use the transforms

$$w(x,t) = v(z), \quad z = a_1 x - a_2 t,$$

in Eq (1.6) to get

$$(a_2^2 - a_1^2)vv'' - (a_2^2 - a_1^2)(v')^2 - v^3 + 1 = 0. \tag{1.7}$$

2) Letting $u = lnw$ and applying the transforms

$$w(x,t) = v(z), \quad z = a_3 x - a_4 t,$$

to Eq (1.2) yields

$$-a_3 a_4 vv'' + a_3 a_4 (v')^2 + \rho v^3 + \sigma v + \gamma = 0. \tag{1.8}$$

3) Inserting

$$u(x,t) = v(z), \quad z = a_5 x - a_6 t,$$

into Eq (1.3), and integrating it, we have

$$(4a_5^3 - a_6 a_5^2)(vv'' - (v')^2) + (a_6 - 4a_5)v^4 + \iota = 0, \tag{1.9}$$

where ι is an integrable constant.

4) Apply the transform

$$w(x,t) = e^{-u},$$

to Eq (1.4) to obtain

$$-ww_{xt} + w_x w_t - w^4 - w^3 = 0. \tag{1.10}$$

Use the transforms

$$w(x,t) = v(z), \quad z = a_7 x - a_8 t,$$

in Eq (1.10) to get

$$a_7 a_8 vv'' - a_7 a_8 (v')^2 - v^4 - v^3 = 0. \tag{1.11}$$

The double sine-Gordon (DSG) equation is a special type of NLDE provided by physics and engineering. We now consider about the mDSG equation as follows.

5) Using the transforms

$$u(x,t) = u(z), \quad z = a_9 x - a_{10} t,$$

and

$$\sin 2u = \frac{v^2 - v^{-2}}{2i}, \ \sin u = \frac{v - v^{-1}}{2i}, \ v = e^{iu},$$

into Eq (1.5) yields

$$-a_9a_{10}u'' = \frac{bv^2 - bv^{-2}}{2i} + \frac{cv - cv^{-1}}{2i}. \tag{1.12}$$

Using the transforms

$$v = e^{iu},\ u' = \frac{v'}{iv},\ u'' = -\frac{1}{i}\left(\frac{(v')^2}{v^2} - \frac{v''}{v}\right),$$

in Eq (1.12) yields

$$2a_9a_{10}vv'' - 2a_9a_{10}(v')^2 + bv^4 + cv^3 - cv - b = 0. \tag{1.13}$$

Through observation, it can be found that the five NLDEs to be solved in this paper share certain commonalities. We can encompass these five NLDEs through the following Eq (1.14), thereby enabling a unified solution.

$$A[vv'' - (v')^2] + Bv^n + Cv^{n-1} + Dv + E = 0, \tag{1.14}$$

where $A, B, \cdots, E$ are arbitrary constants, and when they are selected with appropriate values, the mentioned five NLDEs can be derived. In this study, we utilize the $\exp(-\vartheta(z))$-expansion approach and the complex method to execute meromorphic exact solutions to these nonlinear PDEs.

2. Bifurcation behaviors

In this section, to investigate the dynamic behavior of Eq (1.14), we will use Hamiltonian systems to list the effects of bifurcations via plotting the phase portraits of the system. Assume that,

$$\frac{\mathrm{d}v}{\mathrm{d}z} = U, \tag{2.1}$$

then, we can obtain that the Hamiltonian system is

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}z} = U, \\ \frac{\mathrm{d}U}{\mathrm{d}z} = \frac{U^2}{v} - \frac{1}{Av}(Bv^n + Cv^{n-1} + Dv + E). \end{cases} \tag{2.2}$$

Below, we will focus on the case of $n = 3$ as an example for further discussion, in which the Hamiltonian system becomes

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}z} = U, \\ \frac{\mathrm{d}U}{\mathrm{d}z} = \frac{U^2}{v} - \frac{1}{Av}(Bv^3 + Cv^2 + Dv + E). \end{cases} \tag{2.3}$$

At this point, the Hamiltonian of system (2.3) is

$$H = \frac{2B\,v^5 + 3C\,v^4 + 6\mathrm{D}v^3 + 6E\ln(v)\,v^2 + 3U^2A}{6v^2A}. \tag{2.4}$$

Under the real number condition, system (2.3) has only one equilibrium point

$$M_1 = (\frac{\left(-108E\,B^2 + 36DCB + 12\sqrt{3}\,\sqrt{27B^2E^2 - 18BCDE + 4BD^3 + 4C^3E - C^2D^2}\,B - 8C^3\right)^{\frac{1}{3}}}{6B}$$

$$-\frac{2\left(3DB-C^2\right)}{3B\left(-108E\,B^2+36DCB+12\sqrt{3}\,\sqrt{27B^2E^2-18BCDE+4BD^3+4C^3E-C^2D^2}\,B-8C^3\right)^{\frac{1}{3}}}-\frac{C}{3B},0).$$

The Jacobian matrix of system (2.3) is

$$J=\begin{vmatrix} 0 & 1 \\ -\frac{U^2}{v^2}-\frac{3B\,v^2+2Cv+D}{Av}+\frac{B\,v^3+C\,v^2+Dv+E}{A\,v^2} & \frac{2U}{v}\end{vmatrix}=\frac{U^2}{v^2}+\frac{3B\,v^2+2Cv+\mathrm{D}}{Av}-\frac{B\,v^3+C\,v^2+\mathrm{D}v+E}{A\,v^2}.$$

We then have

$$\begin{aligned} J|_{M_1}=&\frac{\left(-108E\,B^2+36DCB+12\sqrt{3}\,\sqrt{27B^2E^2-18BCDE+4BD^3+4C^3E-C^2D^2}\,B-8C^3\right)^{\frac{1}{3}}}{6B}\\ &-\frac{2\left(3BD-C^2\right)}{3B\left(-108E\,B^2+36BCD+12\sqrt{3}\,\sqrt{27E^2B^2-18EDCB+4BD^3+4C^3E-C^2D^2}\,B-8C^3\right)^{\frac{1}{3}}}-\frac{C}{3B}.\end{aligned}$$

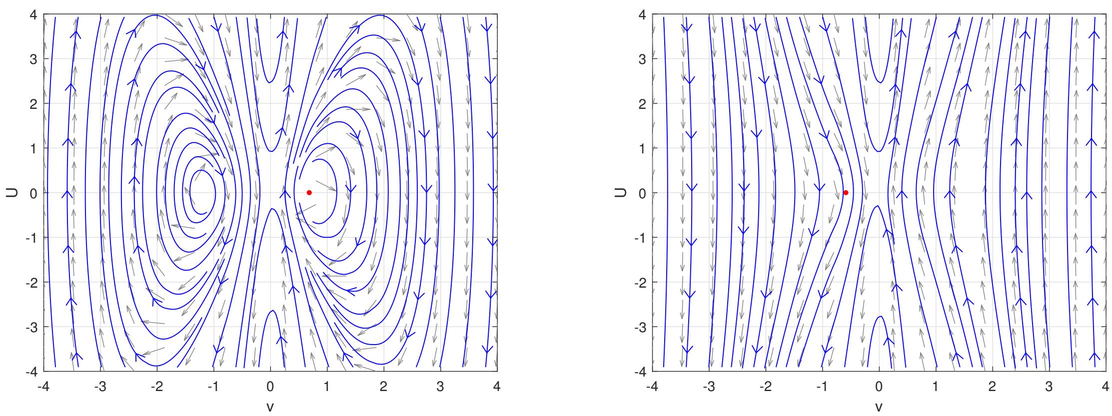
Generally speaking, the equilibrium point M_1 of the Hamiltonian system can be analyze in three cases. Therefore, we obtain the following bifurcation behavior of system (2.3):

Case 1: When $J|_{M_1}>0$, M_1 is a center;

Case 2: When $J|_{M_1}<0$, M_1 is a saddle;

Case 3: When $J|_{M_1}=0$, M_1 is a cuspid.

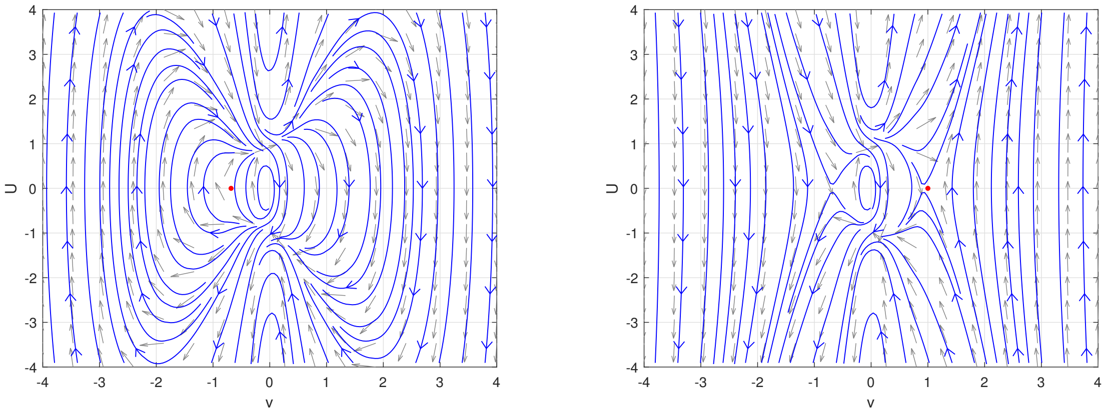
For the three cases above, we have selected three sets of specific parameters and used MATLAB to plot their phase portraits, resulting in Figures 1–3. The red dots in this pictures represent the equilibrium points. It should be noted that the parameters A, B, C, and D, we chose are all fixed in the subplots (a) of Figures 1–3. As is clearly evident from the expression for $J|_{M_1}$, when $A=1$, $B=1$, $C=0$, and $D=1$, the sign of E determines the sign of $J|_{M_1}$. Moreover, the expression also shows that E is not the sole factor determining the sign of the Jacobian determinant; rather, it is jointly influenced by the parameters A, B, C, D, and E. To further clarify this distinction, we selected the subplots (b) of Figures 1–3. From these figures, it can be observed that the bifurcation behavior of the system is closely related to the parameters, and the dynamic behavior is also significantly affected by the bifurcations.



(a) Phase portraits of $A=1$, $B=1$, $C=0$, $D=1$, and $E=-1$.

(b) Phase portraits of $A=-1$, $B=2$, $C=0$, $D=1$, and $E=1$.

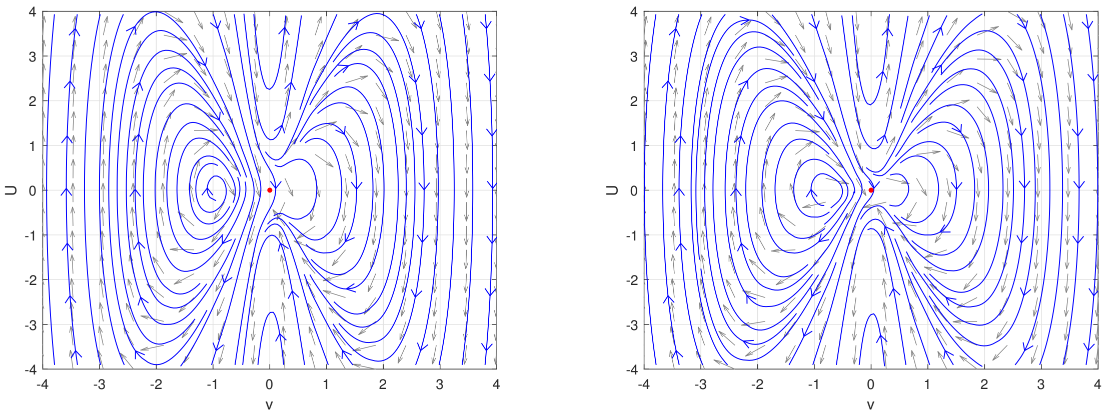
Figure 1. Phase portraits of Case 1.



(a) Phase portraits of $A = 1$, $B = 1$, $C = 0$, $D = 1$, and $E = 1$.

(b) Phase portraits of $A = -1$, $B = 2$, $C = 0$, $D = -1$, and $E = -1$.

Figure 2. Phase portraits of Case 2.



(a) Phase portraits of $A = 1$, $B = 1$, $C = 0$, $D = 1$, and $E = 0$.

(b) Phase portraits of $A = 3$, $B = 2$, $C = 0$, $D = 1$, and $E = 0$.

Figure 3. Phase portraits of Case 3.

3. Methodologies for searching for meromorphic exact solutions

The class W encompasses meromorphic functions f that satisfy at least one of these forms: A rational function of z, being expressible as rational combinations of $e^{\alpha z}$ with $\alpha \in \mathbb{C}$, or an elliptic function.

Set $m \in \mathbb{N} := \{1, 2, 3, ...\}$, $r_i \in \{0, 1, 2, ...\}$, $i = 0, 1, 2, ..., m$, $r = (r_0, r_1, ..., r_m)$, and

$$K_r[v](z) := \prod_{i=0}^{m} [v^{(i)}(z)]^{r_i}.$$

We define the degree of $K_r[v]$ by $d(r) := \sum_{i=0}^{m} r_i$. The differential polynomial is given as

$$P(v, v', \cdots, v^{(m)}) := \sum_{r\in J} a_r K_r[v],$$

in which J is a finite index set, and $\deg P(v, v', \cdots, v^{(m)}) := \max_{r\in J}\{d(r)\}$ is the degree of $P(v, v', \cdots, v^{(m)})$.

Considering the following ODE:

$$P(v, v', \cdots, v^{(m)}) = av^n + \delta, \tag{3.1}$$

where $a \neq 0, \delta$ are constants and $n \in \mathbb{N}$.

Set $p,\ q \in \mathbb{N}$, and assume that meromorphic solution v of Eq (3.1) has at least one pole. Substitute the Laurent series

$$v(z) = \sum_{\tau=-q}^{\infty} \beta_\tau z^\tau, q > 0, \beta_{-q} \neq 0, \tag{3.2}$$

into Eq (3.1). If we obtain p unequal Laurent singular parts

$$\sum_{\tau=-q}^{-1} \beta_\tau z^\tau,$$

then we say that the weak $\langle p, q\rangle$ condition of Eq (3.1) holds.

The Fuchs index is given by the following indicial equation:

$$P(i) = \lim_{z\to 0} z^{-i+J(\lambda)} \hat{D}'(z, v) z^{i-\lambda} = 0,$$

where $\hat{D}'(z, v)$ is the following linear operator:

$$\hat{D}'(z, v) = \lim_{\kappa\to 0} \frac{\hat{D}(z, v + \kappa w) - \hat{D}(z, v)}{\kappa w},$$

$\hat{D}(z, v)$ is the dominant part containing every dominant term with a multiplicity κ of the meromorphic solution.

The Weierstrass elliptic function $\wp(z) := \wp(z, g_2, g_3)$, characterized by its double periodicity, obeys the differential equation

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

which admits an addition formula [41]

$$\wp(z - z_0) = -\wp(z) + \frac{1}{4}\left[\frac{\wp'(z) + \wp'(z_0)}{\wp(z) - \wp(z_0)}\right]^2 - \wp(z_0).$$

Eremenko et al. [42] studied the m-order Briot–Bouquet (BB) equation

$$P(v, v^{(m)}) = \sum_{j=0}^{n} P_j(v)(v^{(m)})^j = 0,$$

in which $m \in \mathbb{N}$ and $P_j(v), (j = 0, 1, ..., n)$ are constant coefficient polynomials.

Lemma 3.1 ([22,41,43]). *Set* $p,\ s,\ n,\ m \in \mathbb{N}$, *and* $\deg P(v, v^{(m)}) < n$. *If a m-order BB equation*

$$P(v, v^{(m)}) = av^n + d$$

admits the weak $\langle p, q\rangle$ *condition, then any meromorphic solution* $v(z)$ *belongs to the class* W. *Assuming that for certain parameter values, a solution* v *exists, all other meromorphic solutions constitute a single-parameter family of the form* $v(z - z_0)$, $z_0 \in \mathbb{C}$. *Moreover, any elliptic solution* $v(z)$ *with a pole at zero has the form*

$$\begin{aligned} v(z) = &\sum_{i=1}^{s-1}\sum_{j=2}^{q}\frac{(-1)^j\beta_{-ij}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\left(\frac{1}{4}\left[\frac{\wp'(z)+D_i}{\wp(z)-B_i}\right]^2-\wp(z)\right)\\ &+\sum_{i=1}^{s-1}\frac{\beta_{-i1}}{2}\frac{\wp'(z)+D_i}{\wp(z)-B_i}+\sum_{j=2}^{q}\frac{(-1)^j\beta_{-sj}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\wp(z)+\beta_0, \end{aligned} \tag{3.3}$$

where β_{-ij} *can be determined by (3.2),* $D_i^2 = 4B_i^3 - g_2B_i - g_3$, *and* $\sum_{i=1}^{s}\beta_{-i1} = 0$.

Every rational function solution is

$$R(z) = \sum_{i=1}^{s}\sum_{j=1}^{q}\frac{\beta_{ij}}{(z-z_i)^j} + \beta_0, \tag{3.4}$$

with $s(\leq p)$ *distinct poles which multiplicity is* q.

Every simply periodic solution is expressed as a rational function of the exponential variable $\eta = e^{\alpha z}(\alpha \in \mathbb{C})$ *and assumes the general form*

$$R(\eta) = \sum_{i=1}^{s}\sum_{j=1}^{q}\frac{\beta_{ij}}{(\eta-\eta_i)^j} + \beta_0. \tag{3.5}$$

3.1. Complex analysis method

The main steps are given below.

Mechanism 1. Convert the PDE to an ODE through appropriate transformations.

Mechanism 2. Calculate the weak $\langle p, q\rangle$ condition of the ODE and verify that the equation passes the Painlevé test through the Fuchs index.

Mechanism 3. Substitute the indeterminate forms (3.3)–(3.5) into the ODE to obtain its meromorphic solutions.

Mechanism 4. Substitute the inverse transform into meromorphic solutions to get meromorphic exact solutions of the studied PDE.

3.2. Exp$(-\vartheta(z))$*-expansion method*

For this expansion method [19, 20], the main steps are as follows.

Mechanism 1. Using appropriate transformations into the PDE yields the ODE.

Mechanism 2. Assume that the ODE has the following exact solutions:

$$v(z) = \sum_{\varsigma=0}^{m} \gamma_\varsigma (\exp(-\vartheta(z)))^\varsigma, \tag{3.6}$$

in which γ_ς $(0 \le \varsigma \le n)$ are constants that will be calculated later, such that $\gamma_m \neq 0$ and $\vartheta = \vartheta(z)$ admits

$$\vartheta'(z) = l + \exp(-\vartheta(z)) + k \exp(\vartheta(z)). \tag{3.7}$$

The following are the solutions of Eq (3.7).

When $l^2 - 4k > 0$, $k \neq 0$, we have

$$\vartheta(z) = \ln\left(\frac{-\sqrt{(l^2-4k)}\tanh(\frac{\sqrt{l^2-4k}}{2}(z+b)) - l}{2k}\right), \tag{3.8}$$

$$\vartheta(z) = \ln\left(\frac{-\sqrt{(l^2-4k)}\coth(\frac{\sqrt{l^2-4k}}{2}(z+b)) - l}{2k}\right). \tag{3.9}$$

When $l^2 - 4k < 0$, $k \neq 0$, we have

$$\vartheta(z) = \ln\left(\frac{\sqrt{(4k-l^2)}\tan(\frac{\sqrt{(4k-l^2)}}{2}(z+b)) - l}{2k}\right), \tag{3.10}$$

$$\vartheta(z) = \ln\left(\frac{\sqrt{(4k-l^2)}\cot(\frac{\sqrt{(4k-l^2)}}{2}(z+b)) - l}{2k}\right). \tag{3.11}$$

When $l^2 - 4k > 0$, $l \neq 0$, $k = 0$, we have

$$\vartheta(z) = -\ln\left(\frac{l}{\exp(l(z+b)) - 1}\right). \tag{3.12}$$

When $l^2 - 4k = 0$, $l \neq 0$, $k \neq 0$, we have

$$\vartheta(z) = \ln\left(-\frac{2(l(z+b)+2)}{l^2(z+b)}\right). \tag{3.13}$$

When $l^2 - 4k = 0$, $l = 0$, $k = 0$, we have

$$\vartheta(z) = \ln(z+b). \tag{3.14}$$

In Eqs (3.8)–(3.14), $\gamma_\varsigma \neq 0, l, k, b$ are arbitrary constants. The positive integer m is computed through applying the homogeneous balance principle between the nonlinear terms and highest-order derivatives in the converted ODE.

Mechanism 3. Upon substitution of Eq (3.6) into the ODE, the expression is rearranged into a polynomial form with respect to $\exp(-\vartheta(z))$. A set of algebraic equations is obtained by enforcing the condition that the coefficient for each order of $\exp(-\vartheta(z))$ must vanish independently. The solution of this system specifies the parameters $\gamma_\varsigma \neq 0, l, k$. Their subsequent substitution into Eq (3.6) and Eqs (3.8)–(3.14) leads to the definitive analytical solutions for the governing PDE.

4. Meromorphic exact solutions of five PDEs via complex analysis method

Theorem 4.1. *When $n > 4$, Eq (1.14) cannot pass the Painlevé test. When $n = 4$, $AB \neq 0$, Eq (1.14) has the following types of meromorphic solutions.*

(I) The rational function solutions

$$v_{r1}(z) = \pm\frac{\sqrt{-AB}}{B}\left(\frac{1}{z-z_0} \mp \frac{C}{2\sqrt{-AB}}\right),$$

$$v_{r2}(z) = \pm\frac{\sqrt{-AB}}{B}\left(\frac{1}{z-z_0} - \frac{1}{z-z_1-z_0}\right),$$

and

$$v_{r3}(z) = \pm\frac{\sqrt{-AB}}{B}\left(\frac{1}{z-z_0} - \frac{1}{z-z_1-z_0} - \frac{3}{z_1}\right),$$

in which $z_0 \in \mathbb{C}$. Here, $D = -\frac{C^3}{4B^2}$, and $E = -\frac{C^4}{16B^3}$ in the first case; $C = \frac{2AB}{\sqrt{-AB}z_1}$, $D = 0$, $E = 0$, and $z_1 \neq 0$ in the second case; and $C = -\frac{4AB}{\sqrt{-AB}z_1}$, $D = 0$, $E = \frac{27A^2}{Bz_1^4}$, and $z_1 \neq 0$ in the third case.

(II) The simply periodic solutions

$$v_{s1}(z) = \pm\frac{\sqrt{-AB}}{B}\frac{\alpha^2}{e^{\alpha(z-z_0)}-1},$$

$$v_{s2}(z) = \mp\frac{\sqrt{-AB}}{B}\frac{\alpha^2}{e^{\alpha(z-z_0)}+1},$$

and

$$v_{s3}(z) = \pm\frac{\sqrt{-AB}}{B}\frac{\alpha^2}{e^{\alpha(z-z_0)}-1} \mp \frac{\sqrt{-AB}}{B}\frac{\alpha^2}{e^{\alpha(z-z_0)}+1},$$

where $z_0 \in \mathbb{C}$.Here $C = -\frac{AB\alpha^2}{\sqrt{-AB}}$, $D = 0$, and $E = 0$ in the first two cases, and $C = -\frac{2AB\alpha^2}{\sqrt{-AB}}$, $D = 0$, $E = 0$ in the third situation.

(III) The elliptic function solution

$$v_d(z) = \pm\frac{\sqrt{-AB}}{2B}\frac{\wp'(z-z_0, g_2, 0)}{\wp(z-z_0, g_2, 0)},$$

where $z_0 \in \mathbb{C}$, $g_3 = 0, F^2 = 4G^3 - g_2G$, and G and g_2 are arbitrary.

Proof. Let $v(z)$ be the meromorphic solution of Eq (1.14), and if $v(z)$ has a simple movable pole at $z = 0$, then in a neighborhood of $z = 0$, the Laurent series of v are in terms of $v(z) = \sum_{\tau=-q}^{\infty}\beta_\tau z^\tau$ ($\beta_{-q} \neq 0, q > 0$). Equation (1.14) has the leading members as follows:

$$A[vv'' - (v')^2] + Bv^n = 0.$$

Inserting the Laurent series into the leading members above yields the values of the pole $q = \frac{2}{n-2}$ for the general solutions. If $n > 4$, then $q < 1$. It means that Eq (1.14) cannot pass the Painlevé test with this situation. While $n = 4$, we get $\beta_{-1} = \pm\frac{\sqrt{-AB}}{B}$, $\lambda = 1, J(\lambda) = 4\lambda$, and

$$P(i) = \lim_{z\to 0} z^{-i+J(\lambda)}\hat{D}'(z, \beta_{-\lambda}z^{-\lambda})z^{i-\lambda}$$

$$= \lim_{z\to 0} z^{-i+4\lambda}[A(\beta_{-\lambda}z^{-\lambda})\frac{d^2}{dz^2} + A(\beta_{-\lambda}z^{-\lambda})'' - 2Av'\frac{d}{dz} + 4B(\beta_{-\lambda}z^{-\lambda})^3]z^{i-\lambda}$$
$$= A\beta_{-\lambda}(i-\lambda)(i-\lambda-1) + A\beta_{-\lambda}(\lambda+1)\lambda + 2A\beta_{-\lambda}\lambda(i-\lambda) + 4B(\beta_{-\lambda})^3$$
$$= 0. \tag{4.1}$$

Substitute $\lambda = 1, \beta_{-1} = \frac{\sqrt{-A}}{B}$ into the indicial equation (4.1) above to achieve $i_1 = -1$, $i_2 = 2$. With the Fuchs indices above, Eq (1.14) passes the Painlevé test when $n = 4$, and has meromorphic closed-form solutions.

By (3.4) and the weak $\langle 2, 1\rangle$ condition, we know that the form of Eq (1.14) with a pole at $z = 0$ is

$$R_1(z) = \frac{\beta_{11}}{z} + \frac{\beta_{12}}{z - z_1} + \beta_0. \tag{4.2}$$

We substitute $R_1(z)$ into Eq (1.14) to derive

$$\sum_{i=0}^{8} \frac{\mu_{1i}z^{4-i}}{(z-z_1)^4} = 0, \tag{4.3}$$

where

$$\mu_{10} = B\beta_0{}^4 + C\beta_0{}^3 + D\beta_0 + E,$$
$$\mu_{11} = 4B\beta_0{}^4z_1 + 4B\beta_0{}^3\beta_{11} + 4B\beta_0{}^3\beta_{12} - 4C\beta_0{}^3z_1 + 3C\beta_0{}^2\beta_{11}$$
$$+ 3C\beta_0{}^2\beta_{12} - 4D\beta_0 z_1 + D\beta_{11} + D\beta_{12} - 4Ez_1,$$
$$\mu_{12} = 6B\beta_0{}^4z_1{}^2 - 16B\beta_0{}^3\beta_{11}z_1 - 12B\beta_0{}^3\beta_{12}z_1 + 6C\beta_0{}^3z_1{}^2 + 6B\beta_0{}^2\beta_{11}{}^2 + 12B\beta_0{}^2\beta_{11}\beta_{12} + 6B\beta_0{}^2\beta_{12}{}^2$$
$$- 12C\beta_0{}^2\beta_{11}z_1 - 9C\beta_0{}^2\beta_{12}z_1 + 3C\beta_0\beta_{11}{}^2 + 6C\beta_0\beta_{11}\beta_{12} + 3C\beta_0\beta_{12}{}^2$$
$$+ 6D\beta_0 z_1{}^2 - 4D\beta_{11}z_1 - 3D\beta_{12}z_1 + 6Ez_1{}^2,$$
$$\mu_{13} = -4B\beta_0{}^4z_1{}^3 + 24B\beta_0{}^3\beta_{11}z_1{}^2 + 12B\beta_0{}^3\beta_{12}z_1{}^2 - 24B\beta_0{}^2\beta_{11}{}^2z_1 - 4C\beta_0{}^3z_1{}^3 + 2A\beta_0\beta_{11}$$
$$- 12B\beta_0{}^2\beta_{12}{}^2z_1 + 18C\beta_0{}^2\beta_{11}z_1{}^2 + 9C\beta_0{}^2\beta_{12}z_1{}^2 + 4B\beta_0\beta_{11}{}^3 + 12B\beta_0\beta_{11}{}^2\beta_{12} + 12B\beta_0\beta_{11}\beta_{12}{}^2$$
$$+ 4B\beta_0\beta_{12}{}^3 - 12C\beta_0\beta_{11}{}^2z_1 - 18C\beta_0\beta_{11}\beta_{12}z_1 - 6C\beta_0\beta_{12}{}^2z_1 - 4D\beta_0 z_1{}^3 + C\beta_{11}{}^3 + 3C\beta_{11}{}^2\beta_{12}$$
$$+ 3C\beta_{11}\beta_{12}{}^2 + C\beta_{12}{}^3 + 6D\beta_{11}z_1{}^2 + 3D\beta_{12}z_1{}^2 - 4Ez_1{}^3 - 36B\beta_0{}^2\beta_{11}\beta_{12}z_1 + 2A\beta_0\beta_{12},$$
$$\mu_{14} = B\beta_0{}^4z_1{}^4 - 16B\beta_0{}^3\beta_{11}z_1{}^3 - 4B\beta_0{}^3\beta_{12}z_1{}^3 + C\beta_0{}^3z_1{}^4 + 36B\beta_0{}^2\beta_{11}{}^2z_1{}^2 + 6B\beta_0{}^2\beta_{12}{}^2z_1{}^2$$
$$+ 36B\beta_0{}^2\beta_{11}\beta_{12}z_1{}^2 - 12C\beta_0{}^2\beta_{11}z_1{}^3 - 3C\beta_0{}^2\beta_{12}z_1{}^3 + A\beta_{11}{}^2 - 36B\beta_0\beta_{11}{}^2\beta_{12}z_1$$
$$- 24B\beta_0\beta_{11}\beta_{12}{}^2z_1 - 4B\beta_0\beta_{12}{}^3z_1 + 18C\beta_0\beta_{11}{}^2z_1{}^2 + Ez_1{}^4 + 3C\beta_0\beta_{12}{}^2z_1{}^2 + D\beta_0 z_1{}^4$$
$$+ B\beta_{11}{}^4 + 4B\beta_{11}{}^3\beta_{12} + 6B\beta_{11}{}^2\beta_{12}{}^2 + 4B\beta_{11}\beta_{12}{}^3 + B\beta_{12}{}^4 - 4C\beta_{11}{}^3z_1 - 9C\beta_{11}{}^2\beta_{12}z_1$$
$$- 6C\beta_{11}\beta_{12}{}^2z_1 - C\beta_{12}{}^3z_1 - 4D\beta_{11}z_1{}^3 - D\beta_{12}z_1{}^3 + 18C\beta_0\beta_{11}\beta_{12}z_1{}^2 - 8A\beta_0\beta_{11}z_1$$
$$- 2A\beta_0\beta_{12}z_1 - 16B\beta_0\beta_{11}{}^3z_1 + 2A\beta_{11}\beta_{12} + A\beta_{12}{}^2,$$
$$\mu_{15} = 4B\beta_0{}^3\beta_{11}z_1{}^4 - 24B\beta_0{}^2\beta_{11}{}^2z_1{}^3 - 12B\beta_0{}^2\beta_{11}\beta_{12}z_1{}^3 + 3C\beta_0{}^2\beta_{11}z_1{}^4 + 24B\beta_0\beta_{11}{}^3z_1{}^2$$
$$+ 36B\beta_0\beta_{11}{}^2\beta_{12}z_1{}^2 + 12B\beta_0\beta_{11}\beta_{12}{}^2z_1{}^2 - 12C\beta_0\beta_{11}{}^2z_1{}^3 - 6C\beta_0\beta_{11}\beta_{12}z_1{}^3 - 4B\beta_{11}{}^4z_1$$
$$- 4A\beta_{11}{}^2z_1 - 12B\beta_{11}{}^2\beta_{12}{}^2z_1 - 4B\beta_{11}\beta_{12}{}^3z_1 + 6C\beta_{11}{}^3z_1{}^2 + 9C\beta_{11}{}^2\beta_{12}z_1{}^2$$
$$+ 3C\beta_{11}\beta_{12}{}^2z_1{}^2 + D\beta_{11}z_1{}^4 + 12A\beta_0\beta_{11}z_1{}^2 - 12B\beta_{11}{}^3\beta_{12}z_1 - 4A\beta_{11}\beta_{12}z_1,$$
$$\mu_{16} = 6B\beta_0{}^2\beta_{11}{}^2z_1{}^4 - 16B\beta_0\beta_{11}{}^3z_1{}^3 - 12B\beta_0\beta_{11}{}^2\beta_{12}z_1{}^3 + 3C\beta_0\beta_{11}{}^2z_1{}^4 + 6B\beta_{11}{}^4z_1{}^2$$

$$+ 12\, B\beta_{11}{}^3\beta_{12}\, z_1{}^2 + 6\, B\beta_{11}{}^2\beta_{12}{}^2 z_1{}^2 - 4\, C\beta_{11}{}^3 z_1{}^3 - 3\, C\beta_{11}{}^2\beta_{12}\, z_1{}^3 - 8\, A\beta_0\beta_{11}\, z_1{}^3$$
$$+ 6\, A\beta_{11}{}^2 z_1{}^2 + 4\, A\beta_{11}\beta_{12}\, z_1{}^2,$$
$$\mu_{17} = 4\, B\beta_0\beta_{11}{}^3 z_1{}^4 - 4\, B\beta_{11}{}^4 z_1{}^3 - 4\, B\beta_{11}{}^3\beta_{12}\, z_1{}^3 + C\beta_{11}{}^3 z_1{}^4 + 2\, A\beta_0\beta_{11}\, z_1{}^4 - 4\, A\beta_{11}{}^2 z_1{}^3 - 2\, A\beta_{11}\beta_{12}\, z_1{}^3,$$
$$\mu_{18} = B\beta_{11}{}^4 z_1{}^4 + A\beta_{11}{}^2 z_1{}^4.$$

If we let all coefficients with same powers to z of Eq (4.3) equate to zero, then we achieve

$$\mu_{1i} = 0, i = 0, 2, \cdots, 8.$$

Calculate the foregoing algebraic equations, then

$$\beta_{11} = \pm\frac{\sqrt{-AB}}{B}, \beta_{12} = 0, \beta_0 = -\frac{C}{2B},$$

$$\beta_{11} = \pm\frac{\sqrt{-AB}}{B}, \beta_{12} = \mp\frac{\sqrt{-AB}}{B}, \beta_0 = 0,$$

and

$$\beta_{11} = \pm\frac{\sqrt{-AB}}{B}, \beta_{12} = \mp\frac{\sqrt{-AB}}{B}, \beta_0 = \mp\frac{3\sqrt{-AB}}{Bz_1}.$$

The values of β_{11}, β_{12}, and β_0 are given above, substituting them into $R_1(z)$ to yield rational solutions to Eq (1.14) with the pole at $z = 0$ as shown below:

$$R_{11,1}(z) = \pm\frac{\sqrt{-AB}}{B}\left(\frac{1}{z} \mp \frac{C}{2\sqrt{-AB}}\right),$$

where $D = -\frac{C^3}{4B^2}$ and $E = -\frac{C^4}{16B^3}$

$$R_{11,2}(z) = \pm\frac{\sqrt{-AB}}{B}\left(\frac{1}{z} - \frac{1}{z - z_1}\right),$$

where $C = \frac{2AB}{\sqrt{-AB}z_1}$, $D = 0$, $E = 0$, and $z_1 \neq 0$

$$R_{11,3}(z) = \pm\frac{\sqrt{-AB}}{B}\left(\frac{1}{z} - \frac{1}{z - z_1} - \frac{3}{z_1}\right),$$

where $C = -\frac{4AB}{\sqrt{-AB}z_1}$, $D = 0$, $E = \frac{27A^2}{Bz_1^4}$, and $z_1 \neq 0$.

Therefore, the rational solutions of Eq (1.14) are

$$v_{r1}(z) = \pm\frac{\sqrt{-AB}}{B}\left(\frac{1}{z - z_0} \mp \frac{C}{2\sqrt{-AB}}\right),$$

$$v_{r2}(z) = \pm\frac{\sqrt{-AB}}{B}\left(\frac{1}{z - z_0} - \frac{1}{z - z_1 - z_0}\right),$$

$$v_{r3}(z) = \pm\frac{\sqrt{-AB}}{B}\left(\frac{1}{z - z_0} - \frac{1}{z - z_1 - z_0} - \frac{3}{z_1}\right),$$

in which $z_0 \in \mathbb{C}$. Here, $D = -\frac{C^3}{4B^2}$, $E = -\frac{C^4}{16B^3}$ in the first case; $C = \frac{2AB}{\sqrt{-AB}z_1}$, $D = 0$, $E = 0$, and $z_1 \neq 0$ in the second care; and $C = -\frac{4AB}{\sqrt{-AB}z_1}$, $D = 0$, $E = \frac{27A^2}{Bz_1^4}$, and $z_1 \neq 0$ in the third care.

Let $\eta = e^{\alpha z}$. To search for simply periodic solutions, we insert $v = R(\eta)$ into Eq (1.14), then

$$A\alpha^2 R(\eta R' + \eta^2 R'') - A\alpha^2 (R'\eta)^2 + Bv^4 + Cv^3 + Dv + E = 0.$$

Substituting $R_2(\eta)$ into the equation above, and using similar procedures to rational solutions, we obtain

$$R_{12,1}(\eta) = \pm\frac{\sqrt{-AB}}{B}\frac{\alpha^2}{\eta - 1}, \tag{4.4}$$

where $C = -\frac{AB\alpha^2}{\sqrt{-AB}}$, $D = 0$, and $E = 0$; and

$$R_{12,2}(\eta) = \mp\frac{\sqrt{-AB}}{B}\frac{\alpha^2}{\eta + 1}, \tag{4.5}$$

where $C = -\frac{AB\alpha^2}{\sqrt{-AB}}$, $D = 0$, and $E = 0$; and

$$R_{12,3}(\eta) = \pm\frac{\sqrt{-AB}}{B}\frac{\alpha^2}{\eta - 1} \mp \frac{\sqrt{-AB}}{B}\frac{\alpha^2}{\eta + 1}, \tag{4.6}$$

where $C = -\frac{2AB\alpha^2}{\sqrt{-AB}}$, $D = 0$, and $E = 0$.

Substituting $\eta = e^{\alpha z}$ into (4.4)–(4.6), we achieve simply periodic solutions to Eq (1.14) containing a pole at $z = 0$

$$v_{s10}(z) = \pm\frac{\sqrt{-AB}}{B}\frac{\alpha^2}{e^{\alpha z} - 1},$$

$$v_{s20}(z) = \mp\frac{\sqrt{-AB}}{B}\frac{\alpha^2}{e^{\alpha z} + 1},$$

and

$$v_{s30}(z) = \pm\frac{\sqrt{-AB}}{B}\frac{\alpha^2}{e^{\alpha z} - 1} \mp \frac{\sqrt{-AB}}{B}\frac{\alpha^2}{e^{\alpha z} + 1}.$$

Thus, Eq (1.14) has simply periodic solutions as follows:

$$v_{s1}(z) = \pm\frac{\sqrt{-AB}}{B}\frac{\alpha^2}{e^{\alpha(z-z_0)} - 1},$$

$$v_{s2}(z) = \mp\frac{\sqrt{-AB}}{B}\frac{\alpha^2}{e^{\alpha(z-z_0)} + 1},$$

and

$$v_{s3}(z) = \pm\frac{\sqrt{-AB}}{B}\frac{\alpha^2}{e^{\alpha z} - 1} \mp \frac{\sqrt{-AB}}{B}\frac{\alpha^2}{e^{\alpha(z-z_0)} + 1},$$

where $z_0 \in \mathbb{C}$. Here, $C = -\frac{AB\alpha^2}{\sqrt{-AB}}$, $D = 0$, and $E = 0$ in the first two cases, whereas $C = -\frac{2AB\alpha^2}{\sqrt{-AB}}$, $D = 0$, and $E = 0$ in the third case.

From (3.3), the indeterminate form of the elliptic solution to Eq (1.14) with a pole at zero is given by

$$v_{d0}(z) = \frac{\beta_{-1}}{2}\frac{\wp'(z) + D_1}{\wp(z) - B_1} + \beta_{20},$$

where $D_1^2 = 4B_1^3 - g_2 B_1 - g_3$. We consider the results obtained above, and deduce $g_3 = 0, \beta_{20} = 0$, and $D_1 = B_1 = 0$. Thus

$$v_{d0}(z) = \pm\frac{\sqrt{-AB}}{2B}\frac{\wp'(z)}{\wp(z)},$$

where $g_3 = 0$.

Therefore, the elliptic function solution of Eq (1.14) is

$$v_d(z) = \pm\frac{\sqrt{-AB}}{2B}\frac{\wp'(z - z_0, g_2, 0)}{\wp(z - z_0, g_2, 0)},$$

where $z_0 \in \mathbb{C}$, $g_3 = 0$, $F^2 = 4G^3 - g_2 G$, and G and g_2 are arbitrary. □

Theorem 4.2. *When $n = 3$, $AB \neq 0$, Eq (1.14) has the following types of meromorphic solutions.*

(I) The rational function solution

$$w_r(z) = -\frac{2A}{B(z - z_0)^2} + \sqrt{\frac{-D}{3B}},$$

where $E = -2\sqrt{\frac{-D^3}{27B}}$, C=0, $z_0 \in \mathbb{C}$.

(II) The simply periodic solutions

$$w_{s1}(z) = -\frac{A}{2B}\alpha^2 \coth^2\frac{\alpha}{2}(z - z_0) + \frac{A(3\alpha^2 - 1)}{3B},$$

$$w_{s2}(z) = -\frac{A}{2B}\alpha^2 \tanh^2\frac{\alpha}{2}(z - z_0) + \frac{A(3\alpha^2 - 1)}{3B}$$

and

$$w_{s3}(z) = -\frac{A}{2B}\alpha^2 \tanh^2\frac{\alpha}{2}(z - z_0) - \frac{A}{2B}\alpha^2 \coth^2\frac{\alpha}{2}(z - z_0) + \frac{2A(3\alpha^2 - 1)}{3B},$$

where $z_0 \in \mathbb{C}$. Here $C = \frac{A^3+36ABD-216B^2E}{6A^2}$, $D = \frac{A^2(2\alpha^2-1)}{12B}$, and $E = \frac{A^3(3\alpha^2-1)}{108B^2}$ in the first two cases; $C = \frac{A^3+9ABD-27B^2E}{6A^2}$, $D = \frac{A^2(4\alpha^2-1)}{3B}$, and $E = \frac{2A^3(6\alpha^2-1)}{27B^2}$ in the third situation.

(III) The elliptic function solution

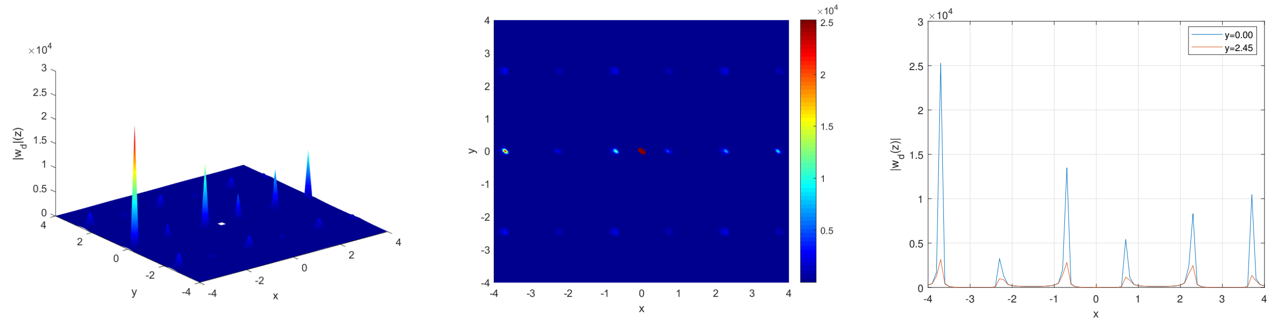
$$w_d(z) = -\frac{2A}{B}\left[-\wp(z) + \frac{1}{4}(\frac{\wp'(z) + D_2}{\wp(z) - B_2})^2 - B_2\right]$$

where $D_2 = 4B_2^3 - g_2 B_2 - g_3$, $C = 0$, $g_2 = \frac{BD}{A^2}$, and $g_3 = -\frac{B^2E}{4A^3}$.

Remark 4.1. *Analog to the process of Theorem 4.1, it is easy to derive Theorem 4.2.*

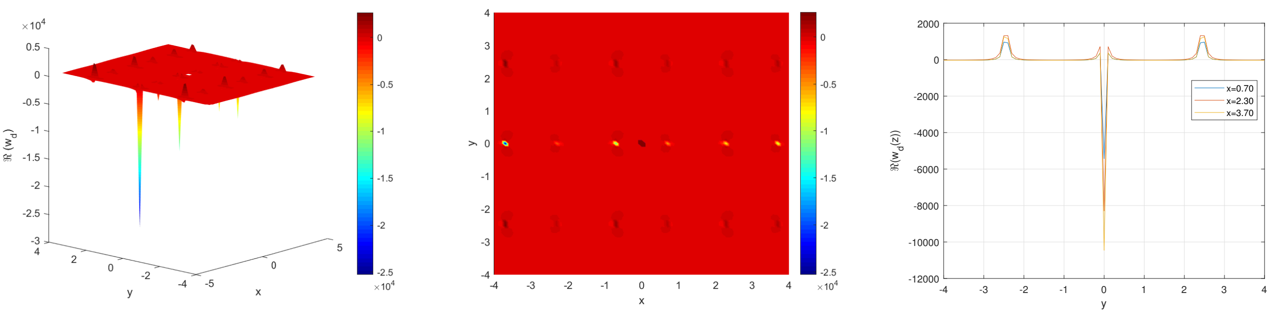
Remark 4.2. *With Theorems 4.1 and 4.2, we can achieve meromorphic solutions of Eqs (1.7)–(1.11) and (1.13) by selecting appropriate values for the coefficients of Eq (1.14). For example, if we let $n = 3$, $A = -a_3a_4$, $B = p$, $C = 0$, $D = q$, and $E = r$ in Eq (1.14), then meromorphic solutions of Eq (1.8) can be easily obtained. Through inverse transformation, we can acquire meromorphic exact solutions for the ZS equation. Here, we do not show them in detail for simplicity. The selected values should satisfy the constraint conditions corresponding to the solutions of the theorems. If the constraint conditions are not met, it means that the solution does not exist.*

The functions deduced from Theorem 4.2, namely $|w_d(z)|, \Re(w_d(z)), \Im(w_d(z))$, $|w_{s1}(z)|$, $|w_{s2}(z)|$, $|w_{s3}(z)|$, and $|w_r(z)|$, have been correspondingly represented in Figures 4–10, where $\Re(w_d(z))$ denotes the real part of $w_d(z)$ and $\Im(w_d(z))$ denotes the imaginary part of $w_d(z)$. The parameters' values are $A = E = \alpha = 1, B = B_2 = D = 2$, and $z_0 = 0$ in Figures 4–6; $A = B = \alpha = 1$, and $z_0 = 0$ in Figures 7–9; and $A = B = 1$, and $z_0 = 0$ in Figure 10.



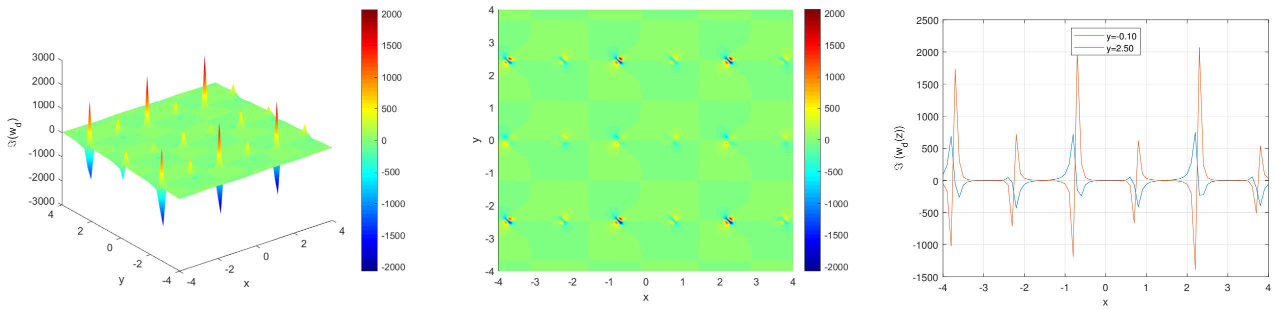
(a) (b) (c)

Figure 4. The representations of $|w_d(z)|, z = x + yi$ in (a) a three-dimensional (3D) plot, (b) a contour image, and (c) a two-dimensional (2D) graph.



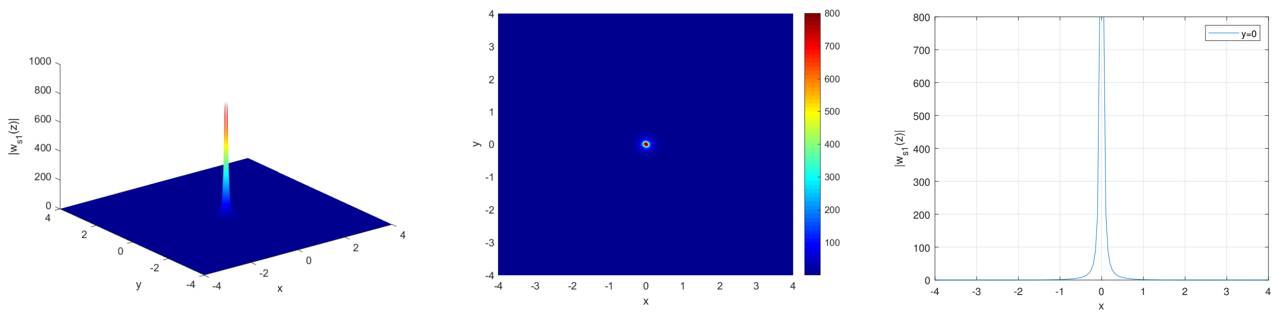
(a) (b) (c)

Figure 5. The representations of $\Re(w_d(z)), z = x + yi$ in (a) a 3D plot, (b) a contour image, and (c) a 2D graph.



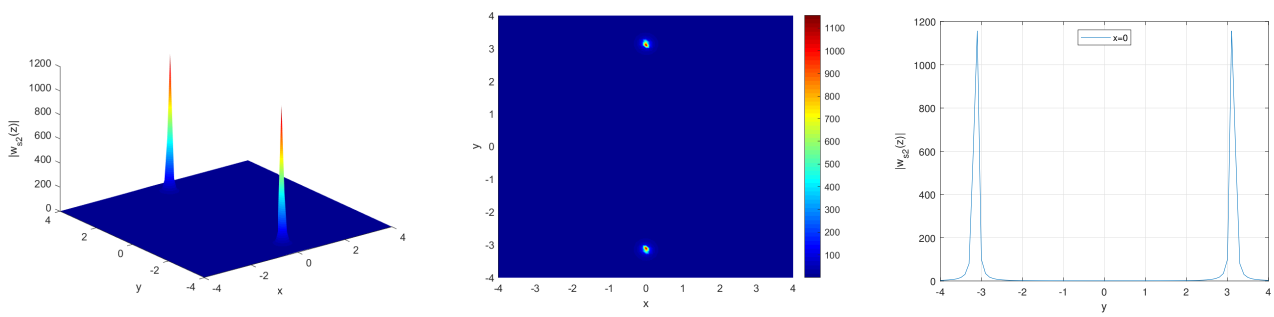
(a) **(b)** **(c)**

Figure 6. The representations of $\Im(w_d(z)), z = x + yi$ in (a) a 3D plot, (b) a contour image, and (c) a 2D graph.



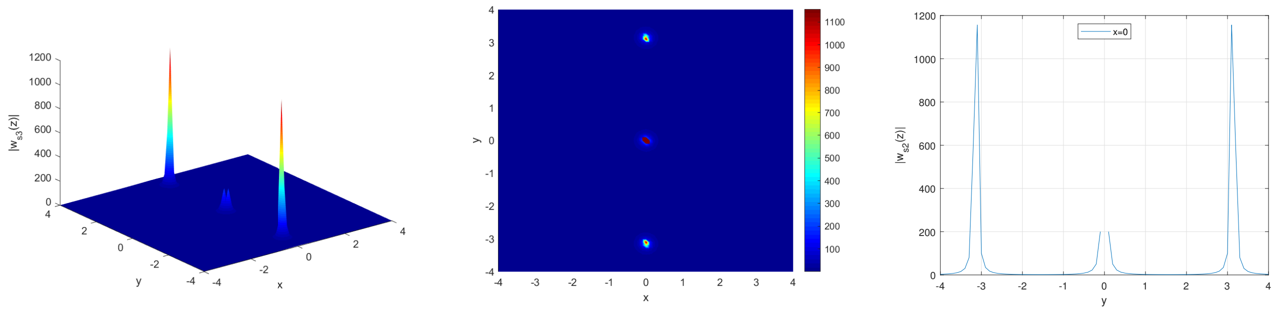
(a) **(b)** **(c)**

Figure 7. The representations of $|w_{s1}(z)|, z = x + yi$ in (a) a 3D plot, (b) a contour image, and (c) a 2D graph.



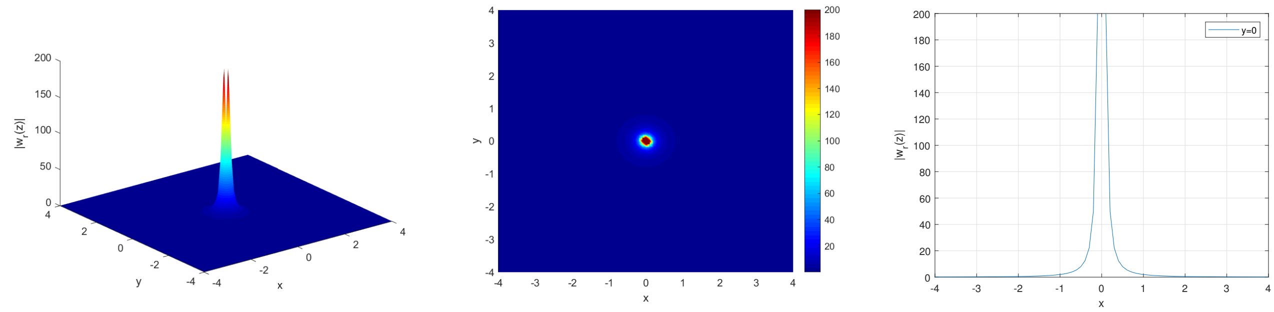
(a) **(b)** **(c)**

Figure 8. The representations of $|w_{s2}(z)|, z = x + yi$ in (a) a 3D plot, (b) a contour image, and (c) a 2D graph.



(a) **(b)** **(c)**

Figure 9. The representations of $|w_{s3}(z)|, z = x + yi$ in (a) a 3D plot, (b) a contour image, and (c) a 2D graph.



(a) **(b)** **(c)**

Figure 10. The representations of $|w_r(z)|, z = x + yi$ in (a) a 3D plot, (b) a contour image, and (c) a 2D graph.

5. Meromorphic exact solutions of five PDEs through exp$(-\vartheta(z))$-expansion method

Case I: $n = 4$ in Eq (1.14).

Applying the homogeneous balance to v^4, vv'' and $(v')^2$ in Eq (1.14), we obtain

$$v(z) = \gamma_0 + \gamma_1 \exp(-\vartheta(z)), \tag{5.1}$$

where $\gamma_1 \neq 0$ and γ_0 are constants.

Substituting vv'', $(v')^2$, v^4, v, and v^3 into Eq (1.14) and setting the coefficients of $\exp(-\vartheta(z))$ to be zero yields

$$e^{0(-\vartheta(z))}:$$

$$-A{\gamma_1}^2k^2 + A\gamma_0\gamma_1 kl + B{\gamma_0}^4 + C{\gamma_0}^3 + D\gamma_0 + E = 0,$$

$$e^{1(-\vartheta(z))}:$$

$$-Akl{\gamma_1}^2 + Al^2\gamma_0\gamma_1 + 4B{\gamma_0}^3\gamma_1 + 2A\gamma_0\gamma_1 k + 3C{\gamma_0}^2\gamma_1 + D\gamma_1 = 0,$$

$$e^{2(-\vartheta(z))}:$$

$$6\,B\gamma_0{}^2\gamma_1{}^2 + 3\,Al\gamma_0\,\gamma_1 + 3\,C\gamma_0\,\gamma_1{}^2 = 0,$$
$$e^{3(-\vartheta(z))}:$$
$$4\,B\gamma_0\,\gamma_1{}^3 + Al\gamma_1{}^2 + C\gamma_1{}^3 + 2\,A\gamma_0\,\gamma_1 = 0,$$
$$e^{4(-\vartheta(z))}:$$
$$B\gamma_1{}^4 + A\gamma_1{}^2 = 0.$$

Solving the algebraic equations above yields

$$\gamma_1 = \pm\frac{\sqrt{-BA}}{B}, \gamma_0 = -\frac{C\sqrt{-BA} \pm ABl}{2B\sqrt{-BA}}, E = -\frac{\left(-BAl^2 + 4\,BAk - C^2\right)^2}{16B^3}, \tag{5.2}$$

where k and l are arbitrary.

Substituting Eq (5.2) into Eq (5.1) yields

$$v(z) = -\frac{C\sqrt{-BA} \pm ABl}{2B\sqrt{-BA}} \pm \frac{\sqrt{-BA}}{B}\exp(-\vartheta(z)), \tag{5.3}$$

Putting Eqs (3.8)–(3.14) into Eq (5.3) derives exact solutions of the equations, respectively, as shown below.

Family 1.1: When $l^2 - 4k > 0$, $k \neq 0$, we have

$$v_{11}(z) = -\frac{C\sqrt{-BA} \pm ABl}{2B\sqrt{-BA}} \mp \frac{2k\sqrt{-BA}}{B\sqrt{(l^2 - 4k)}\tanh\left(\frac{\sqrt{l^2-4k}}{2}(z+b)\right) + Bl},$$

$$v_{12}(z) = -\frac{C\sqrt{-BA} \pm ABl}{2B\sqrt{-BA}} \mp \frac{2k\sqrt{-BA}}{B\sqrt{(l^2 - 4k)}\coth\left(\frac{\sqrt{l^2-4k}}{2}(z+b)\right) + Bl}.$$

Family 1.2: When $l^2 - 4k < 0$, $k \neq 0$, we have

$$v_{13}(z) = -\frac{C\sqrt{-BA} \pm ABl}{2B\sqrt{-BA}} \pm \frac{2k\sqrt{-BA}}{B\sqrt{(4k - l^2)}\tan\left(\frac{\sqrt{4k-l^2}}{2}(z+b)\right) - Bl},$$

$$v_{14}(z) = -\frac{C\sqrt{-BA} \pm ABl}{2B\sqrt{-BA}} \pm \frac{2k\sqrt{-BA}}{B\sqrt{(4k - l^2)}\cot\left(\frac{\sqrt{4k-l^2}}{2}(z+b)\right) - Bl}.$$

Family 1.3: When $l^2 - 4k > 0$, $l \neq 0$, and $k = 0$, we have

$$v_{15}(z) = -\frac{C\sqrt{-BA} \pm ABl}{2B\sqrt{-BA}} \pm \frac{\sqrt{-BA}l}{B\exp(l(z+b)) - B}.$$

Family 1.4: When $l^2 - 4k = 0$, $l \neq 0$, and $k \neq 0$, we have

$$v_{16}(z) = -\frac{C\sqrt{-BA} \pm ABl}{2B\sqrt{-BA}} \mp \frac{\sqrt{-BA}l^2(z+b)}{2B(l(z+b)+2)}.$$

Family 1.5: When $l^2 - 4k = 0$, $l = 0$, and $k = 0$, we have

$$v_{17}(z) = -\frac{C}{2B} \pm \frac{\sqrt{-BA}}{B(z+b)}.$$

Case II: $n = 3$ in Eq (1.14).
According to similar steps to Case I, the following solutions to Eq (1.14) can be obtained.
Family 2.1: When $l^2 - 4k > 0$ and $k \neq 0$, we have

$$\begin{aligned} v_{21}(z) =& \frac{-Al^2 - 8\,Ak + \sqrt{A^2\,(4\,k - l^2)^2 - 12\,BD}}{6B} + \frac{4kAl}{B\sqrt{(l^2-4k)}\tanh\left(\frac{\sqrt{l^2-4k}}{2}(z+b)\right) + Bl} \\ & - \frac{8k^2A}{B\left(\sqrt{(l^2-4k)}\tanh\left(\frac{\sqrt{l^2-4k}}{2}(z+b)\right) + l\right)^2}, \\ v_{22}(z) =& \frac{-Al^2 - 8\,Ak + \sqrt{A^2\,(4\,k - l^2)^2 - 12\,BD}}{6B} + \frac{4kAl}{B\sqrt{(l^2-4k)}\coth\left(\frac{\sqrt{l^2-4k}}{2}(z+b)\right) + Bl} \\ & - \frac{8k^2A}{B\left(\sqrt{(l^2-4k)}\coth\left(\frac{\sqrt{l^2-4k}}{2}(z+b)\right) + l\right)^2}. \end{aligned}$$

Family 2.2: When $l^2 - 4k < 0$ and $k \neq 0$, we have

$$\begin{aligned} v_{23}(z) =& \frac{-Al^2 - 8\,Ak + \sqrt{A^2\,(4\,k - l^2)^2 - 12\,BD}}{6B} - \frac{4kAl}{B\sqrt{(4k-l^2)}\tan\left(\frac{\sqrt{4k-l^2}}{2}(z+b)\right) - Bl} \\ & + \frac{8k^2A}{B\left(\sqrt{(4k-l^2)}\tan\left(\frac{\sqrt{4k-l^2}}{2}(z+b)\right) - l\right)^2}, \\ v_{24}(z) =& \frac{-Al^2 - 8\,Ak + \sqrt{A^2\,(4\,k - l^2)^2 - 12\,BD}}{6B} - \frac{4kAl}{B\sqrt{(4k-l^2)}\cot\left(\frac{\sqrt{4k-l^2}}{2}(z+b)\right) - Bl} \\ & + \frac{8k^2A}{B\left(\sqrt{(4k-l^2)}\cot\left(\frac{\sqrt{4k-l^2}}{2}(z+b)\right) - l\right)^2}. \end{aligned}$$

Family 2.3: When $l^2 - 4k > 0$, $l \neq 0$, and $k = 0$, we have

$$v_{25}(z) = \frac{-Al^2 + \sqrt{A^2 l^4 - 12\,BD}}{6B} - \frac{2Al}{B\exp(l(z+b)) - B} - \frac{2Al}{B\,(\exp(l(z+b)) - 1)^2}.$$

Family 2.4: When $l^2 - 4k = 0$, $l \neq 0$, and $k \neq 0$, we have

$$v_{26}(z) = \frac{-Al^2 - 8\,Ak + \sqrt{A^2 - 12\,BD}}{6B} + \frac{Al^3(z+b)}{B(l(z+b)+2)} - \frac{Al^4(z+b)^2}{2B(l(z+b)+2)^2}.$$

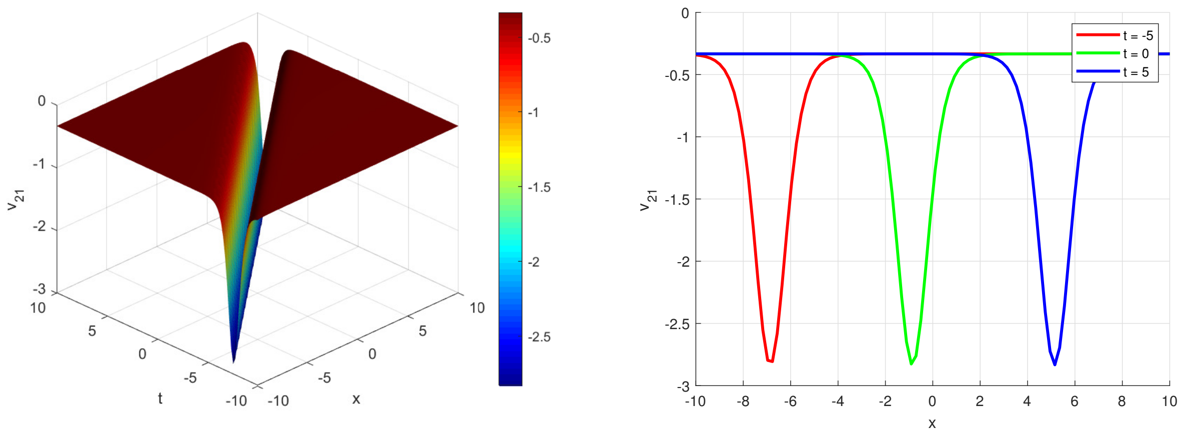
Family 2.5: When $l^2 - 4k = 0$, $l = 0$, and $k = 0$, we have

$$v_{27}(z) = \frac{\sqrt{-12\,BD}}{6B} - \frac{2A}{B(z+b)^2}.$$

Remark 5.1. *Through choosing appropriate values for A, B, C, D, and E, meromorphic exact solutions of the five nonlinear PDEs can be simplicity obtained.*

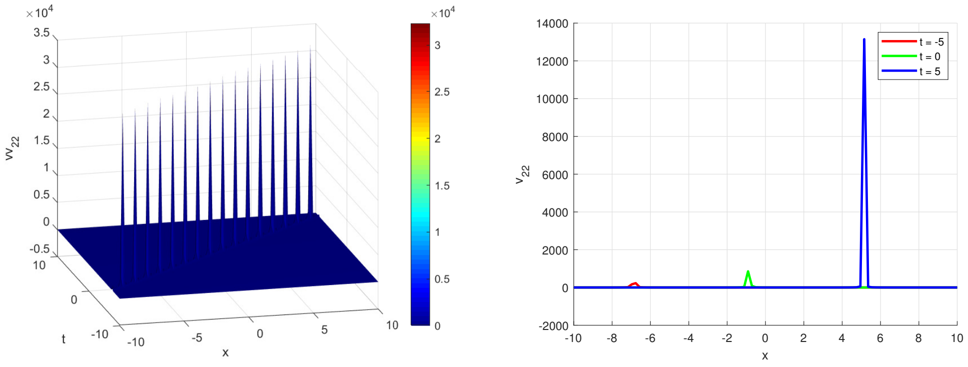
We selected the solutions for v_{21} to v_{24} and v_{27}, and plotted their graphs using MATLAB, thus obtaining the corresponding Figures 11–15. To better investigate the differences among these solutions, the four parameters (A, B, C, D), a_1, and a_2 were fixed in our selection, and the specific parameter values are presented in the figure captions.

As can be clearly seen from Figures 11–15, with most parameters held the same, the solutions differ remarkably when the values of the parameters l and k satisfying the corresponding conditional requirements are selected. It can be seen that the properties of the resulting solutions are manifested distinctly in the corresponding graphs. In Figure 11, the variation in the peaks of the solution follows a linear trend. In Figures 12 and 15, the peaks of the solutions also vary along a line but exhibit an obvious undulation. Notably, in Figures 13 and 14, the solutions are highly sensitive to the variations of x and t under the same parameter settings, with the peaks changing in a very dense manner.



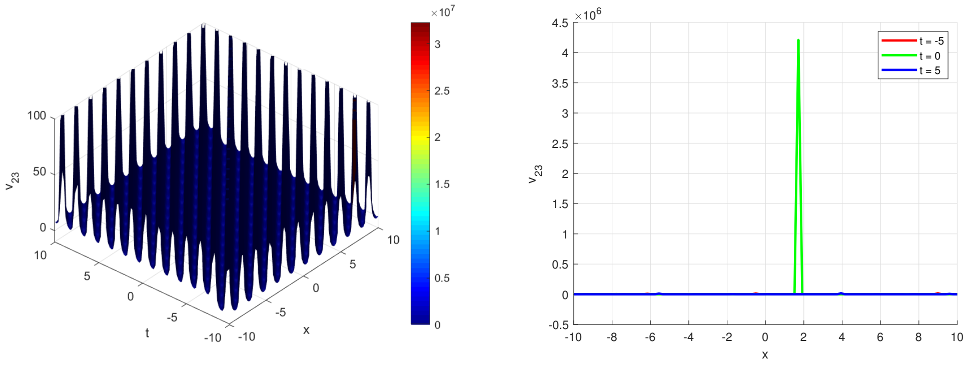
(a) **(b)**

Figure 11. v_{21} of $A = 1, B = -1, C = 1, b = 0, D = 2, a1 = 1; a2 = 1.2, l = 3, k = 1$: (a) 3D graph; (b) 2D graph.



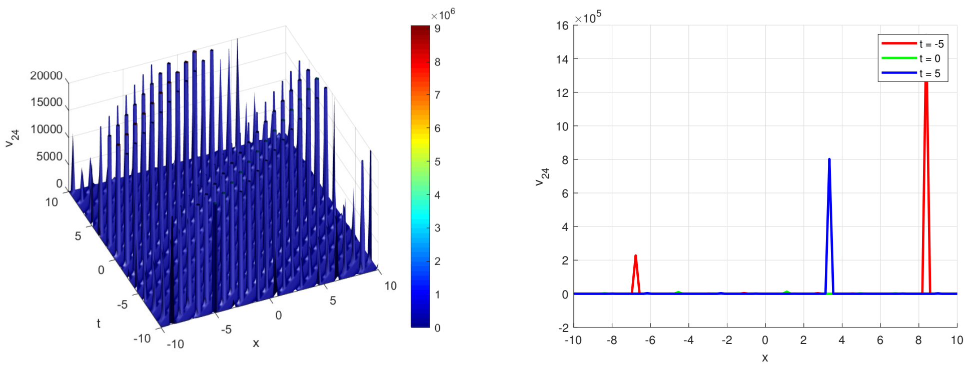
(a) **(b)**

Figure 12. v_{22} of $A = 1, B = -1, C = 1, b = 0, D = 2, a1 = 1; a2 = 1.2, l = 3, k = 1$: (a) 3D graph; (b) 2D graph.



(a) **(b)**

Figure 13. v_{23} of $A = 1, B = -1, C = 1, b = 0, D = 2, a1 = 1; a2 = 1.2, l = 1, k = 3$: (a) 3D graph; (b) 2D graph.



(a) **(b)**

Figure 14. v_{24} of $A = 1, B = -1, C = 1, b = 0, D = 2, a1 = 1; a2 = 1.2, l = 1, k = 3$: (a) 3D graph; (b) 2D graph.

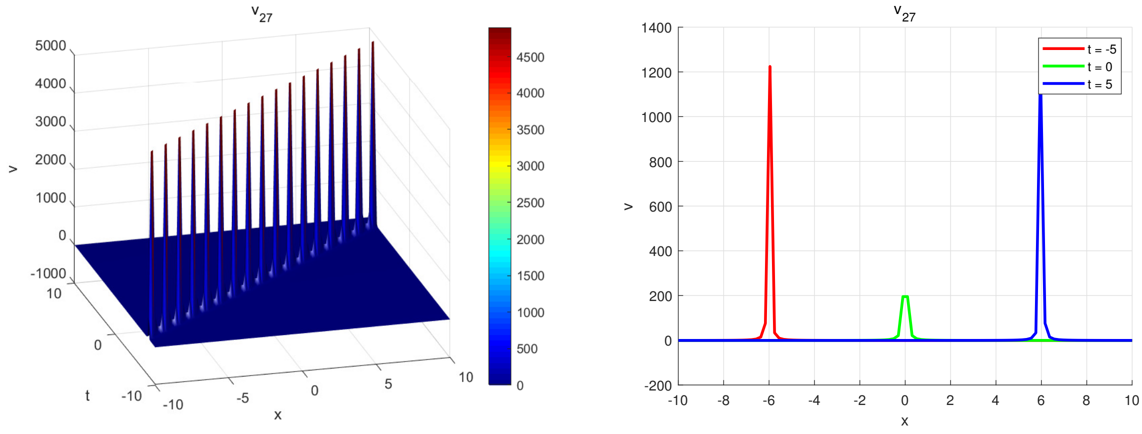


Figure 15. v_{27} of $A = 1, B = -1, C = 1, b = 0, D = 2, a1 = 1; a2 = 1.2, l = 0, k = 0$: (a) 3D graph; (b) 2D graph.

6. Conclusions

In this study, we have analyzed bifurcation behaviors and achieved meromorphic exact solutions for five nonlinear PDEs, simultaneously. These five equations (i.e., the Tzitzeica equation, the ZS equation, the mKdV equation, the TDB equation, and the mDSG equation) represent canonical integrable models that have shaped nonlinear mathematical physics, playing foundational roles across soliton theory, the geometry of surfaces, field-theoretic constructions, and the structure of integrable systems. By using the complex method together with the $\exp(-\vartheta(z))$-expansion approach, meromorphic exact solutions of the abovementioned equations are obtained, including rational function, hyperbolic function, elliptic function, trigonometric function and exponential function solutions. The dynamic behaviors of the meromorphic exact solutions are demonstrated by some graphs. This may have potential applications in fields such as engineering and physics. To our current understanding, the findings presented in this work are new and have not been documented in prior literature.

Author contributions

Hongqiang Tu: Conceptualization, validation, writing—original draft, writing—review and editing; Yongyi Gu: Methodology, software, writing—original draft, writing—review and editing. All authors have read and approved the final version of the manuscript for publication.

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Use of Generative AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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