AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(7): 19567–19592.
DOI: 10.3934/math.2026795
Received: 08 March 2026
Revised: 28 May 2026
Accepted: 10 June 2026
Published: 02 July 2026

Research article

Two-step inertial Tseng's extragradient method for variational inequality and hierarchical fixed-point problems

Ghadah ALbeladi[1], **Rehan Ali**[2] **and Mohammad Farid**[3,*]

[1] Department of Mathematics, College of Sciences & Arts, King Abdulaziz University, Rabigh 21911, Saudi Arabia

[2] Department of Mathematics, Central University of Kashmir, Ganderbal, Jammu and Kashmir-191131, India

[3] Department of Mathematics, College of Science, Qassim University, Saudi Arabia

* **Correspondence:** Email: m.jameel@qu.edu.sa.

Abstract: We developed novel inertial Tseng-type extragradient methods, in both two-step and one-step formulations based on the Krasnosel'skiĭ-Mann iteration framework. The proposed schemes were designed to solve variational inequality problems (VIPs) and hierarchical fixed-point problems (HFPPs) governed by nonexpansive and quasi-nonexpansive operators in Hilbert spaces. The obtained sequences were shown to converge weakly to a shared solution, together with additional supporting results, and its applicability was demonstrated through an example. This formulation generalizes and brings together several known methods and outcomes in the literature.

Keywords: hierarchical fixed-point problem; variational inequality; iterative methods; Tseng's extragradient method; quasi-nonexpansive mapping
Mathematics Subject Classification: 47H05, 47H10, 49J35, 47J25

1. Introduction

Consider a real Hilbert space $\mathcal{H}$ equipped with $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. Suppose $\mathcal{Q} \neq \emptyset$ is a closed convex subset of $\mathcal{H}$. A map $\mathcal{U}$ on $\mathcal{Q}$ is termed nonexpansive if $\|\mathcal{U}u - \mathcal{U}w\| \leq \|u - w\|$, for all $u, w \in \mathcal{Q}$. The associated fixed-point set $\mathcal{F}(\mathcal{U}) = \{v \in \mathcal{Q} : \mathcal{U}v = v\}$, whenever nonempty, is closed and convex in $\mathcal{Q}$. The proper map $g : \mathcal{H} \to (-\infty, +\infty]$, and its subdifferential at a point $p \in \mathcal{H}$ is defined as $\partial g(p) = \{u \in \mathcal{H} : \langle y-p, u\rangle + g(p) \leq g(y),\ \forall y \in \mathcal{H}\}$. If $\partial g(p) \neq \emptyset$, then g is said to be subdifferentiable at p. For a convex set Q, its indicator function $\psi_Q : \mathcal{H} \to (-\infty, +\infty]$ is given by

$$\partial\psi_Q(v) = \begin{cases} 0, & v \in Q, \\ \infty, & \text{otherwise.} \end{cases}$$

Whenever Q is convex, ψ_Q becomes convex. Moreover, for $v \in Q$, the subdifferential $\partial\psi_Q(v)$ coincides with the normal cone of Q at v and, in particular, contains 0.

Moudafi et al. [29] examined a HFPP of finding $\bar{u} \in \mathcal{F}(\mathcal{U})$ for which

$$\langle \bar{u} - \mathcal{W}\bar{u}, \bar{u} - q \rangle \leq 0, \quad \forall q \in \mathcal{F}(\mathcal{U}), \tag{1.1}$$

with $\mathcal{U}, \mathcal{W} : Q \to Q$ being nonexpansive. Assume Φ as a solution set of (1.1).

If $\bar{u} \in \mathcal{F}(\mathcal{U})$, then (1.1) $\Leftrightarrow \langle -(I - \mathcal{W})\bar{u}, q - \bar{u} \rangle + \psi_{\mathcal{F}(\mathcal{U})}(\bar{u}) \leq \psi_{\mathcal{F}(\mathcal{U})}(q) \Leftrightarrow -(I - \mathcal{W})\bar{u} \in \partial\psi_{\mathcal{F}(\mathcal{U})}(\bar{u})$. Thus, problem (1.1) is equivalent to the following variational inclusion:

$$0 \in (I - \mathcal{U})\bar{u} + N_{\mathcal{F}(\mathcal{U})}(\bar{u}), \tag{1.2}$$

$N_{\mathcal{F}(\mathcal{U})}(\bar{u})$ stands for the normal cone associated with $\mathcal{F}(\mathcal{U})$ at $\bar{u}$, defined as

$$N_{\mathcal{F}(\mathcal{U})}(\bar{u}) = \partial\psi_{\mathcal{F}(\mathcal{U})}(\bar{u}) = \begin{cases} \{v \in \mathcal{H} : \langle q - \bar{u}, v \rangle \leq 0, \ \forall q \in \mathcal{F}(\mathcal{U})\}, & \text{if } \bar{u} \in \mathcal{F}(\mathcal{U}), \\ \emptyset, & \text{else.} \end{cases}$$

For $\mathcal{W} = I$, the set Φ reduces to $\mathcal{F}(\mathcal{U})$. We observe that HFPP(1.1) encompasses several important problems, including monotone variational inequalities on fixed-point sets, minimization problems subject to equilibrium constraints, and hierarchical minimization problems; see, for example, [9, 46].

Moudafi [30] presented an iterative scheme of Krasnosel'skiĭ-Mann type to solve HFPP (1.1). Starting from an initial point $x_0 \in Q$, the sequence $\{l_m\}$ is generated by

$$l_{m+1} = (1 - \alpha_m)l_m + \alpha_m(\sigma_m \mathcal{W} l_m + (1 - \sigma_m)\mathcal{U} l_m), \qquad \forall m \geq 0, \tag{1.3}$$

where the parameters satisfy $\{\alpha_m\}, \{\sigma_m\} \subset (0, 1)$. This iterative method unifies and extends a number of classical fixed-point algorithms frequently used in signal and image processing. Moreover, it provides a common framework for studying their convergence properties. Moudafi [29,31] introduced an iterative method for solving hierarchical fixed-point problems. More recently, Kazmi *et al.* [25, 26] proposed an improved iterative algorithm along with a convergence analysis and numerical experiments (see also [47, 48]). Additionally, Meanwhile, Byrne [8] explored iterative techniques and illustrated their practical utility in signal and image processing applications.

Here, we study the variational inequality (VI) introduced by Hartman and Stampacchia [22]: Find $\bar{u} \in Q$ satisfies

$$\langle \mathcal{A}(\bar{u}), v - \bar{u} \rangle \geq 0, \qquad \forall v \in Q, \tag{1.4}$$

with $\mathcal{A} : \mathcal{H} \to \mathcal{H}$. The solution set of (1.4) is represented by Sol(VI(1.4)).

Variational inequality theory has attracted considerable attention in applied mathematics due to its wide-ranging applications in optimization, economics, mechanics, network analysis, and image processing. Variational inequality problems provide an effective framework for modeling numerous real-world phenomena, including obstacle problems, traffic equilibrium, contact mechanics, and optimal control systems [16,19,33]. Because of their strong theoretical and computational significance, variational inequalities have become an established and active area of nonlinear analysis [10, 34].

The projected gradient scheme

$$l_{m+1} = \mathcal{P}_Q(l_m - \mu\mathcal{A}l_m), \tag{1.5}$$

with step-size $\mu > 0$ and $\mathcal{P}_Q$ denoting the metric projection, is a standard tool for VI (1.4). Its convergence, however, demands inverse strong monotonicity (or strong monotonicity) of $\mathcal{A}$, a condition that can be too restrictive in practice.

To overcome this limitation, Korpelevich [27] proposed the extragradient method:

$$\begin{cases} x_m &= \mathcal{P}_Q(l_m - \mu\mathcal{A}l_m), \\ l_{m+1} &= \mathcal{P}_Q(l_m - \mu\mathcal{A}x_m), \end{cases} \tag{1.6}$$

where $\mu \in (0, \frac{1}{\mathcal{L}})$ and $\mathcal{L} > 0$ is the Lipschitz constant of $\mathcal{A}$. A recent study by Farid et al. [18] examined the extragradient method incorporating an inertial iterative technique (see also [2]). Meanwhile, Censor et al. [11] investigated the subgradient extragradient method and established its convergence properties (see, e.g., [1, 17, 23]).

Because the extragradient algorithm requires computing two projections that may be computationally expensive, Censor et al. [11–13] developed a subgradient–extragradient method. In this approach, the second projection onto Q is replaced by one onto a half-space. Specifically, the iteration takes the form

$$\begin{cases} x_m &= \mathcal{P}_Q(l_m - \mu\mathcal{A}l_m), \\ l_{m+1} &= \mathcal{P}_{R_m}(l_m - \mu\mathcal{A}x_m), \end{cases} \tag{1.7}$$

where

$$R_m = \{u \in \mathcal{H} \ : \ \langle l_m - \mu\mathcal{A}l_m - x_m,\ u - x_m\rangle \leq 0\}.$$

For a monotone $\mathcal{L}$-Lipschitz operator $\mathcal{A}$, taking $\mu \in (0, 1/\mathcal{L})$ yields weak convergence of $\{l_m\}$ to a solution of (1.4).

Takahashi and Toyoda [45] presented an iterative algorithm to compute a common element of $\mathcal{F}(\mathcal{U}) \cap \mathrm{Sol(VI(1.4))}$:

$$\begin{cases} x_m &= \mathcal{P}_Q(l_m - \mu_m\mathcal{A}l_m), \\ l_{m+1} &= (1 - \alpha_m)l_m + \alpha_m\mathcal{U}x_m, \end{cases} \tag{1.8}$$

and established the weak convergence of $\{l_m\}$ to some $\bar{u} \in \mathcal{F}(\mathcal{U}) \cap \mathrm{Sol(VI(1.4))}$. The method described above may fail to converge, however, if $\mathcal{A}$ is simply monotone and $\mathcal{L}$-Lipschitz, without the reinforcement of stronger monotonicity assumptions. Motivated by Korpelevich's extragradient technique, Nadezhkina and Takahashi [32] introduced the modified scheme:

$$\begin{cases} x_m &= \mathcal{P}_Q(l_m - \mu_m\mathcal{A}l_m), \\ l_{m+1} &= (1 - \alpha_m)l_m + \alpha_m\mathcal{U}\mathcal{P}_Q(l_m - \mu_m\mathcal{A}x_m), \end{cases} \tag{1.9}$$

with parameters satisfying $\mu_m \in (0, 1/\mathcal{L})$ and $\alpha_m \in (0, 1)$. Under these assumptions, weak convergence $l_m \rightharpoonup \bar{u}$ holds for some $\bar{u}$ in $\mathcal{F}(\mathcal{U}) \cap \mathrm{Sol(VI(1.4))}$.

Tseng's extragradient method (TEGM), introduced in [41], is designed for variational inequalities as follows:

$$\begin{cases} x_m &= \mathcal{P}_Q(l_m - \mu \mathcal{A} l_m), \\ l_{m+1} &= x_m + \mu(\mathcal{A} l_m - \mathcal{A} x_m), \end{cases} \tag{1.10}$$

where the step size satisfies $\mu \in (0, \frac{1}{L})$. An important property of TEGM, as well as several subgradient-type approaches, is that each iteration involves just one projection onto the feasible set Q. The computational efficiency of TEGM, requiring just one projection compared to two in standard extragradient methods, has made it a foundation for numerous extensions; see, for example, [42, 43]. Recent developments have incorporated inertial terms, adaptive step sizes, and extensions to handle pseudo-monotone and quasi-monotone operators, broadening its applicability to nonsmooth optimization and modern data science problems.

In contrast, the concept of inertial extrapolation originally inspired by heavy-ball methods for second-order dynamical systems with friction was introduced by Polyak [36] to accelerate the minimization of smooth convex functions. These inertial techniques are known to improve the convergence speed of iterative procedures. Because of their effectiveness in enhancing convergence rates, inertial-type algorithms have been widely adopted by various authors; examples include [23, 28, 35].

A generalization of the heavy ball methodology to the setting of maximal monotone operators was developed by Alvarez and Attouch [4], using a proximal point algorithm as the foundational structure. The resulting method, termed the inertial proximal point algorithm, takes the form:

$$\begin{cases} x_m &= l_m + \theta_m(l_m - l_{m-1}), \\ l_{m+1} &= (I + \mu_m \mathcal{A})^{-1} x_m, \end{cases} \tag{1.11}$$

where $(I + \mu_m \mathcal{A})^{-1}$ denotes the resolvent of the maximal monotone operator $\mathcal{A}$ with parameter μ_m. Under the assumptions that μ_m is non-decreasing and $\theta_m \in [0, 1)$ satisfies $\sum_{m=1}^{\infty} \theta_m \|l_m - l_{m-1}\|^2 < +\infty$, $\{l_m\}$ converges weakly.

To refine the results of Tseng [41] and develop a more efficient procedure, Thong and Hieu [44] introduced an inertial Tseng-type extragradient method, proving its weak convergence in real Hilbert spaces.

Poon and Liang [38] contributed a Douglas–Rachford splitting scheme that includes double inertial components as:

$$l_{m+1} = \mathcal{A}(l_m + \theta(l_m - l_{m-1}) + \eta(l_{m-1} - l_{m-2})). \tag{1.12}$$

This two-step inertial scheme exhibits faster convergence than its single-step counterpart and is employed to locate zeros of maximal monotone operators [24]. Furthermore, extragradient-based algorithms have also been investigated for quasi-monotone operators, which constitute a weaker class than monotone mappings, as discussed in [10, 15].

Recent research has explored multi-step inertial methods, as illustrated in [14]. One limitation of single-step inertial ADMM, highlighted in [39], has motivated the development of adaptive acceleration strategies. Earlier observations by Polyak [37] also indicate that incorporating multiple inertial steps can substantially enhance optimization speed, suggesting that utilizing more than two past iterates may lead to improved performance in certain scenarios.

Building upon the foundational work on inertial methods [4], Krasnosel'skiĭ–Mann iterations [30], and Tseng's projection technique [41], we propose a novel Krasnosel'skiĭ–Mann–type inertial Tseng extragradient method in both one-step and two-step variants. This algorithm synthesizes concepts from the inertial forward-backward scheme (1.3), Tseng's method (1.10), and the hybrid iterative framework (1.12). We show that the iterates generated by our algorithm converge weakly to a point that solves both the HFPP (1.1) and VIP (1.4) simultaneously. The proposed method extends and unifies several earlier results, including those on inertial proximal methods [4], relaxed Tseng-type algorithms [38], and hierarchical fixed-point iterations [30, 32]. Its practical efficiency is demonstrated through numerical experiments.

2. Preliminaries

In the following, $l_m \to \bar{u}$ (resp. $l_m \rightharpoonup \bar{u}$) indicates strong (resp. weak) convergence of $\{l_m\}$ to $\bar{u} \in \mathcal{H}$. The set of all weak cluster points of $\{l_m\}$ is denoted by $\omega_w\{l_m\}$, that is, $\omega_w\{l_m\} := \{\bar{u} \in \mathcal{H} : \exists \{l_{m_k}\} \subset \{l_m\}$ with $l_{m_k} \rightharpoonup \bar{u}\}$. We now recall several standard notions from convex and nonlinear analysis that will be used throughout the paper.
The following holds true:

(i)

$$\|\alpha u + (1-\alpha)w\|^2 = \alpha\|u\|^2 + (1-\alpha)\|w\|^2 - \alpha(1-\alpha)\|u-w\|^2, \ \forall u, w \in \mathcal{H};\ 0 \le \alpha \le 1; \tag{2.1}$$

(ii) Opial's condition holds: Take $\{w_m\} \subset \mathcal{H}$ with $w_m \rightharpoonup u$, we have $\liminf_{m\to\infty} \|w_m - u\| < \liminf_{m\to\infty} \|w_m - v\|,\ \forall v \neq u$.

To each $u \in \mathcal{H}$, $\exists$ a unique point $\mathcal{P}_Q u \in Q$, known as the metric projection of u onto Q, with $\|u - \mathcal{P}_Q u\| \le \|u - w\|,\ \forall w \in Q$. Thus, $\mathcal{P}_Q$ denotes the metric projection mapping from $\mathcal{H}$ onto Q.
$\mathcal{P}_Q$ is nonexpansive and obeys [21]

$$\langle u - w, \mathcal{P}_Q u - \mathcal{P}_Q w\rangle \ge \|\mathcal{P}_Q u - \mathcal{P}_Q w\|^2, \ \ \forall u \in \mathcal{H}. \tag{2.2}$$

Additionally, $\mathcal{P}_Q u$ is uniquely determined by the conditions $\mathcal{P}_Q u \in Q$, together with

$$\langle u - \mathcal{P}_Q u, w - \mathcal{P}_Q u\rangle \le 0, \tag{2.3}$$

$$\|u - w\|^2 \ge \|u - \mathcal{P}_Q u\|^2 + \|w - \mathcal{P}_Q u\|^2, \ \ \forall u \in \mathcal{H},\ w \in Q. \tag{2.4}$$

Lemma 1. [21] *Let $u, w, z \in \mathcal{H}$ and $c_1, c_2 \in \mathbb{R}$, then*

(i) $2\langle u, w\rangle = \|u\|^2 + \|w\|^2 - \|u - w\|^2 = \|u + w\|^2 - \|u\|^2 - \|w\|^2$;

(ii)

$$\begin{aligned} \|(1+c)u - (c-d)w - dz\|^2 &= (1+c)\|u\|^2 - (c-d)\|w\|^2 - d\|z\|^2 - d(c-d)\|w-z\|^2 \\ &\quad +(1+c)(c-d)\|u-w\|^2 + d(1+c)\|u-z\|^2; \end{aligned} \tag{2.5}$$

(iii) $\|cu + (1-c)w\|^2 = c\|u\|^2 + (1-c)\|w\|^2 - c(1-c)\|u-w\|^2$.

Definition 1. [7] *Let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be an operator. Then:*

(i) *M is monotone if*

$$\langle z - v, u - w\rangle \geq 0,\ \textit{whenever } z \in M(u),\ v \in M(w);$$

(ii) *M is termed maximal monotone if it is monotone and there is no monotone operator M' such that* graph$(M) \subset$ graph(M') *with proper inclusion, where* graph$(M) := \{(u, w) \in \mathcal{H} \times \mathcal{H} : w \in M(u)\}$.

Lemma 2. [8,20] *Consider an operator $\mathcal{U}$ on $\mathcal{H}$. Then,*

(i) *it is $\mathcal{L}$-Lipschitz continuous if $\|\mathcal{U}u - \mathcal{U}w\| \leq \mathcal{L}\|u - w\|,\ \ \forall\ u, w \in \mathcal{H}, \mathcal{L} > 0$;*
(ii) *$\mathcal{U}$ possesses the quasi-nonexpansive if $\|\mathcal{U}u - w\| \leq \|u - w\|,\ \ \forall\ u \in \mathcal{F}(\mathcal{U}),\ w \in \mathcal{H}$;*
(iii) *the mapping $I - \mathcal{U}$ is a maximal monotone;*
(iv) *it is demiclosed on $\mathcal{H}$, i.e., for any sequence $\{u_n\}$ with $u_n \rightharpoonup u$ and $u_n - \mathcal{U}u_n \to 0$, one has $u \in \mathcal{F}(\mathcal{U})$.*

Lemma 3. [5] *Let $\{\psi_m\}$, $\{\delta_m\}$, and $\{\eta_m\}$ be the sequences in $[0, \infty)$ such that $\psi_{m+1} \leq \psi_m + \eta_m(\psi_m - \psi_{m-1}) + \gamma_m$, $\forall m \geq 1$, $\sum\limits_{m=1}^{\infty} \gamma_m < +\infty$ and there is a number η with $0 \leq \eta_m \leq \eta < 1$, $\forall m \geq 1$. Then the following hold:*

(a) $\sum\limits_{m=1}^{\infty} [\psi_m - \psi_{m-1}]_+ < +\infty$, where $[r]_+ := \max\{r, 0\}$;
(b) *there is a $\psi^* \in [0, \infty)$* such that $\lim\limits_{m\to\infty} \psi_m = \psi^*$.

Lemma 4. [40] *Let $\{u_m\} \subset \mathcal{H}$. If*

(i) $\lim\limits_{m\to\infty} \|u_m - u\|$ *exists for every $u \in Q$;*
(ii) $\{u_m\}$ *has all its weak cluster points in Q,*

then $\{u_m\}$ converges weakly to some element of Q.

3. Krasnosel'skiĭ–Mann type two-step inertial Tseng's extragradient method

We propose a Krasnosel'skiĭ–Mann type two-step inertial extragradient method, rooted in Tseng's approach, for solving the HFPP(1.1) and VI(1.4).

Lemma 5. *Under Algorithm 1, the iterates $\{s_m\}$ form a monotonically decreasing sequence with lower bound* $\min\left\{\frac{\mu}{\mathcal{L}}, s_0\right\}$.

Proof. Because $\mathcal{A}$ is Lipschitz continuous with $\mathcal{L} > 0$, the step-size sequence $\{s_m\}$ initiated by Algorithm 1 is monotonically decreasing. Indeed, whenever $\mathcal{A}w_m - \mathcal{A}y_m \neq 0$, the Lipschitz condition yields

$$\frac{\mu\|w_m - y_m\|}{\|\mathcal{A}w_m - \mathcal{A}y_m\|} \geq \frac{\mu\|w_m - y_m\|}{\mathcal{L}\|w_m - y_m\|} = \frac{\mu}{\mathcal{L}}.$$

Consequently, each s_m satisfies $s_m \geq \min\left\{\frac{\mu}{\mathcal{L}},\ s_0\right\} > 0$, so the sequence $\{s_m\}$ is bounded below and therefore converges to a positive limit $s = \lim\limits_{m\to\infty} s_m > 0$. □

Algorithm 1

Initialization: Take $s_0 > 0,\ \mu \in (0,1),\ \theta \in (0,1),\ \eta \in (-1,0],\ \alpha \in (0,1),\ \sigma \in [c,d],\ c,d \in (0,1)$. Select an arbitrary starting point $l_{-1}, l_0, l_1 \in \mathcal{H}$:

Iterative steps: Given the current iterates l_{m-2}, l_{m-1}, l_m:

Step 1. Compute

$$\begin{aligned} w_m &= l_m + \theta(l_m - l_{m-1}) + \eta(l_{m-1} - l_{m-2}),\\ y_m &= \mathcal{P}_Q(w_m - s_m\mathcal{A}w_m). \end{aligned}$$

Step 2. Compute

$$\begin{aligned} z_m &= y_m - s_m(\mathcal{A}y_m - \mathcal{A}w_m),\\ l_{m+1} &= (1-\alpha)w_m + \alpha(\sigma\mathcal{W}z_m + (1-\sigma)\mathcal{U}z_m),\\ s_{m+1} &= \begin{cases} \min\left\{\dfrac{\mu\|w_m - y_m\|}{\|\mathcal{A}w_m - \mathcal{A}y_m\|}, s_m\right\} & \text{if } \mathcal{A}w_m - \mathcal{A}y_m \neq 0,\\ s_m, & \text{otherwise.} \end{cases} \end{aligned}$$

Step 3. Update $m \leftarrow m+1$ and go back to Step 1.

Theorem 1. *Assume that $\mathcal{A}$ is monotone and $\mathcal{L}$-Lipschitz, $\mathcal{U}$ is nonexpansive and $\mathcal{W}$ continuous and quasi-nonexpansive, and $I - \mathcal{W}$ is monotone. Suppose that $\Lambda = \mathrm{Sol}(\mathrm{VI}(1.4)) \bigcap \Phi \bigcap \mathcal{F}(\mathcal{W}) \neq \emptyset$ and that the below requirements are satisfied:*

(i) $\theta \in (0, \frac{1}{3})$;

(ii) $\dfrac{3\theta - 1}{3 + 4\theta} < \eta < 0$.

Then, the sequences $\{w_m\}$, $\{y_m\}$, $\{z_m\}$, and $\{l_m\}$ generated by Algorithm 1 converge weakly to $\bar{u} \in \Lambda$, where $\bar{u} = \mathcal{P}_\Lambda l_0$.

Proof. The proof proceeds through the following steps:

Step I. Let $\{z_m\}$ be as in Algorithm 1. Then

$$\begin{aligned} \|z_m - \mathfrak{f}\|^2 &\leq \|w_m - \mathfrak{f}\|^2 - \left(1 - \mu^2\frac{s_m^2}{s_{m+1}^2}\right)\|y_m - w_m\|^2.\\ &\leq \|w_m - \mathfrak{f}\|^2. \end{aligned}$$

As $\mathfrak{f} \in \mathrm{Sol}(\mathrm{VI}(1.4)) \subset Q$ and $z_m = y_m - s_m(\mathcal{A}y_m - \mathcal{A}w_m)$, we have

$$\begin{aligned} \|z_m - \mathfrak{f}\|^2 &= \|y_m - \tau_m(\mathcal{A}y_m - \mathcal{A}w_m) - \mathfrak{f}\|^2\\ &= \|y_m - \mathfrak{f}\|^2 + s_m^2\|\mathcal{A}y_m - \mathcal{A}w_m\|^2 - 2s_m\langle \mathcal{A}y_m - \mathcal{A}w_m, y_m - \mathfrak{f}\rangle\\ &= \|y_m - w_m\|^2 + \|w_m - \mathfrak{f}\|^2 + 2\langle y_m - w_m, w_m - \mathfrak{f}\rangle + s_m^2\|\mathcal{A}y_m - \mathcal{A}w_m\|^2\\ &\quad -2s_m\langle \mathcal{A}y_m - \mathcal{A}w_m, y_m - \mathfrak{f}\rangle\\ &= \|w_m - \mathfrak{f}\|^2 + \|y_m - w_m\|^2 - 2\langle y_m - w_m, y_m - w_m\rangle + 2\langle y_m - w_m, y_m - \mathfrak{f}\rangle\\ &\quad + s_m^2\|\mathcal{A}y_m - \mathcal{A}w_m\|^2 - 2s_m\langle \mathcal{A}y_m - \mathcal{A}w_m, y_m - \mathfrak{f}\rangle \end{aligned}$$

$$
\begin{aligned}
&= \|w_m - \mathfrak{f}\|^2 - \|y_m - w_m\|^2 + 2\langle y_m - w_m, y_m - \mathfrak{f}\rangle + s_m^2\|\mathcal{A}y_m - \mathcal{A}w_m\|^2 \\
&\quad -2s_m\langle \mathcal{A}y_m - \mathcal{A}w_m, y_m - \mathfrak{f}\rangle. \qquad (3.1)
\end{aligned}
$$

Since $y_m = \mathcal{P}_Q(w_m - s_m\mathcal{A}w_m)$, by (2.3), we get $\langle y_m - w_m + s_m\mathcal{A}w_m, y_m - \mathfrak{f}\rangle \leq 0$ or equivalently

$$
\langle y_m - w_m, y_m - \mathfrak{f}\rangle \leq -s_m\langle \mathcal{A}w_m, y_m - \mathfrak{f}\rangle. \qquad (3.2)
$$

This implies from (3.1) and (3.2) that

$$
\begin{aligned}
\|z_m - \mathfrak{f}\|^2 &\leq \|w_m - \mathfrak{f}\|^2 - \|y_m - w_m\|^2 - 2s_m\langle \mathcal{A}w_m, y_m - \mathfrak{f}\rangle + s_m^2\|\mathcal{A}y_m - \mathcal{A}w_m\|^2 \\
&\quad -2s_m\langle \mathcal{A}y_m - \mathcal{A}w_m, y_m - \mathfrak{f}\rangle \\
&= \|w_m - \mathfrak{f}\|^2 - \|y_m - w_m\|^2 + s_m^2\|\mathcal{A}y_m - \mathcal{A}w_m\|^2 - 2s_m\langle y_m - \mathfrak{f}, \mathcal{A}y_m\rangle \\
&\leq \|w_m - \mathfrak{f}\|^2 - \|y_m - w_m\|^2 + \mu^2\frac{s_m^2}{s_{m+1}^2}\|y_m - w_m\|^2 - 2s_m\langle y_m - \mathfrak{f}, \mathcal{A}y_m\rangle \\
&= \|w_m - \mathfrak{f}\|^2 - \|y_m - w_m\|^2 + \mu^2\frac{s_m^2}{s_{m+1}^2}\|y_m - w_m\|^2 - 2s_m\langle y_m - \mathfrak{f}, \mathcal{A}y_m - \mathcal{A}\mathfrak{f}\rangle \\
&\quad -2s_m\langle y_m - \mathfrak{f}, \mathcal{A}\mathfrak{f}\rangle \\
&\leq \|w_m - \mathfrak{f}\|^2 - \left(1 - \mu^2\frac{s_m^2}{s_{m+1}^2}\right)\|y_m - w_m\|^2. \qquad (3.3)
\end{aligned}
$$

It follows from Lemma 5 that

$$
\lim_{n\to\infty}\left(1 - \mu^2\frac{s_m^2}{s_{m+1}^2}\right) = 1 - \mu^2 > 0, \qquad (3.4)
$$

which yields that $\exists N > 0$ with $\left(1 - \mu^2\frac{s_m^2}{s_{m+1}^2}\right) > 0, \ \ \forall\, m \geq N$. Hence, by (3.3)

$$
\|z_m - \mathfrak{f}\|^2 \leq \|w_m - \mathfrak{f}\|^2. \qquad (3.5)
$$

Step II. The sequences $\{w_m\}$, $\{y_m\}$, $\{z_m\}$, and $\{l_m\}$ from Algorithm 1 are bounded.

Assume for any $\mathfrak{f} \in \Lambda$ and $T_\sigma := \sigma\mathcal{W} + (1 - \sigma)\mathcal{U}$. It follows from Algorithm 1 and (3.5) that

$$
\begin{aligned}
\|l_{m+1} - \mathfrak{f}\|^2 &= \|(1 - \alpha)w_m + \alpha T_\sigma z_m - \mathfrak{f}\|^2 \\
&\leq (1 - \alpha)\|w_m - \mathfrak{f}\|^2 + \alpha\|T_\sigma z_m - \mathfrak{f}\|^2 - \alpha(1 - \alpha)\|T_\sigma z_m - w_m\|^2 \\
&\leq (1 - \alpha)\|w_m - \mathfrak{f}\|^2 + \alpha\|z_m - \mathfrak{f}\|^2 - \alpha\sigma(1 - \sigma)\|\mathcal{W}z_m - \mathcal{U}z_m\|^2 \\
&\quad -\alpha(1 - \alpha)\|T_\sigma z_m - w_m\|^2 \qquad (3.6) \\
&\leq \|w_m - \mathfrak{f}\|^2 - \alpha\sigma(1 - \sigma)\|\mathcal{W}z_m - \mathcal{U}z_m\|^2 \\
&\quad -\alpha(1 - \alpha)\|T_\sigma z_m - w_m\|^2 \qquad (3.7) \\
&\leq \|w_m - \mathfrak{f}\|^2 - \alpha(1 - \alpha)\|T_\sigma z_m - w_m\|^2 \\
&\leq \|w_m - \mathfrak{f}\|^2 - \|T_\sigma z_m - w_m\|^2. \qquad (3.8)
\end{aligned}
$$

Next, it follows from Algorithm 1 and Lemma 1 (ii) that

$$
\|w_m - \mathfrak{f}\|^2 = \|l_m + \theta(l_m - l_{m-1}) + \eta(l_{m-1} - l_{m-2}) - \mathfrak{f}\|^2
$$

$$
\begin{aligned}
&= \|(1+\theta)(l_m-\mathfrak{f})-(\theta-\eta)(l_{m-1}-\mathfrak{f})-\eta(l_{m-2}-\mathfrak{f})\|^2 \\
&= (1+\theta)\|l_m-\mathfrak{f}\|^2-(\theta-\eta)\|l_{m-1}-\mathfrak{f}\|^2-\eta\|l_{m-2}-\mathfrak{f}\|^2 \\
&\quad +(1+\theta)(\theta-\eta)\|l_m-l_{m-1}\|^2+\eta(1+\theta)\|l_m-l_{m-2}\|^2 \\
&\quad -\eta(\theta-\eta)\|l_{m-1}-l_{m-2}\|^2. \qquad (3.9)
\end{aligned}
$$

Since $l_{m+1}=(1-\alpha)w_m+\alpha T_\sigma z_m$, this implies that

$$T_\sigma z_m - w_m = \frac{l_{m+1}-w_m}{\alpha}, \qquad (3.10)$$

and hence

$$
\begin{aligned}
\|l_{m+1}-w_m\|^2 &= \|l_{m+1}-l_m-\theta(l_m-l_{m-1})-\eta(l_{m-1}-l_{m-2})\|^2 \\
&= \|l_{m+1}-l_m\|^2+\theta^2\|l_{m-1}-l_m\|^2+\eta^2\|l_{m-1}-l_{m-2}\|^2 \\
&\quad -2\theta\langle l_{m+1}-l_m, l_m-l_{m-1}\rangle-2\eta\langle l_{m+1}-l_m, l_{m-1}-l_{m-2}\rangle \\
&\quad +2\theta\eta\langle l_m-l_{m-1}, l_{m-1}-l_{m-2}\rangle. \qquad (3.11)
\end{aligned}
$$

From Cauchy–Schwarz inequality, we have

$$2\langle a_1,a_2\rangle \le 2\|a_1\|\|a_2\| \le \|a_1\|^2+\|a_2\|^2, \quad a_1,a_2\in\mathcal{H}.$$

Also, for $\theta>0$ and $\eta<0$, we have

$$2\eta\langle a_1,a_2\rangle \ge 2\eta\|a_1\|\|a_2\| \ge \eta(\|a_1\|^2+\|a_2\|^2). \qquad (3.12)$$

$$-2\theta\langle a_1,a_2\rangle \ge -2\theta\|a_1\|\|a_2\| \ge -\theta(\|a_1\|^2+\|a_2\|^2). \qquad (3.13)$$

It follows from (3.11)–(3.13) that

$$
\begin{aligned}
\|l_{m+1}-w_m\|^2 &\ge \|l_{m+1}-l_m\|^2+\theta^2\|l_{m-1}-l_m\|^2+\eta^2\|l_{m-1}-l_{m-2}\|^2 \\
&\quad -\theta\|l_{m+1}-l_m\|^2-\theta\|l_m-l_{m-1}\|^2+\theta\eta\|l_m-l_{m-1}\|^2 \\
&\quad +\theta\eta\|l_{m-1}-l_{m-2}\|^2+\eta\|l_{m+1}-l_m\|^2-\eta\|l_{m-1}-l_m\|^2 \\
&= (1+\eta-\theta)\|l_{m+1}-l_m\|^2+(\theta^2+\theta\eta-\theta)\|l_{m-1}-l_m\|^2 \\
&\quad +(\eta^2+\theta\eta+\eta)\|l_{m-1}-l_{m-2}\|^2. \qquad (3.14)
\end{aligned}
$$

Since $\alpha\in(0,1)$, it follows from (3.8)–(3.10), and (3.14) that

$$
\begin{aligned}
\|l_{m+1}-\mathfrak{f}\|^2 &= (1+\theta)\|l_m-\mathfrak{f}\|^2-(\theta-\eta)\|l_{m-1}-\mathfrak{f}\|^2-\eta\|l_{m-2}-\mathfrak{f}\|^2 \\
&\quad +(1+\theta)(\theta-\eta)\|l_m-l_{m-1}\|^2-\eta(\theta-\eta)\|l_{m-1}-l_{m-2}\|^2 \\
&\quad -(1+\eta-\theta)\|l_{m+1}-l_m\|^2-(\theta^2+\theta\eta-\theta)\|l_{m-1}-l_m\|^2 \\
&\quad -(\eta^2+\theta\eta+\eta)\|l_{m-1}-l_{m-2}\|^2. \qquad (3.15)
\end{aligned}
$$

Define $P_m := \|l_m-\mathfrak{f}\|^2-\theta\|l_{m-1}-\mathfrak{f}\|^2-\eta\|l_{m-2}-\mathfrak{f}\|^2+(1-\theta+\eta)\|l_m-l_{m-1}\|^2+D_1\|l_{m-1}-l_{m-2}\|^2$, where $D_1=1-3\theta+2\eta(1+\theta)$. Now, (3.15) can be written as

$$P_{m+1} \le P_m - D_2\|l_{m-1}-l_{m-2}\|^2, \qquad (3.16)$$

where $D_2 = 1 + \eta(3 + 4\theta) - 3\theta$.
Next, we prove that $D_1, D_2 \geq 0$ and $P_m \geq 0$. Since $D_2 = 1+\eta(3+4\theta)-3\theta > 0$ if $\eta > \dfrac{3\theta - 1}{3 + 4\theta}$. Therefore, by condition (ii), we have $D_2 > 0$. Also, by condition (i), $3\theta - 1 < 0$. This implies that

$$\frac{3\theta - 1}{2 + 2\theta} < \frac{3\theta - 1}{3 + 4\theta}. \tag{3.17}$$

By condition (ii) together with (3.17), we obtain $\eta > \dfrac{3\theta - 1}{2 + 2\theta}$, which is equivalently written as $1 - 3\theta + 2\eta(1 + \theta) > 0$. Consequently, it follows that $D_1 > 0$. We then proceed to consider

$$\begin{aligned} P_m \;&:=\; \|l_m - \mathfrak{f}\|^2 - \theta\|l_{m-1} - \mathfrak{f}\|^2 - \eta\|l_{m-2} - \mathfrak{f}\|^2 + (1 - \theta + \eta)\|l_m - l_{m-1}\|^2 \\ &\quad + D_1\|l_{m-1} - l_{m-2}\|^2 \\ &\geq\; \|l_m - \mathfrak{f}\|^2 - \theta\|l_{m-1} - \mathfrak{f}\|^2 + (1 - \theta + \eta)\|l_m - l_{m-1}\|^2. \end{aligned} \tag{3.18}$$

The last inequality follows as $D_1 > 0$ and $\eta < 0$. Also

$$-\theta\|l_{m-1} - \mathfrak{f}\|^2 \geq -2\theta\|l_m - l_{m-1}\|^2 - 2\theta\|l_m - \mathfrak{f}\|^2. \tag{3.19}$$

It follows from (3.18) and (3.19) that

$$P_m \geq (1 - 2\theta)\|l_m - \mathfrak{f}\|^2 + (1 - 3\theta + \eta)\|l_m - l_{m-1}\|^2. \tag{3.20}$$

Clearly $1-\theta > 0$ and by condition (ii) and (i), we have $1-3\theta+\eta > 0$, consequently $\eta > \dfrac{3\theta - 1}{2 + 2\theta} > 3\theta-1$. Therefore, we deduce that $P_m \geq 0$. As $D_2 \geq 0$, by (3.16), we have $0 \leq P_{m+1} \leq P_m$. Thus, $\{P_m\}$ is a monotonically non-increasing sequence that is bounded below. Thus, $\{P_m\}$ is a convergent sequence. By using (3.16), we have

$$P_{m+1} - P_m \leq -D_2\|l_{m-1} - l_{m-2}\|^2. \tag{3.21}$$

Taking $m \to \infty$ in (3.21), we have

$$\lim_{m\to\infty} \|l_{m+1} - l_m\| = 0. \tag{3.22}$$

Since $\lim_{m\to\infty} P_m$ exists and from (3.20) and (3.22), we conclude that

$$\lim_{m\to\infty} P_m \geq (1 - 2\theta) \lim_{m\to\infty} \|l_m - \mathfrak{f}\|^2. \tag{3.23}$$

This implies that $\{l_m\}$ is bounded. Let

$$\begin{aligned} b_m &= \|l_m - \mathfrak{f}\|^2 - \eta\|l_{m-2} - \mathfrak{f}\|^2 - \theta\|l_{m-1} - \mathfrak{f}\|^2. \\ c_m &= \langle l_{m-1} - l_m, l_m - \mathfrak{f}\rangle + \|l_m - l_{m-1}\|^2. \\ d_m &= \langle l_{m-2} - l_m, l_m - \mathfrak{f}\rangle + \|l_m - l_{m-2}\|^2. \end{aligned}$$

This implies that

$$P_m = b_m + (1 - \theta + \eta)\|l_m - l_{m-1}\|^2 + [1 - 3\theta + \eta(1 - \theta)]\|l_{m-1} - l_{m-2}\|^2. \tag{3.24}$$

Taking $m \to \infty$ in (3.24), and from $\lim_{m\to\infty} \|l_m - l_{m-1}\| = 0$, we have

$$\lim_{m\to\infty} P_m = \lim_{m\to\infty} b_m. \tag{3.25}$$

Therefore, $\lim_{m\to\infty} b_m$ exists. Note that $(1-\theta+\eta)\|l_m - \mathfrak{f}\|^2 = b_m + \theta c_m + \eta d_m$. This implies that $\lim_{m\to\infty} c_m = \lim_{m\to\infty} d_m = 0$ and $\lim_{m\to\infty} b_m$ exists, we have $\lim_{m\to\infty} \|l_m - \mathfrak{f}\|$ exists. Therefore, $\{l_m\}$ is bounded. Next, by the definition of w_m in Algorithm 1, we have

$$\|w_m - l_m\| \le \theta\|l_m - l_{m-1}\| + |\eta|\|l_{m-1} - l_{m-2}\|,$$

which implies

$$\lim_{m\to\infty} \|w_m - l_m\| = 0, \tag{3.26}$$

hence $\{w_m\}$ is bounded and thus $\{z_m\}$ and $\{y_m\}$ are bounded.

Step III. $\bar{u} \in \mathcal{F}(\mathcal{U}) \bigcap \mathcal{F}(\mathcal{W})$.

Since

$$\|w_m - l_{m+1}\| \le \|w_m - l_m\| + \|l_m - l_{m+1}\|, \tag{3.27}$$

it follows from (3.22), (3.26), and (3.27) that

$$\lim_{m\to\infty} \|w_m - l_{m+1}\| = 0. \tag{3.28}$$

From (3.7), (3.28), and $\alpha,\ \sigma \in (0,1)$, we have

$$\begin{aligned} \alpha\sigma(1-\sigma)\|\mathcal{W}z_m - \mathcal{U}z_m\|^2 &= \|w_m - \mathfrak{f}\|^2 - \|l_{m+1} - \mathfrak{f}\|^2 \\ &\le \|w_m - l_{m+1}\|(\|w_m - \mathfrak{f}\| + \|l_{m+1} - \mathfrak{f}\|) \\ &= \|w_m - l_{m+1}\| M_1, \end{aligned}$$

where $M_1 := \sup_m \{\|w_m - \mathfrak{f}\| + \|l_{m+1} - \mathfrak{f}\|\}$. Hence, it follow that

$$\lim_{k\to\infty} \|\mathcal{W}z_m - \mathcal{U}z_m\| = 0. \tag{3.29}$$

Since $\alpha,\ \sigma \in (0,1)$, $\lim_{n\to\infty} \left(1 - \mu^2 \frac{s_m^2}{s_{m+1}^2}\right) = 1 - \mu^2 > 0$, it follows from (3.3) and (3.6) that

$$\begin{aligned} \left(1 - \mu^2 \frac{s_m^2}{s_{m+1}^2}\right)\|y_m - w_m\|^2 &= \|w_m - \mathfrak{f}\|^2 - \|l_{m+1} - \mathfrak{f}\|^2 \\ &\le \|w_m - l_{m+1}\| M_1, \end{aligned}$$

and it follows from (3.28) that

$$\lim_{k\to\infty} \|y_m - w_m\| = 0. \tag{3.30}$$

Note that

$$\|z_m - y_m\| = \|y_m - s_m(\mathcal{A}y_m - \mathcal{A}w_m) - y_m\| \le s_m\|\mathcal{A}w_m - \mathcal{A}y_m\| \le s_0\mathcal{L}\|y_m - w_m\|, \tag{3.31}$$

it follows from (3.30) and (3.31) that

$$\lim_{m\to\infty} \|z_m - y_m\| = 0. \tag{3.32}$$

Since

$$\|w_m - z_m\| \leq \|w_m - y_m\| + \|y_m - z_m\|, \tag{3.33}$$

it follows from (3.30), (3.32), and (3.33) that

$$\lim_{m\to\infty} \|w_m - z_m\| = 0. \tag{3.34}$$

It follows from (3.28) and (3.34) that

$$\lim_{m\to\infty} \|w_m - l_{m+1} - \alpha(w_m - z_m)\| = 0. \tag{3.35}$$

Further, we have

$$\alpha\|\mathcal{U}z_m - z_m\| \leq \|l_{m+1} - w_m\| + \alpha\|w_m - z_m\| + \alpha\sigma_m\|\mathcal{U}z_m - \mathcal{W}z_m\|,$$

$$\|\mathcal{U}z_m - z_m\| \leq \frac{1}{\alpha}\|l_{m+1} - w_m\| + \|w_m - z_m\| + \sigma_m\|\mathcal{U}z_m - \mathcal{W}z_m\|. \tag{3.36}$$

Since $\alpha \in (0,1)$, it follows from (3.28), (3.29), (3.34), and (3.36) that

$$\lim_{m\to\infty} \|\mathcal{U}z_m - z_m\| = 0. \tag{3.37}$$

From (3.29) and (3.37), we have

$$\lim_{m\to\infty} \|\mathcal{W}z_m - z_m\| = 0. \tag{3.38}$$

Since $\{l_m\}$ is bounded, there exists a subsequence $\{l_{m_k}\}$ of $\{l_m\}$ with $l_{m_k} \rightharpoonup \bar{u}$. Further, it follows from (3.26), (3.30), and (3.32) that there exists a subsequence $\{w_{m_k}\}$ of $\{w_m\}$, $\{y_{m_k}\}$ of $\{y_m\}$ and $\{z_{m_k}\}$ of $\{z_m\}$ such that $w_{m_k}, y_{m_k}, z_{m_k} \rightharpoonup \bar{u}$. Applying Lemma 4(ii) together with (3.37) and (3.38) yields $\bar{u} \in \mathcal{F}(\mathcal{U})$ and $\bar{u} \in \mathcal{F}(\mathcal{W})$.

Step IV. Claim that $\bar{u} \in \mathrm{Sol}(\mathrm{VI}(1.4)) \bigcap \Phi$.

Let us first confirm that $\bar{u} \in \Phi$. By the construction in Algorithm 1,

$$\frac{1}{\alpha\sigma}\left(w_m - l_{m+1} - \alpha(w_m - z_m)\right) = (I - \mathcal{W})z_m + \left(\frac{1-\sigma}{\sigma}\right)(I - \mathcal{U})z_m.$$

Hence, given $z \in \mathcal{F}(\mathcal{U})$ and employing the monotonicity of $\mathcal{I} - \mathcal{W}$, it follows that

$$\begin{aligned} \left\langle \frac{1}{\alpha\sigma}\left(w_m - l_{m+1} - \alpha(w_m - z_m)\right), z_m - z \right\rangle &= \langle (I - \mathcal{W})z_m - (I - \mathcal{W})z, z_m - z\rangle \\ &\quad + \langle (I - \mathcal{W})z, z_m - z\rangle \\ &\quad + \frac{1-\sigma}{\sigma}\langle z_m - \mathcal{U}z_m, z_m - z\rangle \\ &\geq \langle (I - \mathcal{W})z, z_m - z\rangle \end{aligned}$$

$$+\quad \frac{1-\sigma}{\sigma}\langle z_m - \mathcal{U}z_m, z_m - z\rangle. \tag{3.39}$$

Since $\alpha,\ \sigma \in (0,1)$, it follows from (3.35), (3.37), and (3.39) that

$$\limsup_{m\to\infty}\langle z - \mathcal{W}z, z_m - z\rangle \le 0,\quad \forall z \in \mathcal{F}(\mathcal{U}). \tag{3.40}$$

As $z_m \rightharpoonup \bar{u}$, then

$$\langle (\mathcal{I} - \mathcal{W})q, \bar{u} - q\rangle \le 0,\quad \forall q \in \mathcal{F}(\mathcal{U}). \tag{3.41}$$

As $\eta q + (1-\eta)\bar{u} \in \mathcal{F}(\mathcal{U})$, $\eta \in (0,1)$ and due to the convexity of $\mathcal{F}(\mathcal{U})$, we get

$$\begin{aligned}\langle (\mathcal{I} - \mathcal{W})(\eta q + (1-\eta)\bar{u}), \bar{u} - (\eta q + (1-\eta)\bar{u})\rangle &= \lambda\langle (\mathcal{I} - \mathcal{W})(\eta q + (1-\eta)\bar{u}), \bar{u} - q\rangle \\ &\le 0,\quad \forall q \in \mathcal{F}(\mathcal{U}),\end{aligned} \tag{3.42}$$

which yields

$$\langle (\mathcal{I} - \mathcal{W})(\eta q + (1-\eta)\bar{u}), \bar{u} - q\rangle \le 0,\quad \forall q \in \mathcal{F}(\mathcal{U}).$$

Taking limits $\eta \to 0_+$, we get

$$\langle (\mathcal{I} - \mathcal{W})\bar{u}, \bar{u} - q\rangle \le 0,\quad \forall q \in \mathcal{F}(\mathcal{U}). \tag{3.43}$$

Hence, $\bar{u} \in \Phi$.

Next, establish that $\bar{u} \in \mathrm{Sol(VI(1.4))}$. Set

$$\mathcal{M}\mathfrak{f} = \begin{cases} \mathcal{A}\mathfrak{f} + \mathcal{N}_Q(\mathfrak{f}), & \text{if } \mathfrak{f} \in Q\,; \\ \emptyset, & \text{if } \mathfrak{f} \notin Q\,, \end{cases}$$

where $\mathcal{N}_Q(\mathfrak{f})$ denotes the normal cone to Q at $\mathfrak{f} \in Q$. Because $\mathcal{M}$ is maximal monotone, $0 \in \mathcal{M}\mathfrak{f}$ is equivalent to $\mathfrak{f} \in \mathrm{Sol(VI(1.4))}$. Take $(\mathfrak{f}, r) \in \mathrm{graph}(\mathcal{M})$. Then, $r \in \mathcal{M}\mathfrak{f} = \mathcal{A}\mathfrak{f} + \mathcal{N}_Q(\mathfrak{f})$, which implies $r - \mathcal{A}\mathfrak{f} \in \mathcal{N}_Q(\mathfrak{f})$. Consequently, $\langle \mathfrak{f} - v, r - \mathcal{A}\mathfrak{f}\rangle \ge 0$, for all $v \in Q$. Moreover, from $y_m = \mathcal{P}_Q(I - s_m\mathcal{A})w_m$ and $\mathfrak{f} \in Q$, we obtain

$$\langle w_m - s_m\mathcal{A}w_m - y_m, y_m - \mathfrak{f}\rangle \ \ge\ 0.$$

This implies that

$$\langle \mathfrak{f} - y_m, \frac{y_m - w_m}{s_m} + \mathcal{A}w_m\rangle \ \ge\ 0.$$

As $\langle \mathfrak{f} - v, r - \mathcal{A}\mathfrak{f}\rangle \ge 0$, for all $v \in Q$ and $y_{m_k} \in Q$, using monotonicity of $\mathcal{A}$, we have

$$\begin{aligned}\langle \mathfrak{f} - y_{m_k}, w\rangle &\ge \langle \mathfrak{f} - y_{m_k}, \mathcal{A}\mathfrak{f}\rangle \\ &\ge \langle \mathfrak{f} - y_{m_k}, \mathcal{A}\mathfrak{f}\rangle - \left\langle \mathfrak{f} - y_{m_k}, \frac{y_{m_k} - w_{m_k}}{s_m} + \mathcal{A}w_{m_k}\right\rangle \\ &= \langle \mathfrak{f} - y_{m_k}, \mathcal{A}\mathfrak{f} - \mathcal{A}y_{m_k}\rangle + \langle \mathfrak{f} - y_{m_k}, \mathcal{A}y_{m_k} - \mathcal{A}w_{m_k}\rangle - \left\langle \mathfrak{f} - y_{m_k}, \frac{y_{m_k} - w_{m_k}}{s_m}\right\rangle \\ &\ge \langle \mathfrak{f} - y_{m_k}, \mathcal{A}y_{m_k} - \mathcal{A}w_{m_k}\rangle - \left\langle \mathfrak{f} - y_{m_k}, \frac{y_{m_k} - w_{m_k}}{s_m}\right\rangle.\end{aligned}$$

Taking $k \to \infty$ and employing the continuity of $\mathcal{A}$, $\langle \mathfrak{f} - \bar{u}, r\rangle \ge 0$. The maximal monotonicity of $\mathcal{M}$ then implies $\bar{u} \in \mathcal{M}^{-1}0$, so $\bar{u} \in \mathrm{Sol(VI(1.4))}$ and thus $\bar{u} \in \Lambda$. Since $\lim\limits_{m\to\infty}\|w_m - \bar{u}\|$ exists and Lemma 4, $\{w_m\}$ converges weakly to $\bar{u} \in \Lambda$. □

From Theorem 1, we deduce:

With $\mathcal{W} = I$, Algorithm 1 reduces to a two-step inertial subgradient extragradient method of Krasnosel'skiĭ–Mann type for solving $\mathcal{F}(\mathcal{U})$ and VI (1.4).

Algorithm 2

Initialization: Take $s_0 > 0$, $\mu \in (0,1)$, $\theta \in (0,1)$, $\eta \in (-1,0]$, $\alpha \in (0,1)$, $\sigma \in [c,d]$, $c,d \in (0,1)$. Select an arbitrary starting point $l_{-1}, l_0, l_1 \in \mathcal{H}$:

Iterative steps: Given the current iterates l_{m-2}, l_{m-1}, l_m:

Step 1. Compute

$$\begin{aligned} w_m &= l_m + \theta(l_m - l_{m-1}) + \eta(l_{m-1} - l_{m-2}),\\ y_m &= \mathcal{P}_Q(w_m - s_m\mathcal{A}w_m). \end{aligned}$$

Step 2. Compute

$$\begin{aligned} z_m &= y_m - s_m(\mathcal{A}y_m - \mathcal{A}w_m),\\ l_{m+1} &= (1-\alpha)w_m + \alpha(\sigma z_m + (1-\sigma)\mathcal{U}z_m),\\ s_{m+1} &= \begin{cases} \min\left\{\dfrac{\mu\|w_m - y_m\|}{\|\mathcal{A}w_m - \mathcal{A}y_m\|}, s_m\right\} & \text{provided } \mathcal{A}w_m - \mathcal{A}y_m \neq 0,\\ s_m, & \text{otherwise.} \end{cases} \end{aligned}$$

Step 3. Update $m \leftarrow m+1$ and go back to Step 1.

Theorem 2. *Assume that $\mathcal{A}$ is monotone and $\mathcal{L}$-Lipschitz and $\mathcal{U}$ is nonexpansive. Suppose that* $\Lambda_1 = \mathrm{Sol(VI(1.4))} \bigcap \mathcal{F}(\mathcal{U}) \neq \emptyset$ *and that the following hold:*

(i) $\theta \in (0, \frac{1}{3})$;

(ii) $\dfrac{3\theta - 1}{3 + 4\theta} < \eta < 0$.

Then, the sequences $\{w_m\}$, $\{y_m\}$, $\{z_m\}$, and $\{l_m\}$ generated by Algorithm 2 converge weakly to $\bar{u} \in \Lambda_1$, where $\bar{u} = \mathcal{P}_{\Lambda_1} l_0$.

4. Krasnosel'skiĭ–Mann type one-step inertial Tseng's extragradient method

We now introduce a Krasnosel'skiĭ–Mann-type one-step inertial algorithm, using Tseng's extragradient framework, to address HFPP (1.1) and VI (1.4).

Algorithm 3

Initialization: Take $s_0 > 0,\ \mu \in (0,1),\ \{\theta_m\} \subset [0,\theta],\ \forall\ m,$ is nondecreasing with $\theta_1 = 0$ and $0 \le \theta_m < \theta < 1,\ \sigma_m \in [c,d],\ c,d \in (0,1)$ and $\{\alpha_m\}$ is a real sequence with conditions: $\delta > \dfrac{\theta^2(1+\theta)+\theta\sigma}{1-\theta^2}$ and $0 < \alpha \le \alpha_m \le \dfrac{\delta-\theta[\theta(1+\theta)+\theta\delta+\sigma]}{\delta[1+\theta(1+\theta)+\theta\delta+\sigma]}$, where $\alpha,\ \sigma,\ \delta > 0$. Select an arbitrary starting point $l_0, l_1 \in \mathcal{H}$:

Iterative steps: With the current iterates l_{m-1}, l_m:

Step 1. Compute

$$w_m = l_m + \theta_m(l_m - l_{m-1}),$$
$$y_m = \mathcal{P}_Q(w_m - s_m\mathcal{A}w_m).$$

Step 2. Compute

$$\begin{aligned}
&z_m = y_m - s_m(\mathcal{A}y_m - \mathcal{A}w_m),\\
&l_{m+1} = (1-\alpha_m)w_m + \alpha_m(\sigma_m\mathcal{W}z_m + (1-\sigma_m)\mathcal{U}z_m),\\
&s_{m+1} = \begin{cases}\min\left\{\frac{\mu\|w_m-y_m\|}{\|\mathcal{A}w_m-\mathcal{A}y_m\|}, s_m\right\} & \text{provided } \mathcal{A}w_m - \mathcal{A}y_m \ne 0,\\ s_m, & \text{otherwise.}\end{cases}
\end{aligned}$$

Step 3. Update $m \leftarrow m+1$ and go back to Step 1.

Theorem 3. *Assume that $\mathcal{A}$ is monotone and $\mathcal{L}$-Lipschitz, $\mathcal{U}$ is nonexpansive and $\mathcal{W}$ continuous and quasi-nonexpansive, and $I - \mathcal{W}$ monotone. Suppose that $\Lambda = \mathrm{Sol(VI(1.4))} \bigcap \Phi \bigcap \mathcal{F}(\mathcal{W}) \ne \emptyset$. Then, $\{w_m\}$, $\{y_m\}$, $\{z_m\}$, and $\{l_m\}$ generated by Algorithm* 3 *converge weakly to $\bar{u} \in \Lambda$, where $\bar{u} = \mathcal{P}_\Lambda l_0$.*

Proof. The arguments of Steps I–IV parallel those in Theorem 1, with exceptions that will be detailed below:

Step I. Take $\{z_m\}$ to be the iterates from Algorithm 3.

$$\begin{aligned}
\|z_m - \mathfrak{f}\|^2 &\le \|w_m - \mathfrak{f}\|^2 - \left(1 - \mu^2\frac{s_m^2}{s_{m+1}^2}\right)\|y_m - w_m\|^2. &(4.1)\\
&\le \|w_m - \mathfrak{f}\|^2. &(4.2)
\end{aligned}$$

The proof proceeds along the same lines as the proof of Step I in Theorem 1.

Step II. The iterates $\{w_m\}$, $\{y_m\}$, $\{z_m\}$, and $\{l_m\}$ from Algorithm 3 are bounded.

Let for any $\mathfrak{f} \in \Lambda$ and $T_{\sigma_m} := \sigma_m\mathcal{W} + (1-\sigma_m)\mathcal{U}$. It follows from Algorithm 3 and (4.2) that

$$\begin{aligned}
\|l_{m+1} - \mathfrak{f}\|^2 &\le (1-\alpha_m)\|w_m - \mathfrak{f}\|^2 + \alpha_m\|z_m - \mathfrak{f}\|^2 - \alpha_m\sigma_m(1-\sigma_m)\|\mathcal{W}z_m - \mathcal{U}z_m\|^2\\
&\quad -\alpha_m(1-\alpha_m)\|T_{\sigma_m}z_m - w_m\|^2 &(4.3)\\
&\le \|w_m - \mathfrak{f}\|^2 - \alpha_m\sigma_m(1-\sigma_m)\|\mathcal{W}z_m - \mathcal{U}z_m\|^2\\
&\quad -\alpha_m(1-\alpha_m)\|T_{\sigma_m}z_m - w_m\|^2 &(4.4)\\
&\le \|w_m - \mathfrak{f}\|^2 - \alpha_m(1-\alpha_m)\|T_{\sigma_m}z_m - w_m\|^2. &(4.5)
\end{aligned}$$

Next, we estimate

$$\|w_m - \mathfrak{f}\|^2 = \|l_m + \theta_m(l_m - l_{m-1}) - \mathfrak{f}\|^2$$

$$
\begin{aligned}
&= (1+\theta_m)\|l_m - \mathfrak{f}\|^2 - \theta_m\|l_{m-1} - \mathfrak{f}\|^2 \\
&\quad +\theta_m(1+\theta_m)\|l_m - l_{m-1}\|^2. \qquad (4.6)
\end{aligned}
$$

From (4.5) and (4.6), we have

$$
\begin{aligned}
\|l_{m+1} - \mathfrak{f}\|^2 - (1+\theta_m)\|l_m - \mathfrak{f}\|^2 + \theta_m\|l_{m-1} - \mathfrak{f}\|^2 &\le -\alpha_m(1-\alpha_m)\|T_{\sigma_m}z_m - w_m\|^2 \\
&\quad +\theta_m(1+\theta_m)\|l_m - l_{m-1}\|^2. \qquad (4.7)
\end{aligned}
$$

Further, from Algorithm 3, we have

$$
\begin{aligned}
\|T_{\sigma_m}z_m - w_m\|^2 &= \|\frac{1}{\alpha_m}(l_{m+1} - l_m) + \frac{\theta_m}{\alpha_m}(l_{m-1} - l_m)\|^2 \\
&\ge \frac{1}{\alpha_m^2}\|l_{m+1} - l_m\|^2 + \frac{\theta_m^2}{\alpha_m^2}\|l_m - l_{m-1}\|^2 \\
&\quad +\frac{\theta_m}{\alpha_m^2}(-\rho_m\|l_{m+1} - l_m\|^2 - \frac{1}{\rho_m}\|l_m - l_{m-1}\|^2), \qquad (4.8)
\end{aligned}
$$

where $\rho_m := \frac{1}{\theta_m + \delta\alpha_m}$. Thus, it follows from (4.7) and (4.8) that

$$
\begin{aligned}
\|l_{m+1} - \mathfrak{f}\|^2 - (1+\theta_m)\|l_m - \mathfrak{f}\|^2 + \theta_m\|l_{m-1} - \mathfrak{f}\|^2 &\le \frac{(1-\alpha_m)(\theta_m\rho_m - 1)}{\alpha_m}\|l_{m+1} - l_m\|^2 \\
&\quad +\gamma_m\|l_m - l_{m-1}\|^2, \qquad (4.9)
\end{aligned}
$$

where

$$
\gamma_m := \theta_m(1+\theta_m) + \theta_m(1-\alpha_m)\frac{(1-\theta_m\rho_m)}{\alpha_m\rho_m} > 0, \qquad (4.10)
$$

since $\theta_m\rho_m < 1$ and $\alpha_m \in (0,1)$. Using $\delta = \frac{(1-\theta_m\rho_m)}{\alpha_m\rho_m}$ together with (4.10) yields

$$
\gamma_m := \theta_m(1+\theta_m) + \theta_m(1-\alpha_m)\delta \le \theta(1+\theta) + \theta\delta, \ \forall\, m \ge 1. \qquad (4.11)
$$

Next, we define the sequences $\{\phi_m\}$ and $\{\psi_m\}$ by

$$
\phi_m := \|l_m - \mathfrak{f}\|^2, \ \ \psi_m := \phi_m - \theta_m\phi_{m-1} + \gamma_m\|l_m - l_{m-1}\|^2, \ \forall\, m \ge 1. \qquad (4.12)
$$

Using the monotonicity of $\{\theta_m\}$ together with the inequality $\phi_m \ge 0$ ($m \in \mathbb{N}$), we find

$$
\psi_{m+1} - \psi_m \le \phi_{m+1} - (1+\theta_m)\phi_m + \theta_m\phi_{m-1} + \gamma_{m+1}\|l_{m+1} - l_m\|^2 - \gamma_m\|l_m - l_{m-1}\|^2. \qquad (4.13)
$$

Hence, it follows from (4.9) and (4.13) that

$$
\begin{aligned}
\psi_{m+1} - \psi_m &\le \frac{(1-\alpha_m)(\theta_m\rho_m - 1)}{\alpha_m}\|l_{m+1} - l_m\|^2 + \gamma_{m+1}\|l_{m+1} - l_m\|^2 \\
&= \left(\frac{(1-\alpha_m)(\theta_m\rho_m - 1)}{\alpha_m} + \gamma_{m+1}\right)\|l_{m+1} - l_m\|^2. \qquad (4.14)
\end{aligned}
$$

Now, we note that

$$
\frac{(1-\alpha_m)(\theta_m\rho_m - 1)}{\alpha_m} + \gamma_{m+1} \le -\sigma, \ \forall m \ge 1, \qquad (4.15)
$$

see [6]. Using (4.14) and (4.15), we get

$$\psi_{m+1} - \psi_m \leq -\sigma \|l_{m+1} - l_m\|^2. \tag{4.16}$$

Using $\theta_1 = 0$, (4.12) implies $\psi_1 = \phi_1 \geq 0$ and hence (4.16) yields that $\{\psi_m\}$ is bounded. Further, (4.12) and the boundedness of $\{\theta_m\}$ yield

$$-\theta\phi_{m-1} \leq \phi_m - \theta\phi_{m-1} \leq \psi_m \leq \psi_1. \tag{4.17}$$

Hence, we get

$$\phi_m \leq \theta_m \phi_0 + \psi_1 \sum_{j=1}^{m-1} \theta_j \leq \theta_m \phi_0 + \frac{1}{1-\theta}\psi_1. \tag{4.18}$$

Using (4.16)–(4.18) and boundedness of $\{\psi_m\}$, we obtain

$$\sigma \sum_{j=1}^{m} \|l_{j+1} - l_j\|^2 \leq \psi_1 - \psi_{m+1} \leq \psi_1 + \theta\phi_m \leq \psi_1 + \theta_m \phi_0 + \frac{1}{1-\theta}\psi_1, \tag{4.19}$$

which implies that $\sum\limits_{m=1}^{\infty} \|l_{m+1} - l_m\|^2 < +\infty$. From $\theta_m \rho_m < 1$ together with (4.9), (4.11), $\sum\limits_{m=1}^{\infty} \|l_{m+1} - l_m\|^2 < +\infty$, and Lemma 3, we have

$$\lim_{m\to\infty} \|l_m - \mathfrak{f}\| \quad \text{exists and finite,} \tag{4.20}$$

hence $\{l_m\}$ remains bounded. From the summability condition $\sum\limits_{m=1}^{\infty} \|l_{m+1} - l_m\|^2 < +\infty$, we obtain

$$\lim_{m\to\infty} \|l_{m+1} - l_m\| = 0. \tag{4.21}$$

Next, by the definition of w_m in Algorithm 3 and $\theta_m \leq \theta,\ \forall m$, we have

$$\|w_m - l_m\| = \theta_m \|l_m - l_{m-1}\| \leq \theta \|l_m - l_{m-1}\|,$$

which implies that

$$\lim_{m\to\infty} \|w_m - l_m\| = 0. \tag{4.22}$$

Hence, $\{w_m\}$ is bounded, and so are $\{z_m\}$ and $\{y_m\}$.

Step III. $\bar{u} \in \mathcal{F}(\mathcal{U}) \bigcap \mathcal{F}(\mathcal{W})$.

Since

$$\|w_m - l_{m+1}\| \leq \|w_m - l_m\| + \|l_m - l_{m+1}\|, \tag{4.23}$$

it follows from (4.21)–(4.23) that

$$\lim_{m\to\infty} \|w_m - l_{m+1}\| = 0. \tag{4.24}$$

From (4.4), (4.24), and $\alpha_m \in (0,1),\ \sigma_m \in [c,d],\ c,d \in (0,1)$, we get

$$\alpha_m \sigma_m (1-\sigma_m) \|\mathcal{W} z_m - \mathcal{U} z_m\|^2 \quad = \quad \|w_m - q\|^2 - \|l_{m+1} - q\|^2$$

$$\begin{aligned} &\leq \|w_m - l_{m+1}\|(\|w_m - \mathfrak{f}\| + \|l_{m+1} - \mathfrak{f}\|) \\ &= \|w_m - l_{m+1}\| M_1, \end{aligned}$$

where $M_1 := \sup_m\{\|w_m - \mathfrak{f}\| + \|l_{m+1} - \mathfrak{f}\|\}$. Hence, it follow that

$$\lim_{m\to\infty} \|\mathcal{W}z_m - \mathcal{U}z_m\| = 0. \tag{4.25}$$

Since $\alpha_m \in (0,1)$, $\sigma_m \in [c,d]$, $c,d \in (0,1)$, $\lim_{n\to\infty}\left(1-\mu^2\frac{s_m^2}{s_{m+1}^2}\right) = 1-\mu^2 > 0$, it follows from (4.1) and (4.3) that

$$\begin{aligned} \left(1-\mu^2\frac{s_m^2}{s_{m+1}^2}\right)\|y_m - w_m\|^2 &= \|w_m - \mathfrak{f}\|^2 - \|l_{m+1} - \mathfrak{f}\|^2 \\ &\leq \|w_m - l_{m+1}\| M_1. \end{aligned}$$

By (4.24),

$$\lim_{m\to\infty} \|y_m - w_m\| = 0. \tag{4.26}$$

Note that

$$\begin{aligned} \|z_m - y_m\| &= \|y_m - s_m(Ay_m - Aw_m) - y_m\| \\ &\leq s_m\|Aw_m - Ay_m\| \leq s_0 L\|y_m - w_m\|. \end{aligned} \tag{4.27}$$

It follows from (4.26) and (4.27) that

$$\lim_{m\to\infty} \|z_m - y_m\| = 0. \tag{4.28}$$

Since

$$\|z_m - w_m\| \leq \|z_m - y_m\| + \|y_m - w_m\|, \tag{4.29}$$

it follows from (4.26), (4.28), and (4.29) that

$$\lim_{m\to\infty} \|z_m - w_m\| = 0. \tag{4.30}$$

It follows from (4.24) and (4.30) that

$$\lim_{m\to\infty} \|w_m - l_{m+1} - \alpha_m(w_m - z_m)\| = 0. \tag{4.31}$$

Further, we have

$$\alpha_m\|\mathcal{U}z_m - z_m\| \leq \|l_{m+1} - w_m\| + \alpha_m\|w_m - z_m\| + \alpha_m\sigma_m\|\mathcal{U}z_m - \mathcal{W}z_m\|,$$

$$\|\mathcal{U}z_m - z_m\| \leq \frac{1}{\alpha_m}\|l_{m+1} - w_m\| + \|w_m - z_m\| + \sigma_m\|\mathcal{U}z_m - \mathcal{W}z_m\|. \tag{4.32}$$

From the condition $\alpha_m > \alpha > 0$ for all m, together with (4.24), (4.25), (4.30), and (4.32), we obtain

$$\lim_{m\to\infty} \|\mathcal{U}z_m - z_m\| = 0. \tag{4.33}$$

From (3.29) and (4.33), we have

$$\lim_{m\to\infty} \|\mathcal{W} z_m - z_m\| = 0. \tag{4.34}$$

Take $\bar{u}$ to be a weak sequential limit point of $\{l_m\}$. Then there is a subsequence $\{l_{m_k}\}$ with $l_{m_k} \rightharpoonup \bar{u}$ in $\mathcal{H}$. Further, (4.22), (4.26), and (4.28) imply that $\{l_m\}$, $\{w_m\}$, $\{y_m\}$, and $\{z_m\}$ all have the same asymptotic behavior, and hence, there exist subsequences $\{w_{m_k}\}$ of $\{w_m\}$, $\{y_{m_k}\}$ of $\{y_m\}$ and $\{z_{m_k}\}$ of $\{z_m\}$ and such that $\{w_{m_k}\}$, $\{y_{m_k}\}$, and $\{z_{m_k}\}$ all converge weakly to $\bar{u}$. Since Q is closed and $\{w_{m_k}\}$ is in Q, then $\bar{u} \in Q$. Now, Lemma 2, (4.33), and (4.34) imply that $\bar{u} \in \mathcal{F}(\mathcal{U})$ and $\bar{u} \in \mathcal{F}(\mathcal{W})$.

Step IV. Claim that $\bar{u} \in \mathrm{Sol}(\mathrm{VI}(1.4)) \bigcap \Phi$.

Following a similar approach as in Theorem 1, we obtain the result.

□

5. Numerical observation

To support our result, we construct the following numerical example. We illustrate the convergence behavior of the proposed algorithm and compare it with the Nadezhkina et al. scheme [32] and Tseng's extragradient method [41] for distinct settings of parameters.

Example 1. *Consider $\mathcal{H} = Q = \mathbb{R}^m$, $m = 10$ with $\langle u, v\rangle = u^\top v$, and $\|u\| = \sqrt{\langle u, u\rangle}$. Define $\mathcal{A} : \mathbb{R}^{10} \to \mathbb{R}^{10}$ by $\mathcal{A}(u) = Mu$, where $M = \mathrm{diag}(1, 2, 3, \ldots, 10)$. As M is symmetric positive definite, $\mathcal{A}$ is monotone, Lipschitz continuous, and strongly monotone with Lipschitz constant $\mathcal{L} = \max\{1, 2, \ldots, 10\} = 10$. Consider $\mathcal{U}(u) = \beta u$, $\beta = 0.85$ and $\mathcal{W}(u) = \gamma u$, $\gamma = 0.75$. Choose the parameters as: $\sigma = 0.5$, $\mu = 0.9$, $s_0 = \frac{0.9}{\mathcal{L}}$ and for a distinct set of (θ, η, α) with $\theta \in \left(0, \frac{1}{3}\right)$, $\frac{3\theta-1}{3+4\theta} < \eta < 0$. Under these assumptions, we observe that the sequences generated by Algorithm 1 converges to* $\{0\}$.

We use MATLAB R2024a on an HP laptop with Intel Core i7 processor and 8GB RAM for the computation and graphical representation. Tables 1–3 show the convergence performance for a distinct set of double inertial parameters, convex combnation parameter (θ, η, α), and parameter sequences $\{\theta_m\} = \{\frac{4}{5}(1 - \frac{1}{m})\}$, $\{\alpha_m\} = \{0.4 + \frac{0.5}{m+1}\}$, and $\{\sigma_m\} = \{0.3 + \frac{0.6}{m+2}\}$, $\forall$ m. We demonstrate the convergence behavior for a different set of (θ, η, α) in Figures 1–6. From Tables 1–3 and Figures 1–6, we observe that the performance of our Algorithms 1 and 3 is better than that of Nadezhkina et al. [32], *and the Tseng's* [41].

Table 1. Convergence analysis at distinct (θ, η, α).

Algorithms	Iterations	Values	Iterations	Values
$(\theta, \eta, \alpha) = (0.02, -0.09, 0.95)$			$(\theta, \eta, \alpha) = (0.04, -0.08, 0.9)$	
Algorithms 1	72	10^{-8}	77	10^{-8}
Algorithms 3	72	10^{-6}	70	10^{-6}
Nadezhkina et al. [32]	75	0.0001	71	0.0001
Tseng's [41]	72	0.0670	72	0.1250

Table 2. Convergence analysis at distinct (θ, η, α).

Algorithms	Iterations	Values	Iterations	Values
$(\theta, \eta, \alpha) = (0.05, -0.07, 0.9)$			$(\theta, \eta, \alpha) = (0.06, -0.06, 0.85)$	
Algorithms 1	71	10^{-8}	87	10^{-8}
Algorithms 3	71	10^{-6}	83	10^{-7}
Nadezhkina et al. [32]	70	0.0002	80	10^{-5}
Tseng's [41]	80	0.0298	80	0.1058

Table 3. Convergence analysis at distinct (θ, η, α).

Algorithms	Iterations	Values	Iterations	Values
$(\theta, \eta, \alpha) = (0.07, -0.05, 0.85)$			$(\theta, \eta, \alpha) = (0.09, -0.04, 0.8)$	
Algorithms 1	89	10^{-8}	86	10^{-8}
Algorithms 3	86	10^{-7}	82	10^{-6}
Nadezhkina et al. [32]	97	10^{-5}	100	10^{-5}
Tseng's [41]	100	0.0447	100	0.0205

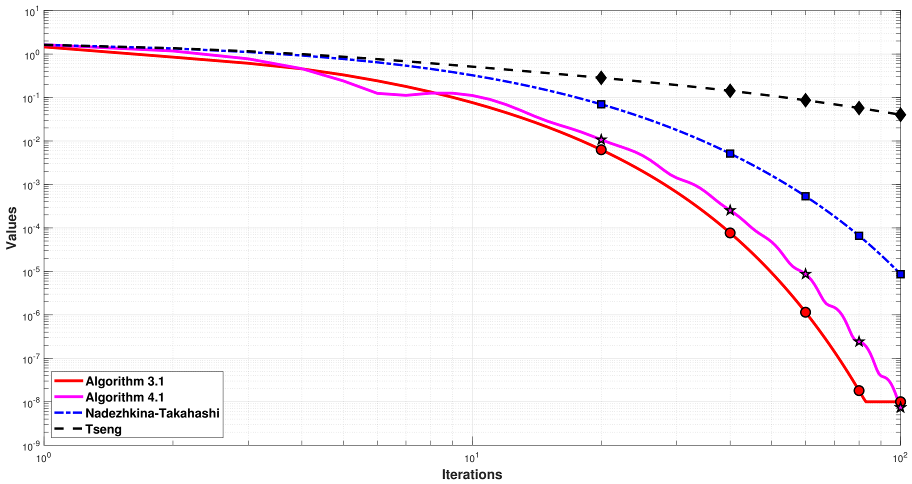

Figure 1. Convergence of sequences for $(\theta, \eta, \alpha) = (0.02, -0.09, 0.95)$.

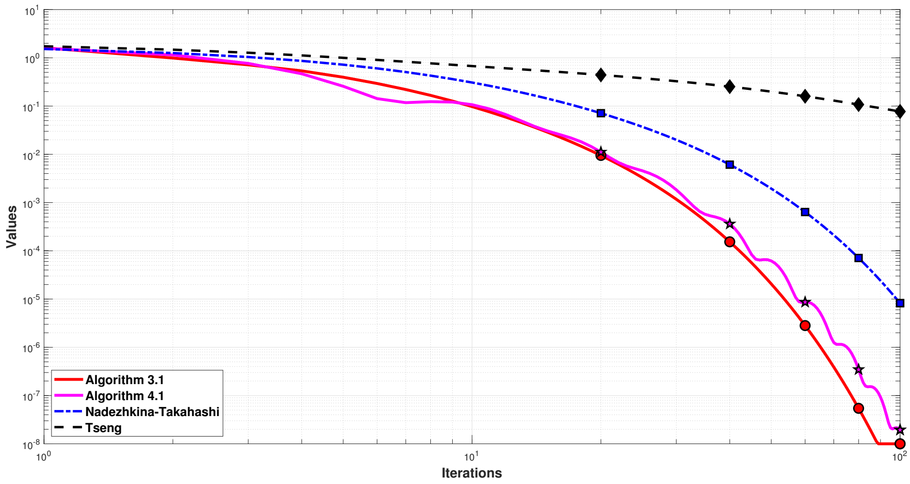

Figure 2. Convergence of sequences for $(\theta, \eta, \alpha) = (0.04, -0.08, 0.9)$.

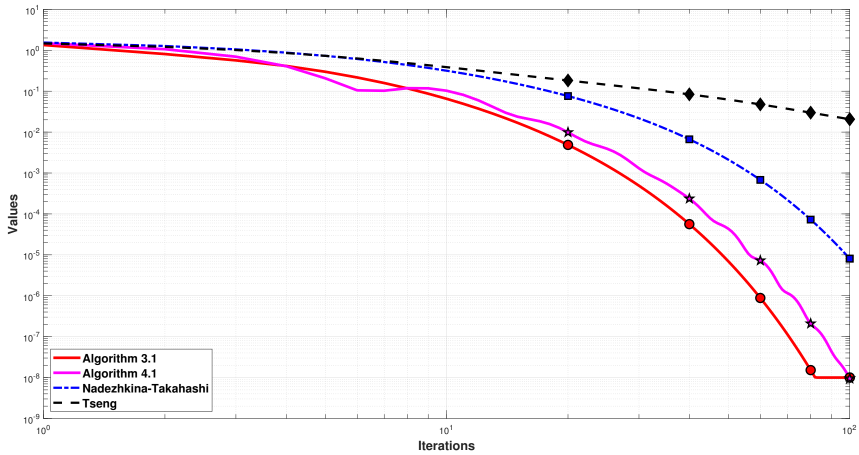

Figure 3. Convergence of sequences for $(\theta, \eta, \alpha) = (0.05, -0.07, 0.9)$.

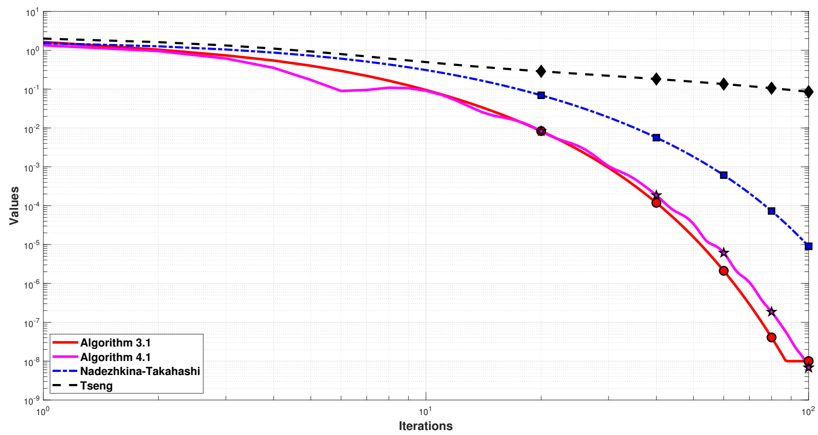

Figure 4. Convergence of sequences for $(\theta, \eta, \alpha) = (0.06, -0.06, 0.85)$.

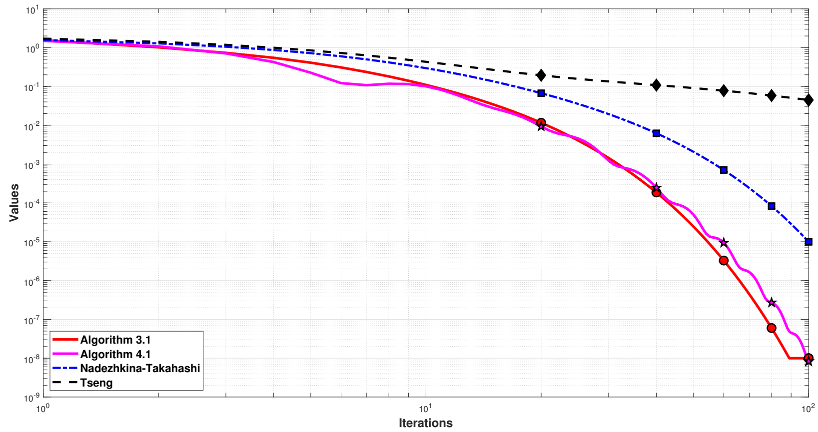

Figure 5. Convergence of sequences for $(\theta, \eta, \alpha) = (0.07, -0.05, 0.85)$.

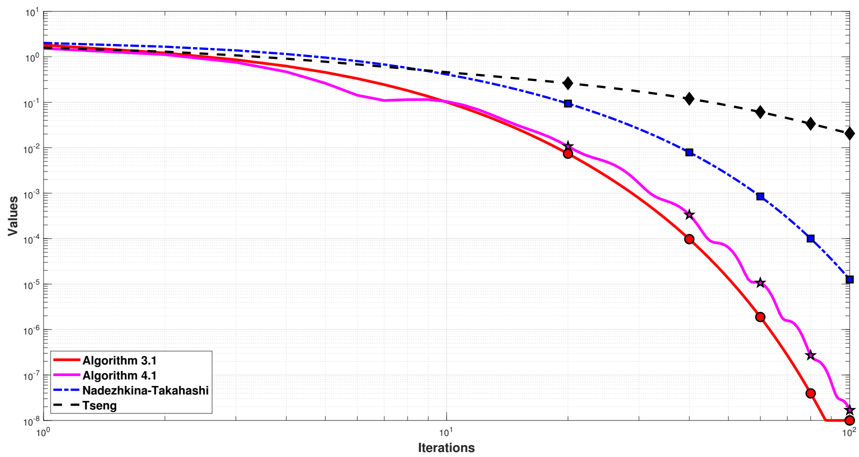

Figure 6. Convergence of sequences for $(\theta, \eta, \alpha) = (0.09, -0.04, 0.8)$.

6. Conclusions

We introduced new two-step and one-step inertial Tseng's extragradient algorithms of Krasnosel'skiĭ–Mann type for VIPs and HFPPs in Hilbert space. Under mild assumptions, it is established that the sequences generated by our methods converge weakly to a common solution, and the effectiveness of the proposed methods is illustrated through a numerical example. The obtained results unify and extend several existing approaches in the literature. By a suitable choice of parameters and operators, Algorithms 1 and 3 reduces to several known algorithms; some notable cases are presented below:

(i) Setting $\theta = \eta = 0$ in Algorithm 1 and $\theta_m = 0$, $\forall m$ in Algorithm 3, then Algorithm 1, respectively, Algorithm 3 reduces to Tseng's extragradient algorithm for solving HFPP(1.1) and VI(1.4).

(ii) Setting $\mathcal{W} = I$, $\mathcal{U} = I$, then Algorithm 1, respectively, Algorithm 3 are reduced to the extension of Algorithm 1.9, Algorithm 1.10, respectively, for solving VI(1.4).

(iii) Setting $\mathcal{W} = I$, $\sigma_m = 0$, $\mathcal{U} = J^B_{\lambda_m} := (I + \lambda_m B)^{-1}$ (where $B : \mathcal{H} \to 2^{\mathcal{H}}$ is maximal monotone and $\lambda_m \in (0, \infty)$), and $\alpha_m = \alpha\ \forall m$, Algorithm 3 takes the following form:

$$\begin{aligned} w_m &= l_m + \theta_m(l_m - l_{m-1}), \\ y_m &= \mathcal{P}_Q(w_m - s_m\mathcal{A}w_m). \end{aligned}$$

$$\begin{aligned} z_m &= y_m - s_m(\mathcal{A}y_m - \mathcal{A}w_m), \\ l_{m+1} &= (1-\alpha)w_m + \alpha J^B_{\lambda_m} z_m, \\ s_{m+1} &= \begin{cases} \min\left\{ \dfrac{\mu\|w_m - y_m\|}{\|\mathcal{A}w_m - \mathcal{A}y_m\|}, s_m \right\} & \text{if } \mathcal{A}w_m - \mathcal{A}y_m \neq 0, \\ s_m, & \text{otherwise,} \end{cases} \end{aligned}$$

which is an extension and an additional error tolerance strategy in [3].

Author contributions

Ghadah ALbeladi: Review and editing, methodology; Rehan Ali: Writing-original draft, conceptualization, methodology; Mohammad Farid: Review and editing, software, conceptualization. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The Researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University (www.qu.edu.sa) for financial support (QU-APC-2026).

Conflict of interest

The authors declare that they have no competing interests.

Funding

The APC was funded by the Deanship of Graduate Studies and Scientific Research at Qassim University (www.qu.edu.sa) for financial support (QU-APC-2026).

References

1. H. Abass, A. Adamu, O. Oyewole, M. Aphane, A golden ratio technique for equilibrium problem in reflexive Banach spaces, *Demonstr. Math.*, **58** (2025), 20250178. http://dx.doi.org/10.1515/dema-2025-0178

2. G. AlNemer, R. Ali, K. R. Kazmi, Inertial KM-type extragradient scheme for solving a variational inequality and a hierarchical fixed point problems, *J. Inequal. Appl.*, **2021** (2021), 38. http://dx.doi.org/10.1186/s13660-021-02565-3
3. F. Alvarez, On the minimizing property of a second order dissipative system in Hilbert spaces, *SIAM J. Control Optim.*, **38** (2000), 1102–1119. http://dx.doi.org/10.1137/S0363012998335802
4. F. Alvarez, H. Attouch, An inertial proximalmethod for maximal monotone operators via discretization of a nonlinear oscillator with damping, *Set-Valued Anal.*, **9** (2001), 3–11. http://dx.doi.org/10.1023/A:1011253113155
5. F. Alvarez, Weak convergence of a relaxed and an inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert space, *SIAM J. Optimiz.*, **14** (2004), 773–782. http://dx.doi.org/10.1137/S1052623403427859
6. R. I. Bot, E. R. Csetnek, C. Hendrich, Inertial Douglas-Rachford splitting for monotone inclusion problems, *Appl. Math. Comput.*, **256** (2015), 472–487. https://doi.org/10.1016/j.amc.2015.01.017
7. H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, Elsever, **5** (1973).
8. C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, *Inverse Probl.*, **20** (2004), 103–120. http://dx.doi.org/10.1088/0266-5611/20/1/006
9. A. Cabot, Proximal point algorithm controlled by a slowly vanishing term: Application to hierarchical minimization, *SIAM J. Optimiz.*, **15** (2005), 555–572. http://dx.doi.org/10.1137/S105262340343467X
10. L. C. Ceng, D. Ghosh, H. U. Rahaman, X. Zhao, Composite Tseng-type extragradient algorithms with adaptive inertial correction strategy for solving bilevel split pseudomonotone VIP under split common fixed-point constraint, *J. Comput. Appl. Math.*, **470** (2025), 116683. http://dx.doi.org/10.1016/j.cam.2025.116683
11. Y. Censor, A. Gibali, S. Reich, The subgradient extragradient method for solving variational inequalities in Hilbert space, *J. Optimiz. Theory App.*, **148** (2011), 318–335. http://dx.doi.org/10.1007/s10957-010-9757-3
12. Y. Censor, A. Gibali, S. Reich, Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space, *Optim. Method. Softw.*, **26** (2011), 827–845. https://doi.org/10.1080/10556788.2010.551536
13. Y. Censor, A. Gibali, S. Reich, Extension of Korpelevich's extragradient method for the variational inequality problem in Euclidean space, *Optimization*, **61** (2012), 1119–1132. https://doi.org/10.1080/02331934.2010.539689
14. P. L. Combettes, L. E. Glaudin, Quasi-nonexpansive iterations on the affine hull of orbits: From Mann's mean value algorithm to inertial methods, *SIAM J. Optim.*, **27** (2017), 2356–2380. http://dx.doi.org/10.1137/17M112806X
15. V. T. Dung, P. K. Anh, D. V. Thong, Convergence of two-step inertial tseng's extragradient methods for quasimonotone variational inequality problems, *Commun. Nonlinear Sci.*, **136** (2024), 108110. http://dx.doi.org/10.1016/j.cnsns.2024.108110
16. F. Facchinei, J. S. Pang, *Finite-dimensional variational inequalities and complementarity problems*, New York: Springer-Verlag, 2003.

17. M. Farid, The subgradient extragradient method for solving mixed equilibrium problems and fixed point problems in Hilbert spaces, *J. Appl. Numer. Optim.*, **1** (2019), 335–345. http://dx.doi.org/10.23952/jano.1.2019.3.10
18. M. Farid, P. Peeyada, R. Ali, W. Cholamjiak, Extragradient method with inertial iterative technique for pseudomonotone split equilibrium and fixed point, *J. Anal.*, **32** (2024), 1463–1485. http://dx.doi.org/10.1007/s41478-023-00695-z
19. G. Fichera, Sul problema elastostatico di signorini con ambigue condizioni al contorno, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat.*, **34** (1963), 138–142.
20. K. Goebel, W. A. Kirk, *Topics in metric fixed point theory, cambridge studies in advanced mathematics*, UK, Cambridge: Cambridge University Press, 1990.
21. K. Goebel, S. Reich, *Uniform convexity, hyperbolic geometry, and nonexpansive mappings*, New York: Marcel Dekker, 1984.
22. P. Hartman, G. Stampacchia, On some non-linear elliptic differential-functional equation, *Acta Math.*, **115** (1996), 271–310. http://dx.doi.org/10.1007/BF02392210
23. Y. Huang, L. You, G. Cai, Q. L. Dong, Subgradient extragradient algorithm with double inertial steps for solving variational inequality problems and fixed point problems in Hilbert spaces, *Rend. Circ. Mat. Palermo*, **74** (2025), 129. http://dx.doi.org/10.1007/s12215-025-01250-4
24. O. S. Iyiola, Y. Shehu, Convergence results of two-step inertial proximal point algorithm, *Appl. Numer. Math.*, **182** (2022), 57–75. https://doi.org/10.1016/j.apnum.2022.07.013
25. K. R. Kazmi, R. Ali, M. Furkan, Krasnoselski-Mann type iterative method for hierarchical fixed point problem and split mixed equilibrium problem, *Numer. Algorithms*, **77** (2018), 289–308. http://dx.doi.org/10.1007/s11075-017-0316-y
26. K. R. Kazmi, R. Ali, M. Furkan, Hybrid iterative method for split monotone variational inclusion problem and hierarchical fixed point problem for a finite family of nonexpansive mappings, *Numer. Algorithms*, **79** (2018), 499–527. http://dx.doi.org/10.1007/s11075-017-0448-0
27. G. M. Korpelevich, The extragradient method for finding saddle points and other problems, *Matecon*, **12** (1976), 747–756.
28. C. A. Micchelli, L. Shen, Y. Xu, Proximity algorithms for image models: Denoising, *Inverse Probl.*, **27** (2011), 045009. http://dx.doi.org/10.1088/0266-5611/27/4/045009
29. A. Moudafi, P. E. Mainge, Towards viscosity approximations of hierarchical fixed-point problems, *Fixed Point Theory A.*, **2006** (2006), 95453. http://dx.doi.org/10.1155/FPTA/2006/95453
30. A. Moudafi, Krasnoselski-Mann iteration for hierarchical fixed-point problems, *Inverse Probl.*, **23** (2007), 1635–1640. http://dx.doi.org/10.1088/0266-5611/23/4/015
31. A. Moudafi, P. E. Mainge, Strong convergence of an iterative method for hierarchical fixed-point problems, *Pac. J. Optim.*, **3** (2007), 529–538.
32. N. Nadezhkina, W. Takahashi, Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings, *J. Optimiz. Theory App.*, **128** (2006), 191–201. http://dx.doi.org/10.1007/s10957-005-7564-z
33. Z. Y. Peng, D. Li, Y. Zhao, R. L. Liang, An accelerated subgradient extragradient algorithm for solving bilevel variational inequality problems involving non-Lipschitz operator, *Commun. Nonlinear Sci.*, **127** (2023), 107549. http://dx.doi.org/10.1016/j.cnsns.2023.107549

34. H. U. Rahman, Z. Y. Peng, J. C. Yao, Approximate subgradient extragradient methods for solving variational inequality problems: Convergence analysis and applications in signal and image processing, *Commun. Nonlinear Sci.*, **152** (2026), 109211. http://dx.doi.org/10.1016/j.cnsns.2025.109211
35. Z. Y. Peng, L. O. Jolaoso, Y. Shehu, J. C. Yao, An extrapolated projection and contraction algorithm with past iterates, *Numer. Algorithms*, 2026. http://dx.doi.org/10.1007/s11075-026-02318-7
36. B. T. Polyak, Some methods of speeding up the convergence of iteration methods, *USSR Comput. Math. Math. Phys.*, **5** (1964), 1–17. https://doi.org/10.1016/0041-5553(64)90137-5
37. B. T. Polyak, *Introduction to optimization*, New York: Optimization Software, Publications Division, 1987.
38. C. Poon, J. Liang, Geometry of first-order methods and adaptive acceleration, *arXiv Preprint*, 2020. Available from: `https://arxiv.org/abs/2003.03910`.
39. C. Poon, J. Liang, Trajectory of alternating direction method of multipliers and adaptive acceleration, *arXiv Preprint*, 2019. Available from: `https://arxiv.org/abs/1906.10114`.
40. Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *B. Am. Math. Soc.*, **73** (1967), 591–597. https://doi.org/10.1090/S0002-9904-1967-11761-0
41. P. Tseng, A modified forward-backward splitting method for maximal monotone mappings, *SIAM J. Control Optim.*, **38** (2000), 431–446. http://dx.doi.org/10.1137/S0363012998338806
42. D. V. Thong, D. V. Hieu, Weak and strong convergence theorems for variational inequality problems, *Numer. Algorithms*, **78** (2018), 1045–1060. http://dx.doi.org/10.1007/s11075-017-0412-z
43. D. V. Thong, D. V. Hieu, Modified subgradient extragradient algorithms for variational inequality problems and fixed point problems, *Optimization*, **67** (2018), 83–102. http://dx.doi.org/10.1080/02331934.2017.1377199
44. D. V. Thong, D. V. Hieu, Modified Tseng's extragradient algorithm for variational inequality problems, *J. Fix. Point Theory A.*, **20** (2018), 152. http://dx.doi.org/10.1007/s11784-018-0634-2
45. W. Takahashi, M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, *J. Optimiz. Theory App.*, **118** (2003), 417–428. http://dx.doi.org/10.1023/A:1025407607560
46. I. Yamada, N. Ogura, Hybrid steepest descent method for the variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings, *Numer. Func. Anal. Opt.*, **25** (2004), 619–655. https://doi.org/10.1081/NFA-200045815
47. Q. Yang, J. Zhao, Generalized KM theorem and their applications, *Inverse Probl.*, **22** (2006), 833–844. http://dx.doi.org/10.1088/0266-5611/22/3/006
48. Y. Yao, Y. C. Liou, Weak and strong convergence of Krasnoselski-Mann iteration for hierarchical fixed-point problems, *Numer. Func. Anal. Opt.*, **24** (2008), 015015. http://dx.doi.org/10.1088/0266-5611/24/1/015015