Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(7): 19516–19544.
DOI: 10.3934/math.2026793
Received: 22 April 2026
Revised: 09 June 2026
Accepted: 24 June 2026
Published: 02 July 2026

Research article

Nonlinear Langevin-type Erdélyi-Kober and quantum-Caputo fractional operators with separated boundary conditions

Chaiyod Kamthorncharoen[1,*], **Sotiris K. Ntouyas**[2] **and Jessada Tariboon**[1]

[1] Intelligent and Nonlinear Dynamic Innovations Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand

[2] Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece

* **Correspondence:** Email: chaiyod.k@sci.kmutnb.ac.th.

Abstract: Fractional Langevin equations have emerged as a powerful tool for modeling systems subject to memory effects, anomalous damping, and non-uniform temporal dynamics. Motivated by the need to capture simultaneously weighted-memory effects - naturally described by the Erdélyi–Kober (EK) fractional operator - and discrete-scale quantum dynamics - described by the quantum-Caputo (q-Caputo) operator - we studied a nonlinear Langevin-type boundary value problem that couples these two structurally distinct fractional operators subject to separated boundary conditions. Our analysis proceeded in three stages. First, we derived an equivalent integral formulation of the boundary value problem in the quantum-EK setting, requiring new mixed EK–q–integral identities of independent interest. Second, we established existence and uniqueness in $C([0,T],\mathbb{R})$ via three fixed-point theorems: the Banach contraction principle (uniqueness under a global Lipschitz condition on f), the Krasnosel'skiĭ theorem (existence for broader nonlinearities via a sum-of-operators decomposition), and the Leray–Schauder nonlinear alternative (existence under polynomial growth without contraction). These approaches are complementary, progressively relaxing hypotheses on f while trading sharpness for generality. Third, we presented numerical experiments based on the spectral collocation method, together with analytical examples verifying that the theoretical conditions can be realized with explicit data.

Keywords: Erdélyi–Kober fractional operator; quantum-Caputo derivative; Langevin-type equation; separated boundary conditions; existence; uniqueness
Mathematics Subject Classification: 34A08, 34B15, 47H10, 26A33, 33D05

---

1. Introduction

Fractional calculus has undergone a remarkable renaissance over the past three decades, evolving from a purely theoretical subject into a powerful modeling framework for complex phenomena that exhibit memory, hereditary properties, and anomalous transport. Unlike their integer-order counterparts, fractional differential operators retain the entire history of a process, making them ideally suited for the description of viscoelastic materials [1], anomalous diffusion [2], control systems with non-integer damping [3], and biological models with long-range interactions [4]. The monographs of Podlubny [1], Kilbas, Srivastava and Trujillo [2], Hilfer [3], and Diethelm [5] remain standard references for the mathematical foundations and physical applications of this field, while the book of Tarasov [4] provides an extensive account of fractional dynamics in particles, fields, and media.

Among the many fractional operators that have been introduced in the literature, the Erdélyi–Kober (EK) operator occupies a distinguished position. Originally introduced independently by Erdélyi [6] and Kober [7] in the 1940s, it is characterized by a power-weight function and two free parameters - an order and a type parameter - which afford a significantly richer structure than the Riemann–Liouville or Caputo operators. Specifically, given parameters $\gamma > -1$ and $\beta > 0$, the EK fractional integral of order $\delta \in (0, 1)$ is defined via a weighted integral kernel, and the corresponding EK fractional derivative ${}^{E}D_{\beta}^{\gamma,\delta}$ is obtained through a suitable composition. This operator has proven invaluable in the analysis of processes with weighted memory and scaling properties, including applications in fluid mechanics, elasticity, and finance; we refer the reader to [2–4] and the references therein for detailed discussions.

A parallel and equally active line of research concerns quantum calculus — the study of q-analogues of classical operators obtained by replacing the usual derivative with a finite difference quotient on a geometric lattice with ratio $0 < q < 1$. The monographs of Kac and Cheung [8] and of Annaby and Mansour [9], together with the foundational work on basic hypergeometric series by Gasper and Rahman [10], provide the theoretical scaffolding for this subject. Of particular relevance to the present work is the quantum-Caputo (q-Caputo) fractional operator of order $\beta \in (0, 1)$, denoted by ${}^{C}D_{q}^{\alpha}$. This operator constitutes a discrete analogue of the classical Caputo derivative and is especially suited to models evolving on non-uniform time scales and quantum mechanical systems. Its advantages - such as the validity of a Leibniz-type rule and the automatic preservation of initial conditions - have spurred an increasing number of existence and uniqueness studies for q-fractional boundary value problems; see, e.g., [11–15] and the references cited therein.

Among the several available definitions of fractional derivatives, including the Riemann–Liouville, Hilfer, Caputo–Fabrizio, and Atangana–Baleanu operators, the Caputo-type derivative is particularly suitable for boundary value problems because it allows the formulation of initial and boundary conditions in terms of classical quantities [5]. In addition, the quantum-Caputo derivative preserves many of the analytical advantages of the classical Caputo operator - notably the vanishing of the derivative of a constant - while incorporating the discrete-scale structure inherent in q-calculus. This makes it a natural choice for the study of nonlinear Langevin-type equations on nonuniform time scales and facilitates the application of fixed-point techniques in the existence and uniqueness analysis.

A third fundamental ingredient of the present work is the Langevin equation. First introduced by Langevin in 1908 to describe Brownian motion, the equation of the form $(D+\lambda)x(t) = f(t, x(t))$ - where D is a differential operator and λ is a real parameter - captures the interplay between a deterministic restoring force and a fluctuating external influence [16,17]. Fractional Langevin equations, obtained by replacing D with a fractional derivative, have attracted substantial attention as models for anomalous diffusion, generalized viscoelastic fluids, and systems under random perturbations with memory; see [16, 17] and the survey [18]. Boundary value problems for fractional differential equations with nonlocal and separated conditions have been studied extensively in recent years; we refer the reader to [18–20] and the references therein for a representative sample. The work in [19] establishes existence results for nonlinear fractional BVPs under multi-point and integral boundary conditions using the Banach and Krasnosel'skiĭ fixed-point theorems, a methodology closely related to that employed in Section 3 of the present paper. [20] considers fractional equations involving non-standard operators and demonstrates how the choice of fractional derivative affects both the solution theory and the qualitative behavior of solutions. [18] investigates the solvability of nonlinear Langevin equations with two fractional orders under Dirichlet conditions, providing an important comparison point for the single-operator case. The present work extends this body of results by coupling the EK and q-Caputo operators in a unified Langevin-type framework. The study of existence, uniqueness, and qualitative properties of solutions to such equations - particularly under nonlocal and separated boundary conditions - represents one of the most active current directions in the theory of fractional differential equations.

From the modeling viewpoint, Langevin-type equations provide a fundamental mathematical framework for describing systems subject to damping, memory effects, and external fluctuations. Fractional Langevin equations have proved particularly effective in the study of anomalous diffusion, viscoelastic materials, stochastic transport processes, and complex dynamical systems exhibiting hereditary behavior. The incorporation of Erdélyi–Kober and quantum-Caputo operators further enriches the model by combining weighted-memory effects with discrete-scale dynamics, thereby allowing a more flexible description of physical processes that cannot be adequately represented by a single fractional operator.

The combination of different fractional operators within a single equation has recently attracted considerable attention, as it leads to models of greater generality and physical fidelity, e.g., [21, 22]. Problems coupling Riemann–Liouville with Caputo operators, or Hilfer-type operators with standard Caputo derivatives, have been studied, for instance, in [21–23]. However, to the best of our knowledge, the coupling of the Erdélyi-Kober operator with the quantum-Caputo operator within a Langevin-type framework has not previously appeared in the literature. This gap is the primary motivation for the present work. The simultaneous presence of both operators endows the model with a weighted-memory component (EK) and a discrete-scale component (q-Caputo), yielding a richer description of systems that exhibit both long-range temporal correlations and scale-discrete quantum effects.

More precisely, in this paper, we investigate the following nonlinear Langevin-type boundary value problem involving the Erdélyi–Kober fractional operator and the quantum-Caputo fractional operator with separated boundary conditions:

$$\begin{cases} {}^{E}D_{\beta}^{\gamma,\delta}\left({}^{C}D_{q}^{\alpha}+\lambda\right)x(t)=f(t,x(t)), \quad t\in[0,T], \\ a_1x(0)+a_2{}^{C}D_{q}^{\alpha-\beta(1+\gamma)}x(0)=a_3, \\ b_1x(T)+b_2\left({}^{C}D_{q}^{\alpha-\beta(1+\gamma)}x\right)(T)=b_3, \end{cases} \tag{1.1}$$

where ${}^{E}D_{\beta}^{\gamma,\delta}$ denotes the Erdélyi–Kober fractional operator of order $\delta\in(0,1)$ and ${}^{C}D_{q}^{\alpha}$ denotes the quantum-Caputo fractional operator of order $\alpha \in (0,1)$. The parameters of the EK operator satisfy $\gamma > -1$ and $\beta > 0$ with $\beta(1+\gamma) < \alpha$, the quantum parameter satisfies $0 < q < 1$, and $\lambda, a_i, b_i,\ i = 1,2,3$, are real constants. The nonlinearity $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is assumed to be continuous. The separated boundary conditions at $t = 0$ and $t = T$ involve a composite quantum-Caputo operator of order $\alpha - \beta(1+\gamma)$, reflecting both the initial and terminal behavior of the system.

Problem (1.1) subsumes and extends several models studied in the recent literature. When $q \to 1$, the quantum-Caputo operator ${}^{C}D_q^{\alpha}$ reduces to the classical Caputo operator ${}^{C}D^{\alpha}$, recovering Langevin-type equations with EK and Caputo operators. When the EK component is absent (i.e., $\delta \to 0$ or the EK part is an identity), the equation reduces to a purely q-fractional Langevin equation, variants of which appear in [11–15]. When both fractional components are taken as Caputo derivatives of the same order and $q \to 1$, the problem further reduces to standard Langevin-type BVPs. Therefore, the present framework is strictly more general than each of these special cases, and the results obtained here yield, as immediate corollaries, several theorems previously established in the literature.

The main contributions of this paper can be summarized as follows.

- We introduce a new class of nonlinear Langevin-type boundary value problems involving a coupled Erdélyi–Kober fractional operator and quantum-Caputo fractional operator under separated boundary conditions, and establish its equivalent integral formulation by combining EK fractional calculus with q-fractional calculus.
- We derive new mixed Erdélyi–Kober/q-fractional integral identities (Lemma 2.14) that do not appear to be available in the existing literature and are of independent interest.
- We prove existence and uniqueness results by employing three complementary fixed-point approaches: the Banach contraction principle (yielding uniqueness under a global Lipschitz condition), Krasnosel'skiĭ's fixed-point theorem (establishing existence for bounded nonlinearities), and the Leray–Schauder nonlinear alternative (providing existence under polynomial growth conditions).
- We show that the proposed framework strictly generalizes several previously studied models, including classical EK–Caputo Langevin equations (as $q \to 1$), purely q-fractional Langevin equations (when the EK component is absent), and standard Langevin-type boundary value problems; the main theorems of those works follow as immediate corollaries.
- We provide analytical examples illustrating each of the three fixed-point theorems and numerical experiments - based on the spectral collocation scheme - that demonstrate the influence of the parameters λ, q, γ, and β on the solution profile.

The interplay between these three approaches gives a layered picture of the solution theory: Banach's theorem provides the sharpest quantitative estimates and admits a natural iteration scheme; Krasnosel'skiĭ's theorem handles a broader class of nonlinearities at the expense of losing uniqueness; and the Leray–Schauder alternative offers the most general existence result under merely polynomial growth. Together, they cover the full spectrum of practically relevant cases. Finally, we illustrate our abstract results with concrete numerical examples that verify the imposed conditions and exhibit the qualitative features of the solutions. The analysis is carried out in the Banach space $C([0,T],\mathbb{R})$ of continuous real-valued functions on $[0,T]$, equipped with the supremum norm. The principal analytical challenge is the treatment of the composition ${}^{E}D_{\beta}^{\gamma,\delta} \circ ({}^{C}D_q^{\alpha} + \lambda)$, whose inversion requires the development of suitable q-EK integral identities that do not appear to be available in the existing literature. We derive these identities in Section 2 as part of the preliminary framework, which we believe to be of independent interest.

It is worth emphasizing that the methodology employed in the present work differs substantially from that used in recent studies on nonlinear dynamical systems, bifurcation phenomena, and controller design (see, e.g., [24,25]). While such works rely on dynamical-system techniques, bifurcation theory, stability analysis, and control-theoretic approaches, our analysis is based on nonlinear functional analysis and fixed-point theory. Moreover, a substantial part of our work is devoted to deriving an equivalent integral representation of the problem, a step that is necessary because of the nonstandard composition of the Erdélyi–Kober and quantum-Caputo operators and which is absent from the aforementioned studies.

The hybrid EK–q-Caputo Langevin framework studied here is particularly suited to physical systems in which two qualitatively different memory mechanisms coexist. The EK operator - characterized by the power-law kernel $s^{\beta\gamma}(t^{\beta} - s^{\beta})^{\delta-1}$ - is well-established as a model for anomalous diffusion in heterogeneous or fractal media, for creep and relaxation in viscoelastic materials, and for transport in systems with weighted long-range correlations. The q-Caputo operator, on the other hand, introduces a discrete geometric time scale $\{t_0 q^k\}$, which is natural in quantum-mechanical settings, lattice field theories, and models of fractal time. The Langevin coupling $({}^{\mathrm{E}}D_{\beta}^{\gamma,\delta})({}^{C}D_q^{\alpha} + \lambda)x = f(t, x)$ simultaneously encodes the power-law weighted memory (EK), the discrete-scale dynamics (q-Caputo), and a linear restoring force (λ), yielding a richer model than either component alone.

The remainder of this paper is organized as follows. Section 2 collects the necessary background on Erdélyi–Kober fractional calculus and quantum-Caputo calculus, derives the key mixed EK–q integral identities, and reformulates problem (1.1) as an equivalent fixed-point problem. Section 3 contains the main existence and uniqueness theorems, proved via the Banach, Krasnosel'skiĭ, and Leray–Schauder fixed-point theorems, respectively. Section 4 presents three concrete analytical examples, each illustrating one of the three theorems under explicitly verified conditions. Section 5 describes the numerical discretization of the integral equation and reports controlled experiments showing how the parameters λ, q, γ, and β individually affect the solution profile. Section 6 concludes with a brief discussion and directions for future research.

2. Preliminaries

In this section, we recall some basic definitions and fundamental results from q-fractional calculus and Erdélyi–Kober fractional calculus that will be used throughout the paper.

2.1. Elements of q-fractional calculus

Let $0 < q < 1$ be a fixed quantum parameter. The q-number is defined by

$$[n]_q = \frac{1 - q^n}{1 - q}.$$

It is well-known that $\lim_{q\to 1}[n]_q = n$. For example, $[4]_q = 1 + q + q^2 + q^3$, and hence $\lim_{q\to 1}[4]_q = 4$. The q-shifted power is defined by

$$(b - a)_q^n = (b - a)(b - qa)(b - q^2 a)\cdots(b - q^{n-1}a) = \prod_{i=0}^{n-1}(b - q^i a),$$

for $n \in \mathbb{Z}^+$, with $(b-a)_q^0 = 1$, where $a, b \in \mathbb{R}$ satisfy $b > a$. For example, $(5-2)_q^3$ can be expressed by $(5-2)(5-2q)(5-2q^2)$, and hence $\lim_{q\to 1}(5-2)_q^3 = (5-2)^3 = 27$. For $n \in \mathbb{R}$, the q-shifted power is defined by

$$(b-a)_q^n = \prod_{i=0}^{\infty} \frac{(b-q^i a)}{(b-q^{n+i}a)}.$$

For example, when $n = \frac{3}{2}$, we obtain

$$(b-a)_q^{\frac{3}{2}} = \frac{(b-a)(b-qa)(b-q^2a)(b-q^3a)\cdots}{(b-q^{\frac{3}{2}}a)(b-q^{\frac{5}{2}}a)(b-q^{\frac{7}{2}}a)(b-q^{\frac{9}{2}}a)\cdots}.$$

The basic concepts of q-calculus can be recalled via the definition of the q-derivative (or q-difference operator) of a function f on $[0, T]$, given by

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \qquad t \neq 0,$$

with $D_q f(0) = \lim_{t\to 0} D_q f(t)$, provided the limit exists. This definition represents a finite difference quotient, measuring the rate of change of f between the points t and qt. This can be viewed as a q-analogue of the classical derivative, where the increment is taken multiplicatively from t to qt. The q-integral of a function f can be interpreted as an infinite sum of the areas of rectangles with width $(1-q)tq^j$ and height $f(tq^j)$, for $j = 0, 1, 2, \ldots$. It is defined by

$$I_q f(t) = \sum_{j=0}^{\infty} (1-q)tq^j f(tq^j).$$

This expression represents a Riemann-type sum with geometrically spaced nodes tq^j. Before introducing fractional q-calculus, we first define the higher-order q-derivative and q-integral as the n-fold iterations of the operators D_q and I_q, respectively, namely,

$$D_q^n f(t) = \underbrace{D_q \cdots D_q}_{n \text{ times}} f(t), \quad I_q^n f(t) = \underbrace{I_q \cdots I_q}_{n \text{ times}} f(t).$$

We now turn to the fractional q-calculus, and the following definitions are based on [11–13].

Definition 2.1. *Let $\alpha \geq 0$ and let h be a function defined on $[0, T]$. The Riemann–Liouville-type fractional q-integral of order α is defined by $(I_q^0 h)(t) = h(t)$ and*

$$(I_q^\alpha h)(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)_q^{(\alpha-1)} h(s)\, d_q s, \qquad \alpha > 0,\ t \in [0, T], \tag{2.1}$$

where the q-gamma function is defined by

$$\Gamma_q(t) = \frac{\prod_{k=0}^{\infty}(1-q^{k+1})}{\prod_{k=0}^{\infty}(1-q^{t+k})}(1-q)^{1-t}, \qquad t \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}.$$

The q-gamma function satisfies $\Gamma_q(t+1) = [t]_q \Gamma_q(t)$ and $\lim_{q\to 1} \Gamma_q(t) = \Gamma(t)$.

Definition 2.2. *The Riemann–Liouville-type fractional q-derivative of order $\alpha \geq 0$ is defined by $(D_q^0 h)(t) = h(t)$ and*

$$(D_q^\alpha h)(t) = (D_q^\ell I_q^{\ell-\alpha} h)(t), \qquad \alpha > 0, \tag{2.2}$$

where $\ell = \lceil \alpha \rceil$ is the smallest integer greater than or equal to α.

Remark 2.3. *Using the definition of the Riemann–Liouville-type fractional q-derivative, one can express $(D_q^\alpha h)(t)$ in the explicit form*

$$(D_q^\alpha h)(t) = \frac{1}{\Gamma_q(\ell - \alpha)} D_q^\ell \int_0^t (t - qs)_q^{(\ell-\alpha-1)} h(s)\, d_q s,$$

where $\ell = \lceil \alpha \rceil$.

Definition 2.4. *Let $\alpha \geq 0$ and let h be a function defined on $[0, T]$ such that $D_q^k h$ exists for $k = 1, 2, \ldots, \ell$, where $\ell = \lceil \alpha \rceil$. The Caputo-type fractional q-derivative of order α is defined by*

$$({}^C D_q^\alpha h)(t) = (I_q^{\ell-\alpha} D_q^\ell h)(t), \qquad \alpha > 0. \tag{2.3}$$

Next, we present the fundamental theorem of fractional q-calculus, which enables us to transform the fractional q-differential equation into an equivalent integral equation.

Theorem 2.5. *[13] Let $\alpha > 0$, let $h \in C([0, T])$, and suppose that $D_q^k h$ exists for $k = 1, \ldots, \ell$. Then the following identities hold:*

(i) $(D_q^\alpha I_q^\alpha h)(t) = h(t), \quad t \in [0, T]$,

(ii) $(I_q^\alpha D_q^\alpha h)(t) = h(t) - \sum_{k=0}^{\ell-1} \frac{t^k}{\Gamma_q(k+1)} (D_q^k h)(0)$,

where $\ell = \lceil \alpha \rceil$ is the smallest integer greater than or equal to α.

2.2. Erdélyi–Kober fractional calculus

In our study of problem (1.1), the Erdélyi–Kober parameters are restricted to the values $\beta > 0$, $\gamma > -1$, and $\delta \in (0, 1)$, with $\beta(1+\gamma) < \alpha$, where $\alpha \in (0, 1)$ represents the order of the q-Caputo operator. However, for completeness, we present in this subsection the general definitions of the Erdélyi–Kober fractional calculus. To avoid confusion with the parameters used in (1.1), we adopt the notation (μ, ζ, ς) in place of (γ, δ, β) throughout this subsection (see Table 1).

Table 1. Notation and meaning of the Erdélyi–Kober parameters.

Local symbol	Symbol in (1.1)	Meaning
μ	γ	Type (weight) parameter; $\gamma > -1$
ζ	δ	Fractional order of the EK operator; $\delta \in (0, 1)$
ς	β	Scaling parameter; $\beta > 0$

The correspondence is $\mu \leftrightarrow \gamma$ (type parameter), $\zeta \leftrightarrow \delta$ (order), and $\varsigma \leftrightarrow \beta$ (scaling parameter). All results stated in terms of (μ, ζ, ς) in this subsection are applied in the rest of the paper with the substitutions $\mu = \gamma$, $\zeta = \delta$, $\varsigma = \beta$.

Definition 2.6. *[2, 26] Let $\zeta > 0$, $\varsigma > 0$, and $\mu \in \mathbb{R}$. The Erdélyi–Kober fractional integral of order ζ with parameters ς and μ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by*

$$({}^E I_\varsigma^{\mu,\zeta} f)(t) = \frac{\varsigma\, t^{-\varsigma(\zeta+\mu)}}{\Gamma(\zeta)} \int_0^t \frac{s^{\varsigma\mu+\varsigma-1} f(s)}{(t^\varsigma - s^\varsigma)^{1-\zeta}}\, ds.$$

If $\mu = 0$ and $\varsigma = 1$, then

$$({}^E I_1^{0,\zeta} f)(t) = t^{-\zeta} ({}^R\mathbb{I}^\zeta f)(t), \qquad t > 0,$$

where ${}^R\mathbb{I}^\zeta$ denotes the Riemann–Liouville fractional integral of order $\zeta > 0$, defined by

$$({}^R\mathbb{I}^\zeta f)(t) = \frac{1}{\Gamma(\zeta)} \int_0^t (t-s)^{\zeta-1} f(s)\, ds, \qquad t > 0.$$

Definition 2.7. *[2, 26] Let $n - 1 < \zeta \le n$, $n \in \mathbb{N}$, $\varsigma > 0$, and $\mu \in \mathbb{R}$. The Erdélyi–Kober fractional derivative of order ζ is defined by*

$$({}^E D_\varsigma^{\mu,\zeta} f)(t) = t^{-\varsigma\mu} \left(\frac{1}{\varsigma t^{\varsigma-1}} \frac{d}{dt} \right)^n t^{\varsigma(n+\mu)} ({}^E I_\varsigma^{\mu+\zeta, n-\zeta} f)(t), \qquad t > 0. \tag{2.4}$$

Remark 2.8. *Using the identity*

$$\prod_{j=1}^n \left(\mu + j + \frac{1}{\varsigma} t \frac{d}{dt} \right)(\cdot) = \left[t^{-\varsigma\mu} \left(\frac{1}{\varsigma t^{\varsigma-1}} \frac{d}{dt} \right)^n t^{\varsigma(n+\mu)} \right](\cdot),$$

the Erdélyi–Kober fractional derivative (2.4) *can be equivalently written as*

$$({}^E D_\varsigma^{\mu,\zeta} f)(t) = \prod_{j=1}^n \left(\mu + j + \frac{1}{\varsigma} t \frac{d}{dt} \right) ({}^E I_\varsigma^{\mu+\zeta, n-\zeta} f)(t), \tag{2.5}$$

see [27, 28].

Remark 2.9. *If $\mu = 0$ and $\varsigma = 1$, then*

$$({}^E D_1^{0,\zeta} f)(t) = ({}^R\mathbb{D}^\zeta [\, t^\zeta f(t)\,])(t),$$

where ${}^R\mathbb{D}^\zeta$ denotes the Riemann–Liouville fractional derivative of order $\zeta > 0$, defined by

$$({}^R\mathbb{D}^\zeta f)(t) = \left(\frac{d}{dt} \right)^n ({}^R\mathbb{I}^{n-\zeta} f)(t), \qquad n - 1 < \zeta \le n, \quad n \in \mathbb{N}.$$

Hence, in this case, the Erdélyi–Kober fractional derivative reduces to the Riemann–Liouville fractional derivative of the weighted function $t^\zeta f(t)$.

Remark 2.10. ***(Physical meaning of the parameters).***

- ***Order*** $\delta \in (0, 1)$ ***(EK fractional order).*** *The parameter δ controls the* strength *of the weighted memory encoded by the EK operator. When $\delta \to 0^+$, the EK integral degenerates to a point evaluation (no memory), while $\delta \to 1^-$ recovers a weighted running average over the full history* $[0, t]$. *In anomalous diffusion, δ is related to the anomalous exponent of the mean-squared displacement.*

- ***Scaling parameter*** $\beta > 0$ ***(EK).*** *The parameter* β *determines the* geometric dilation *of the kernel: it transforms the time variable as* $t \mapsto t^{\beta}$*, compressing or expanding the effective memory window. A larger* β *concentrates the kernel support near the current time* t *(shorter effective memory), as confirmed numerically in Section 5.6. In viscoelastic modeling,* β *characterizes the time-scale separation between the microscopic and macroscopic relaxation processes.*
- ***Type parameter*** $\gamma > -1$ ***(EK).*** *The parameter* γ *weights the contribution of early-time history via the factor* $s^{\beta\gamma}$ *in the EK kernel. Negative values of* γ *amplify the influence of the source function near the origin (long-range past), while positive values suppress it, as illustrated in Figure 4 of Section 5.7. In fluid mechanics,* γ *is related to the spatial exponent in the velocity profile of power-law fluids.*
- ***Quantum parameter*** $q \in (0,1)$***.*** *The parameter* q *determines the* ratio *of the geometric lattice* $\{t_0, t_0q, t_0q^2, \ldots\}$ *on which the q-Caputo operator acts. As* $q \to 1^-$*, the lattice becomes dense and the q-Caputo operator converges to the classical Caputo derivative (continuous-time limit). Smaller values of* q *correspond to a more widely spaced lattice, giving the operator a stronger non-local, discrete-memory character, as illustrated by the smoothing effect observed in Figure 2 of Section 5.5.*

Next, we present the fundamental theorem of the Erdélyi–Kober fractional calculus, which will be used to transform the considered problem into an equivalent integral equation. Let $C_{\epsilon}^{m} = \{f : (0,\infty) \to \mathbb{R} \mid f(x) = x^p f_1(x),\ p > \epsilon,\ f_1 \in C^m([0,\infty))\}$.

Theorem 2.11. *[27] Let* $n-1 < \zeta < n$*,* $n \in \mathbb{N}$*,* $\epsilon > -\varsigma(\mu+1)$*, and* $v \in C_{\epsilon}^{n}[0,T]$*. Then, for* $t > 0$*, the following identity holds:*

$$({}^{E}I_{\varsigma}^{\mu,\zeta}\ {}^{E}D_{\varsigma}^{\mu,\zeta} v)(t) = v(t) - \sum_{i=0}^{n-1} c_i\, t^{-\varsigma(1+\mu+i)}, \tag{2.6}$$

where

$$c_i = \frac{\Gamma(n-i)}{\Gamma(\zeta-i)} \lim_{t\to 0} t^{\varsigma(1+\mu+i)} \prod_{j=i+1}^{n-1} \left(\mu + j + \frac{1}{\varsigma} t \frac{d}{dt}\right) ({}^{E}I_{\varsigma}^{\mu+\zeta, n-\zeta} v)(t), \quad i = 0,1,\ldots,n-1. \tag{2.7}$$

Remark 2.12. *The space* C_{ϵ}^{n} *is required in Theorem 2.11 to guarantee the finiteness of the remainder coefficients* c_i *in (2.6)-(2.7). In the present paper, all functions to which Theorem 2.11 is applied belong to* $C([0,T],\mathbb{R})$*; since* $\gamma > -1$ *is assumed throughout, one has* $-\beta(\gamma+1) < 0$*, so the condition* $\epsilon > -\beta(\gamma+1)$ *is satisfied by taking* $\epsilon \in (-\beta(\gamma+1), 0)$*, and every continuous function on* $[0,T]$ *lies in* $C_{\epsilon}^{1}[0,T]$ *for such* ϵ*. The space* C_{ϵ}^{n} *is therefore not an additional restriction in our setting and is mentioned only for completeness and reference consistency.*

Lemma 2.13. *[17] Let* $\zeta > 0$*,* $\varsigma > 0$*,* $y > 0$*, and* $\mu > -\frac{y+\varsigma}{\varsigma}$*. Then, for* $t > 0$*, we have*

$$({}^{E}I_{\varsigma}^{\mu,\zeta} t^{y})(t) = \frac{\Gamma\left(\mu + \frac{y}{\varsigma} + 1\right)}{\Gamma\left(\mu + \frac{y}{\varsigma} + \zeta + 1\right)} t^{y}. \tag{2.8}$$

2.3. Equivalent integral formulation of the boundary value problem

In this subsection, we transform the boundary value problem (1.1) into an equivalent integral equation by using the fundamental results of fractional q-calculus and Erdélyi–Kober operators.

Lemma 2.14. *Let $0 < \alpha, \delta < 1$ be the orders of the q-Caputo and Erdélyi–Kober fractional differential operators, respectively. Let $\beta > 0$ and $\gamma > -1$ be the Erdélyi–Kober parameters such that $\alpha > \beta(1+\gamma)$. Let $\lambda \in \mathbb{R}$, $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$, and $a_i, b_i \in \mathbb{R}$, $i = 1,2,3$. Then the boundary value problem* (1.1) *is equivalent to the integral equation*

$$x(t) = \frac{1}{a_1}\left[a_3 - a_2\psi_{x,f}(T)\right] - \lambda I_q^{\alpha} x(t) + I_q^{\alpha}({}^{E}I_{\beta}^{\gamma,\delta} f_x)(t) + \frac{t^{\alpha-\beta(1+\gamma)}}{\Gamma_q(\alpha - \beta(1+\gamma) + 1)}\psi_{x,f}(T), \tag{2.9}$$

where $f_x(t) = f(t,x)$ and

$$\begin{aligned} \psi_{x,f}(T) \;=\; & \frac{1}{\Lambda}\Big[a_1 b_3 + a_1\lambda(b_1 I_q^{\alpha} x(T) + b_2 I_q^{\beta(1+\gamma)} x(T)) - a_3 b_1 \\ & - a_1 b_1 I_q^{\alpha}({}^{E}I_{\beta}^{\gamma,\delta} f_x)(T) - a_1 b_2 I_q^{\beta(1+\gamma)}({}^{E}I_{\beta}^{\gamma,\delta} f_x)(T)\Big], \end{aligned} \tag{2.10}$$

with

$$\Lambda = a_1 b_1 \frac{T^{\alpha-\beta(1+\gamma)}}{\Gamma_q(\alpha - \beta(1+\gamma) + 1)} + a_1 b_2 - a_2 b_1,$$

provided that $a_1 \neq 0$ and $\Lambda \neq 0$.

Proof. First, we transform the boundary value problem (1.1) into an equivalent integral formulation by applying the fundamental theorems of the Erdélyi–Kober and q-Caputo fractional operators. By Theorem 2.11, the composition identity

$${}^{E}I_{\beta}^{\gamma,\delta} \circ {}^{E}D_{\beta}^{\gamma,\delta}\, v = v - c_0\, t^{-\beta(1+\gamma)}$$

holds for any function v in the appropriate class, where $c_0 = \frac{\Gamma(1)}{\Gamma(\delta)} \lim_{t\to 0} t^{\beta(1+\gamma)}\, {}^{E}I_{\beta}^{\gamma+\delta,\,1-\delta} v(t)$. Applying this identity with $v = ({}^{C}D_q^{\alpha} + \lambda)x$ and using the linearity of ${}^{E}I_{\beta}^{\gamma,\delta}$, the left-hand side of (1.1) becomes

$${}^{C}D_q^{\alpha} x(t) = c_0 t^{-\beta(1+\gamma)} - \lambda x(t) + {}^{E}I_{\beta}^{\gamma,\delta} f_x(t). \tag{2.11}$$

Next, applying the q-fractional integral operator of order α, namely, I_q^{α}, to both sides of (2.11), and using the fundamental theorem of fractional q-calculus (Theorem 2.5), we obtain

$$x(t) = c_1 - \lambda I_q^{\alpha} x(t) + I_q^{\alpha}({}^{E}I_{\beta}^{\gamma,\delta} f_x)(t) + c_0\left(\frac{\Gamma_q(1 - \beta(1+\gamma))}{\Gamma_q(\alpha - \beta(1+\gamma) + 1)}\right) t^{\alpha-\beta(1+\gamma)}, \tag{2.12}$$

where the last term uses the q-power-function formula $(I_q^{\alpha}\, t^{\mu})(t) = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\mu+\alpha+1)}\, t^{\mu+\alpha}$ with $\mu = -\beta(1+\gamma)$.

Since $\alpha \in (0,1)$, we have $\lceil \alpha \rceil = 1$, so the fundamental theorem of fractional q-calculus (Theorem 2.5(ii)) gives $(I_q^{\alpha} D_q^{\alpha} h)(t) = h(t) - h(0)$. Applying I_q^{α} to (2.11) and using $(I_q^{\alpha} g)(0) = 0$ for any continuous g, the constant of integration $c_1 = x(0)$ is identified as the free parameter to be determined from the boundary conditions.

To incorporate the boundary conditions, we apply the q-Caputo fractional derivative of order $\alpha - \beta(1+\gamma)$ to both sides of (2.12). Using the fundamental theorem of fractional q-calculus and the fact that the Caputo fractional q-derivative of a constant is zero, we obtain

$$
\begin{aligned}
{}^{C}D_q^{\alpha-\beta(1+\gamma)}x(t) &= {}^{C}D_q^{\alpha-\beta(1+\gamma)}c_1 - \lambda {}^{C}D_q^{\alpha-\beta(1+\gamma)}(I_q^{\alpha}x)(t) + {}^{C}D_q^{\alpha-\beta(1+\gamma)}(I_q^{\alpha}({}^{E}I_{\beta}^{\gamma,\delta}f_x))(t) \\
&\quad + c_0\left(\frac{\Gamma_q(1-\beta(1+\gamma))}{\Gamma_q(\alpha-\beta(1+\gamma)+1)}\right){}^{C}D_q^{\alpha-\beta(1+\gamma)}\left(t^{\alpha-\beta(1+\gamma)}\right) \\
&= -\lambda I_q^{\beta(1+\gamma)}x(t) + I_q^{\beta(1+\gamma)}({}^{E}I_{\beta}^{\gamma,\delta}f_x)(t) + c_0\,\Gamma_q(1-\beta(1+\gamma)). \qquad (2.13)
\end{aligned}
$$

By combining the separated boundary conditions in (1.1) with (2.12) and (2.13), and using the fact that $(I_q^{\alpha}g)(0) = 0$ for any continuous function g on $[0, T]$, we obtain a system of two equations for the unknown constants c_0 and c_1 as

$$
\begin{aligned}
a_2\Gamma_q(1-\beta(1+\gamma))\,c_0 + a_1c_1 &= a_3, \\
\left(b_2\Gamma_q(1-\beta(1+\gamma)) \;+\; b_1\left(\frac{\Gamma_q(1-\beta(1+\gamma))}{\Gamma_q(\alpha-\beta(1+\gamma)+1)}\right)T^{\alpha-\beta(1+\gamma)}\right)c_0 + b_1c_1 &= B_3,
\end{aligned}
$$

where

$$
B_3 \;=\; b_3 + \lambda\left(b_1\left(I_q^{\alpha}x\right)(T) + b_2\left(I_q^{\beta(1+\gamma)}x\right)(T)\right) - b_1 I_q^{\alpha}\left({}^{E}I_{\beta}^{\gamma,\delta}f_x\right)(T) - \;b_2\left(I_q^{\beta(1+\gamma)}\left({}^{E}I_{\beta}^{\gamma,\delta}f_x\right)\right)(T).
$$

Solving this system for c_0 and c_1, we obtain

$$
c_0 = \frac{\psi_{x,f}(T)}{\Gamma_q(1-\beta(1+\gamma))}, \qquad c_1 = \frac{1}{a_1}(a_3 - a_2\psi_{x,f}(T)),
$$

where $\psi_{x,f}(T)$ is defined in (2.10). Substituting these expressions for c_0 and c_1 into (2.12), we obtain the equivalent integral equation (2.9).

Conversely, assume that x satisfies the integral equation (2.9). We show that x satisfies the boundary value problem (1.1), which establishes the equivalence between (1.1) and (2.9). Applying the q-Caputo fractional derivative of order $\alpha - \beta(1+\gamma)$ to both sides of (2.9), and using Theorem 2.5 together with the fact that the q-Caputo fractional derivative of a constant is zero, we obtain

$$
{}^{C}D_q^{\alpha-\beta(1+\gamma)}x(t) \;=\; -\lambda I_q^{\beta(1+\gamma)}x(t) + I_q^{\beta(1+\gamma)}({}^{E}I_{\beta}^{\gamma,\delta}f_x)(t) + \;\Gamma_q(1-\beta(1+\gamma))\,\frac{\psi_{x,f}(T)}{\Gamma_q(1-\beta(1+\gamma))},
$$

that is,

$$
{}^{C}D_q^{\alpha-\beta(1+\gamma)}x(t) = -\lambda I_q^{\beta(1+\gamma)}x(t) + I_q^{\beta(1+\gamma)}({}^{E}I_{\beta}^{\gamma,\delta}f_x)(t) + \psi_{x,f}(T). \qquad (2.14)
$$

Next, applying the q-Caputo fractional derivative of order $\beta(1+\gamma)$ to (2.14), or equivalently, the q-Caputo fractional derivative of order α to (2.9), we get

$$
{}^{C}D_q^{\alpha}x(t) = -\lambda x(t) + {}^{E}I_{\beta}^{\gamma,\delta}f_x(t) + \frac{\psi_{x,f}(T)}{\Gamma_q(1-\beta(1+\gamma))}t^{-\beta(1+\gamma)}. \qquad (2.15)
$$

Rewriting (2.15), we obtain

$$
{}^{C}D_q^{\alpha}x(t) + \lambda x(t) = {}^{E}I_{\beta}^{\gamma,\delta}f_x(t) + c_0\,t^{-\beta(1+\gamma)}.
$$

Applying the Erdélyi–Kober fractional derivative ${}^{E}D^{\gamma,\delta,\beta}$ to both sides of (2.15) and using the *left-inverse* identity ${}^{E}D^{\gamma,\delta,\beta}\left({}^{E}I^{\gamma,\delta,\beta}g\right) = g$ (which follows directly from Definitions 2.6-2.7; see also [2], Section 18.2) together with the fact that ${}^{E}D^{\gamma,\delta,\beta}[t^{-\beta(1+\gamma)}] = 0$ (since $t^{-\beta(1+\gamma)}$ belongs to the kernel of ${}^{E}D^{\gamma,\delta,\beta}$), we obtain

$$ {}^{E}D^{\gamma,\delta,\beta}({}^{C}D_q\alpha + \lambda)x(t) = f(t, x(t)), $$

which is the differential equation in (1.1).

It remains to verify the boundary conditions. Setting $t = 0$ in (2.9), and using $I_q^{\alpha}g(0) = 0$ for continuous g, we obtain

$$ x(0) = \frac{1}{a_1}(a_3 - a_2\psi_{x,f}(T)). $$

Also, evaluating (2.14) at $t = 0$, and using $I_q^{\beta(1+\gamma)}g(0) = 0$, we get

$$ {}^{C}D_q^{\alpha-\beta(1+\gamma)}x(0) = \psi_{x,f}(T). $$

Therefore,

$$ a_1x(0) + a_2{}^{C}D_q^{\alpha-\beta(1+\gamma)}x(0) = a_3 - a_2\psi_{x,f}(T) + a_2\psi_{x,f}(T) = a_3. $$

Similarly, evaluating (2.9) and (2.14) at $t = T$, we have

$$ x(T) = \frac{1}{a_1}(a_3 - a_2\psi_{x,f}(T)) - \lambda I_q^{\alpha}x(T) + I_q^{\alpha}({}^{E}I_{\beta}^{\gamma,\delta}f_x)(T) + \frac{T^{\alpha-\beta(1+\gamma)}}{\Gamma_q(\alpha - \beta(1+\gamma) + 1)}\psi_{x,f}(T), $$

and

$$ {}^{C}D_q^{\alpha-\beta(1+\gamma)}x(T) = -\lambda I_q^{\beta(1+\gamma)}x(T) + I_q^{\beta(1+\gamma)}({}^{E}I_{\beta}^{\gamma,\delta}f_x)(T) + \psi_{x,f}(T). $$

Substituting these expressions into

$$ b_1x(T) + b_2{}^{C}D_q^{\alpha-\beta(1+\gamma)}x(T), $$

and using the definition of $\psi_{x,f}(T)$ in (2.10), we obtain

$$ b_1x(T) + b_2{}^{C}D_q^{\alpha-\beta(1+\gamma)}x(T) = b_3. $$

Hence, both boundary conditions in (1.1) are fulfilled. Consequently, x satisfies (1.1), which completes the proof of the equivalence. This completes the proof. □

Remark 2.15. *The non-degeneracy condition* $\Lambda \neq 0$ *is always satisfied in Examples 4.1–4.3; indeed, with the numerical values given there,* $\Lambda \approx 0.008641 \neq 0$. *In the degenerate case* $\Lambda = 0$*, the system for* c_0 *and* c_1 *loses a degree of freedom, signaling a resonance phenomenon; this case is excluded here and may warrant separate study.*

3. Existence and uniqueness results

In this section, we investigate the existence and uniqueness of solutions for the boundary value problem (1.1). By Lemma 2.14, the problem is equivalent to the integral equation (2.9). Therefore, it suffices to study the existence of solutions of (2.9).

Let $\mathcal{X} = C([0,T],\mathbb{R})$ be endowed with the usual supremum norm

$$\|x\| = \sup_{t\in[0,T]} |x(t)|.$$

Then $(\mathcal{X}, \|\cdot\|)$ is a Banach space. Define an operator $\mathcal{A} : \mathcal{X} \to \mathcal{X}$ by

$$(\mathcal{A}x)(t) = \frac{1}{a_1}(a_3 - a_2\psi_{x,f}(T)) - \lambda I_q^{\alpha} x(t) + I_q^{\alpha}({}^{E}I_{\beta}^{\gamma,\delta} f_x)(t) + \frac{t^{\alpha-\beta(1+\gamma)}}{\Gamma_q(\alpha-\beta(1+\gamma)+1)}\psi_{x,f}(T),$$

where $f_x(t) = f(t, x(t))$. It follows that fixed points of $\mathcal{A}$ are solutions of (2.9), and hence solutions of (1.1).

Before proceeding further, we introduce several constants that will be used to simplify the computations. Define

$$\begin{aligned}
\Omega_0 &= \frac{1}{|\Lambda|}\left(\frac{T^{\alpha-\beta(1+\gamma)}}{\Gamma_q(\alpha-\beta(1+\gamma)+1)} + \left|\frac{a_2}{a_1}\right|\right),\\
\Omega_1 &= \left|\frac{a_3}{a_1}\right| + \Omega_0\,|a_1b_3 - a_3b_1|,\\
\Omega_2 &= \frac{T^{\alpha}\Gamma(\gamma+1)}{\Gamma_q(\alpha+1)\Gamma(\delta+\gamma+1)} + \Omega_0\Bigg(|a_1b_1|\,\frac{T^{\alpha}\Gamma(\gamma+1)}{\Gamma_q(\alpha+1)\Gamma(\delta+\gamma+1)}\\
&\quad + |a_1b_2|\,\frac{T^{\beta(1+\gamma)}\Gamma(\gamma+1)}{\Gamma_q(\beta(1+\gamma)+1)\Gamma(\delta+\gamma+1)}\Bigg),\\
\Omega_3 &= |\lambda|\left[\frac{T^{\alpha}}{\Gamma_q(\alpha+1)} + \Omega_0\left(|a_1b_1|\,\frac{T^{\alpha}}{\Gamma_q(\alpha+1)} + |a_1b_2|\,\frac{T^{\beta(1+\gamma)}}{\Gamma_q(\beta(1+\gamma)+1)}\right)\right].
\end{aligned} \tag{3.1}$$

Remark 3.1. *The expressions for Ω_2 and Ω_3 are obtained by applying Lemma 2.11 with $y = \beta\sigma$ (with an appropriate choice of σ) to bound the EK integrals; the q-power-function formula $(I_q^{\nu}t^{\mu})(t) = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\mu+\nu+1)}t^{\mu+\nu}$ is used for the q-fractional integrals.*

Remark 3.2. *The problem under consideration is of Langevin type, involving a sequential fractional operator with a damping term λ. This parameter explicitly appears in the constant Ω_3.*

In the particular case when $\lambda = 0$, we have $\Omega_3 = 0$, and the problem reduces to a sequential fractional equation involving only the Erdélyi–Kober and q-Caputo operators. Thus, the presence of λ reflects the influence of the damping effect in the model.

Note that when $\lambda = 0$, Theorem 3.3 reduces to $L\Omega_2 < 1$ (no damping term) and Theorem 3.4 becomes trivially applicable (since $\Omega_3 = 0 < 1$).

In the following theorem, we apply Banach's fixed-point theorem [29] to the operator $\mathcal{A}$ in order to establish the existence and uniqueness of solutions to (1.1).

Theorem 3.3. *Let $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the following condition:*

(H1) There exists a constant $L > 0$ such that

$$|f(t,x_1) - f(t,x_2)| \le L|x_1 - x_2|, \quad \forall t \in [0,T],\ x_1, x_2 \in \mathbb{R}.$$

Then, the boundary value problem (1.1) *admits a unique solution on* $[0,T]$, *provided that*

$$L\Omega_2 + \Omega_3 < 1.$$

Proof. Let $M_0 = \sup_{t\in[0,T]} |f(t,0)|$ and let $B_\rho = \{x \in \mathcal{X} : \|x\| \le \rho\}$ be a closed ball in $\mathcal{X}$, where

$$\rho \ge \frac{\Omega_1 + M_0\Omega_2}{1 - L\Omega_2 - \Omega_3}.$$

First, we show that $\mathcal{A}(B_\rho) \subseteq B_\rho$. By using condition (H1), we have

$$|f(t,x(t))| \le |f(t,x(t)) - f(t,0)| + |f(t,0)| \le L\|x\| + M_0.$$

For any $x \in B_\rho$, we have

$$\begin{aligned}
|(\mathcal{A}x)(t)| \ \le\ & \left|\frac{a_3}{a_1}\right| + I_q^\alpha({}^E I_\beta^{\gamma,\delta}|f_x|)(t) + \frac{1}{|\Lambda|}\left(\frac{T^{\alpha-\beta(1+\gamma)}}{\Gamma_q(\alpha-\beta(1+\gamma)+1)} + \left|\frac{a_2}{a_1}\right|\right) \\
& \times \Big[\, |a_1b_1|\, I_q^\alpha({}^E I_\beta^{\gamma,\delta}|f_x|)(T) + |a_1b_2|\, I_q^{\beta(1+\gamma)}({}^E I_\beta^{\gamma,\delta}|f_x|)(T)\Big] \\
& + |\lambda|\Bigg[I_q^\alpha|x|(t) + \frac{1}{|\Lambda|}\left(\frac{T^{\alpha-\beta(1+\gamma)}}{\Gamma_q(\alpha-\beta(1+\gamma)+1)} + \left|\frac{a_2}{a_1}\right|\right) \\
& \times \Big(|a_1b_1|\, I_q^\alpha|x|(T) + |a_1b_2|\, I_q^{\beta(1+\gamma)}|x|(T)\Big)\Bigg] \\
\le\ & \left|\frac{a_3}{a_1}\right| + \Omega_0\, |a_1b_3 - a_3b_1| + \frac{T^\alpha\Gamma(\gamma+1)}{\Gamma_q(\alpha+1)\Gamma(\delta+\gamma+1)}\,(L\,\|x\| + M_0) \\
& + \Omega_0\Bigg(|a_1b_1|\,\frac{T^\alpha\Gamma(\gamma+1)}{\Gamma_q(\alpha+1)\Gamma(\delta+\gamma+1)} \\
& + |a_1b_2|\,\frac{T^{\beta(1+\gamma)}\Gamma(\gamma+1)}{\Gamma_q(\beta(1+\gamma)+1)\Gamma(\delta+\gamma+1)}\Bigg)(L\,\|x\| + M_0) \\
& + |\lambda|\,\frac{\|x\|\,T^\alpha}{\Gamma_q(\alpha+1)} + \frac{\|x\|}{|\Lambda|}\Omega_0\left[|a_1b_1|\,\frac{T^\alpha}{\Gamma_q(\alpha+1)} + |a_1b_2|\,\frac{T^{\beta(1+\gamma)}}{\Gamma_q(\beta(1+\gamma)+1)}\right],
\end{aligned}$$

by applying Lemma 2.13 and the well-known formula for the Riemann–Liouville integral of a power function. Then

$$\|\mathcal{A}x\| \le \Omega_1 + (L\|x\| + M_0)\Omega_2 + \|x\|\Omega_3 \le \Omega_1 + (L\rho + M_0)\Omega_2 + \rho\Omega_3 \le \rho.$$

Hence, $\mathcal{A}(B_\rho) \subseteq B_\rho$. We now prove that $\mathcal{A}$ is a contraction on B_ρ. For any $x_1, x_2 \in B_\rho$ and $t \in [0,T]$, we have

$$\begin{aligned}
& |(\mathcal{A}x_1)(t) - (\mathcal{A}x_2)(t)| \\
\le\ & I_q^\alpha({}^E I_\beta^{\gamma,\delta}\left|f_{x_1} - f_{x_2}\right|)(t) + \Omega_0\Big[\, |a_1b_1|\, I_q^\alpha({}^E I_\beta^{\gamma,\delta}\left|f_{x_1} - f_{x_2}\right|)(T) \\
& + |a_1b_2|\, I_q^{\beta(1+\gamma)}({}^E I_\beta^{\gamma,\delta}\left|f_{x_1} - f_{x_2}\right|)(T)\Big] \\
& + |\lambda|\Big[I_q^\alpha\,|x_1 - x_2|\,(t) + \Omega_0\Big[|a_1b_1|\, I_q^\alpha\,|x_1 - x_2|\,(T) + |a_1b_2|\, I_q^{\beta(1+\gamma)}\,|x_1 - x_2|\,(T)\Big]\Big]
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{T^{\alpha}\Gamma(\gamma+1)}{\Gamma_q(\alpha+1)\Gamma(\delta+\gamma+1)}(L\,\|x_1-x_2\|) + \Omega_0\bigg(|a_1b_1|\,\frac{T^{\alpha}\Gamma(\gamma+1)}{\Gamma_q(\alpha+1)\Gamma(\delta+\gamma+1)} \\
&\quad + |a_1b_2|\,\frac{T^{\beta(1+\gamma)}\Gamma(\gamma+1)}{\Gamma_q(\beta(1+\gamma)+1)\Gamma(\delta+\gamma+1)}\bigg)(L\,\|x_1-x_2\|) \\
&\quad + |\lambda|\left[\frac{\|x_1-x_2\|\,T^{\alpha}}{\Gamma_q(\alpha+1)} + \frac{\|x_1-x_2\|}{|\Lambda|}\Omega_0\left\{|a_1b_1|\,\frac{T^{\alpha}}{\Gamma_q(\alpha+1)} + |a_1b_2|\,\frac{T^{\beta(1+\gamma)}}{\Gamma_q(\beta(1+\gamma)+1)}\right\}\right] \\
&= (L\Omega_2+\Omega_3)\,\|x_1-x_2\|\,.
\end{aligned}
$$

Since $L\Omega_2+\Omega_3<1$, the operator $\mathcal{A}$ is a contraction on B_ρ. Hence, by Banach's fixed-point theorem, $\mathcal{A}$ admits a unique fixed point in B_ρ, which, by Lemma 2.14, corresponds to the unique solution of (1.1). □

In the following, we establish an existence result by applying Krasnoselskii's fixed-point theorem [30].

Theorem 3.4. *Assume that $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is continuous and satisfies* (H1). *Suppose further that:*

(H2) there exists a function $\phi\in C([0,T],\mathbb{R})$ such that

$$|f(t,x)|\le\phi(t),\quad \forall (t,x)\in[0,T]\times\mathbb{R}.$$

If $\Omega_3<1$, then the boundary value problem (1.1) *admits at least one solution on* $[0,T]$.

Proof. We define the operators $\mathcal{A}_1,\mathcal{A}_2:\mathcal{X}\to\mathcal{X}$ by

$$
\begin{aligned}
\mathcal{A}_1x(t) &= \frac{a_3}{a_1} + I_q^{\alpha}({}^{E}I_{\beta}^{\gamma,\delta}f_x)(t) + \frac{1}{\Lambda}\left(\frac{t^{\alpha-\beta(1+\gamma)}}{\Gamma_q(\alpha-\beta(1+\gamma)+1)} - \frac{a_2}{a_1}\right) \\
&\quad \times\left[a_1b_3 - a_3b_1 - a_1b_1I_q^{\alpha}({}^{E}I_{\beta}^{\gamma,\delta}f_x)(T) - a_1b_2I_q^{\beta(1+\gamma)}({}^{E}I_{\beta}^{\gamma,\delta}f_x)(T)\right], \\
\mathcal{A}_2x(t) &= -\lambda\bigg[I_q^{\alpha}x(t) + \frac{1}{\Lambda}\left(\frac{t^{\alpha-\beta(1+\gamma)}}{\Gamma_q(\alpha-\beta(1+\gamma)+1)} - \frac{a_2}{a_1}\right) \\
&\quad \times\left[a_1b_1\left(I_q^{\alpha}x\right)(T) + a_1b_2I_q^{\beta(1+\gamma)}x(T)\right]\bigg].
\end{aligned}
$$

Let $\rho>0$ be such that

$$\rho\ge\frac{\Omega_1+\|\phi\|\Omega_2}{1-\Omega_3},$$

and define $B_\rho=\{x\in\mathcal{X}:\|x\|\le\rho\}$. For $x,y\in B_\rho$ and $t\in[0,T]$, we consider

$$
\begin{aligned}
&|\mathcal{A}_1x(t)+\mathcal{A}_2y(t)| \\
&\le \left|\frac{a_3}{a_1}\right| + \Omega_0\,|a_1b_3-a_3b_1| + \frac{T^{\alpha}\Gamma(\gamma+1)}{\Gamma_q(\alpha+1)\Gamma(\delta+\gamma+1)}\,\|\phi\| \\
&\quad + \frac{\|\phi\|}{|\Lambda|}\Omega_0\left(|a_1b_1|\,\frac{T^{\alpha}\Gamma(\gamma+1)}{\Gamma_q(\alpha+1)\Gamma(\delta+\gamma+1)} + |a_1b_2|\,\frac{T^{\beta(1+\gamma)}\Gamma(\gamma+1)}{\Gamma_q(\beta(1+\gamma)+1)\Gamma(\delta+\gamma+1)}\right) \\
&\quad + |\lambda|\left[\frac{\rho T^{\alpha}}{\Gamma_q(\alpha+1)} + \frac{\rho}{|\Lambda|}\Omega_0\left\{|a_1b_1|\,\frac{T^{\alpha}}{\Gamma_q(\alpha+1)} + |a_1b_2|\,\frac{T^{\beta(1+\gamma)}}{\Gamma_q(\beta(1+\gamma)+1)}\right\}\right]
\end{aligned}
$$

$$= \Omega_1 + \|\phi\|\,\Omega_2 + \rho\Omega_3 \le \rho.$$

Therefore, the operator $\mathcal{A}_1 + \mathcal{A}_2$ maps B_ρ into itself.

We next show that $\mathcal{A}_2$ is a contraction mapping on B_ρ. For any $x_1, x_2 \in B_\rho$ and $t \in [0, T]$, we obtain

$$\begin{aligned}
|\mathcal{A}_2 x_1(t) - \mathcal{A}_2 x_2(t)| &\le |\lambda| \left[\frac{T^\alpha}{\Gamma_q(\alpha+1)} + \Omega_0 \left\{ |a_1 b_1| \frac{T^\alpha}{\Gamma_q(\alpha+1)} + |a_1 b_2| \frac{T^{\beta(1+\gamma)}}{\Gamma_q(\beta(1+\gamma)+1)} \right\} \right] \|x_1 - x_2\| \\
&= \Omega_3 \|x_1 - x_2\|.
\end{aligned}$$

Since $\Omega_3 < 1$, the operator $\mathcal{A}_2$ is a contraction on B_ρ.

Furthermore, the continuity of f implies that $\mathcal{A}_1$ is continuous. In addition, for any $x \in B_\rho$, we obtain

$$\|\mathcal{A}_1 x\| \le \Omega_1 + \|\phi\|\,\Omega_2.$$

Therefore, $\mathcal{A}_1(B_\rho)$ is uniformly bounded. We next show that $\mathcal{A}_1$ is compact via the Arzelà–Ascoli theorem. It suffices to verify that $\mathcal{A}_1(B_\rho)$ is equicontinuous. Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. Then

$$\begin{aligned}
&|\mathcal{A}_1 x(t_2) - \mathcal{A}_1 x(t_1)| \\
\le\; & \left| I_q^\alpha({}^E I_\beta^{\gamma,\delta} f_x)(t_2) - I_q^\alpha({}^E I_\beta^{\gamma,\delta} f_x)(t_1) \right| + \frac{1}{|\Lambda|} \left| \frac{t_2^{\alpha-\beta(1+\gamma)} - t_1^{\alpha-\beta(1+\gamma)}}{\Gamma_q(\alpha-\beta(1+\gamma)+1)} \right| \\
& \times \left[|a_1 b_3 - a_3 b_1| + |a_1 b_1|\, I_q^\alpha({}^E I_\beta^{\gamma,\delta} f_x)(T) + |a_1 b_2|\, I_q^{\beta(1+\gamma)}({}^E I_\beta^{\gamma,\delta} f_x)(T) \right] \\
\le\; & \left| \frac{1}{\Gamma_q(\alpha)} \int_0^{t_1} \left[(t_2 - qs)^{\alpha-1} - (t_1 - qs)^{\alpha-1} \right] {}^E I_\beta^{\gamma,\delta} f_x(s) d_q s \right| \\
& + \left| \frac{1}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} \left[(t_2 - qs)^{\alpha-1} \right] {}^E I_\beta^{\gamma,\delta} f_x(s) d_q s \right| + \frac{1}{|\Lambda|} \left| \frac{t_2^{\alpha-\beta(1+\gamma)} - t_1^{\alpha-\beta(1+\gamma)}}{\Gamma_q(\alpha-\beta(1+\gamma)+1)} \right| \\
& \times \left[|a_1 b_3 - a_3 b_1| + |a_1 b_1|\, I_q^\alpha({}^E I_\beta^{\gamma,\delta} f_x)(T) + |a_1 b_2|\, I_q^{\beta(1+\gamma)}({}^E I_\beta^{\gamma,\delta} f_x)(T) \right] \\
\le\; & \frac{\Gamma(\gamma+1)\,\|\phi\|}{\Gamma_q(\alpha+1)\Gamma(\delta+\gamma+1)} \left[2\,|(t_2 - t_1)^\alpha| + \left| t_2^\alpha - t_1^\alpha \right| \right] + \frac{1}{|\Lambda|} \left| \frac{t_2^{\alpha-\beta(1+\gamma)} - t_1^{\alpha-\beta(1+\gamma)}}{\Gamma_q(\alpha-\beta(1+\gamma)+1)} \right| \\
& \times \Bigg[|a_1 b_3 - a_3 b_1| + \frac{|a_1 b_1| \Gamma(\gamma+1) T^\alpha}{\Gamma_q(\alpha+1)\Gamma(\delta+\gamma+1)} \|\phi\| \\
& + \frac{|a_1 b_2|\,\Gamma(\gamma+1) T^{\beta(1+\gamma)}}{\Gamma_q(\beta(1+\gamma)+1)\Gamma(\delta+\gamma+1)} \|\phi\| \Bigg],
\end{aligned}$$

which tends to zero as $t_2 - t_1 \to 0$ uniformly in $x \in B_\rho$. Hence, by the Arzelà–Ascoli theorem, $\mathcal{A}_1$ is completely continuous on B_ρ.

Therefore, by Krasnoselskii's fixed-point theorem, the boundary value problem (1.1) admits at least one solution on $[0, T]$. □

Remark 3.5. *The existence result in the above theorem requires the condition $\Omega_3 < 1$. Since the constant Ω_3 depends explicitly on the parameter λ, this condition imposes a restriction on the magnitude of λ.*

In this sense, the parameter λ acts as a tuning parameter for the existence of solutions to the Langevin-type problem (1.1). *In particular, larger values of λ may violate the condition $\Omega_3 < 1$, while smaller values ensure the applicability of the existence result.*

The final result is established using the Leray–Schauder nonlinear alternative [31].

Theorem 3.6. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that there exist a continuous nondecreasing function $h : [0,\infty) \to (0,\infty)$ and a continuous function $g : [0,T] \to (0,\infty)$ such that*

$$|f(t,u)| \leq g(t)\, h(|u|), \quad \text{for all} \quad (t,u) \in [0,T] \times \mathbb{R}.$$

(By the monotonicity of h and the bound $|x(t)| \leq \|x\|$, this implies $|f(t,x(t))| \leq g(t)\, h(\|x\|)$ for all $x \in X$.) Moreover, assume that there exists a constant $M > 0$ such that

$$\frac{(1-\Omega_3)M}{\Omega_1 + \|g\|\, h(M)\Omega_2} > 1, \tag{3.2}$$

where $\Omega_i, i = 1,2,3,$ are defined in (3.1).

Then, the boundary value problem (1.1) *has at least one solution on* $[0,T]$.

Proof. First, we show that $\mathcal{A}$ maps bounded sets into bounded sets in $\mathcal{X}$. Let $r > 0$ and consider the closed ball $B_r = \{x \in \mathcal{X} : \|x\| \leq r\}$. Then, for any $x \in B_r$ and $t \in [0,T]$, we obtain

$$\begin{aligned} |\mathcal{A}x(t)| &\leq \left|\frac{a_3}{a_1}\right| + \Omega_0\, |a_1 b_3 - a_3 b_1| + \frac{T^\alpha \Gamma(\gamma+1)}{\Gamma_q(\alpha+1)\Gamma(\delta+\gamma+1)}\, \|g\|\, h\,(\|x\|) \\ &\quad + \Omega_0 \left\{ |a_1 b_1| \frac{T^\alpha \Gamma(\gamma+1)}{\Gamma_q(\alpha+1)\Gamma(\delta+\gamma+1)} + |a_1 b_2| \frac{T^{\beta(1+\gamma)}\Gamma(\gamma+1)}{\Gamma_q(\beta(1+\gamma)+1)\Gamma(\delta+\gamma+1)} \right\} \\ &\quad \times \|g\|\, h\,(\|x\|) + |\lambda|\, \|x\| \left[\frac{T^\alpha}{\Gamma_q(\alpha+1)} + \Omega_0 \left\{ |a_1 b_1| \frac{T^\alpha}{\Gamma_q(\alpha+1)} \right.\right. \\ &\quad \left.\left. + \ |a_1 b_2| \frac{T^{\beta(1+\gamma)}}{\Gamma_q(\beta(1+\gamma)+1)} \right\} \right] \\ &= \Omega_1 + \|g\|\, h\,(\|x\|)\, \Omega_2 + \|x\|\, \Omega_3. \end{aligned}$$

Thus, for all $x \in B_r$, it follows that

$$\|\mathcal{A}x\| \leq \Omega_1 + \|g\|\, h(r)\Omega_2 + r\Omega_3.$$

Next, we show that $\mathcal{A}$ maps bounded sets into equicontinuous subsets of $\mathcal{X}$. From the previous theorem, $\mathcal{A}_1$ is equicontinuous. It remains to show that $\mathcal{A}_2$ is equicontinuous.

Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and $x \in B_r$. Then

$$\begin{aligned} |\mathcal{A}_2 x(t_2) - \mathcal{A}_2 x(t_1)| &\leq |\lambda| \left[\left(\frac{T^\alpha}{\Gamma_q(\alpha+1)} \right) \left| t_2^\alpha - t_1^\alpha \right| \right. \\ &\quad \left. + \frac{1}{|\Lambda|} \left\{ |a_1 b_1| \frac{T^\alpha}{\Gamma_q(\alpha+1)} + |a_1 b_2| \frac{T^{\beta(1+\gamma)}}{\Gamma_q(\beta(1+\gamma)+1)} \right\} \frac{\left| t_2^{\alpha-\beta(1+\gamma)} - t_1^{\alpha-\beta(1+\gamma)} \right|}{\Gamma_q(\alpha-\beta(1+\gamma)+1)} \right], \end{aligned}$$

which tends to zero as $t_2 - t_1 \to 0$ uniformly with respect to $x \in B_r$. Therefore, by the Arzelà–Ascoli theorem, $\mathcal{A}(B_r)$ is equicontinuous.

Finally, we show that the set of all solutions to the equation

$$x = \lambda \mathcal{A} x, \quad \lambda \in (0,1),$$

is bounded. Let x be a solution of the above equation. Then, using the previous estimate, we have

$$\|x\| \leq \|\mathcal{A}x\| \leq \Omega_1 + \|g\|\, h(\|x\|)\Omega_2 + \|x\|\Omega_3.$$

Hence,

$$\frac{(1-\Omega_3)\|x\|}{\Omega_1 + \|g\|\, h(\|x\|)\Omega_2} \leq 1.$$

By the assumption, there exists $M > 0$ such that

$$\frac{(1-\Omega_3)M}{\Omega_1 + \|g\|\, h(M)\Omega_2} > 1.$$

Let

$$U = \{x \in \mathcal{X} : \|x\| < M\}.$$

Then $\mathcal{A} : \overline{U} \to \mathcal{X}$ is continuous and completely continuous. Furthermore, there is no $x \in \partial U$ satisfying $x = \lambda\mathcal{A}x$ for any $\lambda \in (0, 1)$.

Hence, by the Leray–Schauder nonlinear alternative, $\mathcal{A}$ admits at least one fixed point in $\overline{U}$, which corresponds to a solution of (1.1). □

Remark 3.7. *The first two theorems require that the nonlinear term f in* (1.1) *satisfies a Lipschitz condition. In contrast, the present result is established under a growth condition that guarantees the boundedness of f.*

Thus, the above results are applicable to different types of nonlinear problems. In particular, the Leray–Schauder approach allows us to treat problems where the Lipschitz condition may fail, but suitable growth conditions are still available.

4. Analytical examples

In this section, we provide three illustrative examples to demonstrate the applicability of the main results of Section 3. By fixing the structural parameters of the boundary value problem and varying only the nonlinear term f together with the damping parameter λ, we exhibit how each of Theorems 3.3–3.6 applies under qualitatively distinct scenarios.

Example 4.1 (Common setting)**.** *Consider the boundary value problem*

$$\begin{cases} {}^{E}D^{\frac{1}{8},\frac{2}{3}}_{\frac{1}{5}}\left({}^{C}D^{\frac{1}{2}}_{\frac{1}{4}} + \lambda\right)x(t) = f(t, x(t)), \quad t \in \left[0, \frac{7}{3}\right], \\ \frac{1}{13}x(0) + \frac{1}{15}{}^{C}D^{\frac{11}{40}}_{\frac{1}{4}}x(0) = \frac{1}{17}, \\ \frac{1}{12}x\left(\frac{7}{3}\right) + \frac{1}{14}\left({}^{C}D^{\frac{11}{40}}_{\frac{1}{4}}x\right)\left(\frac{7}{3}\right) = \frac{1}{16}, \end{cases} \tag{4.1}$$

where the parameters are

$$\gamma = \frac{1}{8}, \quad \delta = \frac{2}{3}, \quad \beta = \frac{1}{5}, \quad \alpha = \frac{1}{2}, \quad q = \frac{1}{4}, \quad T = \frac{7}{3},$$

$$a_1 = \frac{1}{13}, \quad a_2 = \frac{1}{15}, \quad a_3 = \frac{1}{17}, \quad b_1 = \frac{1}{12}, \quad b_2 = \frac{1}{14}, \quad b_3 = \frac{1}{16}.$$

Note that $\alpha - \beta(1+\gamma) = \frac{1}{2} - \frac{1}{5} \cdot \frac{9}{8} = \frac{11}{40} > 0$, *so the parameter constraint of Lemma 2.12 is satisfied. A numerical evaluation of the constants defined in Section 3 gives*

$$\Lambda \approx 0.008641, \quad \Omega_0 \approx 257.3935, \quad \Omega_1 \approx 0.7890, \quad \Omega_2 \approx 6.3062, \quad \Omega_3 \approx 6.2222\,|\lambda|.$$

We examine three sub-cases corresponding to the three fixed-point theorems.

Example 4.2 (Theorem 3.3: Lipschitz nonlinearity, not globally bounded)**.** *Take* $\lambda = \frac{1}{11}$ *and*

$$f(t,x) = \frac{1}{2(t+4)^2}\left(\frac{x^2+2|x|}{1+|x|}\right) + \cos^2 t + t^2 + \frac{1}{3}. \tag{4.2}$$

Verification of (H1). *For all* $t \in [0, \frac{7}{3}]$ *and* $x_1, x_2 \in \mathbb{R}$, *one checks that*

$$\frac{x^2+2|x|}{1+|x|} = \frac{|x|(|x|+2)}{1+|x|},$$

whose derivative with respect to $|x|$ *is* $\frac{|x|^2+2|x|+2}{(1+|x|)^2} \leq 2$. *Since* $\frac{1}{2(t+4)^2} \leq \frac{1}{2\cdot 16} = \frac{1}{32}$ *for* $t \in \left[0, \frac{7}{3}\right]$, *we obtain*

$$|f(t,x_1) - f(t,x_2)| \leq \frac{1}{2(t+4)^2} \cdot 2\,|x_1 - x_2| \leq \frac{1}{(t+4)^2}\,|x_1 - x_2| \leq \frac{1}{16}\,|x_1 - x_2|,$$

so (H1) holds with the Lipschitz constant $L = \frac{1}{16}$.

Note that f *is unbounded (the term* $t^2 + \cos^2 t + \frac{1}{3}$ *is not bounded by a function of* t *alone independent of* x *for all* x*; more precisely,* f *itself is bounded in* x *since* $\frac{x^2+2|x|}{1+|x|} \leq |x| + 1$, *but there is no* $\varphi(t)$ *bounding* $|f(t,x)|$ *uniformly in* x *over all of* $\mathbb{R}$*). Thus condition (H2) of Theorem 3.4 is* not *satisfied, and the present example is specific to Theorem 3.3.*

Verification of the contraction condition. *With* $\lambda = \frac{1}{11}$,

$$\Omega_3 \approx \frac{6.2222}{11} \approx 0.5657,$$

and hence

$$L\Omega_2 + \Omega_3 \approx \frac{6.3062}{16} + 0.5657 \approx 0.3941 + 0.5657 = 0.9598 < 1.$$

All hypotheses of Theorem 3.3 are therefore satisfied. Consequently, problem (4.1) *with nonlinearity* (4.2) *admits a* unique *solution on* $[0, \frac{7}{3}]$.

Example 4.3 (Theorem 3.4: Lipschitz and globally bounded nonlinearity)**.** *Let* $\lambda \in (-\lambda_0, \lambda_0)$, *where* $\lambda_0 = 0.1607$, *and take*

$$f(t,x) = \frac{1}{t^2+3} \cdot \frac{|x|}{1+|x|} + e^{-t^2} + \frac{1}{4}. \tag{4.3}$$

Verification of (H1). *Since* $\dfrac{d}{dx}\frac{|x|}{1+|x|} = \dfrac{1}{(1+|x|)^2} \le 1$*, we get*

$$|f(t,x_1) - f(t,x_2)| \le \frac{1}{t^2+3}\,|x_1 - x_2| \le \frac{1}{3}\,|x_1 - x_2|,$$

so (H1) holds with $L = \dfrac{1}{3}$.

Verification of (H2). *Since* $\dfrac{|x|}{1+|x|} \le 1$ *and* $e^{-t^2} \le 1$ *on* $[0, \frac{7}{3}]$,

$$|f(t,x)| \le \frac{1}{t^2+3} + e^{-t^2} + \frac{1}{4} =: \varphi(t), \quad \forall (t,x) \in [0, \tfrac{7}{3}] \times \mathbb{R},$$

so (H2) holds with $\|\varphi\| \le \dfrac{1}{3} + 1 + \dfrac{1}{4} = \dfrac{19}{12}$.

Verification of $\Omega_3 < 1$***.*** *The constraint* $|\lambda| < \lambda_0 \approx 0.1607$ *is equivalent to* $\Omega_3 = 6.2222\,|\lambda| < 6.2222 \times 0.1607 \approx 1.0001$*; choosing* $|\lambda|$ *strictly inside this interval ensures* $\Omega_3 < 1$*. All hypotheses of Theorem 3.4 are satisfied. Consequently, problem* (4.1) *with nonlinearity* (4.3) *admits* at least one *solution on* $[0, \frac{7}{3}]$.

Example 4.4 (Theorem 3.6: Growth condition, no Lipschitz requirement)**.** *Take* $\lambda = \frac{1}{15}$ *and*

$$f(t,x) = \frac{1}{20(t+2)^2}\left(\frac{x^{2570}}{1+x^{2568}} + \frac{1}{2}\right). \tag{4.4}$$

Note that f does not *satisfy a global Lipschitz condition in x (the factor* $\dfrac{x^{2570}}{1+x^{2568}}$ *grows without bound), so neither Theorem 3.3 nor Theorem 3.4 applies here.*

Verification of the growth condition. *Set* $g(t) = \dfrac{1}{20(t+2)^2}$ *and* $h(r) = r^2 + \dfrac{1}{2}$*. Since* $\dfrac{x^{2570}}{1+x^{2568}} \le x^2$ *for all* $x \ge 0$ *(because* $x^{2570} \le x^2(1+x^{2568})$ *for* $x \ge 0$*), we have*

$$|f(t,u)| \le g(t)\,h(|u|), \quad \forall (t,u) \in [0, \tfrac{7}{3}] \times \mathbb{R}.$$

Moreover $\|g\| = g(0) = \dfrac{1}{80}$.

Verification of the a priori bound condition (3.2)***.*** *With* $\lambda = \dfrac{1}{15}$*, we have* $\Omega_3 \approx \dfrac{6.2222}{15} \approx 0.4148$ *and* $1 - \Omega_3 \approx 0.5852$*. Condition* (3.2) *requires*

$$\frac{(1-\Omega_3)\,M}{\Omega_1 + \|g\|\,h(M)\,\Omega_2} > 1,$$

i.e.,

$$\frac{0.5852\,M}{0.7890 + \frac{1}{80}\left(M^2 + \frac{1}{2}\right) \times 6.3062} > 1.$$

A short calculation shows that this inequality holds for all $M \in (1.9038, 5.5199)$*. Choosing any such M satisfies the hypothesis of Theorem 3.6. Consequently, problem* (4.1) *with nonlinearity* (4.4) *admits* at least one *solution on* $[0, \frac{7}{3}]$.

5. Numerical experiments

The goal of this section is to investigate numerically the behavior of the solution $x(t)$ to the integral equation (2.9) under variation of the four key parameters: the Langevin damping constant λ, the q-deformation parameter q, and the Erdélyi–Kober parameters γ and β. Finding an analytical solution is infeasible owing to the non-local structure of both kernels and their coupling through the boundary conditions; we therefore resort to the spectral collocation method. This approach translates the continuous fractional operators into exact analytical matrices, ensuring a rigorous global approximation and bypassing the numerical instabilities often associated with weakly singular kernels.

5.1. Assembling the discretized integral equation

These theoretical matrix entries map directly back to the integral equation. Consider the continuous damping term from the model:

$$-\lambda I_q^{\alpha} x(t).$$

Within the spectral collocation framework, this continuous functional piece is expressed as a discrete matrix-vector product:

$$-\lambda \mathbf{S}^{(q)} \mathbf{c}.$$

Similarly, to evaluate the boundary conditions at the terminal point $t = T$ (for the boundary functional ψ), we define a single row vector where the nodes t_i are replaced by T. For instance, the term $I_q^{\beta(1+\gamma)} x(T)$ corresponds to the dot product:

$$\sum_{k=0}^{N} c_k \frac{\Gamma_q(k+1)}{\Gamma_q(k+\beta(1+\gamma)+1)} T^{k+\beta(1+\gamma)}.$$

By assembling all these operator matrices into a global left-hand matrix M, the integro-differential problem collapses into a single algebraic system: $M\mathbf{c} = \mathbf{v}$. It is important to note that while the original theoretical model accommodates a nonlinear source term $f(t, x(t))$ - which would yield a nonlinear system of equations requiring iterative root-finding algorithms - we impose a state-independent source function $f(t, x) \equiv f(t)$ for these specific numerical experiments. This simplification strictly linearizes the system, decoupling the source from the state variable and allowing the spectral coefficients to be computed directly via $\mathbf{c} = M^{-1}\mathbf{v}$.

5.2. Global linear system

Substituting the spectral approximation $x(t) \approx \sum_{k=0}^{N} c_k t^k$ into the integral equation (2.9) and evaluating at each collocation point t_i (for $i = 0, 1, \ldots, N$) yields a system of $(N + 1)$ linear equations in terms of the $(N + 1)$ unknown coefficients $\mathbf{c} = (c_0, c_1, \ldots, c_N)^T$:

$$M\,\mathbf{c} = \mathbf{v}. \tag{5.1}$$

Here, M is a dense $(N+1)\times(N+1)$ matrix constructed by applying the left-hand operators to the basis functions t^k for each column k, and $\mathbf{v}$ is a known right-hand vector incorporating the boundary data and the deterministic evaluations of the source function f.

5.3. Baseline parameters and experimental design

All experiments share the following setup unless otherwise stated.

- Fractional orders: $\alpha = 5/7, \ \delta = 1/7$.
- Erdélyi–Kober parameters: $\beta = 2/7, \ \gamma = -3/7$.
- Quantum parameter: $q = 0.5$.
- Damping constant: $\lambda = 1$.
- Time horizon: $T = 3/2$.
- Source function: $f(t, x) \equiv f(t) = 1 + t^3$ (independent of x, so the integral equation is linear).
- Boundary constants: $a_1 = b_1 = 1, \ a_2 = b_2 = 1, \ a_3 = b_3 = 0$ (homogeneous boundary conditions).
- Number of collocation points: $N = 16$.

Each experiment varies a *single* parameter while keeping the remaining three at their baseline values. This allows us to isolate the distinct role of each constant.

5.4. Influence of the damping constant λ

We vary λ over the set $\{1, 2, 3, 4, 5\}$, keeping $q = 0.5$, $\gamma = -3/7$, and $\beta = 2/7$ fixed. The results are shown in Figure 1.

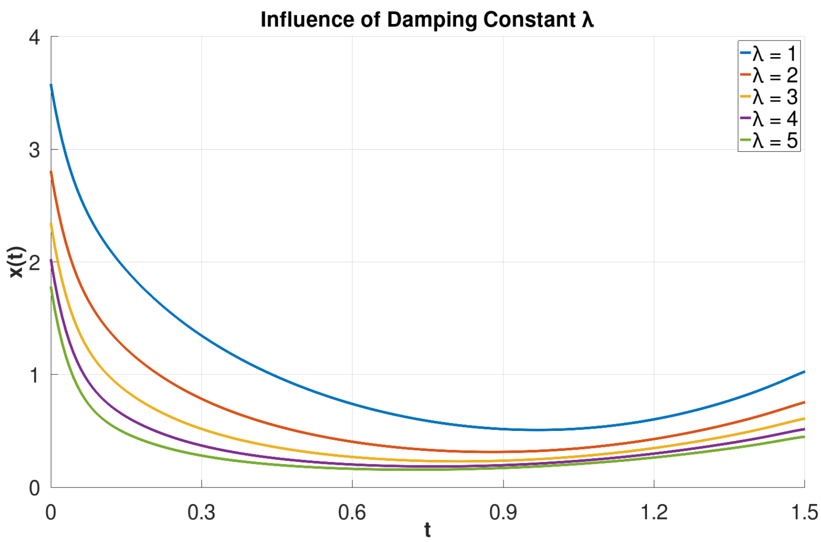

Figure 1. Influence of the damping constant λ on the solution $x(t)$ for $\lambda \in \{1, 2, 3, 4, 5\}$, with baseline values of all other parameters.

It is observed that as the damping coefficient λ increases, the overall amplitude of the solution $x(t)$ is uniformly suppressed across the entire temporal domain $[0, T]$. This behavior is consistent with the role of λ as a damping (or restoring) coefficient in the Langevin structure: a larger restoring force shifts energy toward the initial part of the trajectory in order to satisfy the homogeneous terminal condition.

5.5. *Influence of the quantum parameter q*

We vary q over the set $\{0.1, 0.2, \ldots, 0.9\}$, keeping $\lambda = 1$, $\gamma = -3/7$, and $\beta = 2/7$ fixed. The results are displayed in Figure 2.

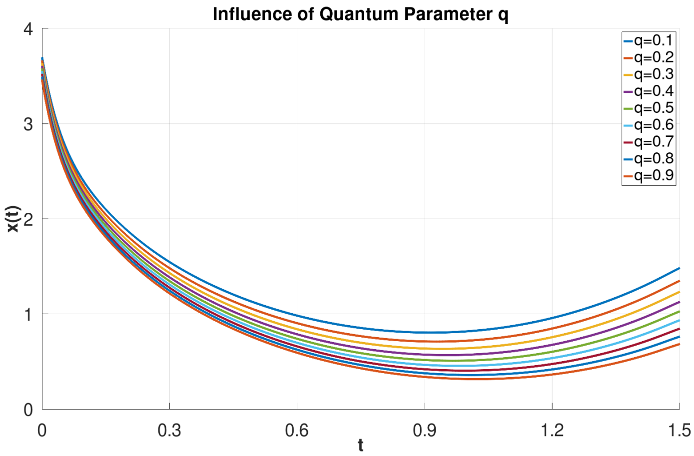

Figure 2. Influence of the quantum parameter q on the solution $x(t)$ for $q \in \{0.1, 0.2, \ldots, 0.9\}$, with baseline values of all other parameters.

Smaller values of q produce a pronounced vertical shift, resulting in a higher overall solution profile. As $q \to 1^-$, the q-Caputo operator transitions from a dilated, discrete-memory structure toward the localized classical Caputo derivative, causing the solution amplitude to decrease. Notably, the solution trajectories run strictly parallel to one another without intersecting.

5.6. *Influence of the Erdélyi–Kober parameter γ*

We vary γ over $\{-0.5, -0.25, 0, 0.25, 0.5\}$, keeping $\lambda = 1$, $q = 0.5$, and $\beta = 2/7$ fixed. The results are shown in Figure 3.

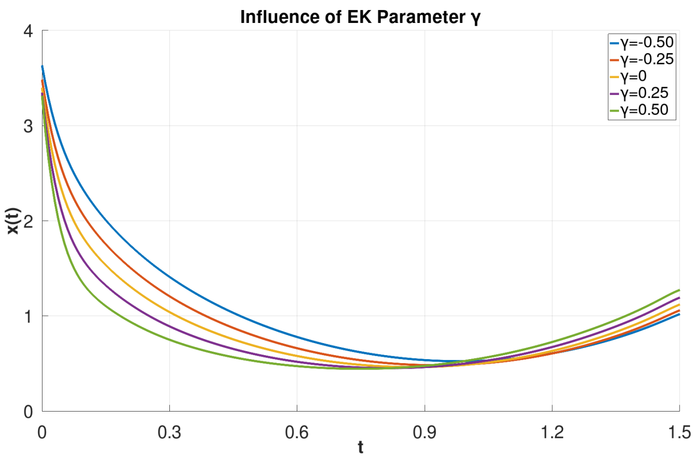

Figure 3. Influence of the Erdélyi–Kober parameter γ on the solution $x(t)$ for $\gamma \in \{-0.5, -0.25, 0, 0.25, 0.5\}$, with baseline values of all other parameters.

Unlike the uniform suppression seen with λ and q, varying γ induces a structural pivot in the trajectory, with all curves intersecting at a well-defined nodal point near $t \approx 1$. The parameter γ controls the power-law weighting of the early-time history within the EK integral kernel. Lower values (e.g., $\gamma = -0.50$) heavily weight the immediate past, amplifying the contribution of the source function near the origin and resulting in higher initial values at $t = 0$. To globally balance the separated boundary conditions, this early-time amplification is compensated by a steeper decay, causing trajectories with lower γ values to terminate at lower states at $t = T$.

5.7. *Influence of the Erdélyi–Kober parameter* β

We vary β over $\{0.1, 0.2, 0.3, 0.4, 0.5\}$, keeping $\lambda = 1$, $q = 0.5$, and $\gamma = -3/7$ fixed. The results are depicted in Figure 4.

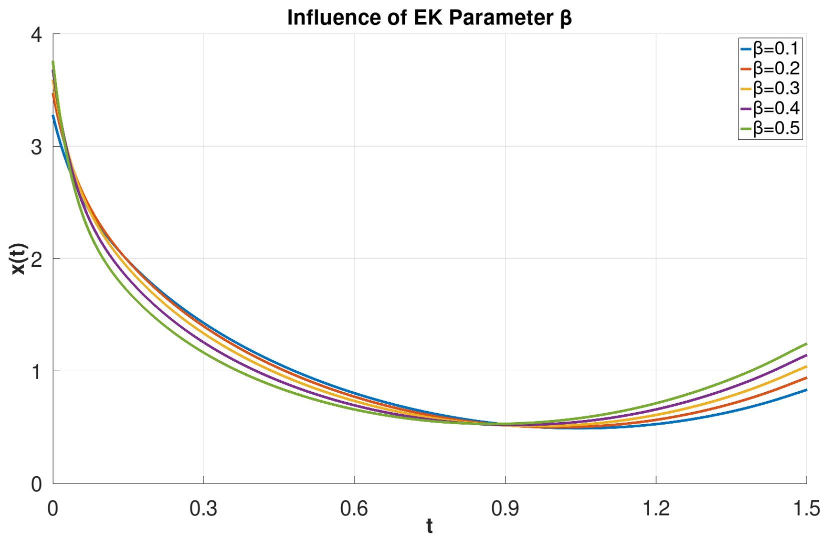

Figure 4. Influence of the Erdélyi–Kober parameter β on the solution $x(t)$ for $\beta \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$, with baseline values of all other parameters.

Similar to γ, the variation of β produces a nodal intersection region near $t \approx 0.9$; however, its geometric influence on the boundaries is inverted. Higher values of β yield higher state values at both the initial boundary $t = 0$ and the terminal boundary $t = T$. Mathematically, β dictates the internal scaling of the independent variable within the EK memory kernel ($u = s^{\beta}$). Increasing β effectively compresses the nonlocal memory window, localizing the influence of the source term. This compression prevents the deep temporal relaxation seen with lower β values, causing the trajectory to rebound more sharply toward the terminal boundary condition.

6. Conclusions

In this work, we have initiated the study of Langevin-type fractional equations in which an Erdélyi–Kober operator acts on a quantum-Caputo Langevin expression, subject to separated boundary conditions. To our knowledge, this is the first paper to treat this particular combination of operators within a boundary value framework, and we hope it will serve as a starting point for a broader investigation of hybrid EK–quantum systems.

The core analytical contribution is the derivation of an equivalent integral representation of the problem. Obtaining this representation is non-trivial: it requires composing EK integral identities with q-analogues of the fundamental theorem of calculus, producing a class of mixed EK–q–integral identities that do not appear elsewhere in the literature and that we believe will find independent applications. Once the integral formulation is in hand, the solution operator on $C([0,T],\mathbb{R})$ can be studied by standard functional-analytic tools, but the verification of the required operator properties - continuity, boundedness, and compactness - demands a careful treatment of the kernel estimates that arise from the combined EK–q-Caputo structure.

On the existence–uniqueness side, we have deliberately employed three distinct fixed-point theorems rather than a single one, and this choice reflects a genuine difference in the hypotheses each approach requires. The Banach contraction principle is the most demanding: it requires a global Lipschitz condition on f with a sufficiently small Lipschitz constant, but in return, it delivers uniqueness and an explicit iteration scheme that is directly implementable. The Krasnosel'skiĭ theorem relaxes the contraction requirement by splitting the solution operator into a contractive part and a compact part; existence is recovered under weaker hypotheses, but uniqueness is no longer guaranteed in general. Finally, the Leray–Schauder alternative demands only that solutions satisfy an a priori bound — a condition implied by mere polynomial growth of f — and yields existence in the broadest regime covered here. Taken together, the three theorems provide a layered and complete picture of the solution theory.

As a by-product of the analysis, several results in the existing literature are recovered as special cases. Setting $q \to 1$ reduces the q-Caputo operator to the classical Caputo derivative, recovering EK–Caputo–Langevin equations studied in recent work. Suppressing the EK component yields purely q-fractional Langevin problems, while taking both components as standard Caputo derivatives recovers classical integer or fractional Langevin BVPs. The present framework is therefore strictly more general, and each specialization can be seen as a corollary of our main theorems.

Several directions for future research naturally suggest themselves.

- *Ulam–Hyers and Ulam–Hyers–Rassias stability.* The integral representation derived in Lemma 2.14 provides a natural starting point for establishing stability in the sense of Ulam–Hyers, a topic of increasing interest for fractional boundary value problems.
- *Resonance and degenerate cases.* When $\Lambda = 0$, the system for the integration constants loses a degree of freedom, signaling a resonance phenomenon. The analysis of this degenerate case, including the conditions for resonance and the multiplicity of solutions, is excluded from the present work and merits separate study.
- *More general boundary conditions.* The extension to multi-point, integral, and nonlocal boundary conditions - widely studied in the Riemann–Liouville and Caputo settings - would broaden the applicability of the theory.
- *Systems of coupled equations.* The generalization to systems of coupled EK–q–Caputo–Langevin equations, which arise in multi-component physical models, is a natural next step.
- *More general fractional operators.* It would be of interest to replace the q-Caputo component by Hilfer, Caputo–Fabrizio, or Atangana–Baleanu operators, or to consider EK operators with non-singular kernels, thereby connecting the present framework with the rapidly growing literature on non-singular fractional calculus.
- *Advanced numerical methods.* The development of higher-order and adaptive q-discretization

schemes that respect the EK structure of the kernel, together with a rigorous convergence analysis, remains an open and practically important problem.
- *Physical and engineering applications.* Concrete applications to anomalous transport, quantum-scale dynamics, and viscoelastic control systems - including parameter estimation from experimental data - are planned for forthcoming work.

We close by remarking on the physical relevance of the model. The EK operator is well-suited to describe processes in which the memory kernel has a power-law weight that decays at a rate governed by the scaling parameter β, a situation that arises in anomalous diffusion in heterogeneous media and in certain viscoelastic constitutive laws. The q-Caputo component, on the other hand, introduces a discrete geometric time scale, which is natural in quantum mechanical settings and in models defined on lattices or trees. The Langevin structure - a composite operator acting on x equal to a nonlinear forcing $f(t,x)$ - encapsulates the balance between a restoring mechanism and a state-dependent perturbation. The separated boundary conditions allow independent specification of the state and its fractional derivative at both endpoints, a flexibility that is important in control and optimal design applications.

The results established in this paper provide a rigorous mathematical foundation for a broad class of hybrid fractional models combining weighted-memory and discrete-scale effects. Such models are relevant to the following areas.

- *Anomalous diffusion.* The EK operator captures power-law memory kernels that are characteristic of sub- and super-diffusive transport in heterogeneous media; the existence and uniqueness results ensure the well-posedness of the corresponding boundary value problems.
- *Viscoelastic systems.* Langevin-type equations with fractional damping arise naturally in the constitutive modeling of viscoelastic materials; the parameter λ acts as a restoring coefficient, and the theoretical results guarantee that the model is mathematically well-posed.
- *Quantum-scale and lattice dynamics.* The q-Caputo operator introduces a geometric time scale that is natural in quantum mechanical systems and in discrete-lattice models; the integral representation derived in Lemma 2.14 facilitates the numerical approximation of such systems.
- *Control and optimal design.* Separated boundary conditions allow independent specification of the state and its fractional derivative at both endpoints, which is relevant to control problems and to the optimal design of systems with memory.

We therefore believe that the model studied here is not merely a mathematical generalization, but a genuinely useful framework, and we hope the rigorous solution theory developed in this paper will encourage its adoption in applied contexts.

Author contributions

Sotiris K. Ntouyas and Jessada Tariboon: Conceptualization, writing—review & editing, supervision; Chaiyod Kamthorncharoen: software, writing—original draft preparation, funding acquisition; Chaiyod Kamthorncharoen, Sotiris K. Ntouyas and Jessada Tariboon: methodology, validation, formal analysis, investigation. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The first author was supported by King Mongkut's University of Technology North Bangkok, Contract No. KMUTNB-68-NEW-08.

Conflict of interest

The authors declare that they have no competing interests.

References

1. I. Podlubny, *Fractional differential equations*, San Diego: Academic Press, 1999.
2. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam: Elsevier, 2006.
3. R. Hilfer, *Applications of fractional calculus in physics*, Singapore: World Scientific, 2000. https://doi.org/10.1142/3779
4. V. E. Tarasov, *Fractional dynamics: applications of fractional calculus to dynamics of particles, fields and media*, Berlin, Heidelberg: Springer, 2010. https://doi.org/10.1007/978-3-642-14003-7
5. K. Diethelm, *The analysis of fractional differential equations*, Berlin, Heidelberg: Springer-Verlag, 2010. https://doi.org/10.1007/978-3-642-14574-2
6. A. Erdélyi, On fractional integration and its application to the theory of Hankel transforms, *Q. J. Math.*, **os-11** (1940), 293–303. https://doi.org/10.1093/qmath/os-11.1.293
7. H. Kober, On fractional integrals and derivatives, *Q. J. Math.*, **os-11** (1940), 193–211. https://doi.org/10.1093/qmath/os-11.1.193
8. V. Kac, P. Cheung, *Quantum calculus*, New York: Springer, 2002. https://doi.org/10.1007/978-1-4613-0071-7
9. M. H. Annaby, Z. S. Mansour, *q-Fractional calculus and equations*, Berlin: Springer, 2012. https://doi.org/10.1007/978-3-642-30898-7
10. G. Gasper, M. Rahman, *Basic hypergeometric series*, 2 Eds., Cambridge: Cambridge University Press, 2009. https://doi.org/10.1017/cbo9780511526251
11. B. Ahmad, S. K. Ntouyas, Existence of solutions for nonlinear fractional q-difference inclusions with nonlocal Robin (separated) conditions, *Mediterr. J. Math.*, **10** (2013), 1333–1351. https://doi.org/10.1007/s00009-013-0258-0
12. R. P. Agarwal, Certain fractional q-integrals and q-derivatives, *Math. Proc. Cambridge Philos. Soc.*, **66** (1969), 365–370. https://doi.org/10.1017/s0305004100045060
13. P. M. Rajković, S. D. Marinković, M. S. Stanković, Fractional integrals and derivatives in q-calculus, *Appl. Anal. Discrete Math.*, **1** (2007), 311–323. https://doi.org/10.2298/aadm0701311r

14. B. Ahmad, S. K. Ntouyas, I. K. Purnaras, Existence results for nonlocal boundary value problems of nonlinear fractional q-difference equations, *Adv. Differ. Equ.*, **2012** (2012), 140. https://doi.org/10.1186/1687-1847-2012-140

15. J. Ma, J. Yang, Existence of solutions for multi-point boundary value problem of fractional q-difference equation, *Electron. J. Qual. Theory Differ. Equ.*, **92** (2011), 1–10. https://doi.org/10.14232/ejqtde.2011.1.92

16. W. T. Coffey, Yu. P. Kalmykov, J. T. Waldron, *The Langevin equation: with applications to stochastic problems in physics, chemistry and electrical engineering*, Singapore: World Scientific, 2004. https://doi.org/10.1142/5343

17. R. Zwanzig, *Nonequilibrium statistical mechanics*, Oxford: Oxford University Press, 2001. https://doi.org/10.1093/oso/9780195140187.001.0001

18. B. Ahmad, J. J. Nieto, Solvability of nonlinear Langevin equation involving two fractional orders with Dirichlet boundary conditions, *Int. J. Differ. Equ.*, **2010** (2010), 1–10. https://doi.org/10.1155/2010/649486

19. H. M. Ahmed, A. S. Ghanem, Existence and uniqueness results for a nonlinear coupled system of nonlinear fractional Langevin equations with a new kind of boundary conditions, *Filomat*, **36** (2022), 5437–5447. https://doi.org/10.2298/FIL2216437A

20. H. M. Ahmed, A. S. Ghanem, Exploring the solvability for coupled system of nonlinear Fractional Langevin equations, *Progr. Fract. Differ. Appl.*, **10** (2024), 73–80. https://doi.org/10.18576/pfda/100107

21. A. Salem, B. Alghamdi, Multi-strip and multi-point boundary conditions for fractional Langevin equation, *Fractal Fract.*, **4** (2020), 18. https://doi.org/10.3390/fractalfract4020018

22. H. Fazli, J. J. Nieto, Fractional Langevin equation with anti-periodic boundary conditions, *Chaos Soliton. Fract.*, **114** (2018), 332–337. https://doi.org/10.1016/j.chaos.2018.07.009

23. A. Wongcharoen, B. Ahmad, S. K. Ntouyas, J. Tariboon, Three-point boundary value problems for Langevin equation with Hilfer fractional derivative, *Adv. Math. Phys.*, **2020** (2020), 1–11. https://doi.org/10.1155/2020/9606428

24. P. Li, R. Qiao, C. Xu, M. Yiao, Y. Qu, S. Ahmad, Exploration of soliton solutions and dynamical analysis of the q-form Zhanbota equation: an application to image encryption, *Phys. A – Stat. Mech. Appl.*, **675** (2025), 130809. https://doi.org/10.1016/j.physa.2025.130809

25. Q. Deng, C. Xu, J. Lin, Y. Zhao, Bifurcation mechanism, speed feedback controller and hybrid controller design in a delayed tumor-immune competitive model, *AIP Adv.*, **15** (2025), 095009. https://doi.org/10.1063/5.0292455

26. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives: theory and applications*, Philadelphia: Gordon and Breach Science Publishers, 1993.

27. Y. Luchko, J. J. Trujillo, Caputo-type modification of the Erdélyi–Kober fractional derivative, *Fract. Calc. Appl. Anal.*, **10** (2007), 249–267.

28. Z. Odibat, D. Baleanu, On a new modification of the Erdélyi–Kober fractional derivative, *Fractal Fract.*, **5** (2021), 121. https://doi.org/10.3390/fractalfract5030121

29. K. Deimling, *Nonlinear functional analysis*, New York, NY, USA: Springer-Verlag, 1985. https://doi.org/10.1007/978-3-662-00547-7

30. M. A. Krasnosel'skiĭ, Two remarks on the method of successive approximations, *Uspekhi Mat. Nauk*, **10** (1955), 123–127.

31. A. Granas, J. Dugundji, *Fixed point theory*, New York, NY, USA: Springer, 2003. https://doi.org/10.1007/978-0-387-21593-8

AIMS Press