Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 11(7): 19488–19515.
DOI: 10.3934/math.2026792
Received: 12 April 2026
Revised: 06 June 2026
Accepted: 17 June 2026
Published: 02 July 2026

Research article

A coupled numerical method for time-fractional convection-diffusion equations based on compressed Legendre polynomials

Hui Zhu[1,2], Lei Yang[1,3,*], Ruimin Zhang[4,2] and Yuntao Jia[2]

1 School of Computer Science and Engineering, Faculty of Innovation Engineering, Macau University of Science and Technology, Macau 999078, China
2 Beijing Institute of Technology, Zhuhai 519088, China
3 Innovation Technology Research Institute, Macau University of Science and Technology, Zhuhai, 519000, Guangdong, China
4 Guangzhou City University of Technology, Guangzhou, 510800

* **Correspondence:** Email: leiyang@must.edu.mo.

Abstract: In this paper, we develop a novel coupled numerical method for time-fractional convection-diffusion equations. First, in the temporal dimension, a derivative-based linear interpolation is utilized to approximate the time partial derivative. Then, we construct a high-order approximate scheme for the temporal fractional derivative, which transforms the original partial differential equation into a semi-discrete ordinary differential system involving only spatial variables. In the spatial dimension, a standard orthogonal basis is constructed by using compressed Legendre polynomials. Combining the concept of ε-approximate solutions, we derive the numerical solution of the semi-discrete scheme. Convergence analysis and numerical experiments demonstrate that the proposed algorithm achieves second-order convergence in time, and the spatial convergence accuracy improves as the Legendre polynomial degree and compression level increase.

Keywords: coupled numerical method; compressed Legendre polynomials; semi-discrete scheme; time fractional convection-diffusion equations
Mathematics Subject Classification: 35R11, 65M12

1. Introduction

In the past few decades, fractional calculus has attracted extensive attention because of its excellent ability to describe complex physical phenomena with memory and nonlocality [1, 2]. Time fractional diffusion equations, reaction-diffusion equations, and diffusion wave equations have become key tools for simulating anomalous transport, viscoelasticity, biological systems, and financial processes [3–5].

However, the nonlocality of the fractional order operator leads to great difficulties in solving fractional diffusion equations. Analytical solutions can be obtained only under a few special coefficients and special boundary conditions, and numerical methods are needed in most cases.

In recent years, scholars worldwide have conducted extensive and in-depth research on the numerical solutions of fractional diffusion equations and achieved a series of significant breakthroughs. Stynes et al. [6] used a spatially uniform, time-graded grid to discretize the reaction-diffusion problem. The Caputo derivative was approximated using the $L1$ scheme, and the spatial variables were approximated using the classical finite difference operator. The $L2$-1σ scheme proposed by Alikhanov [7] achieved an accuracy of $O(\tau^{3-\alpha})$ on a uniform grid, and Roul et al. [8] extended the scheme to a nonuniform grid to solve the two-dimensional variable coefficient time fractional reaction-diffusion equation and established a stability and convergence analysis under the H^1 norm. Li et al. [9] further studied the generalized fractional diffusion equation with variable diffusion coefficients by developing the generalized $L2$ scheme and introducing a time-dependent compact difference operator. Sarumi et al. [10] proposed the generalized exponential time difference (GETD) scheme for the advection-diffusion-reaction equation with variable coefficients. Wang et al. [11] combined the $L1$ scheme with the compact alternating-direction implicit (ADI) method for the two-dimensional time-fractional diffusion equation and gave the point state error estimate. For the fractional convection-diffusion-reaction problem on the metric graph, Wang et al. [12] introduced the ghost point technique to handle the Robin-Kirchhoff condition at the nodes and combined it with the $L1$ scheme to give a fine estimate of the local error.

Besides the finite difference method, spectral methods and spline methods are also important tools for solving fractional-order equations. Sadri et al. [13] used three-variable Jacobi polynomials combined with the collocation method to solve two-dimensional multiterm fractional-order diffusion-wave equations and gave an error estimate. Shafiq et al. [14] combined extended cubic B-splines with finite difference techniques to solve fractional-order diffusion wave equations with exponential kernels and proved the unconditional stability of the method. Wang et al. [15] proposed a numerical scheme that combines the Crank-Nicolson leapfrog scheme with the Riesz fractional-order regenerating kernel method for nonlinear Riesz fractional-order reaction-diffusion equations and analyzed the method's convergence and stability. Chen and Ding [16] addressed the Caputo-Hadamard-type fractional derivative by establishing the $L1$ approximation on a nonuniform grid and proved the global convergence order.

Although remarkable progress has been made in the numerical methods for fractional diffusion equations, traditional numerical methods have problems such as large computational cost and insufficient stability, and there is still room for improvement in the approximation accuracy of fractional operators in the numerical discretization process. On this basis, this paper conducts research on the numerical solution of fractional diffusion equations, aiming to construct a new numerical scheme and enhance computational efficiency and accuracy.

This paper will study the following fractional diffusion equations:

$$\begin{cases} D_t^\alpha u(x,t) = \lambda u_{xx}(x,t) - r u_x(x,t) + f(x,t), \\ u(x,0) = u_0(x), & (x,t) \in \Omega = (0,1) \times (0,T], \\ u(0,t) = g_0(t), u(1,t) = g_1(t), \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$. The constant $\lambda > 0$ denotes the diffusion coefficient, and $r > 0$ denotes the convection coefficient. The source function $f(x,t)$ is taken to be bounded and continuous. To guarantee the smooth behavior of the solution, the initial data $u_0(x)$ should satisfy the compatibility conditions with boundary data $g_0(t)$ and $g_1(t)$. $D_t^\alpha u(x,t)$ is the Caputo fractional derivative. Without loss of generality, we set $T = 1$ for subsequent analysis. This paper proposes a new coupled algorithm to solve Eq (1.1). Different from traditional fully discrete schemes that require refined meshes to achieve high accuracy, this paper constructs a semi-discrete scheme to solve the equation. In the temporal direction, unlike the classical $L1$ and $L2$ schemes, we adopt piecewise linear interpolation on the derivative of function $u(x,t)$ using derivative values to establish an approximation for the time fractional derivative. Accordingly, the original Eq (1.1) is transformed into a system of ordinary differential equations(ODEs) with respect to the spatial variables. Compressed Legendre polynomials exhibit excellent performance in solving ordinary differential equations. In this paper, these polynomials are adopted as basis functions to construct a numerical method for the resulting ODE system. The proposed algorithm achieves high-precision approximation with only a small number of basis functions; it can also effectively suppress the Runge phenomenon induced by high-order polynomial approximation and guarantee numerical stability. By coupling the two approaches, a concise and efficient numerical algorithm is obtained. Both theoretical analysis and numerical verification demonstrate that the proposed method possesses high accuracy, favorable stability, and computational efficiency.

This paper is organized as follows: Section 2 introduces the discretization method for the time variable t, obtaining an approximate expression for the fractional derivative. In Section 3, by substituting the discretized form of the fractional derivative into the original equation, we obtain a semi-discrete scheme in the spatial variables. This semi-discrete system is then solved using compressed Legendre polynomials, yielding the numerical solutions. Section 4 analyzes the convergence and stability of the new algorithm with respect to both time and space variables. Section 5 validates the actual performance of the algorithm through specific examples. The final section provides the conclusions.

2. Discretization of the time variable

First, we employ a piecewise linear interpolation scheme constructed from derivative values to approximate the temporal partial derivative of the function $u(x,t)$.

Divide the interval $[0,1]$ uniformly into n subintervals $[t_{i-1}, t_i], i = 1, \cdots, n$. Set time node $t_i = ih$, and step size $h = 1/n$.

Let $u(x,t)$ be the exact solution to Eq (1.1), and on $[t_{i-1}, t_i], i = 1, \cdots, n$,

$$u_{t,i}(x,t) = u_t(x,t).$$

Assume that the linear interpolation function of $u_{t,i}(x,t)$ on $[t_{i-1}, t_i]$ is $\tilde{u}_{t,i}(x,t)$; we have

$$\tilde{u}_{t,i}(x,t) = u_t(x,t_{i-1})\frac{t_i - t}{h} + u_t(x,t_i)\frac{t - t_{i-1}}{h}. \tag{2.1}$$

Then,

$$\tilde{u}_i(x,t) = \int_{t_0}^{t_1} \tilde{u}_{t,1}(x,t)dt + \int_{t_1}^{t_2} \tilde{u}_{t,2}(x,t)dt + \cdots + \int_{t_{i-1}}^{t} \tilde{u}_{t,i}(x,t)dt + u_0(x). \tag{2.2}$$

On the interval $[t_{i-1}, t_i]$,

$$\begin{aligned}
\tilde{u}_i(x,t_i) &= \int_{t_0}^{t_1} \tilde{u}_{t,1}(x,s)ds + \int_{t_1}^{t_2} \tilde{u}_{t,2}(x,s)ds + \cdots + \int_{t_{i-1}}^{t_i} \tilde{u}_{t,i}(x,s)ds + u_0(x) \\
&= \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} \tilde{u}_{t,j}(x,s)ds + u_0(x) \\
&= \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} (u_t(x,t_{j-1})\frac{t_j - s}{h} + u_t(x,t_j)\frac{s - t_{j-1}}{h})ds + u_0(x), \qquad i = 1, 2, \cdots, n.
\end{aligned}$$

Therefore,

$$\tilde{u}_i(x,t_i) = \sum_{j=0}^{i} \beta_j^i u_t(x,t_j) + u_0(x), \quad i = 1, \cdots, n, \tag{2.3}$$

where

$$\begin{cases}
\beta_0^i = \int_{t_0}^{t_1} \frac{t_1 - s}{h} ds = \frac{h}{2}, \\
\beta_j^i = \int_{t_{j-1}}^{t_j} \frac{s - t_{j-1}}{h} ds + \int_{t_j}^{t_{j+1}} \frac{t_{j+1} - s}{h} ds = h, \qquad 1 \le j \le i-1, i \ge 2, \\
\beta_i^i = \int_{t_{i-1}}^{t_i} \frac{s - t_{i-1}}{h} ds = \frac{h}{2}.
\end{cases}$$

Consequently, we obtain

$$\tilde{u}_{x,i}(x,t_i) = \sum_{j=0}^{i} \beta_j^i u_{tx}(x,t_j) + u_0'(x), \tag{2.4}$$

and

$$\tilde{u}_{xx,i}(x,t_i) = \sum_{j=0}^{i} \beta_j^i u_{txx}(x,t_j) + u_0''(x). \tag{2.5}$$

According to Eq (2.1), the approximate expression of the fractional derivative can be obtained as:

$$D_t^\alpha \tilde{u}(x,t) = \frac{1}{\Gamma(1-\alpha)}[\int_{t_0}^{t_1} \frac{\tilde{u}_{t,1}(x,s)}{(t_i - s)^\alpha} ds + \int_{t_1}^{t_2} \frac{\tilde{u}_{t,2}(x,s)}{(t_i - s)^\alpha} ds + \cdots + \int_{t_{i-1}}^{t} \frac{\tilde{u}_{t,i}(x,s)}{(t_i - s)^\alpha} ds],$$

and

$$\begin{aligned}
D_t^\alpha \tilde{u}(x,t_i) &= \frac{1}{\Gamma(1-\alpha)}[\int_{t_0}^{t_1} \frac{\tilde{u}_{t,1}(x,s)}{(t_i - s)^\alpha} ds + \int_{t_1}^{t_2} \frac{\tilde{u}_{t,2}(x,s)}{(t_i - s)^\alpha} ds + \cdots + \int_{t_{i-1}}^{t_i} \frac{\tilde{u}_{t,i}(x,s)}{(t_i - s)^\alpha} ds] \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} \frac{\tilde{u}_{t,j}(x,s)}{(t_i - s)^\alpha} ds \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} \frac{1}{(t_i - s)^\alpha} (u_t(x,t_{j-1})\frac{t_j - s}{h} + u_t(x,t_j)\frac{s - t_{j-1}}{h}) ds \\
&= \sum_{j=0}^{i} \gamma_j^i u_t(x,t_j), \quad i = 1, \cdots, n.
\end{aligned} \tag{2.6}$$

That is,

$$D_t^\alpha \tilde{u}(x,t_i) = \sum_{j=0}^{i} \gamma_j^i u_t(x,t_j), \qquad i = 1, \cdots, n, \tag{2.7}$$

where

$$\begin{cases} \gamma_0^i = \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_{t_0}^{t_1} \dfrac{1}{(t_i-s)^\alpha} \dfrac{t_1-s}{h} ds, \\ \gamma_j^i = \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_{t_{j-1}}^{t_j} \dfrac{1}{(t_i-s)^\alpha} \dfrac{s-t_{j-1}}{h} ds + \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_{t_j}^{t_{j+1}} \dfrac{1}{(t_i-s)^\alpha} \dfrac{t_{j+1}-s}{h} ds, \qquad 1 \le j \le i-1, i \ge 2, \\ \gamma_i^i = \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_{t_{i-1}}^{t_i} \dfrac{1}{(t_i-s)^\alpha} \dfrac{s-t_{i-1}}{h} ds. \end{cases} \tag{2.8}$$

Direct computation yields

$$\begin{cases} \gamma_0^i = \dfrac{h^{1-\alpha}}{\Gamma(3-\alpha)} \left[(i-1)^{2-\alpha} - i^{2-\alpha} + (2-\alpha) i^{1-\alpha} \right], \\ \gamma_j^i = \dfrac{h^{1-\alpha}}{\Gamma(3-\alpha)} \left[(i-j+1)^{2-\alpha} - 2(i-j)^{2-\alpha} + (i-j-1)^{2-\alpha} \right], \qquad 1 \le j \le i-1, i \ge 2, \\ \gamma_i^i = \dfrac{h^{1-\alpha}}{\Gamma(3-\alpha)}. \end{cases} \tag{2.9}$$

One can easily derive the following conclusion.

Lemma 2.1. If $0 < \alpha < 1$, the coefficient $\gamma_j^i,\ i = 1, 2 \cdots, n, j = 1, 2,\ \cdots, i$ satisfies the following

(1) $\gamma_j^i > 0, \quad j = 1, 2, \cdots,\ i$;

(2) $\gamma_j^i \le \gamma_{j+1}^i, \quad j = 1, 2, \cdots,\ i-2$;

(3) $\sum\limits_{j=1}^{i} \gamma_j^i = \dfrac{(hi)^{1-\alpha}}{\Gamma(2-\alpha)}$.

3. Compressed Legendre algorithm

Substituting $\tilde{u}_i(x,t_i)$ into Eq (1.1) and denoting $u^i(x), u_t^i(x)$ as the numerical solution of $u(x,t_i), u_t(x,t_i)$, Eq (1.1) can be transformed into the following semi-discrete form:

$$\begin{cases} D_t^\alpha u^i = \lambda u_{xx}^i - r u_x^i + f^i(x), \\ u^0(x) = u_0(x), & x \in (0,1), \\ u^i(0) = g_0(t_i), u^i(1) = g_1(t_i), \quad i = 1, \cdots, n, \end{cases} \tag{3.1}$$

where $f^i(x) = f(x,t_i),\ u^i(x) = \sum\limits_{j=0}^{i} \beta_j^i u_t^j(x) + u_0(x)$.

Equation (3.1) can be rewritten in terms of $u_t^i(x)$ as follows:

$$\begin{cases} \sum_{j=0}^{i} \gamma_j^i u_t^j(x) - \lambda \sum_{j=1}^{i} \beta_j^i u_{txx}^j(x) + r \sum_{j=1}^{i} \beta_j^i u_{tx}^j(x) \\ = \lambda u_0''(x) - r u_0'(x) + f^i(x), \\ \sum_{j=0}^{i} \beta_j^i u_t^j(0) + u_0(0) = g_0(t_i), & x \in (0,1). \\ \sum_{j=0}^{i} \beta_j^i u_t^j(1) + u_0(1) = g_1(t_i), \quad i = 1, \cdots, n, \end{cases} \tag{3.2}$$

It can be seen that the above equations are several second-order ordinary differential equations for $u_t^i(x)$. By solving these equations, $u_t^0(x), u_t^1(x), \cdots, u_t^n(x)$ can be obtained iteratively. Furthermore, the approximate expression for $u(x,t)$ can be derived from Eqs (2.1) and (2.2), which can be expressed as

$$u(x,t) \approx \begin{cases} \tilde{u}_1(x,t), & t \in [t_0, t_1], \\ \cdots\cdots \\ \tilde{u}_i(x,t), & t \in [t_{i-1}, t_i], \\ \cdots\cdots \\ \tilde{u}_n(x,t), & t \in [t_{n-1}, t_n], \end{cases}$$

$$= \begin{cases} \int_{t_0}^{t} \left(u_t^0 \frac{t_1 - t}{h} + u_t^1 \frac{t - t_0}{h} \right) dt + u_0(x), & t \in [t_0, t_1], \\ \cdots\cdots \\ \int_{t_0}^{t_1} \left(u_t^0 \frac{t_1 - t}{h} + u_t^1 \frac{t - t_0}{h} \right) dt + \cdots + \int_{t_{i-1}}^{t} \left(u_t^{i-1} \frac{t_i - t}{h} + u_t^i \frac{t - t_{i-1}}{h} \right) dt + u_0(x), & t \in [t_{i-1}, t_i], \\ \cdots\cdots \\ \int_{t_0}^{t_1} \left(u_t^0 \frac{t_1 - t}{h} + u_t^1 \frac{t - t_0}{h} \right) dt + \cdots + \int_{t_{n-1}}^{t} \left(u_t^{n-1} \frac{t_n - t}{h} + u_t^n \frac{t - t_{n-1}}{h} \right) dt + u_0(x), & t \in [t_{n-1}, t_n]. \end{cases} \tag{3.3}$$

Next, Eq (3.2) will be solved using the compressed Legendre polynomial algorithm.

3.1. Compressed Legendre polynomial

Define the reproducing kernel space [17] as

$$W_2^3[0,1] = \left\{ u(x) \,\middle|\, u'' \text{ is absolutely continuous on [0,1]}, u''' \in L^2[0,1] \right\}.$$

The inner product of $W_2^3[0,1]$ is

$$\langle u, v \rangle_{W_2^3[0,1]} = u(0)v(0) + u'(0)v'(0) + u''(0)v''(0) + \int_0^1 u'''v'''dx, \tag{3.4}$$

and inner product of $L^2[0,1]$ is

$$\langle u, v \rangle_{L^2[0,1]} = \int_0^1 uvdx. \tag{3.5}$$

It is well-known that

$$L(m,x)=\sum_{i=0}^{[\frac{m}{2}]}\frac{(-1)^i((2m-2i)!)x^{m-2i}}{2^m i!(m-i)!(m-2i)!},$$

is an orthonormal basis in $L^2[-1,1]$. Obviously,

$$\Phi(m,x)=\sqrt{2m+1}\sum_{i=0}^{[\frac{m}{2}]}\frac{(-1)^i((2m-2i)!)(2x-1)^{m-2i}}{2^m i!(m-i)!(m-2i)!}, \tag{3.6}$$

forms a standard orthonormal basis for $L^2[0,1]$. Using Eq (3.6), we construct an orthonormal basis for $W_2^3[0,1]$. First, we compress the polynomials in Eq (3.6) to generate a modified basis for $L^2[0,1]$.

Lemma 3.1. [18] Let

$$\varphi_{mk}(x)=\begin{cases}\sqrt{p}\Phi_m(px-k+1), & x\in[\frac{k-1}{p},\frac{k}{p}],\ \ k=1,2,\cdots,p, p\in Z^+,\\ 0, & else.\end{cases}$$

Then $\{\varphi_{mk}\}_{m=0,k=1}^{\infty,p}$ is a standard orthonormal basis of $L^2[0,1]$. Define new functions as

$$JJJ\varphi_{mk}(x)=\int_0^x dy\int_0^y ds\int_0^s \varphi_{mk}(t)dt. \tag{3.7}$$

Theorem 3.1. $\{\psi_l(x)\}_{l=1}^{\infty}\overset{\Delta}{=}\left\{1,x,x^2/2\right\}\cup\{JJJ\varphi_{mk}(x)\}_{m=1,k=1}^{\infty,p}$ is a standard orthonormal basis of $W_2^3[0,1]$.

Proof. Note that $\psi_1(x)=1,\psi_2(x)=x,\psi_3(x)=x^2/2$, and $JJJ\phi_{mk}(0)=JJJ\phi'_{mk}(0)=JJJ\phi''_{mk}(0)=0$. According to the definition of the inner product on the function space $W_2^3[0,1]$, we obtain

$$\left\langle\psi_i,\psi_j\right\rangle_{W_2^3[0,1]}=\begin{cases}1, & i=j,\\ 0, & i\neq j,\end{cases}\ \ i=1,2,3,\ j=1,2,3,$$

and

$$\left\langle\psi_i,\psi_j\right\rangle_{W_2^3[0,1]}=0,\ \ i=1,2,3,\ j=4,5,6\cdots$$

Subsequently,

$$\begin{aligned}\left\langle\psi_i,\psi_j\right\rangle_{W_2^3[0,1]}&=\langle JJJ\varphi_{mk},JJJ\varphi_{ns}\rangle_{W_2^3[0,1]}\\&=\sum_{q=0}^{2}JJJ\varphi_{mk}^{(q)}(0)JJJ\varphi_{ns}^{(q)}(0)+\int_0^1(JJJ\varphi_{mk})'''(JJJ\varphi_{ns})'''dt\\&=\int_0^1\varphi_{mk}\varphi_{ns}dx\\&=\begin{cases}1, & m=n,\ k=s,\\ 0, & others.\end{cases}\ \ i=4,5,6,\cdots,j=4,5,6,\cdots\end{aligned}$$

Therefore, $\{\psi_l(x)\}_{l=1}^{\infty}$ are mutually orthogonal.

Now, consider the completeness of $\{\psi_l(x)\}_{l=1}^{\infty}$. $\forall u(x) \in W_2^3[0,1]$; if

$$\langle u(x), \psi_i \rangle_{W_2^3[0,1]} = 0, i = 1, 2, 3,$$

then

$$u(0) = u'(0) = u''(0) = 0.$$

In addition, if $\langle u(x), \psi_i \rangle_{W_2^3[0,1]} = 0, i = 4, 5, 6 \cdots$, then

$$\begin{aligned}
\langle u(x), JJJ\varphi_{mk}(x) \rangle_{W_2^3[0,1]} &= \sum_{q=0}^{2} u^{(q)}(0) JJJ\varphi_{mk}^{(q)}(0) + \int_0^1 u'''(x)(JJJ\varphi_{mk}(x))''' dt \\
&= \int_0^1 u'''(x)\varphi_{mk}(x) dx \\
&= 0, \quad n = 1, 2, 3, \cdots, k = 1, 2, 3, \cdots, p.
\end{aligned}$$

Because $\{\varphi_{mk}\}_{m=0,k=1}^{\infty,p}$ is a standard orthonormal basis of $L^2[0,1]$, we obtain $u'''(x) = 0$. That is, $u(x)$ is a cubic polynomial. Noting $u(0) = u'(0) = u''(0) = 0$, we conclude $u(x) \equiv 0$.

It follows that if $\langle u(x), \psi_i \rangle_{W_2^3[0,1]} = 0, i = 1, 2, 3, \cdots$, then $u(x) \equiv 0$, which proves the completeness [17].

Combined with

$$\|\psi_l\|_{W_2^3[0,1]} = 1, \quad l = 1, 2, 3, \cdots,$$

we conclude that $\{\psi_l\}_{l=1}^{\infty}$ constitutes an orthonormal basis for $W_2^3[0,1]$. □

Next, this set of functions is used to compute the numerical solution $u_t^i(x)$ to $u_t(x, t_i)$.

3.2. ε-approximate solution

First, we determine $u_t^0(x)$ and $u_t^1(x)$. At $i = 1$, Eq (3.2) becomes

$$\begin{cases}
\sum_{j=0}^{1} \gamma_j^1 u_t^j(x) - \lambda \sum_{j=0}^{1} \beta_j^1 u_{txx}^j(x) + r \sum_{j=0}^{1} \beta_j^1 u_{tx}^j(x) \\
= \lambda u_0''(x) - r u_0'(x) + f^1(x), & x \in (0,1). \\
\sum_{j=0}^{1} \beta_j^1 u_t^j(0) + u_0(0) = g_0(t_1), \sum_{j=0}^{1} \beta_j^1 u_t^j(1) + u_0(1) = g_1(t_1),
\end{cases} \tag{3.8}$$

Because there is only one equation associated with $u_t^0(x)$ and $u_t^1(x)$, by choosing an arbitrary $t = t_\xi$ in Eq (2.2) and substituting $\tilde{u}_{t,1}(x, t_\xi)$ into Eq (1.1), we obtain

$$\begin{cases}
\sum_{j=0}^{1} \tilde{\gamma}_j^1 u_t^j(x) - \lambda \sum_{j=0}^{1} \tilde{\beta}_j^1 u_{txx}^j(x) + r \sum_{j=0}^{1} \tilde{\beta}_j^1 u_{tx}^j(x) \\
= \lambda u_0''(x) - r u_0'(x) + f(x, t_\xi), & x \in (0,1), \\
\sum_{j=0}^{1} \tilde{\beta}_j^1 u_t^j(0) + u_0(0) = g_0(t_\xi), \sum_{j=0}^{1} \tilde{\beta}_j^1 u_t^j(1) + u_0(1) = g_1(t_\xi),
\end{cases} \tag{3.9}$$

where $\tilde{\beta}^1_j,\ \tilde{\gamma}^1_j,\ j=0,1$ depend on t_ξ .

Define the linear operators as follows:

$$\mathcal{L}_i : W^3_2[0,1] \to L^2[0,1], \quad i=0,1,$$

and

$$\begin{aligned}
\mathcal{L}_0 u^0_t(x) &= \sum_{j=0}^{1} \tilde{\gamma}^1_j u^j_t(x) - \lambda \sum_{j=0}^{1} \tilde{\beta}^1_j u^j_{txx}(x) + r \sum_{j=0}^{1} \tilde{\beta}^1_j u^j_{tx}(x),\\
\mathcal{L}_1 u^1_t(x) &= \sum_{j=0}^{1} \gamma^1_j u^j_t(x) - \lambda \sum_{j=1}^{1} \beta^1_j u^j_{txx}(x) + r \sum_{j=1}^{1} \beta^1_j u^j_{tx}(x).
\end{aligned}$$

Equations (3.8) and (3.9) can be rewritten in the following operator form:

$$\begin{cases}
\mathcal{L}_0 u^0_t(x) = \tilde{f}^0(x),\\
\mathcal{L}_1 u^1_t(x) = \tilde{f}^1(x),\\
\mathcal{B}^0_0 u^0_t(x) = g_0(t_\xi), \mathcal{B}^0_1 u^0_t(x) = g_1(t_\xi),\\
\mathcal{B}^1_0 u^1_t(x) = g_0(t_1), \mathcal{B}^1_1 u^1_t(x) = g_1(t_1),
\end{cases} \tag{3.10}$$

where

$$\begin{aligned}
\tilde{f}^0(x) &= \lambda u''_0(x) - r u'_0(x) + f(x,t_\xi),\\
\tilde{f}^1(x) &= \lambda u''_0(x) - r u'_0(x) + f(x,t_1),
\end{aligned}$$

and

$$\mathcal{B}^0_0 u^0_t(x) = g_0(t_\xi), \mathcal{B}^0_1 u^0_t(x,t_0) = g_1(t_\xi), \mathcal{B}^1_0 u^1_t(x) = g_0(t_1), \mathcal{B}^1_1 u^1_t(x)(x) = g_1(t_1),$$

are boundary conditions.

We introduce the concept of ε-approximate solutions to solve Eq (3.10).

Definition 3.1. For $y_0(x), y_1(x) \in W^3_2[0,1]$,$\forall \varepsilon > 0$, $y_0(x), y_1(x)$ are called ε-approximate solutions of Eq (3.10) when

$$\sum_{i=0}^{1} \left\| \mathcal{L}_i y_i(x) - \tilde{f}^i \right\|^2_{L^2[0,1]} + \sum_{i=0}^{1} \left[\left(\mathcal{B}^0_i y_i - g_i(t_\xi) \right)^2 \right] + \sum_{i=0}^{1} \left[\left(\mathcal{B}^1_i y_i - g_i(t_1) \right)^2 \right] < \varepsilon.$$

Theorem 3.2. $\forall \varepsilon > 0$, $\exists N$; when $M > N$, $u^0_{t,M}(x) = \sum\limits_{j=1}^{M} c^{0*}_j \psi_j, u^1_{t,M}(x) = \sum\limits_{j=1}^{M} c^{1*}_j \psi_j$ is the ε-approximate solution of Eq (3.10), where $c^{0*}_j, c^{1*}_j, j = 1, \cdots, M$ satisfy

$$\begin{aligned}
&\sum_{i=0}^{1} \left\| \mathcal{L}_i \sum_{j=1}^{M} c^{i*}_j \psi_j(x) - \tilde{f}^i \right\|^2 + \sum_{i=0}^{1} \left[\left(\mathcal{B}^0_i \sum_{j=1}^{M} c^{0*}_j \psi_j(x) - g_i(t_\xi) \right)^2 + \left(\mathcal{B}^1_i \sum_{j=1}^{M} c^{1*}_j \psi_j(x) - g_i(t_1) \right)^2 \right]\\
&= \min_{c^0_j, c^1_j} \sum_{i=0}^{1} \left\| \mathcal{L}_i \sum_{j=1}^{M} c^i_j \psi_j(x) - \tilde{f}^i \right\|^2 + \sum_{i=0}^{1} \left[\left(\mathcal{B}^0_i \sum_{j=1}^{M} c^0_j \psi_j(x) - g_i(t_\xi) \right)^2 + \left(\mathcal{B}^1_i \sum_{j=1}^{M} c^1_j \psi_j(x) - g_i(t_1) \right)^2 \right].
\end{aligned}$$

Proof. Assume that y_0, y_1 is the exact solution of Eq (3.10),

$$y_0 = \sum_{j=1}^{\infty} a_j^0 \psi_j, y_1 = \sum_{j=1}^{\infty} a_j^1 \psi_j.$$

$\forall \varepsilon > 0$, $\exists N$, when $M > N$, such that

$$y_{0M} = \sum_{j=1}^{M} a_j^0 \psi_j, y_{1M} = \sum_{j=1}^{M} a_j^1 \psi_j$$

satisfy

$$\|y_0 - y_{0M}\| < \sqrt{\frac{\varepsilon}{2\left(\|\mathcal{L}_0\|^2 + \sum_{i=0}^{1} \left\|\mathcal{B}_i^0\right\|^2\right)}}, \|y_1 - y_{1M}\| < \sqrt{\frac{\varepsilon}{2\left(\|\mathcal{L}_1\|^2 + \sum_{i=0}^{1} \left\|\mathcal{B}_i^1\right\|^2\right)}}.$$

Then,

$$\begin{aligned}
&\sum_{i=0}^{1} \left\| \mathcal{L}_i \sum_{j=1}^{M} c_j^{i*} \psi_j(x) - \tilde{f}^i \right\|^2 + \sum_{i=0}^{1} \left[\left(\mathcal{B}_i^0 \sum_{j=1}^{M} c_j^{0*} \psi_j(x) - g_i(t_\xi) \right)^2 + \left(\mathcal{B}_i^0 \sum_{j=1}^{M} c_j^{1*} \psi_j(x) - g_i(t_1) \right)^2 \right] \\
&= \min_{c_j^0, c_j^1} \sum_{i=0}^{1} \left\| \mathcal{L}_i \sum_{j=1}^{M} c_j^{i} \psi_j(x) - \tilde{f}^i \right\|^2 + \sum_{i=0}^{1} \left[\left(\mathcal{B}_i^0 \sum_{j=1}^{M} c_j^{0} \psi_j(x) - g_i(t_\xi) \right)^2 + \left(\mathcal{B}_i^1 \sum_{j=1}^{M} c_j^{1} \psi_j(x) - g_i(t_1) \right)^2 \right] \\
&\leq \sum_{i=0}^{1} \left\| \mathcal{L}_i \sum_{j=1}^{M} a_j^{i} \psi_j(x) - \tilde{f}^i \right\|^2 + \sum_{i=0}^{1} \left[\left(\mathcal{B}_i^0 \sum_{j=1}^{M} a_j^{0} \psi_j(x) - g_i(t_\xi) \right)^2 + \left(\mathcal{B}_i^1 \sum_{j=1}^{M} a_j^{1} \psi_j(x) - g_i(t_1) \right)^2 \right] \\
&= \sum_{i=0}^{1} \left\| \mathcal{L}_i y_{iM} - \tilde{f}^i \right\|^2 + \sum_{i=0}^{1} \left[\left(\mathcal{B}_i^0 y_{0M} - g_i(t_\xi) \right)^2 + \left(\mathcal{B}_i^1 y_{1M} - g_i(t_1) \right)^2 \right] \\
&\leq \sum_{i=0}^{1} \|\mathcal{L}_i y_{iM} - \mathcal{L}_i y_i\|^2 + \sum_{i=0}^{1} \left[\left(\mathcal{B}_i^0 y_{0M} - \mathcal{B}_i^0 y_0 \right)^2 + \left(\mathcal{B}_i^1 y_{1M} - \mathcal{B}_i^1 y_1 \right)^2 \right] \\
&\leq \|\mathcal{L}_0\|^2 \|y_{0M} - y_0\|^2 + \|\mathcal{L}_1\|^2 \|y_{1M} - y_1\|^2 + \sum_{i=0}^{1} \left[\left\|\mathcal{B}_i^0\right\|^2 \|y_{0M} - y_0\|^2 + \left\|\mathcal{B}_i^1\right\|^2 \|y_{1M} - y_1\|^2 \right] \\
&\leq \left(\|\mathcal{L}_0\|^2 + \sum_{i=0}^{1} \left\|\mathcal{B}_i^0\right\|^2 \right) \|y_{0M} - y_0\|^2 + \left(\|\mathcal{L}_1\|^2 + \sum_{i=0}^{1} \left\|\mathcal{B}_i^1\right\|^2 \right) \|y_{1M} - y_1\|^2 \\
&\leq \varepsilon.
\end{aligned}$$

□

Let

$$\begin{aligned}
F(c_1^0, \cdots, c_M^0, c_1^1, \cdots, c_M^1) = \min_{c_j^0, c_j^1} &\sum_{i=0}^{1} \left\| \mathcal{L}_i \sum_{j=1}^{M} c_j^{i} \psi_j(x) - \tilde{f}^i \right\|^2 \\
&+ \sum_{i=0}^{1} \left[\left(\mathcal{B}_i^0 \sum_{j=1}^{M} c_j^{0} \psi_j(x) - g_i(t_\xi) \right)^2 + \left(\mathcal{B}_i^1 \sum_{j=1}^{M} c_j^{1} \psi_j(x) - g_i(t_1) \right)^2 \right].
\end{aligned} \tag{3.11}$$

The coefficients $c_1^0, \cdots, c_M^0, c_1^1, \cdots, c_M^1$ are obtained by computing the minimum point of the function $F(c_1^0, \cdots, c_M^0, c_1^1, \cdots, c_M^1)$, from which $u_{t,M}^0(x), u_{t,M}^1(x)$ are determined.

We proceed to solve Eq (3.2) for $i \geq 2$ in a similar fashion.

Define the linear operators as follows:

$$\mathcal{L}_i : W_2^3[0,1] \to L^2[0,1], \quad i = 2, 3, \cdots n,$$

and

$$\mathcal{L}_i u_t^i(x) = \gamma_i^i u_t^i(x) - \lambda \beta_i^i u_{txx}^j(x) + r\beta_i^i u_{tx}^j(x).$$

Equation (3.2) can be expressed as

$$\begin{cases} \mathcal{L}_i u_t^i(x) = \tilde{f}^i(x), \\ \mathcal{B}_0^i u_t^i(x) = g_0(t_i), \mathcal{B}_1^i u_t^i(x) = g_1(t_i), \end{cases} i = 2, 3, \cdots, n, \tag{3.12}$$

where

$$\tilde{f}^i(x) = \lambda u_0''(x) - ru_0'(x) + f(x, t_i) - \sum_{j=0}^{i-1} \gamma_j^i u_t^i(x) + \lambda \sum_{j=1}^{i-1} \beta_j^i u_{txx}^j(x) - r \sum_{j=1}^{i-1} \beta_j^i u_{tx}^j(x),$$

and

$$\mathcal{B}_0^i u_t^i(x) = g_0(t_i), \mathcal{B}_1^i u_t^i(x) = g_1(t_i),$$

are boundary conditions.

Definition 3.2. For $y_i(x) \in W_2^3[0,1]$, $\forall \varepsilon > 0$, $y_i(x)$ are called ε-approximate solutions of Eq (3.12) when

$$\left\| \mathcal{L}_i y_i(x) - \tilde{f}^i \right\|_{L^2[0,1]}^2 + \sum_{j=0}^{1} \left(\mathcal{B}_j^i y_j - g_j(t_i) \right)^2 < \varepsilon.$$

Following a similar argument to the proof of Theorem 3.2, the following theorem can be established.

Theorem 3.3. $\forall \varepsilon > 0$, $\exists N$; when $M > N$, $u_{t,M}^i(x) = \sum\limits_{j=1}^{M} c_j^{i*} \psi_j$ is the ε-approximate solution of Eq (3.12), where c_j^{i*}, $j = 1, \cdots, M$ satisfies

$$\begin{aligned} F(c_1^i, c_2^i, \cdots c_M^i) = \min_{c_j^i} \Bigg[& \left\| \mathcal{L}_i \sum_{j=1}^{M} c_j^i \psi_j(x) - \tilde{f}^i \right\|^2 + \left(\mathcal{B}_0^i \sum_{j=1}^{M} c_j^i \psi_j(x) - g_0(t_i) \right)^2 \\ & + \left(\mathcal{B}_1^i \sum_{j=1}^{M} c_j^i \psi_j(x) - g_1(t_i) \right)^2 \Bigg]. \end{aligned} \tag{3.13}$$

According to Theorem 3.3, the approximate solutions $u_{t,M}^i(x)$ to Eq (3.12) can be obtained by finding the minimizer of $F(c_1^i, c_2^i, \cdots c_M^i)$.

Substituting $u^i_{t,M}(x),\ i = 1, 2, \cdots, n$ into Eq (3.3) yields the approximate solution to the original Eq (1.1) as

$$u(x,t) \approx \begin{cases} \displaystyle\int_{t_0}^{t} \left(u^0_{t,M}\frac{t_1 - t}{h} + u^1_{t,M}\frac{t - t_0}{h}\right) dt + u_0(x), \qquad t \in [t_0, t_1], \\ \cdots\cdots \\ \displaystyle\int_{t_0}^{t_1} \left(u^0_{t,M}\frac{t_1 - t}{h} + u^1_{t,M}\frac{t - t_0}{h}\right) dt + \cdots + \int_{t_{i-1}}^{t} \left(u^{i-1}_{t,M}\frac{t_i - t}{h} + u^i_{t,M}\frac{t - t_{i-1}}{h}\right) dt + u_0(x), \quad t \in [t_{i-1}, t_i], \\ \cdots\cdots \\ \displaystyle\int_{t_0}^{t_1} \left(u^0_{t,M}\frac{t_1 - t}{h} + u^1_{t,M}\frac{t - t_0}{h}\right) dt + \cdots + \int_{t_{n-1}}^{t} \left(u^{n-1}_{t,M}\frac{t_n - t}{h} + u^n_{t,M}\frac{t - t_{n-1}}{h}\right) dt + u_0(x), \quad t \in [t_{n-1}, t_n]. \end{cases} \tag{3.14}$$

Thus, we have constructed the coupled algorithm for solving $u(x,t)$.

Next, we analyze the convergence and stability of the algorithm.

4. Error analysis

In the first step, we perform error analysis for the semi-discrete scheme.

Theorem 4.1. Let $Ru(x,t_i) = D^\alpha_t u(x,t_i) - D^\alpha_t \tilde{u}(x,t_i)$. If $u_{ttt}(x,t)$ is bounded on $[0,1] \times [0,1]$, then $\forall 0 < \alpha < 1$, we have

$$|Ru(x,t_i)| \le M_1 h^2, \ \ i = 1, 2, \cdots, n,$$

where M_1 is a constant.

Proof. According to the Taylor theorem and the boundedness of $u_{ttt}(x,t)$ on $[0,1] \times [0,1]$, we have

$$\begin{aligned} \left|u_{t,i}(x,t) - \tilde{u}_{t,i}(x,t)\right| &= \left|\frac{u_{ttt}(x,\xi_i)}{2}(t - t_i)(t - t_{i-1})\right| \\ &\le M_0 \left|(t - t_i)(t - t_{i-1})\right| \le M_0 h^2, \end{aligned} \tag{4.1}$$

where $\xi_i \in (t_{i-1}, t_i)$, and $M_0 = \max\limits_{\substack{0\le t\le 1 \\ 0\le x\le 1}} |u_{ttt}(x,t)|/2$.

From Eqs (2.6) and (4.1), we obtain

$$\begin{aligned} |R_i u(x,t)| &= \frac{1}{\Gamma(1-\alpha)} \left| \int_{t_0}^{t_i} \frac{u_t(x,t) - \tilde{u}_t(x,t)}{(t_i - s)^\alpha} ds \right| \\ &= \frac{1}{\Gamma(1-\alpha)} \left| \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} \frac{u_{t,j}(x,t) - \tilde{u}_{t,j}(x,t)}{(t_i - s)^\alpha} ds \right| \\ &\le \frac{M_0 h^2}{\Gamma(1-\alpha)} \left| \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} \frac{1}{(t_i - s)^\alpha} ds \right| \\ &\le \frac{M_0 h^2}{\Gamma(1-\alpha)} \left| \frac{(ih)^{1-\alpha}}{1-\alpha} \right| \le \frac{M_0 h^2}{(1-\alpha)\Gamma(1-\alpha)} \\ &\le M_1 h^2, \qquad \left(M_1 = \frac{M_0}{(1-\alpha)\Gamma(1-\alpha)}\right). \end{aligned}$$

□

We then proceed to prove the order of convergence order of the semi-discrete scheme.

Theorem 4.2. Let $u(x,t_i)$ and u^i be the solutions to Eqs (1.1) and (3.1), respectively. Set $e^i = u^i(x) - u(x,t_i)$. If $u_{ttt}(x,t)$ is bounded on $[0,1]\times[0,1]$, then

$$\left\|e^i\right\|_{L^2} = O(h^2), i = 1,2,\cdots,n.$$

Proof. Clearly, $e^0 = 0,\ e^i(0) = e^i(1) = 0$.

From Eqs (1.1) and (3.1), we obtain

$$\sum_{j=0}^{i}\gamma_j^i e_t^j = \lambda e_{xx}^i - re_x^i + Ru(x,t_i), \tag{4.2}$$

where $|Ru(x,t_i)| \le M_1h^2,\ \ i = 1,2,\cdots,n$. Using difference quotients to approximate derivatives, the left-hand side of Eq (4.2) can be rewritten as follows:

$$\begin{aligned}
\sum_{j=0}^{i}\gamma_j^i e_t^j &= \gamma_0^i e_t^0 + \gamma_i^i e_t^i + \sum_{j=1}^{i-1}\gamma_j^i e_t^j \\
&= \gamma_0^i\left(\frac{e^1-e^0}{h} - \frac{e_{tt}(x,\zeta_0)}{2}h\right) + \gamma_i^i\left(\frac{e^i-e^{i-1}}{h} + \frac{e_{tt}(x,\zeta_i)}{2}h\right) \\
&+ \sum_{j=1}^{i-1}\gamma_j^i\left(\frac{e^{j+1}-e^{j-1}}{2h} - \frac{e_{ttt}(x,\zeta_j)}{6}h^2\right) \\
&= \frac{\gamma_0^i e^1}{h} - \frac{\gamma_0^i e_{tt}(x,\zeta_0)}{2}h + \frac{\gamma_i^i e^i}{h} - \frac{\gamma_i^i e^{i-1}}{h} + \frac{\gamma_i^i e_{tt}(x,\zeta_i)}{2}h \\
&+ \frac{1}{2h}\left(\sum_{j=2}^{i-2}\left(\gamma_{j-1}^i - \gamma_{j+1}^i\right)e^j + \gamma_{i-1}^i e^i + \gamma_{i-2}^i e^{i-1} - \gamma_2^i e^1\right) - \sum_{j=1}^{i-1}\frac{\gamma_j^i e_{ttt}(x,\zeta_j)}{6}h^2,
\end{aligned} \tag{4.3}$$

where $\zeta_j \in (t_{j-1},t_j), j = 1,2,\cdots,n$.

Inserting Eq (4.3) into Eq (4.2) gives

$$\begin{aligned}
&\frac{\gamma_0^i e^1}{h} - \frac{\gamma_0^i e_{tt}(x,\zeta_0)}{2}h + \frac{\gamma_i^i e^i}{h} - \frac{\gamma_i^i e^{i-1}}{h} + \frac{\gamma_i^i e_{tt}(x,\zeta_i)}{2}h \\
&+ \frac{1}{2h}\left(\sum_{j=2}^{i-2}\left(\gamma_{j-1}^i - \gamma_{j+1}^i\right)e^j + \gamma_{i-1}^i e^i + \gamma_{i-2}^i e^{i-1} - \gamma_2^i e^1\right) - \sum_{j=1}^{i-1}\frac{\gamma_j^i e_{ttt}(x,\zeta_j)}{6}h^2 \\
=&\lambda e_{xx}^i - re_x^i + Ru(x,t_i).
\end{aligned} \tag{4.4}$$

Taking the inner product of both sides with e^i yields

$$\begin{aligned}
&\left\langle\frac{\gamma_0^i e^1}{h}, e^i\right\rangle - \left\langle\frac{\gamma_0^i e_{tt}(x,\zeta_0)}{2}h, e^i\right\rangle + \left\langle\frac{\gamma_i^i e^i}{h}, e^i\right\rangle - \left\langle\frac{\gamma_i^i e^{i-1}}{h}, e^i\right\rangle + \left\langle\frac{\gamma_i^i e_{tt}(x,\zeta_i)}{2}h, e^i\right\rangle \\
&+ \frac{1}{2h}\left\langle\sum_{j=2}^{i-2}\left(\gamma_{j-1}^i - \gamma_{j+1}^i\right)e^j + \gamma_{i-1}^i e^i + \gamma_{i-2}^i e^{i-1} - \gamma_2^i e^1, e^i\right\rangle - \sum_{j=1}^{i-1}\gamma_j^i\left\langle\frac{e_{ttt}(x,\zeta_j)}{6}h^2, e^i\right\rangle \\
=&\left\langle\lambda e_{xx}^i, e^i\right\rangle - \left\langle re_x^i, e^i\right\rangle + \left\langle Ru(x,t_i), e^i\right\rangle.
\end{aligned} \tag{4.5}$$

Using integration by parts and substituting the boundary conditions, we obtain

$$\left\langle \lambda e^i_{xx}, e^i \right\rangle - \left\langle r e^i_x, e^i \right\rangle = -\frac{\lambda}{2}\left\|e^i_x\right\|^2.$$

For $i = 1$, according to the Cauchy-Schwarz inequality, Eq (4.5) becomes

$$\begin{aligned}\left(\gamma_1^1 + \gamma_0^1\right)\left\|e^1\right\|^2 &\le \frac{\gamma_0^1 M_2 h^2}{2}\left\|e^1\right\| + \frac{\gamma_1^1 M_2 h^2}{2}\left\|e^1\right\| + M_1 h^3\left\|e^1\right\|,\\ &\le \frac{M_2 h^2}{2}\left(\gamma_1^1 + \gamma_0^1\right)\left\|e^1\right\| + M_1 h^3\left\|e^1\right\|,\end{aligned} \tag{4.6}$$

where $M_2 = \max\limits_{\substack{0\le t\le 1\\ 0\le x\le 1}} |e_{tt}(x,t)|$. Because $\gamma_1^1 + \gamma_0^1 = h^{1-\alpha}/\Gamma(2-\alpha)$, we obtain

$$\left\|e^1\right\| \le \frac{M_2 h^2}{2} + \frac{M_1 h^{2+\alpha}}{\Gamma(2-\alpha)} \le C_1 h^2.$$

For $i \ge 2$, Eq (4.5) becomes

$$\begin{aligned}\left(2\gamma_i^i + \gamma_{i-1}^i\right)\left\|e^i\right\|^2 \le{}& 2\gamma_0^i\left\|e^1\right\|\left\|e^i\right\| + \gamma_0^i M_2 h^2\left\|e^i\right\| + 2\gamma_i^i\left\|e^{i-1}\right\|\left\|e^i\right\| + \gamma_i^i M_2 h^2\left\|e^i\right\|\\ &+ \sum_{j=2}^{i-2}\left(\gamma_{j+1}^i - \gamma_{j-1}^i\right)\left\|e^j\right\|\left\|e^i\right\| + \gamma_{i-2}^i\left\|e^{i-1}\right\|\left\|e^i\right\| + \gamma_2^i\left\|e^1\right\|\left\|e^i\right\|\\ &+ \frac{M_3 h^3}{3}\sum_{j=1}^{i-1}\gamma_j^i\left\|e^i\right\| + 2M_1 h^3\left\|e^i\right\|,\end{aligned} \tag{4.7}$$

where $M_3 = \max\limits_{\substack{0\le t\le 1\\ 0\le x\le 1}} |e_{ttt}(x,t)|$.

We will prove the conclusion of the theorem by mathematical induction. Assume that

$$\left\|e^j\right\| \le C_j h^2$$

for $j = 2, \cdots, K-1$, and C_j are constants.

Let $\tilde{C}_K = \max\limits_{j=1,2,\cdots,K-1}\left\{C_j\right\}$. When $j = K$, Eq (4.7) becomes

$$\begin{aligned}\left(2\gamma_K^K + \gamma_{K-1}^K\right)\left\|e^K\right\|^2 \le{}& 2\gamma_0^K\tilde{C}h^2\left\|e^K\right\| + \gamma_0^K M_2 h^2\left\|e^K\right\| + 2\gamma_K^K\tilde{C}h^2\left\|e^K\right\| + \gamma_K^K M_2 h^2\left\|e^K\right\|\\ &+ \sum_{j=2}^{K-2}\left(\gamma_{j+1}^K - \gamma_{j-1}^K\right)\tilde{C}h^2\left\|e^K\right\| + \gamma_{K-2}^K\tilde{C}h^2\left\|e^K\right\| + \gamma_2^K\tilde{C}h^2\left\|e^K\right\|\\ &+ \frac{M_2 h^3}{3}\sum_{j=1}^{K-1}\gamma_j^K\left\|e^K\right\| + 2M_1 h^3\left\|e^K\right\|,\end{aligned} \tag{4.8}$$

which means

$$\begin{aligned}\left\|e^K\right\| \le{}& \frac{2\gamma_0^K}{2\gamma_K^K + \gamma_{K-1}^K}\tilde{C}h^2 + \frac{\gamma_0^K M_2}{2\gamma_K^K + \gamma_{K-1}^K}h^2 + \frac{2\gamma_K^K}{2\gamma_K^K + \gamma_{K-1}^K}\tilde{C}h^2 + \frac{\gamma_K^K M_2}{2\gamma_K^K + \gamma_{K-1}^K}\tilde{C}h^2\\ &\frac{\gamma_{K-1}^K}{2\gamma_K^K + \gamma_{K-1}^K}\tilde{C}h^2 + \frac{2\gamma_{K-2}^K}{2\gamma_K^K + \gamma_{K-1}^K}\tilde{C}h^2 + \frac{M_2 h^3}{3\left(2\gamma_K^K + \gamma_{K-1}^K\right)}\sum_{j=1}^{K-1}\gamma_j^K + \frac{2M_1 h^3}{2\gamma_K^K + \gamma_{K-1}^K}.\end{aligned} \tag{4.9}$$

By Lemma 2.1, we have

$$\sum_{j=1}^{K-1}\gamma_j^K < \sum_{j=0}^{K}\gamma_j^K = \frac{(hK)^{1-\alpha}}{\Gamma(2-\alpha)} = \frac{1}{\Gamma(2-\alpha)}, \gamma_K^K \geq \gamma_0^K.$$

Thus,

$$\begin{aligned}\left\|e^K\right\| &\leq \tilde{C}h^2 + \frac{M_2}{2}h^2 + \tilde{C}h^2 + \frac{M_2}{2}\tilde{C}h^2 + \frac{1}{2}\tilde{C}h^2 + \tilde{C}h^2 \\ &+ \frac{M_2}{6}\frac{\Gamma(3-\alpha)}{\Gamma(2-\alpha)}h^{2+\alpha} + M_1\Gamma(3-\alpha)h^{2+\alpha} \\ &\triangleq C_K h^2.\end{aligned} \tag{4.10}$$

From the above analysis, it follows that for all $i = 1, 2, \cdots, n$,

$$\left\|e^K\right\| = O(h^2).$$

□

Let u_t^i be the exact solution of Eq (3.2) and $u_{t,M}^i = \sum_{j=1}^{M} c_j^{i*}\psi_j$ be the ε-approximate solution of Eq (3.2).

Theorem 4.3. $u_{t,M}^i$ converges uniformly to u_t^i.

Proof. From the definition of an ε-approximate solution, we obtain that

$$\begin{aligned}\left\|u_{t,M}^i(x) - u_t^i(x)\right\|_{W_2^3}^2 &= \left\|\mathcal{L}_i^{-1}\mathcal{L}_i u_{t,M}^i(x) - \mathcal{L}_i^{-1}\mathcal{L}_i u_t^i(x)\right\|_{W_2^3}^2 \\ &\leq \left\|\mathcal{L}_i^{-1}\right\|\left(\left\|\mathcal{L}_i u_{t,M}^i(x) - \mathcal{L}_i u_t^i(x)\right\|^2 + \sum_{j=0}^{1}\left(\mathcal{B}_j^i u_{t,M}^i(x) - g_j(t_i)\right)^2\right) \\ &\leq \left\|\mathcal{L}_i^{-1}\right\|\left(\left\|\mathcal{L}_i u_{t,M}^i(x) - \tilde{f}^i\right\|^2 + \sum_{j=0}^{1}\left(\mathcal{B}_j^i u_{t,M}^i(x) - g_j(t_i)\right)^2\right) \\ &\leq \left\|\mathcal{L}_i^{-1}\right\|\varepsilon.\end{aligned} \tag{4.11}$$

Therefore,

$$\left\|u_{t,M}^i(x) - u_t^i(x)\right\|_{W_2^3} \to 0.$$

Let $R_x(y)$ be the reproducing kernel of W_2^3,

$$\begin{aligned}\left|u_{t,M}^i(x) - u_t^i(x)\right| &= \left|\left\langle u_{t,M}^i(\cdot) - u_t^i(\cdot), R_x(\cdot)\right\rangle\right| \\ &\leq \left\|u_{t,M}^i(x) - u_t^i(x)\right\| \|R_x\|.\end{aligned}$$

Because $R_x(y)$ is bounded on $[0, 1]$, we have

$$\left|u_{t,M}^i(x) - u_t^i(x)\right| \to 0.$$

□

The convergence analysis is studied in the next part.

Let $M = 3 + mp$. According to the composition of $\psi_l(x)$, the approximate solution $u^i_{t,M}(x)$ can also be expressed as

$$u^i_{t,mp}(x) = \sum_{l=1}^{3} c^i_l \psi_l(x) + \sum_{j=0}^{m} \sum_{k=1}^{p} c_{jk} JJJ\varphi_{jk}(x). \tag{4.12}$$

Put $u^i_t(x) = \sum\limits_{l=1}^{3} a_l \psi_l(x) + \sum\limits_{j=0}^{\infty} \sum\limits_{k=1}^{p} a_{jk} JJJ\varphi_{jk}$, $w^i_{t,mp}(x) = \sum\limits_{l=1}^{3} a_l \psi_l(x) + \sum\limits_{j=0}^{m} \sum\limits_{k=1}^{p} a_{jk} JJJ\varphi_{jk}$.

By Eq (4.11), we get

$$\left\| u^i_{t,mp}(x) - u^i_t(x) \right\|_{W^3_2} \le \left\| \mathcal{L}_i^{-1} \right\| \left\| \mathcal{L}_i u^i_{t,mp}(x) - \tilde{f}^i \right\|, \tag{4.13}$$

and

$$\begin{aligned} \left\| u^i_{t,mp}(x) - u^i_t(x) \right\|^2_{W^3_2} &\le \left\| \mathcal{L}_i^{-1} \right\|^2 \left(\left\| \mathcal{L}_i u^i_{t,mp}(x) - \mathcal{L}_i u^i_t(x) \right\|^2 + \sum_{j=0}^{1} \left(\mathcal{B}^i_j u^i_{t,mp}(x) - g_j(t_i) \right)^2 \right) \\ &\le \left\| \mathcal{L}_i^{-1} \right\|^2 \left(\left\| \mathcal{L}_i w^i_{t,mp}(x) - \mathcal{L}_i u^i_t(x) \right\|^2 + \sum_{j=0}^{1} \left(\mathcal{B}^i_j w^i_{t,mp}(x) - \mathcal{B}^i_j u^i_t(x) \right)^2 \right) \\ &\le \left\| \mathcal{L}_i^{-1} \right\|^2 \left[\|\mathcal{L}\|^2 \left(\left\| w^i_{t,mp}(x) - u^i_t(x) \right\|^2 \right) + \sum_{j=0}^{1} \left\| \mathcal{B}^i_j \right\|^2 \left(\left\| w^i_{t,mp}(x) - u^i_t(x) \right\|^2 \right) \right] \\ &\le \left\| \mathcal{L}_i^{-1} \right\|^2 \left(\|\mathcal{L}\|^2 + \left\| \mathcal{B}^i_0 \right\|^2 + \left\| \mathcal{B}^i_1 \right\|^2 \right) \left\| w^i_{t,mp}(x) - u^i_t(x) \right\|^2. \end{aligned} \tag{4.14}$$

Obviously,

$$\left\| u^i_{t,mp}(x) - u^i_t(x) \right\|^2_{W^3_2} \le \tilde{M}_1 \left\| w^i_{t,mp}(x) - u^i_t(x) \right\|^2_{W^3_2}, \tag{4.15}$$

where $\tilde{M}_1 = \left\| \mathcal{L}_i^{-1} \right\|^2 \left(\|\mathcal{L}\|^2 + \left\| \mathcal{B}^i_0 \right\|^2 + \left\| \mathcal{B}^i_1 \right\|^2 \right)$.

By the expressions of $w^i_{t,mp}(x)$ and $u^i_t(x)$,

$$u^i_t(x) - w^i_{t,mp}(x) = \sum_{j=m}^{\infty} \sum_{k=1}^{p} a_{jk} JJJ\varphi_{jk}.$$

This implies that

$$u^i_t(0) - w^i_{t,mp}(0) = \frac{\partial}{\partial x} \left(u^i_t(x) - w^i_{t,mp}(x) \right) |_{x=0} = \frac{\partial}{\partial x^2} \left(u^i_t(x) - w^i_{t,mp}(x) \right) |_{x=0} = 0.$$

Therefore,

$$\begin{aligned} \left\| w^i_{t,mp}(x) - u^i_t(x) \right\|^2_{W^3_2} &= \int_0^1 \left(\left(w^i_{t,mp}(x) \right)''' - \left(u^i_t(x) \right)''' \right)^2 dx \\ &= \left\| \left(w^i_{t,mp}(x) \right)''' - \left(u^i_t(x) \right)''' \right\|^2_{L^2}. \end{aligned} \tag{4.16}$$

Observe that $\left(w^i_{t,mp}(x)\right)''' = \sum\limits_{j=0}^{m}\sum\limits_{k=1}^{p} a_{jk}\varphi_{jk}$, $\left(u^i_t(x)\right)''' = \sum\limits_{j=0}^{\infty}\sum\limits_{k=1}^{p} a_{jk}\varphi_{jk}$, we derive

$$\begin{aligned} a_{jk} &= \left\langle \left(u^i_t(x)\right)''', \varphi_{jk}\right\rangle = \int_0^1 \left(u^i_t(x)\right)''' \varphi_{jk} dx \\ &= \int_{\frac{k-1}{p}}^{\frac{k}{p}} \left(u^i_t(x)\right)''' \sqrt{p}\Phi_j(px-k+1)dx \\ &= \frac{1}{p}\int_0^1 \left(u^i_t(\frac{s+k-1}{p})\right)''' \sqrt{p}\Phi_j(s)ds. \end{aligned} \tag{4.17}$$

Set $V(s,k) = \left(u^i_t((s+k-1)/p)\right)'''$, $V_m(s,k) = \left(w^i_{t,mp}((s+k-1)/p)\right)'''$ so that

$$\begin{aligned} a_{jk} &= \frac{\sqrt{p}}{p}\int_0^1 V(s,k)\Phi_j(s)ds \\ &= \frac{\sqrt{p}}{p}\int_0^1 V(s,k)\Phi_j(s)ds \\ &= \frac{\sqrt{p}}{p} b_{jk}, \end{aligned}$$

where b_{jk} are the coefficients of generalized Fourier expansion on the basis $\left\{\Phi_j(s)\right\}$. Namely,

$$V(s,k) = \sum_{j=0}^{\infty} b_{jk}\Phi_j(s),\ V_m(s,k) = \sum_{j=0}^{m} b_{jk}\Phi_j(s).$$

The approximation error then reduces to

$$\begin{aligned} \left\|\left(u^i_t(x)\right)''' - \left(w^i_{t,mp}(x)\right)'''\right\|^2_{L^2} &= \left\|\left(u^i_t(x)\right)''' - \sum_{j=0}^{m}\sum_{k=1}^{p} a_{jk}\varphi_{jk}\right\|^2_{L^2} \\ &= \left\|\left(u^i_t(x)\right)''' - \frac{\sqrt{p}}{p}\sum_{j=0}^{m}\sum_{k=1}^{p} b_{jk}\varphi_{jk}\right\|^2_{L^2} \\ &= \int_0^1 \left(\left(u^i_t(x)\right)''' - \frac{\sqrt{p}}{p}\sum_{j=0}^{m}\sum_{k=1}^{p} b_{jk}\varphi_{jk}\right)^2 dx \\ &= \sum_{k=1}^{p}\int_{\frac{k-1}{p}}^{\frac{k}{p}} \left(\left(u^i_t(x)\right)''' - \frac{\sqrt{p}}{p}\sum_{j=0}^{m} b_{jk}\sqrt{p}\Phi_j(px-k+1)\right)^2 dx \\ &= \frac{1}{p}\sum_{k=1}^{p}\int_0^1 \left(V(s,k) - \sum_{j=0}^{m} b_{jk}\Phi_j(s)\right)^2 ds \\ &= \frac{1}{p}\sum_{k=1}^{p}\int_0^1 (V(s,k)-V_m(s,k))^2 ds. \end{aligned} \tag{4.18}$$

In accordance with [19], we have

$$\begin{aligned}\left\|\left(u_t^i(x)\right)''' - \left(w_{t,mp}^i(x)\right)'''\right\|_{L^2}^2 &= \frac{1}{p}\sum_{k=1}^{p}\int_0^1 (V(s,k) - V_m(s,k))^2 ds \\ &= \frac{1}{p}\sum_{k=1}^{p}\|V(s,k) - V_m(s,k)\|_{L^2}^2 \\ &\le \frac{1}{p}\sum_{k=1}^{p}\left(C\frac{1}{m^{2\tau}}\sum_{l=\min(\tau,m+1)}^{\tau}\left\|\partial_s^l V(s,k)\right\|_{L^2}^2\right),\end{aligned} \tag{4.19}$$

where C is a constant, and τ makes $\left(u_t^i\right)^{(\tau+3)} \in L^2[0,1]$. As $V(s,k) = \left(u_t^i\left(\frac{s+k-1}{p}\right)\right)'''$,

$$\partial_s^l V(s,k) = \frac{1}{p^l}\left(u_t^i(x)\right)^{(l+2)}. \tag{4.20}$$

It follows that

$$\begin{aligned}\left\|\left(u_t^i(x)\right)''' - \left(w_{t,mp}^i(x)\right)'''\right\|_{L^2}^2 &\le \frac{1}{p}\sum_{k=1}^{p}C\frac{1}{m^{2\tau}}\sum_{l=\min(\tau,m+1)}^{\tau}\frac{1}{p^{2l}}\left\|\left(u_t^i\right)^{(l+2)}\right\|_{L^2}^2 \\ &\le \tilde{C}\frac{1}{m^{2\tau}}\sum_{l=\min(\tau,m+1)}^{\tau}\frac{1}{p^{2l}},\end{aligned} \tag{4.21}$$

where $\tilde{C}$ is a constant.

Equations (4.16) and (4.21) show that

$$\left\|u_t^i(x) - w_{t,mp}^i(x)\right\|_{W_2^3}^2 \le \tilde{C}\frac{1}{m^{2\tau}}\sum_{l=\min(\tau,m+1)}^{\tau}\frac{1}{p^{2l}}.$$

Putting the result into Eq (4.15),

$$\left\|u_{t,mp}^i(x) - u_t^i(x)\right\|_{W_2^3}^2 \le \tilde{M}\frac{1}{m^{2\tau}}\sum_{l=\min(\tau,m+1)}^{\tau}\frac{1}{p^{2l}}, \tag{4.22}$$

where $\tilde{M} = \tilde{M}_1\tilde{C}$. Equation (4.22) implies the error bound of the new approach. Under the assumptions $\left(u_t^i\right)^{(\tau+3)} \in L^2[0,1]$, the conclusion is presented in the following theorem.

Theorem 4.4. If $\left(u_t^i\right)^{(\tau+3)} \in L^2[0,1]$, then

$$\left\|u_{t,mp}^i(x) - u_t^i(x)\right\|_{W_3}^2 \le \tilde{M}\frac{1}{m^{2\tau}}\sum_{l=\min(\tau,m+1)}^{\tau}\frac{1}{p^{2l}}.$$

Theorem 4.4 indicates that the convergence behavior depends on both m and p, and the convergence order increases with rising values of these two parameters.

For the stability of spatial variables, it refers to the proof methods in [18]. Here, we will prove the stability of scheme (3.1). Because $u^i(x), i = 0,1,2,\cdots n$ are obtained through Eq (3.1), assume that there is a disturbance during the calculation process and denote that $v^i(x),\ i = 1, 2, \cdots, n$ satisfies Eq (3.1) under the initial condition $v^0(x)$.

Theorem 4.5. Suppose $u_{ttt}(x,t)$ is bounded on $[0,1] \times [0,1]$; then scheme (3.1) is stable and satisfies

$$\left\|\mathcal{E}^j\right\| \le D_j \left\|\mathcal{E}^0\right\| + O(h^2), i = 1, \cdots, n.$$

Proof. Let $\mathcal{E}^i = u^i(x) - v^i(x)$; we have $\sum_{j=0}^{i} \gamma_j^i \mathcal{E}_t^j = \lambda \mathcal{E}_{xx}^i - r\mathcal{E}_x^i$. By a similar approach to Eq (4.3), we obtain the following equality:

$$\begin{aligned} &\gamma_0^i \left(\frac{\mathcal{E}^1 - \mathcal{E}^0}{h} - \frac{\mathcal{E}_{tt}(x,\vartheta_0)}{2} h \right) + \gamma_i^i \left(\frac{\mathcal{E}^i - \mathcal{E}^{i-1}}{h} + \frac{\mathcal{E}_{tt}(x,\vartheta_i)}{2} h \right) \\ &+ \sum_{j=1}^{i-1} \gamma_j^i \left(\frac{\mathcal{E}^{j+1} - \mathcal{E}^{j-1}}{2h} - \frac{\mathcal{E}_{ttt}(x,\vartheta_j)}{6} h^2 \right) = \lambda \mathcal{E}_{xx}^i - r\mathcal{E}_x^i, \end{aligned} \tag{4.23}$$

where $\vartheta_j \in [t_{j-1}, t_j], j = 1, 2, \cdots, i$.

Taking the inner product of both sides of the equation with $\mathcal{E}^i$ yields

$$\begin{aligned} &\gamma_0^i \left\langle \left(\frac{\mathcal{E}^1 - \mathcal{E}^0}{h} - \frac{\mathcal{E}_{tt}(x,\vartheta_0)}{2} h \right), \mathcal{E}^i \right\rangle + \gamma_i^i \left\langle \left(\frac{\mathcal{E}^i - \mathcal{E}^{i-1}}{h} + \frac{\mathcal{E}_{tt}(x,\vartheta_i)}{2} h \right), \mathcal{E}^i \right\rangle \\ &+ \sum_{j=1}^{i-1} \gamma_j^i \left\langle \left(\frac{\mathcal{E}^{j+1} - \mathcal{E}^{j-1}}{2h} - \frac{\mathcal{E}_{ttt}(x,\vartheta_j)}{6} h^2 \right), \mathcal{E}^i \right\rangle = -\frac{\lambda}{2} \left\|\mathcal{E}_x^i\right\|^2. \end{aligned} \tag{4.24}$$

For $i = 1$,

$$\left(\gamma_1^1 + \gamma_0^1\right) \left\|\mathcal{E}^1\right\|^2 \le \gamma_0^1 \left\|\mathcal{E}^0\right\| \left\|\mathcal{E}^1\right\| + \gamma_1^1 \left\|\mathcal{E}^0\right\| \left\|\mathcal{E}^1\right\| + \frac{\gamma_0^1 M_4 h^2}{2} \left\|\mathcal{E}^1\right\| + \frac{\gamma_0^1 M_4 h^2}{2} \left\|\mathcal{E}^1\right\|,$$

where $M_4 = \max\limits_{\substack{0 \le t \le 1 \\ 0 \le x \le 1}} |e_{tt}(x,t)|$.

Therefore,

$$\left\|\mathcal{E}^1\right\| \le \left\|\mathcal{E}^0\right\| + M_4 h^2 = D_1 \left\|\mathcal{E}^0\right\| + O(h^2), \quad D_1 = 1.$$

For $i \ge 2$, Eq (4.23) becomes

$$\begin{aligned} \left(2\gamma_i^i + \gamma_{i-1}^i\right) \left\|\mathcal{E}^i\right\|^2 \le\ & 2\gamma_0^i \left\|\mathcal{E}^1\right\| \left\|\mathcal{E}^i\right\| + 2\gamma_0^i \left\|\mathcal{E}^0\right\| \left\|\mathcal{E}^i\right\| + \gamma_0^i\ M_4 h^2 \left\|\mathcal{E}^i\right\| + 2\gamma_i^i \left\|\mathcal{E}^{i-1}\right\| \left\|\mathcal{E}^i\right\| \\ &+ \gamma_i^i\ M_4 h^2 \left\|\mathcal{E}^i\right\| + \sum_{j=2}^{i-2} \left(\gamma_{j+1}^i - \gamma_{j-1}^i\right) \left\|\mathcal{E}^j\right\| \left\|\mathcal{E}^i\right\| + \gamma_{i-2}^i \left\|\mathcal{E}^{i-1}\right\| \left\|\mathcal{E}^i\right\| \\ &+ \gamma_2^i \left\|\mathcal{E}^1\right\| \left\|\mathcal{E}^i\right\| + \gamma_1^i \left\|\mathcal{E}^0\right\| \left\|\mathcal{E}^i\right\| + \frac{M_5 h^3}{3} \sum_{j=1}^{i-1} \gamma_j^i \left\|\mathcal{E}^i\right\|, \end{aligned} \tag{4.25}$$

where $M_5 = \max\limits_{\substack{0 \le t \le 1 \\ 0 \le x \le 1}} |e_{ttt}(x,t)|$.

Equation (4.25) can be rewritten as

$$\begin{aligned}\left\|\mathcal{E}^{i}\right\| \le & \frac{2\gamma_0^i}{2\gamma_i^i+\gamma_{i-1}^i}\left\|\mathcal{E}^{1}\right\| + \frac{2\gamma_0^i}{2\gamma_i^i+\gamma_{i-1}^i}\left\|\mathcal{E}^{0}\right\| + \frac{\gamma_0^i\, M_4h^2}{2\gamma_i^i+\gamma_{i-1}^i} + \frac{2\gamma_i^i}{2\gamma_i^i+\gamma_{i-1}^i}\left\|\mathcal{E}^{i-1}\right\| + \frac{\gamma_i^i\, M_4h^2}{2\gamma_i^i+\gamma_{i-1}^i}\\ & + \frac{\gamma_{i-1}^i}{2\gamma_i^i+\gamma_{i-1}^i}\left\|\mathcal{E}^{i-1}\right\| + \frac{2\gamma_{i-2}^i}{2\gamma_i^i+\gamma_{i-1}^i}\left\|\mathcal{E}^{i-2}\right\| + \frac{\gamma_2^i}{2\gamma_i^i+\gamma_{i-1}^i}\left\|\mathcal{E}^{1}\right\| + \frac{\gamma_1^i}{2\gamma_i^i+\gamma_{i-1}^i}\left\|\mathcal{E}^{0}\right\|\\ & + \frac{M_5h^3}{3\left(2\gamma_i^i+\gamma_{i-1}^i\right)}\sum_{j=1}^{i-1}\gamma_j^i\left\|\mathcal{E}^{i}\right\|.\end{aligned} \tag{4.26}$$

We proceed by mathematical induction. Suppose

$$\left\|\mathcal{E}^{j}\right\| \le D_j\left\|\mathcal{E}^{0}\right\| + O(h^2),\ j = 2,\cdots,i-1,$$

where D_j, $j = 2,\cdots,i-1$ are constants. Let $\tilde{D}_i = \max\limits_{j=1,2,\cdots,i-1}\left\{D_j\right\}$; we get

$$\begin{aligned}\left\|\mathcal{E}^{i}\right\| \le & \frac{2\gamma_0^i}{2\gamma_i^i+\gamma_{i-1}^i}\left(\tilde{D}_i\left\|\mathcal{E}^{0}\right\| + O(h^2)\right) + \frac{2\gamma_0^i}{2\gamma_i^i+\gamma_{i-1}^i}\left\|\mathcal{E}^{0}\right\| + \frac{\gamma_0^i\, M_4h^2}{2\gamma_i^i+\gamma_{i-1}^i}\\ & + \frac{2\gamma_i^i}{2\gamma_i^i+\gamma_{i-1}^i}\left(\tilde{D}_i\left\|\mathcal{E}^{0}\right\| + O(h^2)\right) + \frac{\gamma_i^i\, M_4h^2}{2\gamma_i^i+\gamma_{i-1}^i} + \frac{\gamma_{i-1}^i}{2\gamma_i^i+\gamma_{i-1}^i}\left(\tilde{D}_i\left\|\mathcal{E}^{0}\right\| + O(h^2)\right)\\ & + \frac{2\gamma_{i-2}^i}{2\gamma_i^i+\gamma_{i-1}^i}\left(\tilde{D}_i\left\|\mathcal{E}^{0}\right\| + O(h^2)\right) + \frac{\gamma_2^i}{2\gamma_i^i+\gamma_{i-1}^i}\left(\tilde{D}_i\left\|\mathcal{E}^{0}\right\| + O(h^2)\right)\\ & + \frac{\gamma_1^i}{2\gamma_i^i+\gamma_{i-1}^i}\left\|\mathcal{E}^{0}\right\| + \frac{M_5h^3}{3\left(2\gamma_i^i+\gamma_{i-1}^i\right)}\sum_{j=1}^{i-1}\gamma_j^i\left\|\mathcal{E}^{i}\right\|.\end{aligned} \tag{4.27}$$

By Lemma 2.1, it follows that

$$\begin{aligned}\left\|\mathcal{E}^{i}\right\| \le & \left(\tilde{D}_i\left\|\mathcal{E}^{0}\right\| + O(h^2)\right) + \left\|\mathcal{E}^{0}\right\| + \frac{M_4h^2}{2} + \left(\tilde{D}_i\left\|\mathcal{E}^{0}\right\| + O(h^2)\right) + \frac{M_4h^2}{2}\\ & + \left(\tilde{D}_i\left\|\mathcal{E}^{0}\right\| + O(h^2)\right) + \left(\tilde{D}_i\left\|\mathcal{E}^{0}\right\| + O(h^2)\right) + \left(\tilde{D}_i\left\|\mathcal{E}^{0}\right\| + O(h^2)\right)\\ & + \left\|\mathcal{E}^{0}\right\| + \frac{M_5h^{2+\alpha}}{6}\frac{\Gamma(3-\alpha)}{\Gamma(2-\alpha)}\\ \le & D_i\left\|\mathcal{E}^{0}\right\| + O(h^2).\end{aligned} \tag{4.28}$$

□

This completes the proof of the theorem and implies the stability of the semi-discrete scheme.

5. Numerical experiments

In this section, we carry out numerical tests on three representative benchmark examples. These cases have been widely investigated and compared using various algorithms in the existing literature, as documented in [20–23]. Our goal is to verify the effectiveness and accuracy of the proposed method through systematic testing. By comparing our results with those reported in previous studies, we further validate the reliability and advantages of our approach.The absolute error function is $E_M^n = \max\limits_{\substack{1\le i\le n\\ 1\le j\le N}} \left|u_M^i(x_j) - u(x_j,t_i)\right|$ with $N = 10$. The temporal and spatial convergence rates are derived as follows:

$$CR_t = \log_{\frac{n_2}{n_1}} \frac{E_M^{n_1}}{E_M^{n_2}}, CR_{x,m} = \log_{\frac{m_2}{m_1}} \frac{E_{m_1,p}^n}{E_{m_2,p}^n}, CR_{x,p} = \log_{\frac{p_2}{p_1}} \frac{E_{m,p_1}^n}{E_{m,p_2}^n}.$$

Example 5.1. Consider the following convection-diffusion equation on domain $\Omega = (0,1)\times(0,1]$:

$$\begin{cases} \dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \dfrac{\partial^2 u(x,t)}{\partial x^2} + \dfrac{\partial u(x,t)}{\partial x} = \dfrac{\Gamma(4+\alpha)t^3}{6}\sin(\pi x) + \pi^2 t^{3+\alpha}\sin(\pi x) + \pi t^{3+\alpha}\cos(\pi x), \\ u(x,0) = 0, \\ u(0,t) = u(1,t) = 0. \end{cases}$$

We present the exact solution denoted by $u(x,t) = t^{3+\alpha}\sin(\pi x)$ and list the corresponding numerical outcomes when $\alpha = 0.1$ in Table 1. Table 1 shows the profiles of approximate solutions corresponding to different values of m and p. Comparisons with the $L1$ scheme in [20] and the $L1$-2 scheme in [21] demonstrate that our algorithm achieves more accurate numerical results with fewer temporal divisions and a smaller number of basis functions. Let $U(x,t)$ be the approximate solution of $u(x,t)$. Figures 1 and 2 display the absolute error $|U(x,t) - u(x,t)|$ as well as the error distribution on domain $\Omega = (0,1)\times(0,1]$.

Table 1. Comparison of maximum errors for Example 5.1.

Method in [20]			Method in [21]			Present method				
M	n	E_{max}	M	n	E_{max}	(m,p)	n	E_M^n	CR_t	CPU(s)
20	20	1.43e-3	66	5	1.25e-4	(4,5)	5	1.01e-5	-	9.14
40	40	3.67e-4	66	10	1.87e-5	(4,5)	10	1.67e-6	2.60	10.68
80	80	9.15e-5	66	20	2.72e-6	(4,5)	15	4.30e-7	3.35	11.95
160	160	2.31e-5	66	30	8.71e-7	(4,5)	28	7.99e-8	2.70	14.29

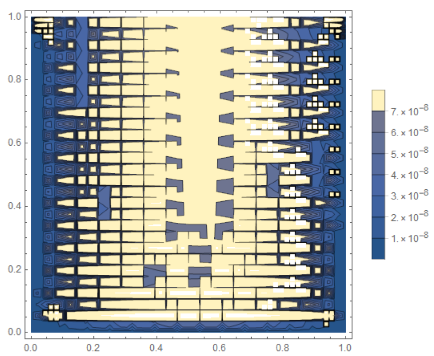

Figure 1. Error distribution of $|U(x,t) - u(x,t)|$ for Example 5.1 ($m = 4, p = 5, n = 28$).

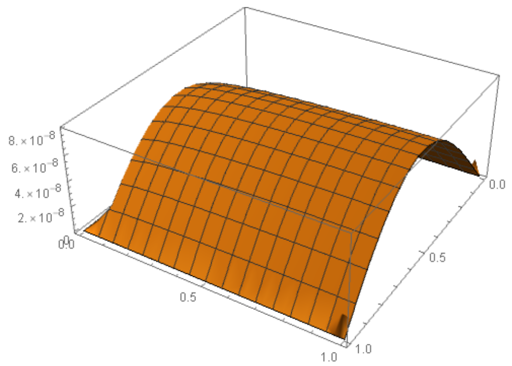

Figure 2. $|U(x,t) - u(x,t)|$for Example 5.1 ($m = 4, p = 5, n = 28$).

Example 5.2. Consider the following diffusion equation on domain $\Omega = (0,1) \times (0,1]$:

$$\begin{cases} \dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \dfrac{\partial^2 u(x,t)}{\partial x^2} = \dfrac{2}{\Gamma(2.1)} t^{1.1} \sin(2\pi x) + 4\pi^2 t^2 \sin(2\pi x), \\ u(x,0) = 0, \\ u(0,t) = u(1,t) = 0. \end{cases}$$

The exact solution is given by $u = t^2 \sin(2\pi x)$. The numerical results when $\alpha = 0.9$ are listed in Table 2. By comparison with the methods developed in [22], it can be observed that the absolute error decreases rapidly with the increase of the parameters t, m, and p. Figures 3 and 4 present the errors between the exact and approximate solutions and their spatial distribution throughout the domain. It can be clearly observed that the numerical solutions are in excellent agreement with the exact solution.

Table 2. Comparison of maximum errors for Example 5.2.

Method in [22]			Present method					
M	n	E_{max}	(m,p)	n	E_M^n	$CR_{x,m}$	$CR_{x,p}$	CPU(s)
19	8	4.75e-5	(5,3)	10	6.27e-8	-	-	9.08
35	8	1.26e-6	(5,5)	10	3.25e-9	-	5.79	9.69
67	8	4.34e-8	(5,8)	10	2.85e-11	-	10.08	11.49
131	8	1.42e-9	(5,12)	10	4.75e-12	-	4.42	12.1
259	8	4.54e-11	(6,12)	10	5.31e-13	12.02	-	15.12

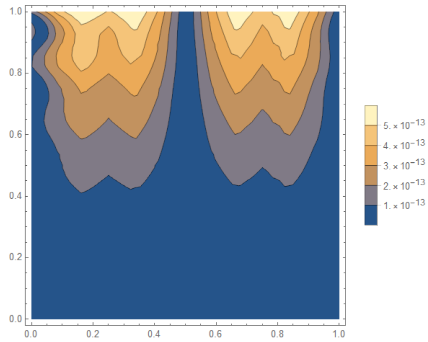

Figure 3. Error distribution of $|U(x,t) - u(x,t)|$ for Example 5.2 ($m = 6, p = 12, n = 10$).

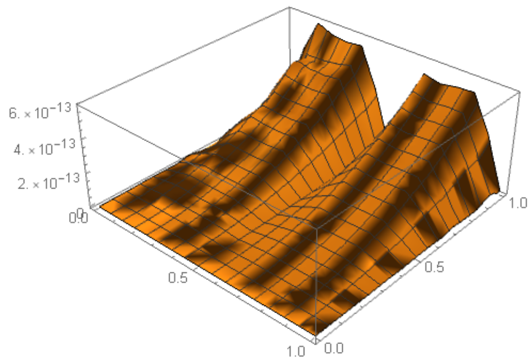

Figure 4. $|U(x,t) - u(x,t)|$ for Example 5.2 ($m = 6, p = 12, n = 10$).

Example 5.3. Consider the following convection-diffusion equation on domain $\Omega = (0,1) \times (0,1]$:

$$\begin{cases} \dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \dfrac{1}{2}\sigma^2 \dfrac{\partial^2 u(x,t)}{\partial x^2} + (r - \dfrac{\sigma^2}{2})\dfrac{\partial u(x,t)}{\partial x} - ru(x,t) + f(x,t), \\ u(x,0) = (1-x)x^2, \\ u(0,t) = u(1,t) = 0, \end{cases}$$

where $f(x,t) = [2t^{2-\alpha}/\Gamma(3-\alpha) + 2t^{1-\alpha}/\Gamma(2-\alpha)](1-x)x^2 - (t+1)^2[(\sigma^2/2)(2-6x) + (r-\sigma^2/2)(2x - 3x^2) - r(1-x)x^2]$. The exact solution is $u(x,t) = x^2(1-x)(1+t)^2$. The maximum absolute errors are calculated in Table 3. Compared with the method proposed in [23], the developed algorithm achieves smaller error and superiority in computational accuracy. It can be seen that the algorithm maintains favorable computational accuracy under various values of parameter α, which fully demonstrates its strong robustness. Figures 5 to 8 plot the three-dimensional error diagrams for varying α.

Table 3. Comparison of maximum errors for Example 5.3.

Method in [23]			Present method		
$\alpha = 0.25$	$\alpha = 0.7$	$\alpha = 0.95$	$\alpha = 0.25$	$\alpha = 0.7$	$\alpha = 0.95$
$N = 4, M = 8$	$N = 4, M = 7$	$N = 4, M = 7$		$m = 2, p = 3, n = 10$	
1.50e-10	7.04e-11	9.45e-13	1.86e-14	4.44e-16	2.88e-15

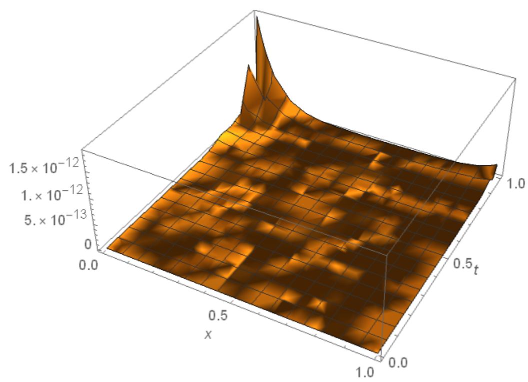

Figure 5. Graph of $|U(x,t) - u(x,t)|$ for Example 5.3 ($m = 2,\ p = 3,\ n = 10,\ \alpha = 0.4$).

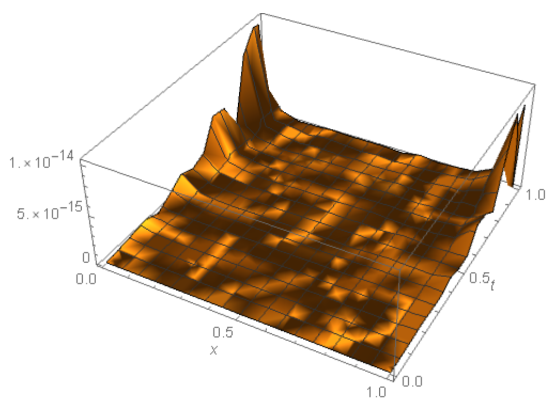

Figure 6. Graph of $|U(x,t) - u(x,t)|$ for Example 5.3 ($m = 2,\ p = 3,\ n = 10,\ \alpha = 0.6$).

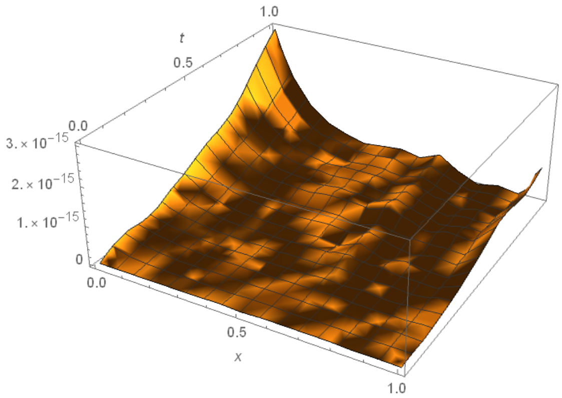

Figure 7. Graph of $|U(x,t) - u(x,t)|$ for Example 5.3 ($m = 2,\ p = 3,\ n = 10,\ \alpha = 0.8$).

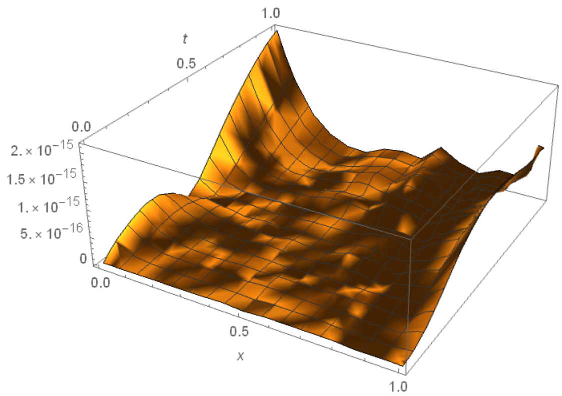

Figure 8. Graph of $|U(x,t) - u(x,t)|$ for Example 5.3 ($m = 2,\ p = 3,\ n = 10,\ \alpha = 0.9$).

6. Conclusions

This paper studies the time-fractional convection-diffusion problem and proposes a novel coupled algorithm. In the algorithm, the derivative values are used to perform piecewise linear interpolation for $u_t(x,t)$, which is then employed to approximate the temporal fractional derivative and construct the semi-discrete scheme (3.1). This scheme reduces to ordinary differential equations involving only the spatial variable. Compressed Legendre polynomials are then employed to construct a standard orthogonal basis, which is combined with the ε-approximate solution approach to compute the numerical solutions of the scheme (3.1). Convergence analysis shows that the algorithm has second-order convergence in the time direction, and the spatial convergence accuracy increases gradually with the growth of m and p. Furthermore, the compressed Legendre polynomials can effectively suppress the Runge phenomenon caused by higher-order polynomials. Numerical results illustrate that the proposed method yields satisfactory accuracy and efficiency with coarser time discretizations and fewer basis functions.

Author contributions

Hui Zhu was responsible for designing the overall algorithm, conducting the convergence proof, and completing the initial manuscript. Lei Yang reviewed the full text. Ruimin Zhang collected the literature. Yuntao Jia performed the numerical example tests. All authors were involved in writing the manuscript. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is partially supported by the Science and Technology Development Fund, Macau SAR (Grant No. 0105/2024/RIB2), the Characteristic Innovative Scientific Research Project of Guangdong Province (Grant No. 2023KTSCX181), the Basic and Applied Basic Research Project of Zhuhai City (Grant No. 2320004002520), and the Zhuhai Social Development Science and Technology Plan Project (Grant No. 2420004000094).

Conflict of interest

The authors declare no conflicts of interest.

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