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Research article

On the number of spanning trees and asymptotic behavior of certain classes of sprocket wheel graphs

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Abstract: The sprocket wheels are specialized, toothed wheels that transfer rotating motion by engaging with chains, belts, or perforated materials. They are frequently seen on equipment, motorcycles, and bicycles. They do not mesh directly with one another like gears do. They are characterized by their teeth count and pitch and are usually composed of metal or sturdy polymers, guaranteeing accurate, long-lasting, and dependable power transfer. In this study, we used block matrices, linear algebraic techniques, and the properties of Chebyshev polynomials to obtain explicit formulas for the number of spanning trees of new classes of sprocket wheel graph families. Additionally, we studied the asymptotic entropy of spanning trees on our graphs. Last, we compared the entropies of our graphs to those of other studied graphs with average degreed of 4 and 5.

Keywords: number of spanning trees; sprocket wheel graph; Chebyshev polynomials; entropy
Mathematics Subject Classification: 05C05, 05C50, 05C99

1. Introduction

Spanning trees are crucial for creating the bare minimum of connections needed to link each component in a network or system in industries like computer networking, telecommunications, and transportation. Network efficiency, redundancy, and stability all depend on this. They prevent loops and broadcast storms in networks by permitting only one active path for data and providing backup paths for fault tolerance [1–3]. Furthermore, the number of spanning trees is counted because it is a fundamental problem in combinatorics and graph theory with applications in physics and engineering, helps design effective routing algorithms and network infrastructure, and provides a sparse subgraph that reflects graph properties [4,5].

A spanning tree is either a maximal subset of the edges without any cycles or a minimal subset of the edges connecting all the vertices in a finite connected graph G.

The history of counting the number of spanning trees, $\tau(G)$ in a graph G, started in 1842 when the physicist Kirchhoff [6] proposed the matrix tree theorem, which was based on the determinants of a specific matrix obtained from the Laplacian matrix, H which is defined by the difference between the degree matrix, $D(G)$ and adjacency matrix, $A(G)$, where $D(G)$ is a diagonal matrix, $D(G) = dig(d_1, d_2, \ldots, d_n)$, corresponding to a graph G with n vertices that has the vertex degree of d_i in the *i*th position of a graph G and $A(G)$ is a matrix with rows and columns labeled by graph vertices, with a 1 or 0 in position (v_i, v_j) depending on whether v_i and v_j are adjacent or not. That is

$$H_{ij} = \begin{cases} d_{i,} & \text{if } i = j, \\ -1, & \text{if } i \neq j \text{ and } i \text{ is adjacent to } j, \\ 0, & \text{otherwise,} \end{cases}$$

where d_i denotes the degree of the vertex v_i. According to the matrix tree theorem, all the H 's cofactors are equal, and the complexity $\tau(G)$ is equal to their common value.

There are straightforward closed formulas that greatly simplify the process of counting the number of related spanning trees for certain unique types of graphs, particularly when those numbers are quite high. A complete graph K_n has $n^{n-2}, n \geq 2$ spanning trees, as demonstrated by Cayley [7].

Another conclusion, $\tau(K_{r,s}) = r^{s-1}s^{r-1}, r, s \geq 1$, where $K_{r,s}$ is the complete bipartite graph with bipartite sets that have r and s vertices, respectively. This is well known, as in, for example, [8].

Sedlacek [9] obtained an additional outcome by obtaining a formula for the wheel on $k+1$ vertices, W_{k+1}. He proved that

$$\tau(W_{k+1}) = \left(\tfrac{3+\sqrt{5}}{2}\right)^k + \left(\tfrac{3-\sqrt{5}}{2}\right)^k - 2, \text{ for } k \geq 3.$$

Furthermore, a formula for the number of spanning trees in a Mobius ladder M_k was later derived by Sedlacek [10],

$$\tau(M_k) = \tfrac{k}{2}[(2+\sqrt{3})^k + (2-\sqrt{3})^k + 2] \text{ for } k \geq 2.$$

Moreover, Boesch and Bogdanowicz [11] employed the matrix tree theorem to obtain the number of spanning trees for a prism graph, R_k with $2k$ vertices,

$$\tau(P_k) = \tfrac{1}{2}[(2+\sqrt{3})^k + (2-\sqrt{3})^k] - 1] \text{ for } k \geq 3.$$

Using the matrix tree theorem, Daoud studied the intricacy of many classes of graphs that are related to cycles [12]. Also using this technique, researchers have focused on counting and maximizing the number of spanning trees for graph families [13–15].

There is another method to compute $\tau(G)$. Assume $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_p = 0$ represent the eigenvalues of the matrix H of a graph G, which has p vertices. According to "Chelnokov" and "Kelmans" [16]:

$$\tau(G) = \tfrac{1}{p}\textstyle\prod_{i=1}^{p-1} \mu_p. \quad (1)$$

The deletion-contraction method is a widely used approach for figuring out complexity. If e is any edge in a graph G, then its complexity $\tau(G)$ equals $\tau(G) = \tau(G-e) + \tau(G/e)$, where G/e is the contraction of e in G and $G-e$ is the deletion of e from G. Recursively determining a graph's complexity is possible in this manner [17,18].

The number of spanning trees can also be determined iteratively by simplifying the graph and tracking changes in the number of weighted spanning trees. Moreover, graph transformations, including merging parallel edges, joining serial edges, and switching between triangle (Δ) and star (Y) configurations, are used in the electrically equivalent transformation approach. This is done by leveraging the relationship between electrical networks, where conductance is represented by edge weights—and the counting spanning tree issue.

Asiri and Daoud [19,20] derived a closed-form formula for counting spanning trees in augmented triangular prism graphs and pyramid graphs based on the Fritsch graph using this technique.

Sujing Cheng, Jun Ge [21] derived formulas for counting the number of spanning trees in generalized K_n-chain/ring graphs.

2. Some properties of Chebyshev polynomials

Several characteristics of Chebyshev polynomials that are crucial to our calculations are listed in this section [22,23]. The definition of the Chebyshev polynomial of the first kind $T_n(z)$ is:

$$T_n(\cos z) = \cos(nz). \tag{2}$$

Or equivalently

$$T_n(z) = \cos(n \cos^{-1} z). \tag{3}$$

Since

$$T_0(z) = \cos(0 \cos^{-1} z) = 1,$$

$$T_1(z) = \cos(1 \cos^{-1} z) = z,$$

$$T_2(z) = \cos(2 \cos^{-1} z) = 2z^2 - 1,$$

$$T_3(z) = \cos(3 \cos^{-1} z) = 4z^3 - 3z,$$

$$T_4(z) = \cos(4 \cos^{-1} z) = 8z^4 - 8z^2 + 1$$

and so on. We can define $T_n(z)$ if we assume $z = \cos\varphi$ in the form

$$\begin{aligned} T_n(z) = \cos(n\varphi) &= \frac{1}{2}[e^{in\varphi} + e^{-in\varphi}] \\ &= \frac{1}{2}[(\cos\varphi + i\sin\varphi)^n + (\cos\varphi - i\sin\varphi)^n] \\ &= \frac{1}{2}[(z + i\sqrt{1-z^2})^n + (z - i\sqrt{1-z^2})^n] \\ &= \frac{1}{2}[(z + \sqrt{z^2-1})^n + (z - \sqrt{z^2-1})^n]. \end{aligned} \tag{4}$$

The definition of the second-kind Chebyshev polynomial $U_n(z)$ is: The Chebyshev polynomial of the second kind $U_n(z)$, which is defined as

$$U_n(z) = \frac{1}{n+1}\frac{d}{dz}T_{n+1}(z) = \frac{sin((n+1)\,cos^{-1} z)}{sin(\,cos^{-1} z)}. \tag{5}$$

Consequently, we deduce that:

$$U_n(z) = \frac{1}{2\sqrt{z^2-1}}[(z+\sqrt{z^2-1})^{n+1} - (z-\sqrt{z^2-1})^{n+1}]. \tag{6}$$

Since $U_{n-1}(cos\frac{k\pi}{n}) = 0, 1 \le k \le n-1$, we obtain

$$U_{n-1}(z) = 2^{n-1}\prod_{k=1}^{n-1}(z - cos\frac{k\pi}{n}). \tag{7}$$

Additionally, observe that

$$U_{n-1}(-z) = 2^{n-1}\prod_{k=1}^{n-1}(z + cos\frac{k\pi}{n}). \tag{8}$$

Identities (7) and (8) give us

$${U_{n-1}}^2(z) = 4^{n-1}\prod_{k=1}^{n-1}(z^2 - cos^2\frac{k\pi}{n}). \tag{9}$$

Therefore,

$$U_{n-1}^2(\sqrt{\frac{z+2}{4}}) = \prod_{k=1}^{n-1}(z - 2\,cos\frac{2k\pi}{n}). \tag{10}$$

We can also see that

$$\prod_{k=1}^{n-1}(2 - 2\,cos\frac{k\pi}{n}) = n, n \ge 2, \tag{11}$$

$$\prod_{k=1}^{n-1}(2 - 2\,cos\frac{2k\pi}{n}) = n^2, n \ge 2. \tag{12}$$

Polynomials $T_n(z)$ and $U_{n-1}(z)$ are connected by the following identity:

$$U_{n-1}^2(z) = \frac{T_n(2z^2-1)-1}{2(z^2-1)}. \tag{13}$$

Lemma 1 [13]**.** Let $A_n(x)$ be $n \times n$ Circulant matrix such that

$$A_n(x) = \begin{pmatrix} x & -1 & 0 & \cdots & 0 & -1 \\ -1 & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \ddots & \ddots & \ddots & \ddots & -1 \\ -1 & 0 & \cdots & 0 & -1 & x \end{pmatrix}.$$

Then, for $n \ge 3, x \ge 4,$ one has $det(A_n(x)) = 2[T_n(\frac{x}{2}) - 1]$**.**

Lemma 2 [24]**.** If $P \in F^{n\times n}$, $Q \in F^{n\times m}$, $R \in F^{m\times n}$, and $T \in F^{m\times m}$. Suppose that P and T are nonsingular matrices, then: $det\begin{pmatrix} P & Q \\ R & T \end{pmatrix} = det\,T\,det(\,P - QT^{-1}R) = det\,P\,det(\,T - RP^{-1}Q)$.

This Lemma simplifies our calculations for the complexity of the graphs examined in this work and gives certain matrices a sort of symmetry.

We also provide the following crucial Lemma to help with our calculations:

Lemma 3. Let H be a four-block matrix with $n \times n$ matrices. Its structure is as follows:

$$H = \begin{pmatrix} a & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & a & 0 & \cdots & \cdots & 0 & -1 \\ -1 & 0 & \cdots & \cdots & 0 & b & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & b \end{pmatrix}.$$

Then for $ab \neq 1, detH = (ab-1)^n$, and

$$H^{-1} = \begin{pmatrix} \frac{b}{ab-1} & 0 & \cdots & \cdots & 0 & \frac{1}{ab-1} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \frac{b}{ab-1} & 0 & 0 & \cdots & 0 & \frac{1}{ab-1} \\ \frac{1}{ab-1} & 0 & \cdots & \cdots & 0 & \frac{a}{ab-1} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \frac{1}{ab-1} & 0 & \cdots & \cdots & 0 & \frac{a}{ab-1} \end{pmatrix}.$$

3. Major results

Sprocket wheels, which are frequently used in drive systems, are shaped wheels with teeth that mesh with chains, belts, or perforated materials to transfer torque. The number of teeth (which determines the speed/torque ratio), bore diameter, and chain pitch (the distance between tooth centers) are crucial factors. In machinery, they are essential for power transmission, reversal, or speed changes, with variations for varying load capacities.

In the field of graphs, it is possible to create many new graphs from a given set of graphs; just as in mathematics, one always attempts to obtain new structures from given ones.

In this work, we define a new set of ten sprocket wheel graph families, $SW_n^{(1)}, SW_n^{(2)}, \ldots, SW_n^{(10)}$, constructed on the wheel graph. Thereafter, we find explicit formulas for the number of spanning trees of these graph families. Finally, we study the asymptotic entropy of these graphs.

Definition 1. For $n \geq 3$, the sprocket wheel graph family, $SW_n^{(1)}$ is a graph constructed from the wheel graph, W_{2n+1}, with vertex set $\{u_0, u_i, v_i, 1 \leq i \leq n\}$, by adding another set of vertices $\{w_i, x_i, 1 \leq i \leq n\}$, so that the edge set of the graph family, $SW_n^{(1)}$, is

$$\{u_0u_i, u_0v_i, u_0x_i, u_iw_i, v_iw_i, u_ix_i, v_ix_i, u_iv_i, 1 \leq i \leq n\} \cup \{v_iu_{i+1}, 1 \leq i \leq n-1\} \cup \{u_1v_n\},$$

as illustrated in Figure 1. This structure gives order $4n+1$, size $9n$, and an average degree 4.5 in the big n limit for the sprocket wheel graph family, $SW_n^{(1)}$.

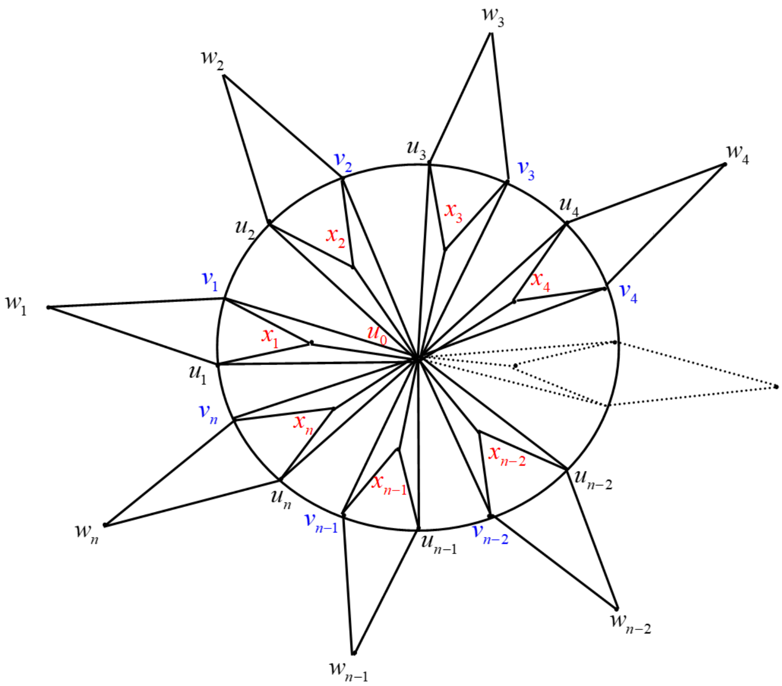

Figure 1. The sprocket wheel graph family, $SW_n^{(1)}$.

Theorem 1. For $n \geq 1$, we have

$$\tau\left(SW_n^{(1)}\right) = (39 + 10\sqrt{14})^n + (39 - 10\sqrt{14})^n - 2 \times 11^n.$$

Proof. By applying the matrix tree theorem and removing the first row and column that includes every value that matches the center vertex, u_0, we get

$$\tau\left(SW_n^{(1)}\right) = det \begin{array}{c}
\begin{matrix} u_1 & u_2 & \cdots & \cdots & u_n & v_1 & v_2 & \cdots & \cdots & v_n & w_1 & w_2 & \cdots & \cdots & w_n & x_1 & x_2 & \cdots & \cdots & x_n \end{matrix} \\
\begin{pmatrix}
5 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots & -1 & \ddots & \ddots & \ddots & 0 & 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 5 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 \\
-1 & -1 & 0 & \cdots & 0 & 5 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 \\
-1 & 0 & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & 5 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 \\
-1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & 0 \\
-1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 & 3 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 3
\end{pmatrix}
\end{array}$$

$$= det \begin{pmatrix} A & B & C & D \\ E & F & G & H \\ P & Q & R & S \\ T & U & V & W \end{pmatrix}.$$

Using Lemma 2, we obtain

$$\tau\left(SW_n^{(1)}\right) = det\begin{pmatrix} R & S \\ V & W \end{pmatrix} \times \det\left[\begin{pmatrix} A & B \\ E & F \end{pmatrix} - \begin{pmatrix} P & Q \\ T & U \end{pmatrix}\begin{pmatrix} R & S \\ V & W \end{pmatrix}^{-1}\begin{pmatrix} C & D \\ G & H \end{pmatrix}\right]$$

$$= det\begin{pmatrix} 2 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 2 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 3 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 3 \end{pmatrix}$$

$$\times det\begin{pmatrix} \frac{25}{6} & 0 & \cdots & \cdots & 0 & -\frac{11}{6} & 0 & \cdots & 0 & -1 \\ 0 & \ddots & \ddots & \ddots & \vdots & -1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \frac{25}{6} & 0 & \cdots & 0 & -1 & -\frac{11}{6} \\ -\frac{11}{6} & -1 & 0 & \cdots & 0 & \frac{25}{6} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & 0 \\ -1 & 0 & \cdots & 0 & -\frac{11}{6} & 0 & \cdots & \cdots & 0 & \frac{25}{6} \end{pmatrix}.$$

$$= 6^n \times \det\begin{pmatrix} X & Z \\ Z^t & X \end{pmatrix}.$$

Reapplying Lemma 2, we obtain

$$\tau\left(SW_n^{(1)}\right) = 6^n \times \det X \det(X - Z^t X^{-1} Z)$$

$$= 6^n \times (\frac{25}{6})^n \times det\begin{pmatrix} \frac{78}{25} & -\frac{11}{25} & 0 & \cdots & 0 & -\frac{11}{25} \\ -\frac{11}{25} & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & -\frac{11}{6} \\ -\frac{11}{25} & 0 & \cdots & 0 & -\frac{11}{25} & \frac{78}{25} \end{pmatrix}$$

$$= 11^n \times det\begin{pmatrix} \frac{78}{11} & -1 & 0 & \cdots & 0 & -1 \\ -1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & -1 \\ -1 & 0 & \cdots & 0 & -1 & \frac{78}{11} \end{pmatrix}.$$

Using Lemma 1, we obtain

$$\tau\left(SW_n^{(1)}\right) = 11^n \times 2\left[T_n\left(\frac{39}{11}\right) - 1\right].$$

The result follows from Eq (4).

Definition 2. For $n \geq 3$, the sprocket wheel graph family, $SW_n^{(2)}$, is a graph constructed from the wheel graph, W_{2n+1}, with vertex set $\{u_0, u_i, v_i, 1 \leq i \leq n\}$ by adding another set of vertices $\{w_i, x_i, 1 \leq i \leq n\}$, so that the edge set of the graph family, $SW_n^{(2)}$, is

$$\{u_0u_i, u_0v_i, u_iw_i, v_iw_i, u_ix_i, v_ix_i, u_iv_i, w_ix_i, 1 \leq i \leq n\} \cup \{v_iu_{i+1}, 1 \leq i \leq n-1\} \cup \{u_1v_n\},$$

as illustrated in Figure 2. This structure gives order $4n+1$, size $9n$, and an average degree 4.5 in the large n limit for the sprocket wheel graph family, $SW_n^{(2)}$.

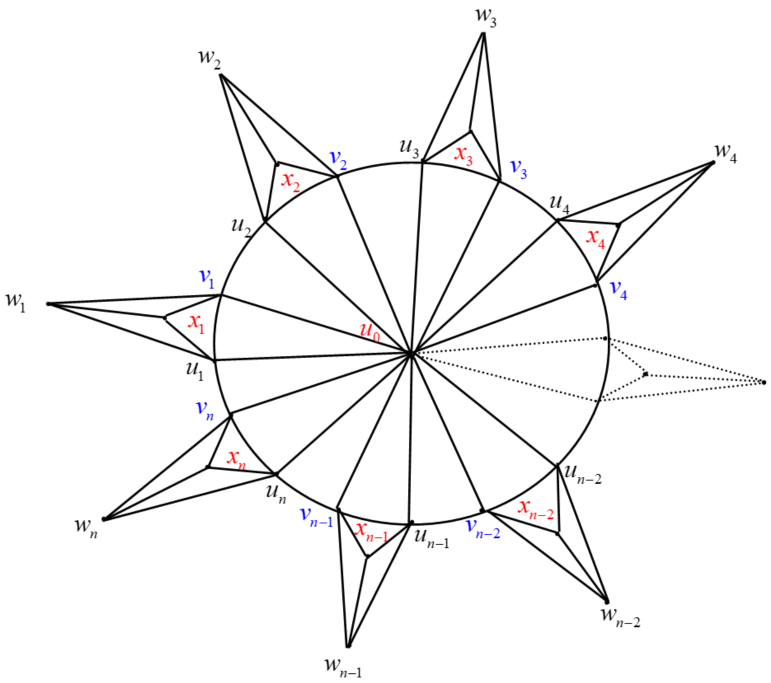

Figure 2. The sprocket wheel graph family, $SW_n^{(2)}$.

Theorem 2. For $n \geq 1$, we have

$$\tau(SW_n^{(2)}) = 4^n[(11+\sqrt{105})^n + (11-\sqrt{105})^n - 2^{2n+1}].$$

Proof. By applying the matrix tree theorem and removing the first row and column that includes every value that matches the center vertex, u_0, we get

$$
\tau\left(SW_n^{(2)}\right) = det \begin{array}{c}
\begin{array}{cccccccccccccccccccc} u_1 & u_2 & \dots & \dots & u_n & v_1 & v_2 & \dots & \dots & v_n & w_1 & w_2 & \dots & \dots & w_n & x_1 & x_2 & \dots & \dots & x_n \end{array} \\
\begin{pmatrix}
5 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots & -1 & \ddots & \ddots & \ddots & 0 & 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 5 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 \\
-1 & -1 & 0 & \cdots & 0 & 5 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 \\
-1 & 0 & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & 5 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 \\
-1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & 3 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & 3 & 0 & \cdots & \cdots & 0 & -1 \\
-1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & 3 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & 3
\end{pmatrix}
\end{array}
$$

$$
= det \begin{pmatrix} A & B & C & D \\ E & F & G & H \\ P & Q & R & S \\ T & U & V & W \end{pmatrix}.
$$

Using Lemma 2, we obtain

$$
\tau\left(SW_n^{(2)}\right) = det \begin{pmatrix} R & S \\ V & W \end{pmatrix} \times \det \left[\begin{pmatrix} A & B \\ E & F \end{pmatrix} - \begin{pmatrix} P & Q \\ T & U \end{pmatrix} \begin{pmatrix} R & S \\ V & W \end{pmatrix}^{-1} \begin{pmatrix} C & D \\ G & H \end{pmatrix} \right]
$$

$$
= det \begin{pmatrix}
3 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 3 & 0 & \cdots & 0 & -1 \\
-1 & 0 & \cdots & 0 & 3 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 3
\end{pmatrix}
$$

$$
\times det \begin{pmatrix}
4 & 0 & \cdots & \cdots & 0 & -2 & 0 & \cdots & 0 & -1 \\
0 & \ddots & \ddots & \ddots & \vdots & -1 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 4 & 0 & \cdots & 0 & -1 & -2 \\
-2 & -1 & 0 & \cdots & 0 & 4 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & 0 \\
-1 & 0 & \cdots & 0 & -2 & 0 & \cdots & \cdots & 0 & 4
\end{pmatrix}.
$$

Applying Lemma 3 yields

$$
\tau\left(SW_n^{(2)}\right) = 8^n \times \det \begin{pmatrix} X & Z \\ Z^t & X \end{pmatrix}.
$$

Reapplying Lemma 2, we obtain

$$\tau\left(SW_n^{(2)}\right) = 8^n \times \det X \det(X - Z^t X^{-1} Z)$$

$$= 8^n \times 4^n det \begin{pmatrix} \frac{11}{4} & -\frac{1}{2} & 0 & \cdots & 0 & -\frac{1}{2} \\ -\frac{1}{2} & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \cdots & 0 & -\frac{1}{2} & \frac{11}{4} \end{pmatrix}$$

$$= 16^n \times det \begin{pmatrix} \frac{11}{2} & -1 & 0 & \cdots & 0 & -1 \\ -1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & -1 \\ -1 & 0 & \cdots & 0 & -1 & \frac{11}{2} \end{pmatrix}.$$

Lemma 1 gives us $\tau\left(SW_n^{(2)}\right) = 16^n \times 2[T_n(\frac{11}{4}) - 1]$. The result follows from Eq (4).

Definition 3. For $n \geq 3$, the sprocket wheel graph family, $SW_n^{(3)}$, is a graph built from sprocket wheel graph family, $SW_n^{(2)}$, by adding the set of edges $\{u_0 x_i, 1 \leq i \leq n\}$, as illustrated in Figure 3. This structure gives order $4n + 1$, size $10n$, and an average degree 5 in the large n limit for the sprocket wheel graph family, $SW_n^{(3)}$.

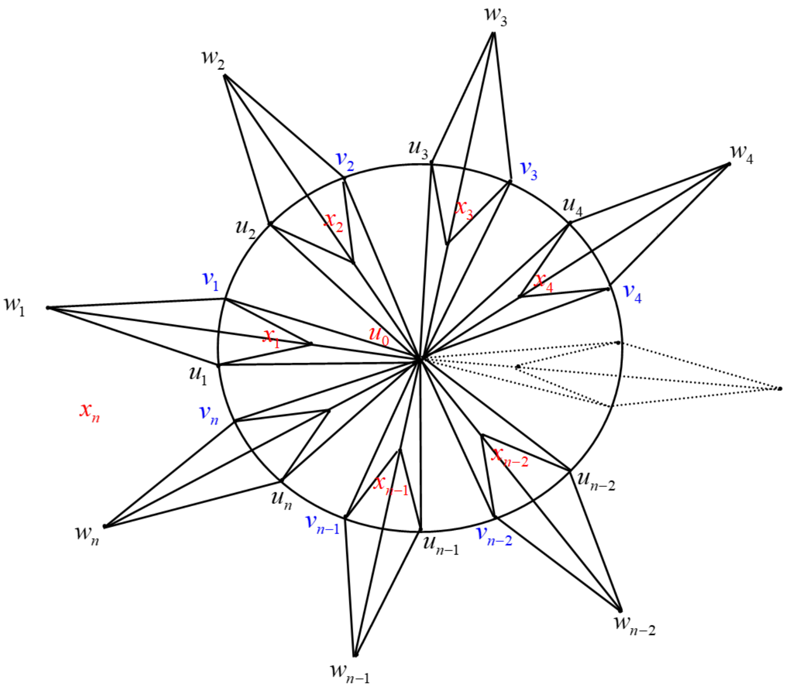

Figure 3. The sprocket wheel graph family, $SW_n^{(3)}$.

Theorem 3. For $n \geq 1$, we have

$$\tau(SW_n^{(3)}) = \left(\frac{5}{2}\right)^n [(29+\sqrt{777})^n + (29-\sqrt{777})^n - 2^{3n+1}].$$

Proof. By applying the matrix tree theorem and removing the first row and column that includes every value that matches the center vertex, u_0, we get

$$\tau\left(SW_n^{(3)}\right) = det \begin{array}{c} \begin{matrix} u_1 & u_2 & \cdots & \cdots & u_n & v_1 & v_2 & \cdots & \cdots & v_n & w_1 & w_2 & \cdots & \cdots & w_n & x_1 & x_2 & \cdots & \cdots & x_n \end{matrix} \\ \begin{pmatrix}
5 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots & -1 & \ddots & \ddots & \ddots & 0 & 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 5 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 \\
-1 & -1 & 0 & \cdots & 0 & 5 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 \\
-1 & 0 & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & 5 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 \\
-1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & 3 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & 3 & 0 & \cdots & \cdots & 0 & -1 \\
-1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & 4 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & 4
\end{pmatrix} \end{array}$$

$$= det \begin{pmatrix} A & B & C & D \\ E & F & G & H \\ P & Q & R & S \\ T & U & V & W \end{pmatrix}.$$

Using Lemma 2, we obtain

$$\tau\left(SW_n^{(3)}\right) = det \begin{pmatrix} R & S \\ V & W \end{pmatrix} \times \det\left[\begin{pmatrix} A & B \\ E & F \end{pmatrix} - \begin{pmatrix} P & Q \\ T & U \end{pmatrix} \begin{pmatrix} R & S \\ V & W \end{pmatrix}^{-1} \begin{pmatrix} C & D \\ G & H \end{pmatrix}\right]$$

$$= det \begin{pmatrix}
3 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 3 & 0 & \cdots & \cdots 0 & -1 \\
-1 & 0 & \cdots & 0 & 4 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 4
\end{pmatrix} det \begin{pmatrix}
\frac{46}{11} & 0 & \cdots & \cdots & \cdots & 0 & -\frac{20}{11} & 0 & \cdots & \cdots & 0 & -1 \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & -1 & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & \cdots & 0 & \frac{46}{11} & 0 & \cdots & \cdots & 0 & -1 & -\frac{20}{11} \\
-\frac{20}{11} & -1 & 0 & \cdots & \cdots & 0 & \frac{46}{11} & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
-1 & 0 & \cdots & \cdots & 0 & -\frac{20}{11} & 0 & \cdots & \cdots & \cdots & 0 & \frac{46}{11}
\end{pmatrix}.$$

Applying Lemma 3 yields

$$\tau\left(SW_n^{(3)}\right) = 11^n \times \det \begin{pmatrix} X & Z \\ Z^t & X \end{pmatrix}.$$

Applying Lemma 2 once more, we get

$$\tau\left(SW_n^{(3)}\right) = 11^n \times \det X \det(X - Z^t X^{-1} Z)$$

$$= 11^n \times (\frac{46}{11})^n \times det \begin{pmatrix} \frac{145}{46} & -\frac{10}{23} & 0 & \cdots & 0 & -\frac{10}{23} \\ -\frac{10}{23} & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & -\frac{10}{23} \\ -\frac{10}{23} & 0 & \cdots & 0 & -\frac{10}{23} & \frac{145}{46} \end{pmatrix}$$

$$= 20^n \times det \begin{pmatrix} \frac{29}{4} & -1 & 0 & \cdots & 0 & -1 \\ -1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & -1 \\ -1 & 0 & \cdots & 0 & -1 & \frac{29}{4} \end{pmatrix}.$$

By applying Lemma 1, we obtain $\tau\left(SW_n^{(3)}\right) = 20^n \times 2[T_n(\frac{29}{8}) - 1]$. The result follows from Eq (4).

Definition 4. For $n \geq 3$, the sprocket wheel graph family, $SW_n^{(4)}$, is a graph constructed from the wheel graph, W_{2n+1}, with vertex set $\{u_0, u_i, v_i, 1 \leq i \leq n\}$ by adding another set of vertices $\{w_i, x_i, 1 \leq i \leq n\}$, so that the edge set of the graph family, $SW_n^{(4)}$, is $\{u_0u_i,\ u_0v_i,\ u_0x_i, u_iv_i,\ u_iw_i,\ v_iw_i, v_ix_i, 1 \leq i \leq n\} \cup \{v_iu_{i+1},\ x_iu_{i+1}, 1 \leq i \leq n-1\} \cup \{u_1v_n, u_1x_n\}$, as shown in Figure 4. This structure gives order $4n+1$, size $9n$, and an average degree 4.5 in the large n limit for the sprocket wheel graph family, $SW_n^{(4)}$.

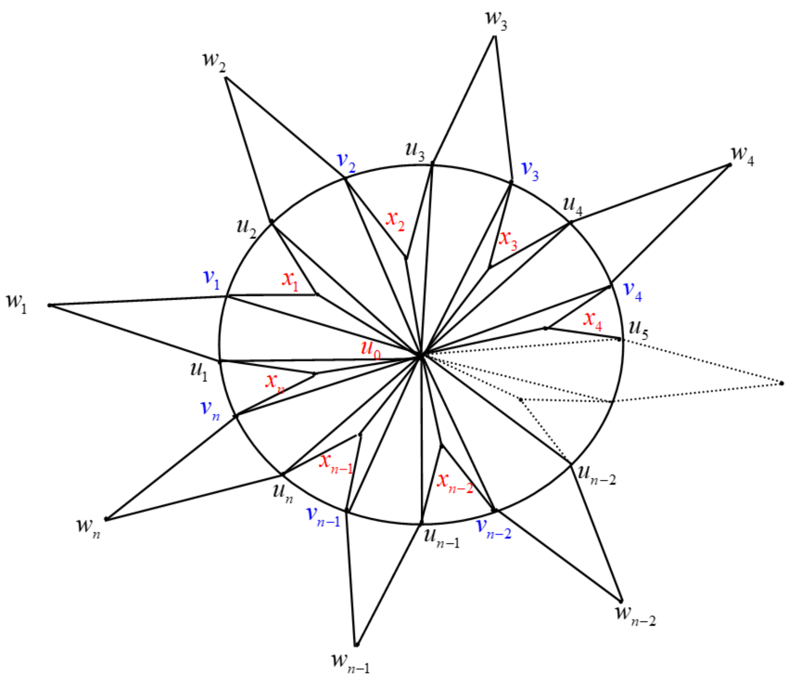

Figure 4. The sprocket wheel graph family, $SW_n^{(4)}$.

Theorem 4. For $n \geq 1$, we have $\tau\left(SW_n^{(4)}\right) = 4^n[(10+\sqrt{91})^n + (10-\sqrt{91})^n - 2\times 3^n]$.

Proof. By applying the matrix tree theorem and removing the first row and column that includes every value that matches the center vertex, u_0, we get

$$\tau\left(SW_n^{(4)}\right) = det \begin{array}{c} \begin{array}{cccccccccccccccccccc} u_1 & u_2 & \dots & \dots & u_n & v_1 & v_2 & \dots & \dots & v_n & w_1 & w_2 & \dots & \dots & w_n & x_1 & x_2 & \dots & \dots & x_n \end{array} \\ \begin{pmatrix} 5 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & -1 \\ 0 & \ddots & \ddots & \ddots & \vdots & -1 & \ddots & \ddots & \ddots & 0 & 0 & \ddots & \ddots & \ddots & \vdots & -1 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 5 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 & -1 & 0 \\ -1 & -1 & 0 & \cdots & 0 & 5 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 \\ -1 & 0 & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & 5 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 \\ -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 & 3 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ -1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 3 \end{pmatrix} \end{array}$$

$$= det \begin{pmatrix} A & B & C & D \\ E & F & G & H \\ P & Q & R & S \\ T & U & V & W \end{pmatrix}.$$

Using Lemma 2, we obtain

$$\tau\left(SW_n^{(4)}\right) = det \begin{pmatrix} R & S \\ V & W \end{pmatrix} \times \det\left[\begin{pmatrix} A & B \\ E & F \end{pmatrix} - \begin{pmatrix} P & Q \\ T & U \end{pmatrix}\begin{pmatrix} R & S \\ V & W \end{pmatrix}^{-1}\begin{pmatrix} C & D \\ G & H \end{pmatrix}\right]$$

$$= det \begin{pmatrix} 2 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 2 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 3 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 3 \end{pmatrix}$$

$$\times det \begin{pmatrix} \frac{25}{6} & 0 & \cdots & \cdots & 0 & -\frac{3}{2} & 0 & \cdots & 0 & -\frac{4}{3} \\ 0 & \ddots & \ddots & \ddots & \vdots & -\frac{4}{3} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \frac{25}{6} & 0 & \cdots & 0 & -\frac{4}{3} & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{4}{3} & 0 & \cdots & 0 & \frac{25}{6} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & -\frac{4}{3} & \vdots & \ddots & \ddots & \ddots & 0 \\ -\frac{4}{3} & 0 & \cdots & 0 & -\frac{3}{2} & 0 & \cdots & \cdots & 0 & \frac{25}{6} \end{pmatrix} = 6^n \times \det\begin{pmatrix} X & Z \\ Z^t & X \end{pmatrix}.$$

Reapplying Lemma 2, we obtain:

$$\tau\left(SW_n^{(4)}\right) = 6^n \times \det X \det(X - Z^t X^{-1} Z)$$

$$= 6^n \times (\frac{25}{6})^n \times det \begin{pmatrix} \frac{16}{5} & -\frac{12}{25} & 0 & \cdots & 0 & -\frac{12}{25} \\ -\frac{12}{25} & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & -\frac{12}{25} \\ -\frac{12}{25} & 0 & \cdots & 0 & -\frac{12}{25} & \frac{16}{5} \end{pmatrix}$$

$$= 12^n \times det \begin{pmatrix} \frac{20}{3} & -1 & 0 & \cdots & 0 & -1 \\ -1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & -1 \\ -1 & 0 & \cdots & 0 & -1 & \frac{20}{3} \end{pmatrix}.$$

Using Lemma 1, we get $\tau\left(SW_n^{(4)}\right) = 12^n \times 2[T_n(\frac{10}{3}) - 1]$. The result follows from Eq (4).

Definition 5. For $n \geq 3$, the sprocket wheel graph family, $SW_n^{(5)}$, is a graph built from sprocket wheel graph family, $SW_n^{(4)}$, by adding the set of edges $\{u_0 w_i, 1 \leq i \leq n\}$, as illustrated in Figure 5. This structure gives order $4n + 1$, size $10n$, and an average degree 5 in the large n limit for the sprocket wheel graph family, $SW_n^{(5)}$.

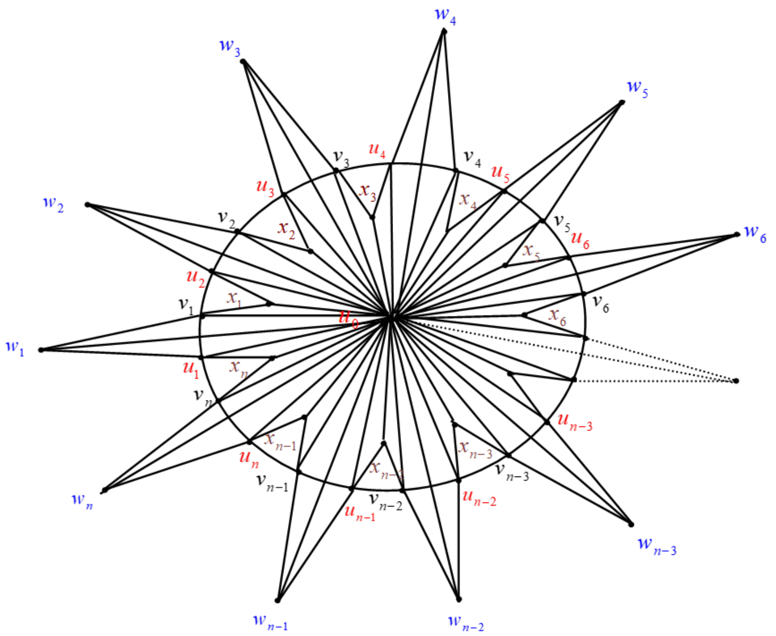

Figure 5. The sprocket wheel graph family, $SW_n^{(5)}$.

Theorem 5. For $n \geq 1$, we have

$$\tau(SW_n^{(5)}) = \frac{1}{2^n}[(137+13\sqrt{105})^n + (137-13\sqrt{105})^n - 2^{5n+1}].$$

Proof. By applying the matrix tree theorem and removing the first row and column that includes every value that matches the center vertex, u_0, we get

$$\tau\left(SW_n^{(5)}\right) = det \begin{array}{c} \begin{matrix} u_1 & u_2 & \cdots & \cdots & u_n & v_1 & v_2 & \cdots & \cdots & v_n & w_1 & w_2 & \cdots & \cdots & w_n & x_1 & x_2 & \cdots & \cdots & x_n \end{matrix} \\ \begin{pmatrix} 5 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & -1 \\ 0 & \ddots & \ddots & \ddots & \vdots & -1 & \ddots & \ddots & \ddots & 0 & 0 & \ddots & \ddots & \ddots & \vdots & -1 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 5 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 & -1 & 0 \\ -1 & -1 & 0 & \cdots & 0 & 5 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 \\ -1 & 0 & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & 5 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 \\ -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & 3 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & 3 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 & 3 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ -1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 3 \end{pmatrix} \end{array}$$

$$= det \begin{pmatrix} A & B & C & D \\ E & F & G & H \\ P & Q & R & S \\ T & U & V & W \end{pmatrix}.$$

Using Lemma 2, we obtain

$$\tau\left(SW_n^{(5)}\right) = det \begin{pmatrix} R & S \\ V & W \end{pmatrix} \times \det\left[\begin{pmatrix} A & B \\ E & F \end{pmatrix} - \begin{pmatrix} P & Q \\ T & U \end{pmatrix}\begin{pmatrix} R & S \\ V & W \end{pmatrix}^{-1}\begin{pmatrix} C & D \\ G & H \end{pmatrix}\right]$$

$$= det \begin{pmatrix} 3 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 3 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 3 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 3 \end{pmatrix}$$

$$\times det \begin{pmatrix} \frac{13}{3} & 0 & \cdots & \cdots & 0 & -\frac{4}{3} & 0 & \cdots & 0 & -\frac{4}{3} \\ 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \frac{13}{3} & 0 & \cdots & 0 & -\frac{4}{3} & -\frac{4}{3} \\ -\frac{4}{3} & -\frac{4}{3} & 0 & \cdots & 0 & \frac{13}{3} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & -\frac{4}{3} & \vdots & \ddots & \ddots & \ddots & 0 \\ -\frac{4}{3} & 0 & \cdots & 0 & -\frac{4}{3} & 0 & \cdots & \cdots & 0 & \frac{13}{3} \end{pmatrix} = 9^n \times \det\begin{pmatrix} X & Z \\ Z^t & X \end{pmatrix}.$$

Applying Lemma 2 once more yields

$$\tau\left(SW_n^{(5)}\right) = 9^n \times \det X \det(X - Z^t X^{-1} Z)$$

$$= 9^n \times (\frac{13}{3})^n \times det \begin{pmatrix} \frac{137}{39} & -\frac{16}{39} & 0 & \cdots & 0 & -\frac{16}{39} \\ -\frac{16}{39} & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & -\frac{16}{39} \\ -\frac{16}{39} & 0 & \cdots & 0 & -\frac{16}{39} & \frac{137}{39} \end{pmatrix}$$

$$= 16^n \, det \begin{pmatrix} \frac{137}{16} & -1 & 0 & \cdots & 0 & -1 \\ -1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & -1 \\ -1 & 0 & \cdots & 0 & -1 & \frac{137}{16} \end{pmatrix}.$$

Using Lemma 1, we obtain

$$\tau(SW_n^{(5)}) = 16^n \times 2[T_n(\tfrac{137}{32}) - 1].$$

The result follows from Eq (4).

Definition 6. For $n \geq 3$, the sprocket wheel graph family, $SW_n^{(6)}$, is a graph constructed from the wheel graph, W_{n+1}, with vertex set $\{u_0, u_i, 1 \leq i \leq n\}$ by adding another set of vertices $\{v_i, w_i, x_i, 1 \leq i \leq n\}$, so that the edge set of the graph family, $SW_n^{(6)}$, is

$$\{u_0u_i, u_0x_i, u_iv_i, u_iw_i, v_iw_i, u_ix_i, 1 \leq i \leq n\} \cup \{u_{i+1}v_i, u_{i+1}w_i, u_{i+1}x_i, u_iu_{i+1}, 1 \leq i \leq n-1\} \\ \cup \{u_1v_n, u_1w_n, u_1x_n, u_1u_n\},$$

as illustrated in Figure 6.

This structure gives order $4n + 1$, size $10n$, and an average degree 5 in the large n limit for the sprocket wheel graph family, $SW_n^{(6)}$.

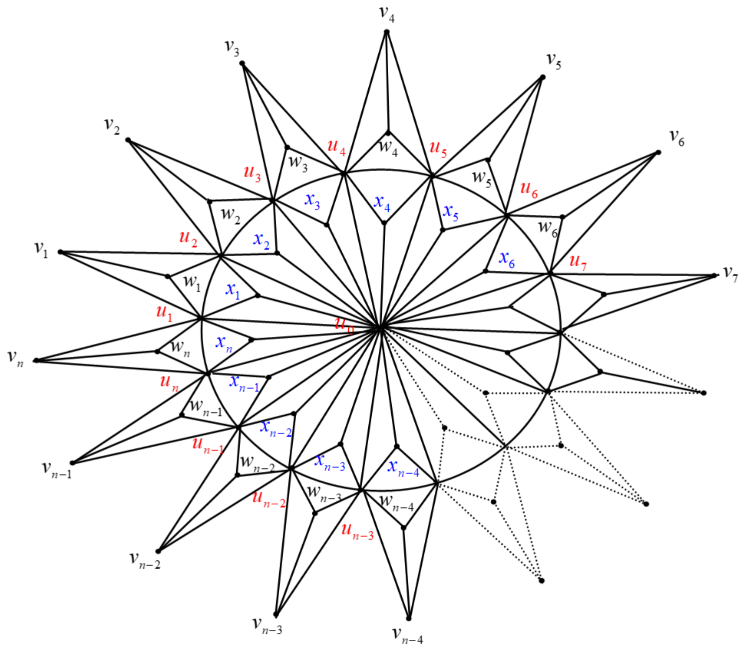

Figure 6. The sprocket wheel graph family, $SW_n^{(6)}$.

Theorem 6. For $n \geq 1$, we have $\tau\left(SW_n^{(6)}\right) = 4^n[(19+\sqrt{165})^n + (19-\sqrt{165})^n - 2 \times 14^n]$.

Proof. By applying the matrix tree theorem and removing the first row and column that includes every value that matches the center vertex, u_0, we get

$$\tau(SW_n^{(6)}) = det \begin{array}{c} \begin{array}{cccccccccccccccccccccccc} u_1 & u_2 & \cdots & \cdots & \cdots & u_n & v_1 & v_2 & \cdots & \cdots & \cdots & v_n & w_1 & w_2 & \cdots & \cdots & \cdots & w_n & x_1 & x_2 & \cdots & \cdots & \cdots & x_n \end{array} \\ \begin{pmatrix}
9 & -1 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -1 \\
-1 & \ddots & \ddots & \ddots & \vdots & 0 & -1 & \ddots & \ddots & \ddots & \ddots & 0 & -1 & \ddots & \ddots & \ddots & \ddots & 0 & -1 & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
-1 & 0 & \cdots & 0 & -1 & 9 & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -1 & -1 \\
-1 & -1 & 0 & \cdots & \cdots & 0 & 3 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
-1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 3 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
-1 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 3 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
-1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots 0 & 3 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
-1 & -1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 3 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
-1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 & 3
\end{pmatrix} \end{array}$$

$$= det \begin{pmatrix} A & B & C & D \\ E & F & G & H \\ P & Q & R & S \\ T & U & V & W \end{pmatrix}.$$

Using Lemma 2, we obtain

$$\tau\left(SW_n^{(6)}\right) = det\begin{pmatrix} R & S \\ V & W \end{pmatrix} \times \det\left[\begin{pmatrix} A & B \\ E & F \end{pmatrix} - \begin{pmatrix} P & Q \\ T & U \end{pmatrix}\begin{pmatrix} R & S \\ V & W \end{pmatrix}^{-1}\begin{pmatrix} C & D \\ G & H \end{pmatrix}\right]$$

$$= det\begin{pmatrix}
3 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 3 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 3 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 3
\end{pmatrix}$$

$$\times det\begin{pmatrix}
\frac{23}{3} & -\frac{5}{3} & 0 & \cdots & 0 & -\frac{5}{3} & -\frac{4}{3} & 0 & \cdots & \cdots & 0 & -\frac{4}{3} \\
-\frac{5}{3} & \ddots & \ddots & \ddots & \ddots & 0 & -\frac{4}{3} & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & -\frac{5}{3} & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
-\frac{5}{3} & 0 & \cdots & 0 & -\frac{5}{3} & \frac{23}{3} & 0 & \cdots & \cdots & 0 & -\frac{4}{3} & -\frac{4}{3} \\
-\frac{4}{3} & -\frac{4}{3} & 0 & \cdots & \cdots & 0 & \frac{8}{3} & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & -\frac{4}{3} & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
-\frac{4}{3} & 0 & \cdots & \cdots & 0 & -\frac{4}{3} & 0 & \cdots & \cdots & \cdots & 0 & \frac{8}{3}
\end{pmatrix}$$

$$= 9^n \times \det\begin{pmatrix} X & Z \\ Z^t & Y \end{pmatrix}.$$

Applying Lemma 2 once more yields

$$\tau\left(SW_n^{(6)}\right) = 9^n \times \det Y \det(X - Z^t\, Y^{-1} Z)$$

$$= 9^n \times (\frac{8}{3})^n \times det\begin{pmatrix}
\frac{19}{3} & -\frac{7}{3} & 0 & \cdots & 0 & -\frac{7}{3} \\
-\frac{7}{3} & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & -\frac{7}{3} \\
-\frac{7}{3} & 0 & \cdots & 0 & -\frac{7}{3} & \frac{19}{3}
\end{pmatrix}$$

$$= 56^n \times det\begin{pmatrix}
\frac{19}{7} & -1 & 0 & \cdots & 0 & -1 \\
-1 & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & -1 \\
-1 & 0 & \cdots & 0 & -1 & \frac{19}{7}
\end{pmatrix}.$$

By applying Lemma 1, we get

$$\tau\left(SW_n^{(6)}\right) = (56)^n \times 2\left[T_n\left(\frac{19}{14}\right) - 1\right].$$

The result follows from Eq (4).

Definition 7. For $n \geq 3$, the sprocket wheel graph family, $SW_n^{(7)}$, is a graph constructed from the wheel graph, W_{n+1} with vertex set $\{u_0, u_i, 1 \leq i \leq n\}$ by adding another set of vertices $\{v_i, w_i, x_i, 1 \leq i \leq n\}$, so that the edge set of the graph family, $SW_n^{(7)}$, is

$$\{u_0u_i, u_0x_i, u_iv_i, u_iw_i, u_ix_i, w_ix_i, 1 \leq i \leq n\} \cup \{u_iv_{i+1}, u_iw_{i+1}, u_ix_{i+1}, u_iu_{i+1}, 1 \leq i \leq n-1\}$$
$$\cup \{u_nv_1, u_nx_1, u_nw_1, u_nu_1\},$$

as illustrated in Figure 7. This structure gives order $4n + 1$, size $10n$, and an average degree 5 in the large n limit for the sprocket wheel graph family, $SW_n^{(7)}$.

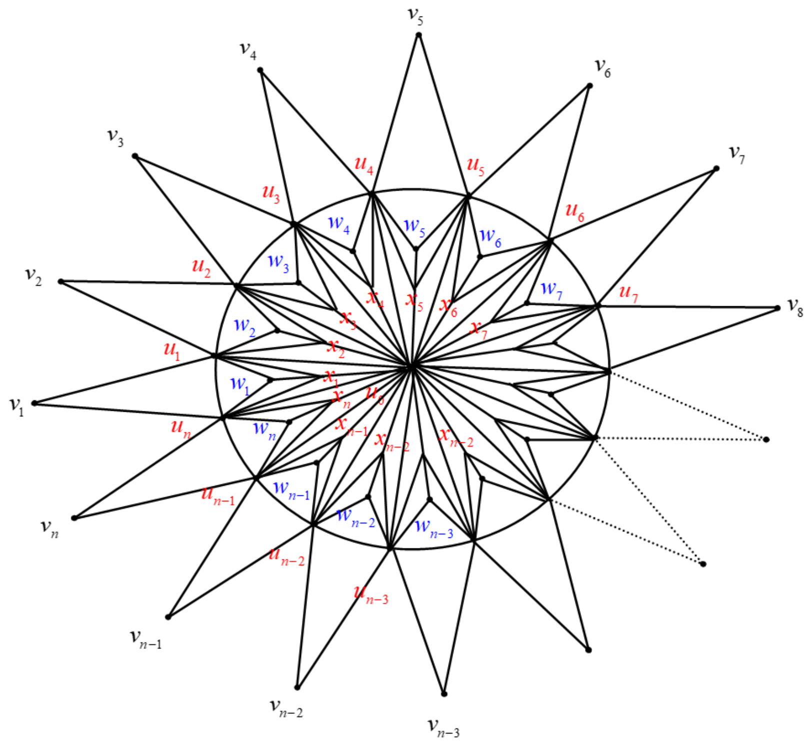

Figure 7. The sprocket wheel graph family, $SW_n^{(7)}$.

Theorem 7. For $n \geq 1$, we have

$$\tau(SW_n^{(7)}) = (70 + 11\sqrt{19})^n + (70 - 11\sqrt{19})^n - 2 \times 51^n.$$

Proof. By applying the matrix tree theorem and removing the first row and column that includes every value that matches the center vertex, u_0, we get

$$
\tau(SW_n^{(7)}) = det \begin{array}{c}
\begin{matrix} u_1 & u_2 & \cdots & \cdots & \cdots & u_n & v_1 & v_2 & \cdots & \cdots & \cdots & v_n & w_1 & w_2 & \cdots & \cdots & \cdots & w_n & x_1 & x_2 & \cdots & \cdots & \cdots & x_n \end{matrix} \\
\begin{pmatrix}
9 & -1 & 0 & \cdots & 0 & -1 & -1 & -1 & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -1 & -1 & \cdots & \cdots & \cdots & 0 \\
-1 & \ddots & \ddots & \ddots & \vdots & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & -1 & 0 & \ddots & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & \ddots & -1 & 0 & \ddots & \ddots & \ddots & \ddots & -1 \\
-1 & 0 & \cdots & 0 & -1 & 9 & -1 & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -1 \\
-1 & 0 & \cdots & \cdots & 0 & -1 & 2 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
-1 & \ddots & \ddots & \ddots & \ddots & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
-1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 3 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\
-1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots 0 & 3 & 0 & \cdots & \cdots & \cdots & 0 & -1 \\
-1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 4 & 0 & \cdots & \cdots & \cdots & 0 \\
-1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 4
\end{pmatrix}
\end{array}
$$

$$
= det \begin{pmatrix} A & B & C & D \\ E & F & G & H \\ P & Q & R & S \\ T & U & V & W \end{pmatrix}.
$$

Using Lemma 2, we obtain

$$
\tau(SW_n^{(7)}) = det \begin{pmatrix} R & S \\ V & W \end{pmatrix} \times \det [\begin{pmatrix} A & B \\ E & F \end{pmatrix} - \begin{pmatrix} P & Q \\ T & U \end{pmatrix} \begin{pmatrix} R & S \\ V & W \end{pmatrix}^{-1} \begin{pmatrix} C & D \\ G & H \end{pmatrix}]
$$

$$
= det \begin{pmatrix}
3 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 3 & 0 & \cdots & 0 & -1 \\
-1 & 0 & \cdots & 0 & 4 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 4
\end{pmatrix}
$$

$$
\times det \begin{pmatrix}
\frac{81}{11} & -\frac{20}{11} & 0 & \cdots & 0 & -\frac{20}{11} & -1 & -1 & 0 & \cdots & \cdots & 0 \\
-\frac{20}{11} & \ddots & \ddots & \ddots & \ddots & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & -\frac{20}{11} & 0 & \ddots & \ddots & \ddots & \ddots & -1 \\
-\frac{20}{11} & 0 & \cdots & 0 & -\frac{20}{11} & \frac{81}{11} & -1 & 0 & \cdots & \cdots & 0 & -1 \\
-1 & 0 & \cdots & \cdots & 0 & -1 & 2 & 0 & \cdots & \cdots & \cdots & 0 \\
-1 & \ddots & \ddots & \ddots & \ddots & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 2
\end{pmatrix}.
$$

Applying Lemma 3 yields $\tau\left(SW_n^{(7)}\right) = 11^n \times \det \begin{pmatrix} X & Z \\ Z^t & Y \end{pmatrix}$. Reapplying Lemma 2, we obtain

$$\tau\left(SW_n^{(7)}\right) = 11^n \times \det\, Y \det(X - Z^t Y^{-1} Z)$$

$$= 11^n \times 2^n \times det \begin{pmatrix} \frac{70}{11} & -\frac{51}{22} & 0 & \cdots & 0 & -\frac{51}{22} \\ -\frac{51}{22} & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & -\frac{51}{22} \\ -\frac{51}{22} & 0 & \cdots & 0 & -\frac{51}{22} & \frac{70}{11} \end{pmatrix}$$

$$= 51^n\, det \begin{pmatrix} \frac{140}{51} & -1 & 0 & \cdots & 0 & -1 \\ -1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & -1 \\ -1 & 0 & \cdots & 0 & -1 & \frac{140}{51} \end{pmatrix}.$$

Using Lemma 1, we obtain $\tau\left(SW_n^{(7)}\right) = 51^n \times 2\left[T_n\left(\frac{70}{51}\right) - 1\right]$. The result follows from Eq (4).

Definition 8. For $n \geq 3$, the sprocket wheel graph family, $SW_n^{(8)}$, is a graph constructed from the wheel graph, W_{n+1}, with vertex set $\{u_0, u_i, 1 \leq i \leq n\}$ by adding another set of vertices $\{v_i, w_i, x_i, 1 \leq i \leq n, 1 \leq i \leq n\}$, so that the edge set of the graph family, $SW_n^{(8)}$, is

$$\{u_0u_i, u_iv_i, u_iw_i, u_ix_i, v_iw_i, w_ix_i, 1 \leq i \leq n\} \cup \{u_iv_{i+1}, u_iw_{i+1}, u_ix_{i+1}, u_iu_{i+1}, 1 \leq i \leq n-1\}$$
$$\cup \{u_nv_1, u_nx_1, u_nw_1, u_nu_1\},$$

as illustrated in Figure 8. This structure gives order $4n + 1$, size $10n$, and an average degree 5 in the large n limit for the sprocket wheel graph family, $SW_n^{(8)}$.

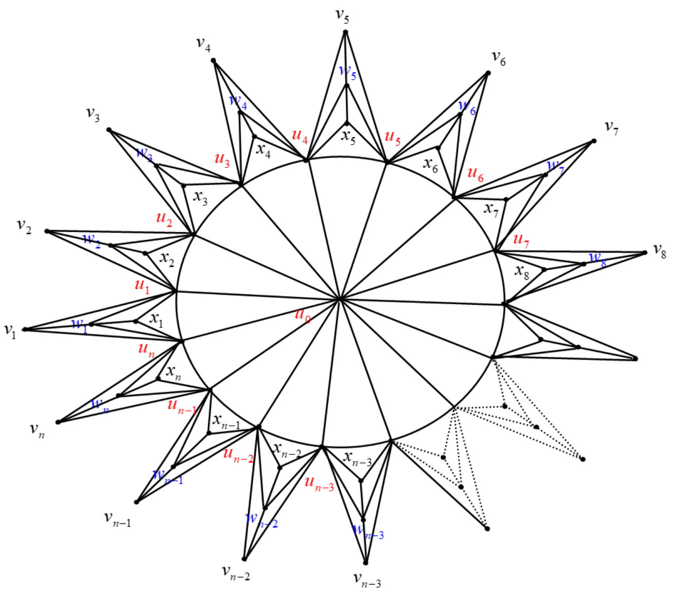

Figure 8. The sprocket wheel graph family, $SW_n^{(8)}$.

Theorem 8. For $n \geq 1$, we have $\tau(SW_n^{(8)}) = 15^n[(6 + \sqrt{11})^n + (6 - \sqrt{11})^n - 2 \times 5^n]$.

Proof. By applying the matrix tree theorem and removing the first row and column that includes every value that matches the center vertex, u_0, we get

$$\tau\left(SW_n^{(8)}\right)=det\begin{array}{c}
\begin{array}{cccccccccccccccccccccccc}
u_1 & u_2 & \cdots & \cdots & \cdots & u_n & v_1 & v_2 & \cdots & \cdots & \cdots & v_n & w_1 & w_2 & \cdots & \cdots & \cdots & w_n & x_1 & x_2 & \cdots & \cdots & \cdots & x_n
\end{array}\\
\begin{pmatrix}
9 & -1 & 0 & \cdots & 0 & -1 & -1 & -1 & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0\\
-1 & \ddots & \ddots & \ddots & \vdots & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots\\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0\\
0 & \ddots & \ddots & \ddots & \ddots & -1 & 0 & \ddots & \ddots & \ddots & \ddots & -1 & 0 & \ddots & \ddots & \ddots & \ddots & -1 & 0 & \ddots & \ddots & \ddots & \ddots & -1\\
-1 & 0 & \cdots & 0 & -1 & 9 & -1 & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -1\\
-1 & 0 & \cdots & \cdots & 0 & -1 & 3 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0\\
-1 & \ddots & \ddots & \ddots & \ddots & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0\\
0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 3 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0\\
-1 & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 4 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0\\
-1 & \ddots & \ddots & \ddots & \ddots & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots\\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0\\
0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots 0 & 4 & 0 & \cdots & \cdots & \cdots & 0 & -1\\
-1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 3 & 0 & \cdots & \cdots & \cdots & 0\\
-1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots\\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0\\
0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 3
\end{pmatrix}
\end{array}$$

$$= det\begin{pmatrix} A & B & C & D\\ E & F & G & H\\ P & Q & R & S\\ T & U & V & W\end{pmatrix}.$$

Using Lemma 2, we obtain

$$\tau(SW_n^{(8)}) = det\begin{pmatrix} R & S\\ V & W\end{pmatrix}\times \det\left[\begin{pmatrix} A & B\\ E & F\end{pmatrix}-\begin{pmatrix} P & Q\\ T & U\end{pmatrix}\begin{pmatrix} R & S\\ V & W\end{pmatrix}^{-1}\begin{pmatrix} C & D\\ G & H\end{pmatrix}\right]$$

$$= det\begin{pmatrix}
4 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0\\
0 & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0\\
0 & \cdots & 0 & 4 & 0 & \cdots & 0 & -1\\
-1 & 0 & \cdots & 0 & 3 & 0 & \cdots & 0\\
0 & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0\\
0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 3
\end{pmatrix}$$

$$\times det\begin{pmatrix}
\frac{81}{11} & -\frac{20}{11} & 0 & \cdots & 0 & -\frac{20}{11} & -\frac{15}{11} & -\frac{15}{11} & 0 & \cdots & \cdots & 0\\
-\frac{20}{11} & \ddots & \ddots & \ddots & \ddots & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots\\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0\\
0 & \ddots & \ddots & \ddots & \ddots & -\frac{20}{11} & 0 & \ddots & \ddots & \ddots & \ddots & -\frac{15}{11}\\
-\frac{20}{11} & 0 & \cdots & 0 & -\frac{20}{11} & \frac{81}{11} & -\frac{15}{11} & 0 & \cdots & \cdots & 0 & -\frac{15}{11}\\
-\frac{15}{11} & 0 & \cdots & \cdots & 0 & -\frac{15}{11} & \frac{30}{11} & 0 & \cdots & \cdots & \cdots & 0\\
-\frac{15}{11} & \ddots & \ddots & \ddots & \ddots & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots\\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0\\
0 & \cdots & \cdots & 0 & -\frac{15}{11} & -\frac{15}{11} & 0 & \cdots & \cdots & \cdots & 0 & \frac{30}{11}
\end{pmatrix}.$$

Applying Lemma 3 yields $\tau\left(SW_n^{(8)}\right) = 11^n \times \det\begin{pmatrix} X & Z\\ Z^t & Y\end{pmatrix}$. Applying Lemma 2 once more yields

$$\tau\left(SW_n^{(8)}\right) = 11^n \times \det Y \det(X - Z^t Y^{-1} Z)$$

$$= 11^n \times (\frac{30}{11})^n \times det \begin{pmatrix} 6 & -\frac{5}{2} & 0 & \cdots & 0 & -\frac{5}{2} \\ -\frac{5}{2} & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & -\frac{5}{2} \\ -\frac{5}{2} & 0 & \cdots & 0 & -\frac{5}{2} & 6 \end{pmatrix}$$

$$= 75^n \times det \begin{pmatrix} \frac{12}{5} & -1 & 0 & \cdots & 0 & -1 \\ -1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & -1 \\ -1 & 0 & \cdots & 0 & -1 & \frac{12}{5} \end{pmatrix}.$$

Using Lemma 1, we obtain $\tau\left(SW_n^{(8)}\right) = 75^n \times 2\left[T_n\left(\frac{6}{5}\right) - 1\right]$. The result follows from Eq (4).

Definition 9. For $n \geq 3$, the sprocket wheel graph family, $SW_n^{(9)}$, is a graph constructed from the wheel graph, W_{n+1}, with vertex set $\{u_0, u_i, 1 \leq i \leq n\}$ by adding another set of vertices $\{v_i, w_i, x_i, 1 \leq i \leq n, 1 \leq i \leq n\}$, so that the edge set of the graph family, $SW_n^{(9)}$,

$$\{u_0u_i, u_0w_i, u_0x_i, u_iv_i, w_ix_i, u_iw_i, 1 \leq i \leq n\} \cup \{u_{i+1}v_i, u_{i+1}x_i, u_iu_{i+1}, 1 \leq i \leq n-1\}$$
$$\cup \{u_1v_n,\ u_1x_n, u_1u_n\},$$

as illustrated in Figure 9. This structure gives order $4n + 1$, size $9n$, and an average degree 4.5 in the large n limit for the sprocket wheel graph family, $SW_n^{(9)}$.

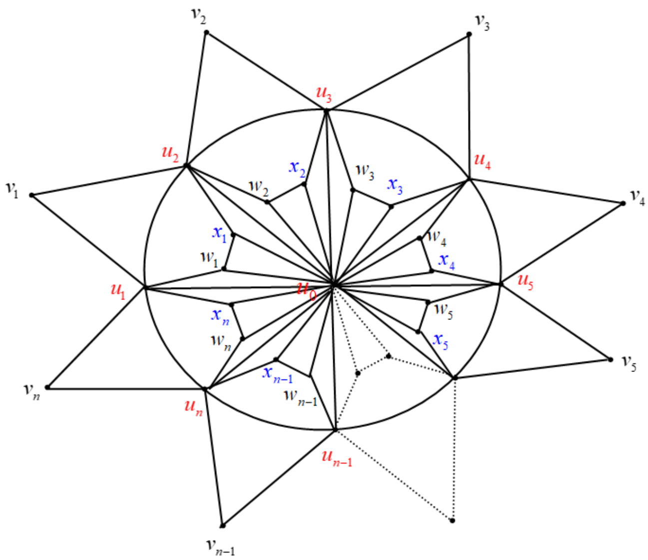

Figure 9. The sprocket wheel graph family, $SW_n^{(9)}$.

Theorem 9. For $n \geq 1$, we have $\tau(SW_n^{(9)}) = 2^n[(21 + 4\sqrt{17})^n + (21 - 4\sqrt{17})^n - 2 \times 13^n]$.

Proof. By applying the matrix tree theorem and removing the first row and column that includes every value that matches the center vertex, u_0, we get

$$\begin{array}{c}
\begin{array}{cccccccccccccccccccccccc} u_1 & u_2 & \cdots & \cdots & \cdots & u_n & v_1 & v_2 & \cdots & \cdots & \cdots & v_n & w_1 & w_2 & \cdots & \cdots & \cdots & w_n & x_1 & x_2 & \cdots & \cdots & \cdots & x_n \end{array}
\end{array}$$

$$\tau(SW_n^{(9)}) = det\begin{pmatrix}
7 & -1 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 & -1 \\
-1 & \ddots & \ddots & \ddots & \vdots & 0 & -1 & \ddots & \ddots & \ddots & \ddots & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & -1 & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
-1 & 0 & \cdots & 0 & -1 & 7 & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 \\
-1 & -1 & 0 & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
-1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
-1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 3 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots 0 & 3 & 0 & \cdots & \cdots & \cdots & 0 & -1 \\
0 & -1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 3 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
-1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 3
\end{pmatrix}$$

$$= det\begin{pmatrix} A & B & C & D \\ E & F & G & H \\ P & Q & R & S \\ T & U & V & W \end{pmatrix}.$$

Using Lemma 2, we obtain

$$\tau\left(SW_n^{(9)}\right) = det\begin{pmatrix} R & S \\ V & W \end{pmatrix} \times \det\left[\begin{pmatrix} A & B \\ E & F \end{pmatrix} - \begin{pmatrix} P & Q \\ T & U \end{pmatrix}\begin{pmatrix} R & S \\ V & W \end{pmatrix}^{-1}\begin{pmatrix} C & D \\ G & H \end{pmatrix}\right]$$

$$= det\begin{pmatrix}
3 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 3 & 0 & \cdots & 0 & -1 \\
-1 & 0 & \cdots & 0 & 3 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 3
\end{pmatrix}$$

$$\times\, det\begin{pmatrix}
\frac{25}{4} & -\frac{9}{8} & 0 & \cdots & 0 & -\frac{9}{8} & -1 & 0 & \cdots & \cdots & 0 & -1 \\
-\frac{9}{8} & \ddots & \ddots & \ddots & \ddots & 0 & -1 & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & -\frac{9}{8} & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
-\frac{9}{8} & 0 & \cdots & 0 & -\frac{9}{8} & \frac{25}{4} & 0 & \cdots & \cdots & 0 & -1 & -1 \\
-1 & -1 & 0 & \cdots & \cdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
-1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 2
\end{pmatrix}.$$

Applying Lemma 3 yields $\tau\left(SW_n^{(9)}\right) = 8^n \times \det\begin{pmatrix} X & Z \\ Z^t & Y \end{pmatrix}$. Reapplying Lemma 2, we obtain

$$\tau\left(SW_n^{(9)}\right) = 8^n \times \det\ Y \det(X - Z^t\, Y^{-1} Z)$$

$$= 8^n \times 2^n \times det \begin{pmatrix} \frac{21}{4} & -\frac{13}{8} & 0 & \cdots & 0 & -\frac{13}{8} \\ -\frac{13}{8} & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & -\frac{13}{8} \\ -\frac{13}{8} & 0 & \cdots & 0 & -\frac{13}{8} & \frac{21}{4} \end{pmatrix}$$

$$= 13^n \times det \begin{pmatrix} \frac{42}{13} & -1 & 0 & \cdots & 0 & -1 \\ -1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & -1 \\ -1 & 0 & \cdots & 0 & -1 & \frac{42}{13} \end{pmatrix}.$$

Using Lemma 1, we get $\tau\left(SW_n^{(9)}\right) = 26^n \times 2[T_n(\frac{21}{13}) - 1]$. The result follows from Eq (4).

Definition 10. For $n \geq 3$, the sprocket wheel graph family, $SW_n^{(10)}$, is a graph that is made from sprocket wheel graph family, $SW_n^{(9)}$, by adding the set of edges $\{v_i w_i, v_i x_i, 1 \leq i \leq n\}$, as illustrated in Figure 10. This structure gives order $4n + 1$, size $11n$, and an average degree 5.5 in the large n limit for the sprocket wheel graph family, $SW_n^{(10)}$.

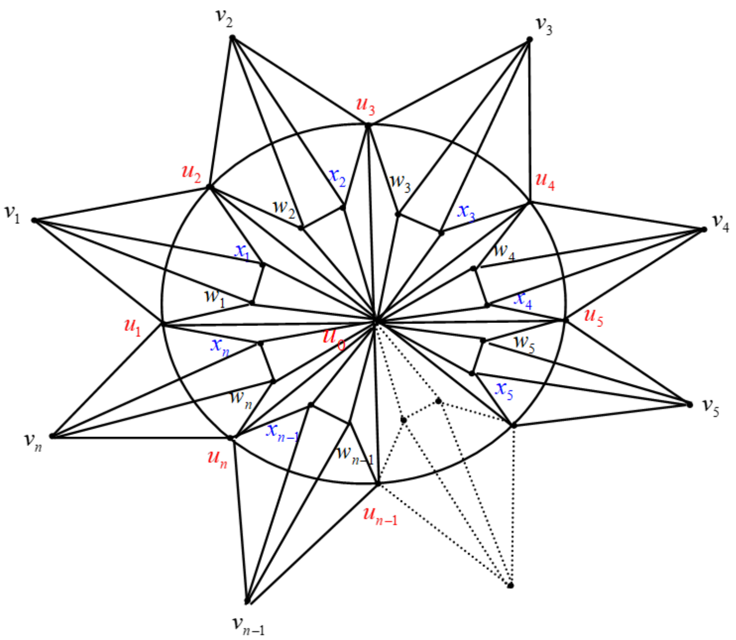

Figure 10. The sprocket wheel graph family, $SW_n^{(10)}$.

Theorem 10. For $n \geq 1$, we have $\tau(SW_n^{(10)}) = 5^n[(27+\sqrt{473})^n + (27-\sqrt{473})^n - 2^{4n+1}]$.

Proof. By applying the matrix tree theorem and removing the first row and column that includes every value that matches the center vertex, u_0, we get

$$\begin{array}{rc}
 & \begin{matrix} u_1 & u_2 & \cdots & \cdots & \cdots & u_n & v_1 & v_2 & \cdots & \cdots & \cdots & v_n & w_1 & w_2 & \cdots & \cdots & \cdots & w_n & x_1 & x_2 & \cdots & \cdots & \cdots & x_n \end{matrix} \\
\tau(SW_n^{(10)}) = det & \begin{pmatrix}
7 & -1 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 & -1 \\
-1 & \ddots & \ddots & \ddots & \vdots & 0 & -1 & \ddots & \ddots & \ddots & \ddots & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & -1 & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
-1 & 0 & \cdots & 0 & -1 & 7 & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 \\
-1 & -1 & 0 & \cdots & \cdots & 0 & 4 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
-1 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 4 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & -1 \\
-1 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 4 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots 0 & 4 & 0 & \cdots & \cdots & \cdots & 0 & -1 \\
0 & -1 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 4 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
-1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 & 4
\end{pmatrix}
\end{array}$$

$$= det\begin{pmatrix} A & B & C & D \\ E & F & G & H \\ P & Q & R & S \\ T & U & V & W \end{pmatrix}.$$

Using Lemma 2, we obtain

$$\tau(SW_n^{(10)}) = det\begin{pmatrix} R & S \\ V & W \end{pmatrix} \times \det[\begin{pmatrix} A & B \\ E & F \end{pmatrix} - \begin{pmatrix} P & Q \\ T & U \end{pmatrix}\begin{pmatrix} R & S \\ V & W \end{pmatrix}^{-1}\begin{pmatrix} C & D \\ G & H \end{pmatrix}]$$

$$= det\begin{pmatrix}
4 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 4 & 0 & \cdots & 0 & -1 \\
-1 & 0 & \cdots & 0 & 4 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & -1 & 0 & \cdots & \cdots & 4
\end{pmatrix}$$

$$\times det\begin{pmatrix}
\frac{97}{15} & -\frac{16}{15} & 0 & \cdots & 0 & -\frac{16}{15} & -\frac{4}{3} & 0 & \cdots & \cdots & 0 & -\frac{4}{3} \\
-\frac{16}{15} & \ddots & \ddots & \ddots & \ddots & 0 & -\frac{4}{3} & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & -\frac{16}{15} & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
-\frac{16}{15} & 0 & \cdots & 0 & -\frac{16}{15} & \frac{97}{15} & 0 & \cdots & \cdots & 0 & -\frac{4}{3} & -\frac{4}{3} \\
-\frac{4}{3} & -\frac{4}{3} & 0 & \cdots & \cdots & 0 & \frac{10}{3} & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & -\frac{4}{3} & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
-\frac{4}{3} & 0 & \cdots & \cdots & 0 & -\frac{4}{3} & 0 & \cdots & \cdots & \cdots & 0 & \frac{10}{3}
\end{pmatrix}.$$

Applying Lemma 3 yields

$$\tau\left(SW_n^{(10)}\right) = 15^n \times \det \begin{pmatrix} X & Z \\ Z^t & Y \end{pmatrix}.$$

Applying Lemma 2 once more yields

$$\tau\left(SW_n^{(10)}\right) = 15^n \times \det Y \det(X - Z^t Y^{-1} Z)$$

$$= 15^n \times (\frac{10}{3})^n \times det \begin{pmatrix} \frac{27}{5} & -\frac{8}{5} & 0 & \cdots & 0 & -\frac{8}{5} \\ -\frac{8}{5} & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & -\frac{8}{5} \\ -\frac{8}{5} & 0 & \cdots & 0 & -\frac{8}{5} & \frac{27}{5} \end{pmatrix}$$

$$= 80^n \times det \begin{pmatrix} \frac{27}{8} & -1 & 0 & \cdots & 0 & -1 \\ -1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & -1 \\ -1 & 0 & \cdots & 0 & -1 & \frac{27}{8} \end{pmatrix}.$$

Using Lemma 1, we get

$$\tau\left(SW_n^{(10)}\right) = 80^n \times 2[T_n(\frac{27}{16}) - 1].$$

The result follows from Eq (4).

4. Numerical results

Tables 1 and 2 display some numbers of spanning trees in the Sprocket wheel graph families $SW_n^{(1)}$, $SW_n^{(2)}$, $SW_n^{(3)}$, $SW_n^{(4)}$, $SW_n^{(5)}$, $SW_n^{(6)}$, $SW_n^{(7)}$, $SW_n^{(8)}$, $SW_n^{(9)}$, and $SW_n^{(10)}$, respectively.

Table 1. Some values of the spanning trees in the sprocket wheel graph families, $SW_n^{(1)}$, $SW_n^{(2)}$, $SW_n^{(3)}$, $SW_n^{(4)}$, and $SW_n^{(5)}$.

n	$\tau(SW_n^{(1)})$	$\tau(SW_n^{(2)})$	$\tau(SW_n^{(3)})$	$\tau(SW_n^{(4)})$	$\tau(SW_n^{(5)})$
1	56	56	105	56	105
2	5,600	6,720	19,425	5,824	17,745
3	443,576	605,696	2,858,625	473,984	2,457,945
4	34,070,400	52,039,680	408,410,625	37,273,600	333,055,905
5	2,605,458,296	4,431,773,696	58,109,690,625	2,915,956,736	45,013,187,625

Table 2. Some values of the spanning trees in the Sprocket wheel graph families, $SW_n^{(6)}$, $SW_n^{(7)}$, $SW_n^{(8)}$, $SW_n^{(9)}$, and $SW_n^{(10)}$.

n	$\tau(SW_n^{(6)})$	$\tau(SW_n^{(7)})$	$\tau(SW_n^{(8)})$	$\tau(SW_n^{(9)})$	$\tau(SW_n^{(10)})$
1	40	38	30	32	110
2	10,560	9,196	9,900	4,352	47,300
3	1,730,560	1,386,278	1,950,750	387,200	13,475,000
4	243,978,240	180,241,600	320,760,000	30,707,712	3,448,170,000
5	32,444,416,000	22,142,270,198	48,662,268,750	2,346,947,072	853,777,100,000

5. Spanning tree entropy

Sprocket wheels (or other graph families) are used to measure the structural complexity, robustness, and dependability of vast networks as they expand toward infinity by counting the asymptotic entropy of the number of spanning trees. Counting the number of spanning trees is vital for figuring out key communication links, assessing network stability, and comprehending structural constraints. How securely a network stays connected if links break is determined by the number of spanning trees.

The spanning tree entropy, a fascinating variable that describes the network topology and is defined in [25,26] as a finite number, may be calculated as:

$$\mathrm{Z(G)} = \lim_{n\to\infty} \frac{\ln \tau(\mathrm{G})}{|\mathrm{V(G)}|}. \tag{14}$$

The asymptotic entropy of sprocket wheel graph families under study is now computed as:

$$\mathrm{Z}\left(SW_n^{(1)}\right) = \frac{1}{4} ln\left[39 + 10\sqrt{14}\right] = 1.084,$$

$$\mathrm{Z}\left(SW_n^{(2)}\right) = \frac{1}{4} ln\left[4\left(11 + \sqrt{105}\right)\right] = 1.111,$$

$$\mathrm{Z}\left(SW_n^{(3)}\right) = \frac{1}{4} ln\left[\frac{5}{2}\left(29 + \sqrt{777}\right)\right] = 1.240,$$

$$\mathrm{Z}\left(SW_n^{(4)}\right) = \frac{1}{4} ln\left[4\left(10 + \sqrt{91}\right)\right] = 1.090,$$

$$\mathrm{Z}\left(SW_n^{(5)}\right) = \frac{1}{4} ln\left[\frac{1}{2}\left(137 + 13\sqrt{105}\right)\right] = 1.230,$$

$$\mathrm{Z}\left(SW_n^{(6)}\right) = \frac{1}{4} ln\left[4\left(19 + \sqrt{165}\right)\right] = 1.212,$$

$$\mathrm{Z}\left(SW_n^{(7)}\right) = \frac{1}{4} ln\left[70 + 11\sqrt{19}\right] = 1.193,$$

$$\mathrm{Z}\left(SW_n^{(8)}\right) = \frac{1}{4} ln\left[15\left(6 + \sqrt{11}\right)\right] = 1.235,$$

$$\mathrm{Z}\left(SW_n^{(9)}\right) = \frac{1}{4} ln\left[42 + 8\sqrt{17}\right] = 1.080,$$

$$\mathrm{Z}\left(SW_n^{(10)}\right) = \frac{1}{4} ln\left[5\left(27 + \sqrt{473}\right)\right] = 1.374.$$

According to our estimates, we have:

- The sprocket wheel graph family with the lowest entropy is $SW_n^{(9)}$, which has an average degree of 4.5, whereas the sprocket wheel graph family with the highest entropy is $SW_n^{(10)}$, which has an average degree of 5.5.
- The sprocket wheel graph families, $SW_n^{(1)}, SW_n^{(2)}, SW_n^{(4)}$, and $SW_n^{(9)}$, all have the same average degree of 4.5; the entropy of $SW_n^{(2)}$ is the highest among them, while that of $SW_n^{(9)}$ is the lowest.
- The sprocket wheel graph families $SW_n^{(3)}, SW_n^{(5)}, SW_n^{(6)}, SW_n^{(7)}$, and $SW_n^{(8)}$ have the same average degree of 5; the entropy of $SW_n^{(3)}$ is the highest among them, while that of $SW_n^{(7)}$ is the lowest.
- Compared to the fractal scale-free lattice [27] and the two-dimensional Sierpinski gasket [28], which have entropies of 1.049 and 1.040, respectively, with the same average degree of 4, all ten sprocket wheel graph families $SW_n^{(i)}$, $SW_n^{(\mathrm{i})}, 1 \leq \mathrm{i} \leq 10$, have a higher entropy.
- Although the sprocket wheel graph families $SW_n^{(1)}, SW_n^{(2)}, SW_n^{(4)}$, and $SW_n^{(9)}$ have an average degree of 4.5, their entropy is lower than that of the square lattice [29], which has an average degree of 4 and an entropy of 1.166. This seemingly counterintuitive result can be attributed to the structural irregularity inherent in the sprocket wheel graph families, particularly the presence of the high-degree hub vertex u_0, which is adjacent to all outer vertices u_i, $1 \leq \mathrm{i} \leq \mathrm{n}$. Such a hub vertex constrains the combinatorial diversity of spanning trees by forcing most of them to pass through u_0, thereby reducing the overall count of independent spanning trees and, consequently, lowering the asymptotic entropy. In contrast, the square lattice exhibits a perfectly regular degree distribution, where every vertex has exactly degree 4, which promotes a richer variety of spanning tree configurations and yields a higher entropy despite its lower average degree. This observation highlights the fact that asymptotic entropy is not solely determined by the average degree of a graph but is also significantly influenced by the regularity and homogeneity of its degree distribution.
- The Apollonian graph [30], which has an entropy of 1.354 and an average degree of 5, has a higher entropy than all five sprocket wheel graph families $SW_n^{(3)}, SW_n^{(5)}, SW_n^{(6)}, SW_n^{(7)}$, and $SW_n^{(8)}$, which share the same average degree.

6. Conclusions

Sprocket wheels are crucial toothed parts of mechanical systems that are necessary for chaining between shafts or converting rotary motion to linear motion. They guarantee accurate, high-torque power transfer, which is essential for industrial machines, vehicles, and conveyors. They also improve efficiency, durability, and operating speed by reducing slippage.

Through the analysis of the correlation between the Kirchhoff matrix and the established features of Chebyshev polynomials, we were able to provide precise formulas for several new graph families derived from wheel graphs (sprocket wheel graphs). Furthermore, when compared to other graphs with comparable average degrees, our findings demonstrate a relationship between entropy and these graphs' average degree.

Author contributions

Salama Daoud: Conceptualization, Methodology, Formal analysis, Writing – original draft, Writing – review & editing; Ahmad Asiri: Conceptualization, Methodology, Supervision, Validation, Writing – review& editing. All authors have read and approved the final version of the manuscript for

publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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