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Research article

The Laplace transform method for a scalar differential equation with symmetry/reflection

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Abstract: This work investigated a scalar differential equation involving proportional arguments of the form $\phi'(t) = \alpha\phi(\gamma t) + \beta\phi(-\gamma t)$. The terms $\phi(\gamma t)$ and $\phi(-\gamma t)$ represented proportional delay and reflection (time-reversal) effects, respectively, which arose in models exhibiting scaling and symmetry properties. A constructive analytical approach based on the Laplace transform was developed to derive a series representation of the solution. The method converted the original scalar equation into a recursive sequence of algebraic relations in the Laplace variable, allowing the systematic computation of successive terms. Explicit formulas for the series components were obtained, revealing a clear structure separating even and odd contributions. The convergence of the resulting series was rigorously established, showing that the solution existed globally and defined an entire function. Furthermore, the series representation was summed into a closed form expressed through hyperbolic functions, providing a compact analytical expression of the solution. Numerical illustrations confirmed the effectiveness of the derived formulation.

Keywords: scalar differential equation; Laplace transform; series; exact solution
Mathematics Subject Classification: 34K06, 34K07, 65L03

1. Introduction

Scalar differential equations (SDEs) with symmetry or reflection proportional delay constitute an important class of functional differential equations in which the derivative at a given time depends on the state of the solution evaluated at a transformed argument, typically of the form $y(\gamma t)$, $y(-\gamma t)$, or other symmetric mappings. Such equations arise naturally in applications where time-reversal effects, scaling laws, or mirror symmetries play a role, including control theory, population dynamics, viscoelasticity, and mathematical physics. The presence of proportional or reflection delays introduces nonlocal behavior that fundamentally distinguishes these models from classical ordinary differential equations (ODEs) and motivates the development of specialized analytical and numerical techniques for their investigation [1–3].

SDEs with symmetry or reflection proportional delay have gained increasing relevance in mathematical neuroscience, particularly in modeling the transmission of nerve impulses along axons. The propagation of action potentials is inherently influenced by delayed feedback mechanisms arising from ion channel kinetics, spatial signal propagation, and interactions between different segments of the nerve fiber. Proportional delays naturally capture scale-dependent signal transmission along axons, where the electrical state at a given location depends on earlier states at proportionally related spatial or temporal positions. These models describe the wave propagation, attenuation, and the stability of neural signals while preserving essential physiological features [4]. Motivated by these challenges, this study considers the equation given by

$$\phi'(t) = \alpha\phi(\gamma t) + \beta\phi(-\gamma t),\ \phi(0) = \lambda, \tag{1.1}$$

and develops a systematic procedure to obtain its solution.

An increasing attention was given for studying scientific models with proportional delays [5–7] including the homogeneous pantograph equations (HPE) [8–10] and its the stability analysis [11,12] in addition to a special case known as the Ambartsumian equation (AE) [13–15]. In this context, various numerical techniques [3, 16–18] and analytical procedures [19–21] have been proposed in addition to the Laplace transform (LT) [22]. Moreover, some variable forms of the HPE and the AE have been investigated in Refs. [23, 24]. The present approach relies on the LT, which converts the functional relation into an equation involving scaled Laplace variables. This transformation naturally leads to a recursive decomposition of the solution into a sequence of terms. By iterating the resulting recurrence, a structured series solution is derived, where parity properties (even–odd behavior) play a fundamental role. This structure allows identification of general formulas for the series terms and facilitates a rigorous convergence analysis. The obtained series is then recognized as polynomial and hyperbolic expansions, leading to closed-forms representation. The main contributions of this work include the development of a LT approach for SDEs with reflective arguments, the derivation of a recursive algebraic structure in the Laplace domain, and obtaining explicit solution in terms of a convergent series. The proposed methodology provides a theoretical insight for handling a broader class of SDEs with symmetry/reflective arguments.

2. Laplace transform approach

Applying the LT on Eq (1.1) gives

$$s\Phi(s) - \phi(0) = \frac{\alpha}{\gamma}\Phi\left(\frac{s}{\gamma}\right) - \frac{\beta}{\gamma}\Phi\left(-\frac{s}{\gamma}\right), \tag{2.1}$$

where $\mathcal{L}\{\phi(t)\} = \Phi(s)$, hence

$$\Phi(s) = \frac{\lambda}{s} + \frac{\alpha}{\gamma s}\Phi\left(\frac{s}{\gamma}\right) - \frac{\beta}{\gamma s}\Phi\left(-\frac{s}{\gamma}\right). \tag{2.2}$$

Assume Adomian's series representation

$$\Phi(s) = \sum_{n=0}^{\infty} \Phi_n(s), \tag{2.3}$$

then the following recurrence relation holds:

$$\begin{cases} \Phi_0(s) = \frac{\lambda}{s}, \\ \Phi_n(s) = \frac{\alpha}{\gamma s}\,\Phi_{n-1}\left(\frac{s}{\gamma}\right) - \frac{\beta}{\gamma s}\,\Phi_{n-1}\left(-\frac{s}{\gamma}\right),\ n \geq 1. \end{cases} \tag{2.4}$$

For $n = 1$, we have

$$\begin{aligned} \Phi_1(s) &= \frac{\alpha}{\gamma s}\Phi_0\left(\frac{s}{\gamma}\right) - \frac{\beta}{\gamma s}\Phi_0\left(-\frac{s}{\gamma}\right) = \frac{\alpha}{\gamma s}\left(\frac{\lambda\gamma}{s}\right) - \frac{\beta}{\gamma s}\left(-\frac{\lambda\gamma}{s}\right), \\ &= \frac{\alpha\lambda}{s^2} + \frac{\beta\lambda}{s^2} = \frac{\lambda(\alpha+\beta)}{s^2}. \end{aligned} \tag{2.5}$$

For $n = 2$, we obtain

$$\begin{aligned} \Phi_2(s) &= \frac{\alpha}{\gamma s}\Phi_1\left(\frac{s}{\gamma}\right) - \frac{\beta}{\gamma s}\Phi_1\left(-\frac{s}{\gamma}\right) = \frac{\alpha}{\gamma s}\left(\frac{\lambda(\alpha+\beta)\gamma^2}{s^2}\right) - \frac{\beta}{\gamma s}\left(\frac{\lambda(\alpha+\beta)\gamma^2}{s^2}\right), \\ &= \frac{\lambda(\alpha+\beta)\gamma(\alpha-\beta)}{s^3} = \frac{\lambda\gamma(\alpha^2-\beta^2)}{s^3}. \end{aligned} \tag{2.6}$$

Since Φ_2 is odd, then $\Phi_2\left(-\frac{s}{\gamma}\right) = -\Phi_2\left(\frac{s}{\gamma}\right)$. Thus, the case $n = 3$ gives

$$\begin{aligned} \Phi_3(s) &= \frac{\alpha}{\gamma s}\Phi_2\left(\frac{s}{\gamma}\right) - \frac{\beta}{\gamma s}\Phi_2\left(-\frac{s}{\gamma}\right), \\ &= \frac{\alpha+\beta}{\gamma s}\Phi_2\left(\frac{s}{\gamma}\right). \end{aligned} \tag{2.7}$$

Equation (2.6) implies

$$\Phi_2\left(\frac{s}{\gamma}\right) = \frac{\lambda\gamma(\alpha^2-\beta^2)}{(s/\gamma)^3} = \frac{\lambda\gamma^4(\alpha^2-\beta^2)}{s^3}. \tag{2.8}$$

Therefore,

$$\Phi_3(s) = \frac{\lambda\gamma^3(\alpha^2-\beta^2)(\alpha+\beta)}{s^4}. \tag{2.9}$$

Higher order terms of the series $\Phi(s)$ can be also obtained using the recurrence relation (2.4). Since $\Phi_3(s)$ is even, then

$$\Phi_4(s) = \frac{\alpha-\beta}{\gamma s}\Phi_3\left(\frac{s}{\gamma}\right). \tag{2.10}$$

Thus,

$$\Phi_4(s) = \frac{\alpha-\beta}{\gamma s}\left[\frac{\lambda\gamma^3(\alpha^2-\beta^2)(\alpha+\beta)}{(s/\gamma)^4}\right] = \frac{\lambda\gamma^6(\alpha^2-\beta^2)^2}{s^5}. \tag{2.11}$$

Similarly, we get

$$\Phi_5(s) = \frac{\alpha+\beta}{\gamma s}\Phi_4\left(\frac{s}{\gamma}\right) = \frac{\lambda\gamma^{10}(\alpha^2-\beta^2)^2(\alpha+\beta)}{s^6}, \tag{2.12}$$

and

$$\Phi_6(s) = \frac{\alpha-\beta}{\gamma s}\Phi_5\left(\frac{s}{\gamma}\right) = \frac{\lambda\gamma^{15}(\alpha^2-\beta^2)^3}{s^7}, \tag{2.13}$$

and

$$\Phi_7(s) = \frac{\alpha+\beta}{\gamma s}\Phi_6\left(\frac{s}{\gamma}\right) = \frac{\lambda\gamma^{21}(\alpha^2-\beta^2)^3(\alpha+\beta)}{s^8}. \tag{2.14}$$

3. Series representation and inverse LT

From the previous results, the LT can be written as a sum of even and odd order terms. The even-order terms are

$$\Phi_{2n}(s) = \frac{\lambda\gamma^{n(2n-1)}(\alpha^2-\beta^2)^n}{s^{2n+1}}, \quad n \geq 0, \tag{3.1}$$

while the odd-order terms are

$$\Phi_{2n+1}(s) = \frac{\lambda\gamma^{n(2n+1)}(\alpha^2-\beta^2)^n(\alpha+\beta)}{s^{2n+2}}, \quad n \geq 0. \tag{3.2}$$

Hence

$$\Phi(s) = \sum_{n=0}^{\infty}\frac{\lambda\gamma^{n(2n-1)}(\alpha^2-\beta^2)^n}{s^{2n+1}} + \sum_{n=0}^{\infty}\frac{\lambda\gamma^{n(2n+1)}(\alpha^2-\beta^2)^n(\alpha+\beta)}{s^{2n+2}}. \tag{3.3}$$

The inverse LT of Eq (3.3) reads

$$\phi(t) = \lambda\sum_{n=0}^{\infty}\gamma^{n(2n-1)}(\alpha^2-\beta^2)^n\frac{t^{2n}}{(2n)!} + \lambda\sum_{n=0}^{\infty}\gamma^{n(2n+1)}(\alpha^2-\beta^2)^n(\alpha+\beta)\frac{t^{2n+1}}{(2n+1)!}. \tag{3.4}$$

3.1. Closed-form representation

One can rewrite the obtained series (3.4) as

$$\phi(t) = \lambda\sum_{n=0}^{\infty}\gamma^{2n^2}\left(\frac{\alpha^2-\beta^2}{\gamma}\right)^n\frac{t^{2n}}{(2n)!} + \lambda\sum_{n=0}^{\infty}\gamma^{2n^2}\left(\gamma(\alpha^2-\beta^2)\right)^n(\alpha+\beta)\frac{t^{2n+1}}{(2n+1)!}. \tag{3.5}$$

Let

$$\sigma = \sqrt{\frac{\alpha^2-\beta^2}{\gamma}}, \tag{3.6}$$

then the solution (3.5) becomes

$$\phi(t) = \lambda\sum_{n=0}^{\infty}\frac{\gamma^{2n^2}(\sigma t)^{2n}}{(2n)!} + \lambda\sqrt{\frac{\gamma(\alpha+\beta)}{\alpha-\beta}}\sum_{n=0}^{\infty}\frac{\gamma^{2n(n+1)}(\sigma t)^{2n+1}}{(2n+1)!}. \tag{3.7}$$

Define

$$u(t) = \sum_{n=0}^{\infty} \frac{\gamma^{2n^2}(\sigma t)^{2n}}{(2n)!}, \qquad v(t) = \sum_{n=0}^{\infty} \frac{\gamma^{2n(n+1)}(\sigma t)^{2n+1}}{(2n+1)!}. \tag{3.8}$$

Hence,

$$\phi(t) = \lambda \left[u(t) + \sqrt{\frac{\gamma(\alpha+\beta)}{\alpha-\beta}}\, v(t) \right], \tag{3.9}$$

provided that $\gamma(\alpha^2-\beta^2) > 0$.

Remark 1. *While Eq (1.1) is indeed a first-order ODE, it has been selected deliberately as a prototype model to clearly demonstrate the essential features of the method, including the derivation of the recursive structure in the Laplace domain and the explicit construction of the solution as a convergent infinite series. It is worth noting that the nontrivial aspect of the problem lies not in the order of the equation itself, but in the presence of the functional structure (e.g., symmetry/reflective-arguments), which makes the analytical treatment more involved and provides a meaningful test-bed for the method.*

3.2. Convergence analysis

Assume that

$$u_n(t) = \frac{\gamma^{2n^2}(\sigma t)^{2n}}{(2n)!}, \qquad v_n(t) = \frac{\gamma^{2n(n+1)}(\sigma t)^{2n+1}}{(2n+1)!}. \tag{3.10}$$

Using the ratio test, then

$$\left| \frac{u_{n+1}(t)}{u_n(t)} \right| = \frac{|\gamma^{4n+2}\sigma^2 t^2|}{(2n+2)(2n+1)} = \frac{|\gamma^{4n+1}(\alpha^2-\beta^2)| t^2}{(2n+2)(2n+1)}, \tag{3.11}$$

and

$$\left| \frac{v_{n+1}(t)}{v_n(t)} \right| = \frac{|\gamma^{4n+4}\sigma^2 t^2|}{(2n+3)(2n+2)} = \frac{|\gamma^{4n+3}(\alpha^2-\beta^2)| t^2}{(2n+3)(2n+2)}. \tag{3.12}$$

For $|\gamma| \le 1$, Eqs (3.11) and (3.12) lead to

$$\lim_{n\to\infty} \left| \frac{u_{n+1}(t)}{u_n(t)} \right| = 0, \quad \lim_{n\to\infty} \left| \frac{v_{n+1}(t)}{v_n(t)} \right| = 0. \tag{3.13}$$

Therefore, by the ratio test, the series $u(t)$ and $v(t)$ converge absolutely for all $t \in \mathbb{R}$ and so does the series solution $\phi(t)$ given by Eq (3.9).

Remark 2. *In fact, the proposed approach is not inherently restricted to first-order equations. The same LT-procedure, combined with the recursive decomposition technique, can be systematically extended to linear higher-order ODEs and certain classes of partial differential equations (PDEs), provided that appropriate initial and/or boundary conditions are specified. A full treatment of higher-order ODEs or PDEs is indeed an interesting direction for future work and deserves subsequent studies.*

4. Special case and exact solution

4.1. $\gamma = 1$

This section shows that the series solution (3.9) reduces to the same exact solution in the literature at $\gamma = 1$. Inserting this value into Eq (3.6) implies $\sigma = \sqrt{\alpha^2 - \beta^2}$. Accordingly, the two series in Eq (3.8) become

$$u(t) = \sum_{n=0}^{\infty} \frac{\left(\sqrt{\alpha^2 - \beta^2} t\right)^{2n}}{(2n)!} = \cosh\left(\sqrt{\alpha^2 - \beta^2} t\right), \tag{4.1}$$

and

$$v(t) = \sum_{n=0}^{\infty} \frac{\left(\sqrt{\alpha^2 - \beta^2} t\right)^{2n+1}}{(2n+1)!} = \sinh\left(\sqrt{\alpha^2 - \beta^2} t\right). \tag{4.2}$$

Substituting Eqs (4.1) and (4.2) into Eq (3.9), then the solution reads

$$\phi(t) = \lambda\left[\cosh\left(\sqrt{\alpha^2 - \beta^2} t\right) + \sqrt{\frac{\alpha+\beta}{\alpha-\beta}} \sinh\left(\sqrt{\alpha^2 - \beta^2} t\right)\right] \quad \alpha > \beta, \tag{4.3}$$

which agrees with the obtained solution in Ref. [25] for the SDE with reflection; $\phi'(t) = \alpha\phi(t) + \beta\phi(-t)$.

4.2. $\alpha + \beta = 0$

Equation (3.6) reveals that σ vanishes under the constraint $\alpha + \beta = 0$, i.e., $\beta = -\alpha$. This means that all terms of the series $u(t)$ and $v(t)$ violate except the initial term of the series $u(t)$ (equal to λ). Consequently, $\phi(t) = \lambda$ is a constant/trivial solution for the SDE:

$$\phi'(t) = \alpha[\phi(\gamma t) - \phi(-\gamma t)], \quad \phi(0) = \lambda, \tag{4.4}$$

whatever the parameters α and γ are.

4.3. $\alpha - \beta = 0$

From Section 2, one can see that all higher-terms of the transformed components $\Phi_n(s)$ $\forall\, n \geq 2$ under the relation $\alpha - \beta = 0$, i.e, $\beta = \alpha$. Thus, $\Phi_n(s)$ consists of only the first two components $\Phi_0(s)$ and $\Phi_1(s)$ and, hence,

$$\Phi(s) = \frac{\lambda}{s} + \frac{\lambda(\alpha+\beta)}{s^2} = \frac{\lambda}{s} + \frac{\lambda(2\alpha)}{s^2}. \tag{4.5}$$

Applying the inverse LT on the above equation yields the exact solution

$$\phi(t) = \lambda(1 + 2\alpha t), \tag{4.6}$$

for the SDE:

$$\phi'(t) = \alpha[\phi(\gamma t) + \phi(-\gamma t)], \quad \phi(0) = \lambda. \tag{4.7}$$

It can be noticed that the exact solution (4.6) for the SDE (4.7) is independent of γ.

Remark 3. *While the conditions $\alpha+\beta = 0$ and $\alpha-\beta = 0$ can indeed be compactly expressed as $|\alpha| = |\beta|$, we have chosen to retain the separate cases in this section for clarity, as they lead to structurally different forms of the solution (symmetric versus anti-symmetric behavior).*

5. Symmetry/Reflection-structure

This section aims to confirm some theoretical aspects about the symmetry/reflection-structure of the SDE (1.1). This target is to be accomplished through a direct theoretical analysis based on the structure of the model itself which confirms the obtained results in Section 4. Let us begin with evaluating Eq (1.1) at $-t$, which yields

$$\phi'(-t) = \alpha\phi(-\gamma t) + \beta\phi(\gamma t). \tag{5.1}$$

Define the even and odd parts of $\phi(t)$ by

$$\phi_e(t) = \frac{\phi(t) + \phi(-t)}{2}, \qquad \phi_o(t) = \frac{\phi(t) - \phi(-t)}{2}, \tag{5.2}$$

then $\phi(t) = \phi_e(t) + \phi_o(t)$. Adding and subtracting the equations for $\phi'(t)$ and $\phi'(-t)$, we obtain

$$\begin{cases} \phi'(t) + \phi'(-t) = (\phi(t) - \phi(-t))' = (\alpha + \beta)\left[\phi(\gamma t) + \phi(-\gamma t)\right], \\ \phi'(t) - \phi'(-t) = (\phi(t) + \phi(-t))' = (\alpha - \beta)\left[\phi(\gamma t) - \phi(-\gamma t)\right]. \end{cases} \tag{5.3}$$

Dividing both sides by two, the system (38) can be rewritten as

$$\begin{cases} \left(\frac{\phi(t)-\phi(-t)}{2}\right)' = (\alpha + \beta)\left[\frac{\phi(\gamma t)+\phi(-\gamma t)}{2}\right], \\ \left(\frac{\phi(t)+\phi(-t)}{2}\right)' = (\alpha - \beta)\left[\frac{\phi(\gamma t)-\phi(-\gamma t)}{2}\right]. \end{cases} \tag{5.4}$$

In view of Eqs (5.2) and (5.4), we obtain the coupled system

$$\begin{cases} \phi_o'(t) = (\alpha + \beta)\,\phi_e(\gamma t), \quad \phi_o(0) = 0, \\ \phi_e'(t) = (\alpha - \beta)\,\phi_o(\gamma t), \quad \phi_e(0) = \lambda. \end{cases} \tag{5.5}$$

If $\beta = -\alpha$, the system (5.5) reduces to

$$\begin{cases} \phi_o'(t) = 0, & \phi_o(0) = 0, \\ \phi_e'(t) = 2\alpha\,\phi_o(\gamma t), & \phi_e(0) = \lambda. \end{cases} \tag{5.6}$$

Solving this system yields $\phi_o(t) = 0$ and $\phi_e(t) = \lambda$, thus $\phi(t) = \lambda$ which agrees with the obtained result in Section 4.2.

Moreover, if $\beta = \alpha$, the system (5.5) reduces to

$$\begin{cases} \phi_o'(t) = 2\alpha\,\phi_e(\gamma t), & \phi_o(0) = 0, \\ \phi_e'(t) = 0, & \phi_e(0) = \lambda. \end{cases} \tag{5.7}$$

The solution of this system is $\phi_e(t) = \lambda$ and $\phi_o(t) = 2\alpha\lambda t$. This gives $\phi(t) = \lambda(1 + 2\alpha t)$ which is the same exact solution given in Eq (4.6). Additionally, the choice $\gamma = 1$ transforms the system (5.5) to

$$\begin{cases} \phi_o'(t) = (\alpha + \beta)\,\phi_e(t), \quad \phi_o(0) = 0, \\ \phi_e'(t) = (\alpha - \beta)\,\phi_o(t), \quad \phi_e(0) = \lambda. \end{cases} \tag{5.8}$$

This system can be converted to the following equivalent system of second-order initial value problems (IVPs)

$$\begin{cases}\phi_o''(t) - (\alpha^2 - \beta^2)\,\phi_o(t) = 0, & \phi_o(0) = 0,\ \phi_o'(0) = \lambda(\alpha + \beta),\\ \phi_e''(t) - (\alpha^2 - \beta^2)\,\phi_e(t) = 0, & \phi_e(0) = \lambda,\ \phi_e'(0) = 0.\end{cases} \tag{5.9}$$

The general solutions for $\phi_o(t)$ and $\phi_e(t)$ in Eq (5.9) are

$$\begin{cases}\phi_o(t) = C_1 \sinh\left[\sqrt{\alpha^2 - \beta^2}\, t\right] + C_2 \cosh\left[\sqrt{\alpha^2 - \beta^2}\, t\right],\\ \phi_e(t) = C_3 \sinh\left[\sqrt{\alpha^2 - \beta^2}\, t\right] + C_4 \cosh\left[\sqrt{\alpha^2 - \beta^2}\, t\right], \quad \alpha > \beta,\end{cases}$$

where C_i $(i = 1, 2, 3, 4)$ are unknown constants. Applying the initial conditions given in Eq (5.9), we obtain

$$\begin{cases}C_1 = \lambda\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}, & C_2 = 0,\\ C_3 = 0, & C_4 = \lambda.\end{cases}$$

Therefore, the final solutions of the two IVPs (5.9) are given as

$$\begin{cases}\phi_o(t) = \lambda\sqrt{\frac{\alpha+\beta}{\alpha-\beta}} \sinh\left[\sqrt{\alpha^2 - \beta^2}\, t\right],\\ \phi_e(t) = \lambda \cosh\left[\sqrt{\alpha^2 - \beta^2}\, t\right], \quad \alpha > \beta.\end{cases} \tag{5.10}$$

Adding $\phi_o(t)$ to $\phi_e(t)$ in Eq (5.10) leads to the same solution $\phi(t)$ given by Eq (4.3) at the special case $\gamma = 1$.

6. Exploration via power series method (PSM)

Due to the constant coefficients of equation, the solution admits a power series expansion in the form

$$\phi(t) = \sum_{n=0}^{\infty} a_n t^n. \tag{6.1}$$

Substituting Eq (6.1) into Eq (1.1) and equating coefficients gives

$$(n + 1)a_{n+1} = \alpha a_n \gamma^n + \beta a_n (-\gamma)^n, \qquad n \geq 0, \tag{6.2}$$

with

$$a_{n+1} = \frac{\alpha\gamma^n + \beta(-\gamma)^n}{n + 1} a_n, \quad a_0 = \lambda. \tag{6.3}$$

If $\alpha = \beta$, then

$$a_{n+1} = \frac{\alpha\gamma^n(1 + (-1)^n)}{n + 1} a_n. \tag{6.4}$$

Thus, $a_1 = 2\alpha\lambda$ and $a_n = 0\ \forall\, n \geq 2$ leading to the solution $\phi(t) = a_0 + a_1 t = \lambda(1 + 2\alpha t)$ which agrees with the result in Eq (4.6). Similarly, if $\alpha = -\beta$, then

$$a_{n+1} = \frac{\alpha\gamma^n(1 - (-1)^n)}{n + 1} a_n. \tag{6.5}$$

So $a_n = 0\ \forall\, n \geq 1$ which gives the trivial/constant solution $\phi(t) = \lambda$. For arbitrary α, β and γ, the recurrence scheme (6.3) can be implemented to derive the closed-series solution given by Eq (3.4). Furthermore, a growth estimate for the coefficient a_n can be established as follows. From (6.3), we have

$$|a_{n+1}| \leq \frac{(|\alpha| + |\beta|)|\gamma|^n}{n+1}|a_n|, \tag{6.6}$$

which gives by induction that

$$|a_n| \leq |\lambda|\frac{(|\alpha| + |\beta|)^n|\gamma|^{\frac{n(n-1)}{2}}}{n!}. \tag{6.7}$$

This confirms the absolute convergence of the series (6.1) for all $t \in \mathbb{R}$.

7. Numerical results

The objective to derive a series solution for the SDE (1.1) was accomplished in Section 3. The series solution was given in Eq (3.9) under the condition that $\gamma(\alpha^2 - \beta^2) > 0$. It was also shown that $|\gamma| < 1$ is a necessary condition for the series to converge. Combining the above two conditions implies that $|\beta/\alpha| < 1$ if $0 < \gamma < 1$ and $|\alpha/\beta| < 1$ if $-1 < \gamma < 0$. Figures 1 and 2 display samples of valid values for α and β at $\gamma = 1/2$ (Figure 1) and at $\gamma = -1/2$ (Figure 2). It was shown in Section 4 that the exact solution of the present model is available at some special cases of the involved parameters. In the absence of such cases, the approximate solution denoted by $\Theta_m(t)$ is only available as

$$\Theta_m(t) = \lambda\left[u_m(t) + \sqrt{\frac{\gamma(\alpha+\beta)}{\alpha-\beta}}\, v_m(t)\right], \tag{7.1}$$

where

$$u_m(t) = \sum_{n=0}^{m}\frac{\gamma^{2n^2}(\sigma t)^{2n}}{(2n)!}, \qquad v_m(t) = \sum_{n=0}^{m}\frac{\gamma^{2n(n+1)}(\sigma t)^{2n+1}}{(2n+1)!}. \tag{7.2}$$

It may be important to clarify that the method remains valid in the limiting cases $\alpha = 0$ and $\beta = 0$ in which the series solutions could have simplified forms. The residuals accompanied with of the approximations $\Theta_m(t)$ are denoted by $RE_m(t)$ and defined as

$$RE_m(t) = \left|\Theta'_m(t) - \alpha\Theta_m(\gamma t) - \beta\Theta_m(-\gamma t)\right|, \quad m \geq 1. \tag{7.3}$$

The approximations $\Theta_m(t)$ and the corresponding residuals $RE_m(t)$ are depicted in Figures 3 and 4 at $\alpha = \sqrt{2}$, $\beta = 1$, and $\gamma = \frac{1}{2}$ for different $m = 4, 5, 6, 7$ while $\lambda = 1$ is fixed. Figure 3 indicates the convergence of the approximations $\Theta_m(t)$ using a few number of terms. Moreover, the approximations grow exponentially, approaching a certain curve in a wide range of the problem's domain as the number of terms increases. Figure 4 displays the residuals $RE_m(t)$ at the same set of the parameters values which shows acceptable results. To confirm this point, additional plots are depicted in Figures 5 and 6 for the approximations $\Theta_m(t)$ and the corresponding residuals $RE_m(t)$, respectively, at $\alpha = 2$, $\beta = \sqrt{5}$, and $\gamma = -\frac{1}{2}$. These figures confirm the above concept regarding the rapid convergence of the obtained approximations in addition to the acceptable accuracy. The rest of Figures 7 and 8 also confirm the above conclusion. Finally, although the present approach was analytic in nature, there are other effective numerical methods [26, 27] that can be further applied to deal with the present model or higher-order equations of the same type involving reflective/symmetry arguments.

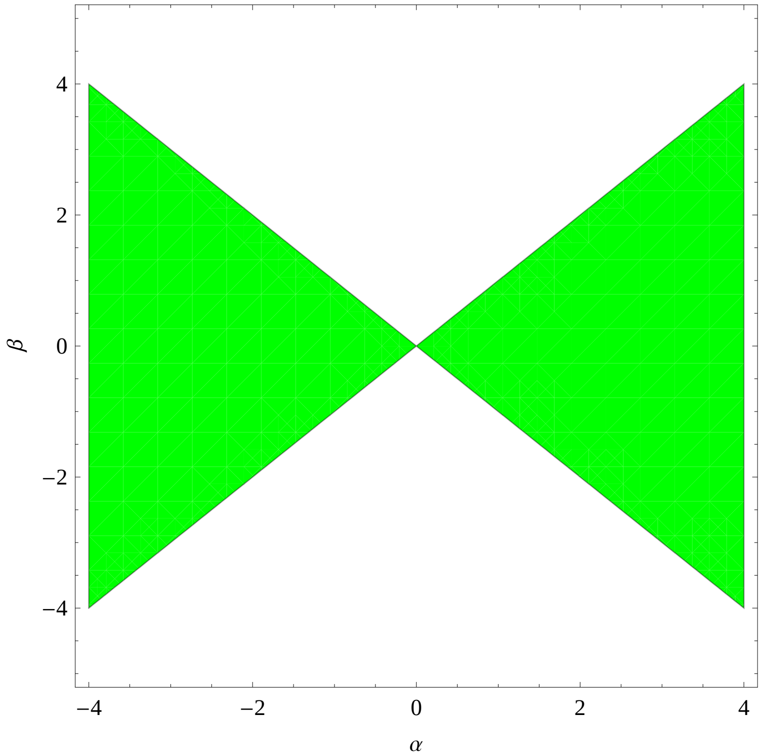

Figure 1. Samples of valid values for α and β at $\gamma = 1/2$.

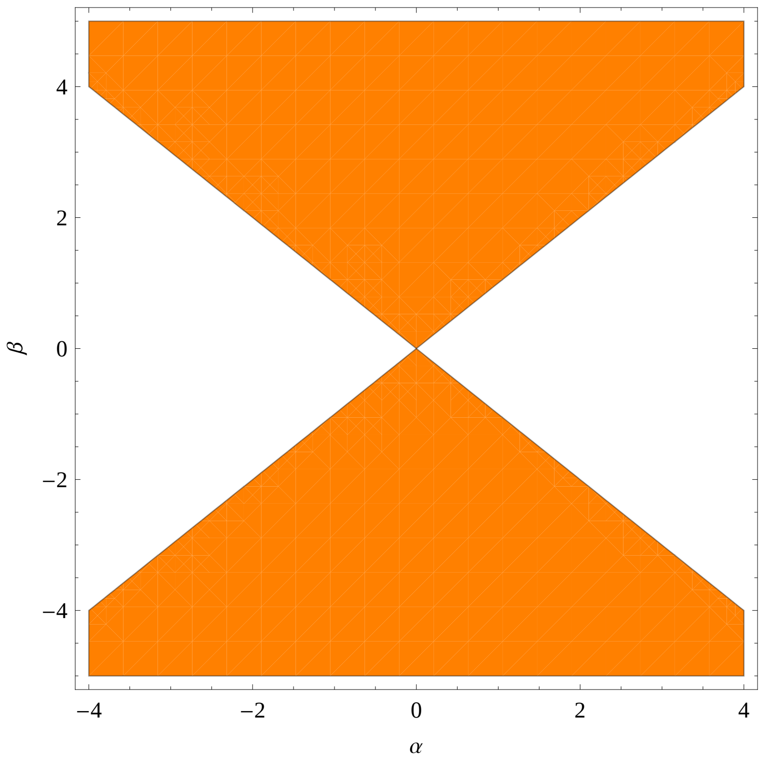

Figure 2. Samples of valid values for α and β at $\gamma = -1/2$.

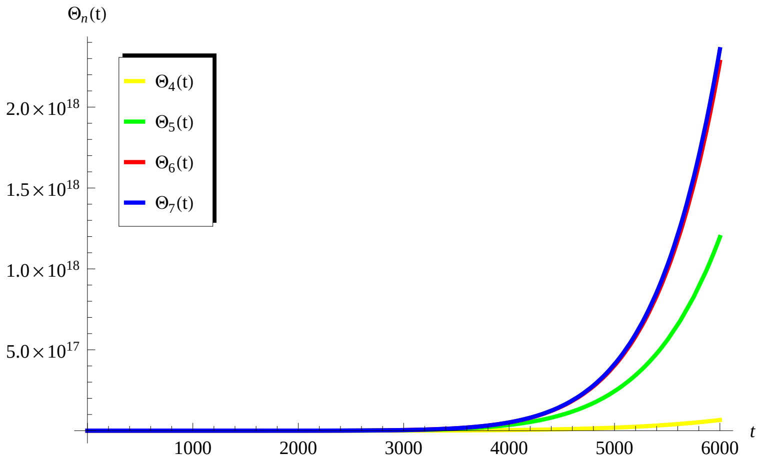

Figure 3. Plots of the approximations $\Theta_m(t)$ $(m = 4, 5, 6, 7)$ at $\alpha = \sqrt{2}, \beta = 1$, and $\gamma = \frac{1}{2}$.

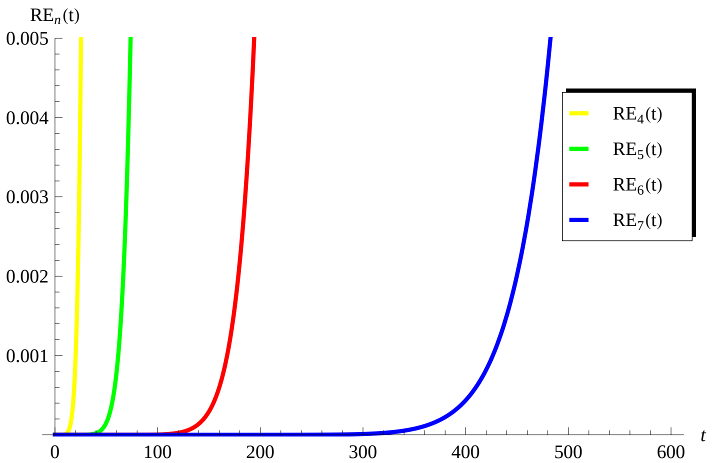

Figure 4. Plots of the residuals $RE_m(t)$ $(m = 4, 5, 6, 7)$ at $\alpha = \sqrt{2}, \beta = 1$, and $\gamma = \frac{1}{2}$.

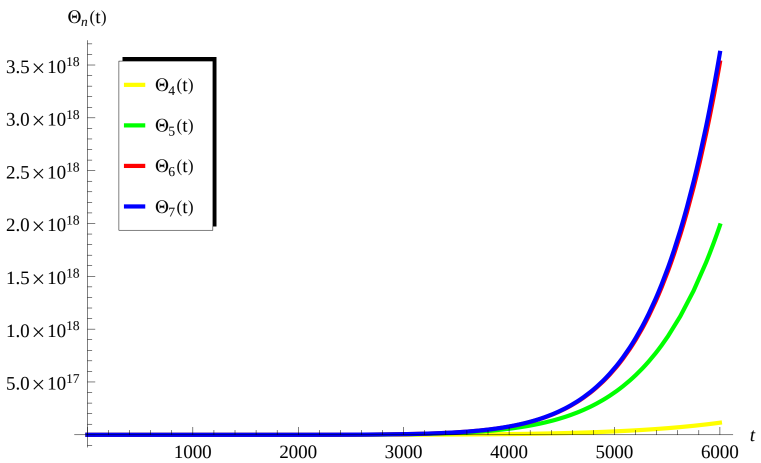

Figure 5. Plots of the approximations $\Theta_m(t)$ $(m = 4, 5, 6, 7)$ at $\alpha = 2, \beta = \sqrt{5}$, and $\gamma = -\frac{1}{2}$.

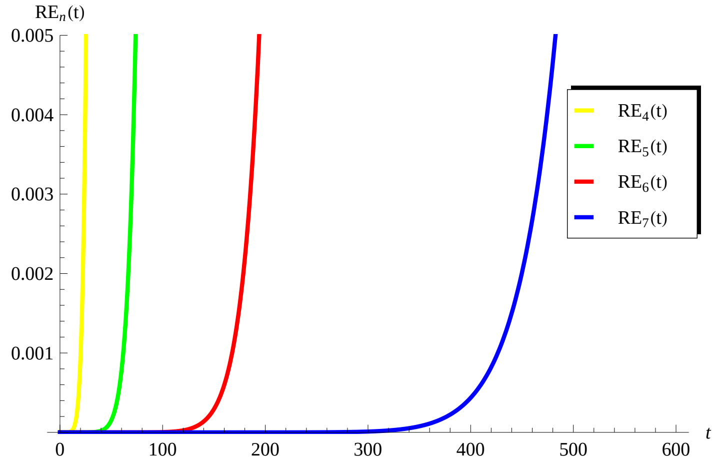

Figure 6. Plots of the residuals $RE_m(t)$ $(m = 4, 5, 6, 7)$ at $\alpha = 2, \beta = \sqrt{5}$, and $\gamma = -\frac{1}{2}$.

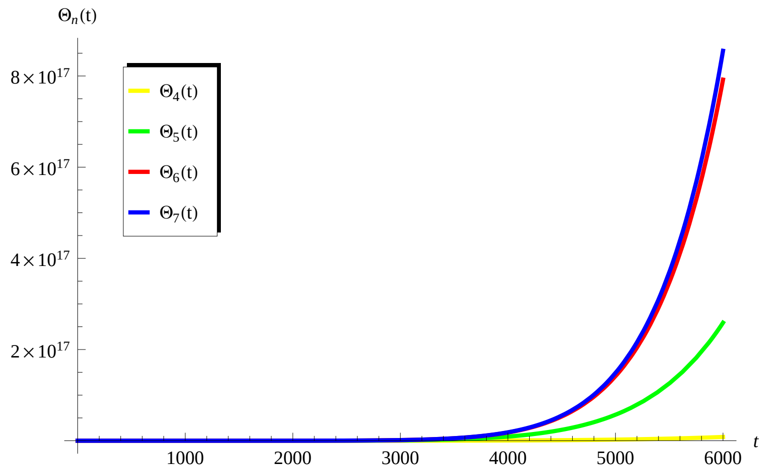

Figure 7. Plots of the approximations $\Theta_m(t)$ $(m = 4, 5, 6, 7)$ at $\alpha = 2, \beta = -\sqrt{5}$, and $\gamma = -\frac{1}{2}$.

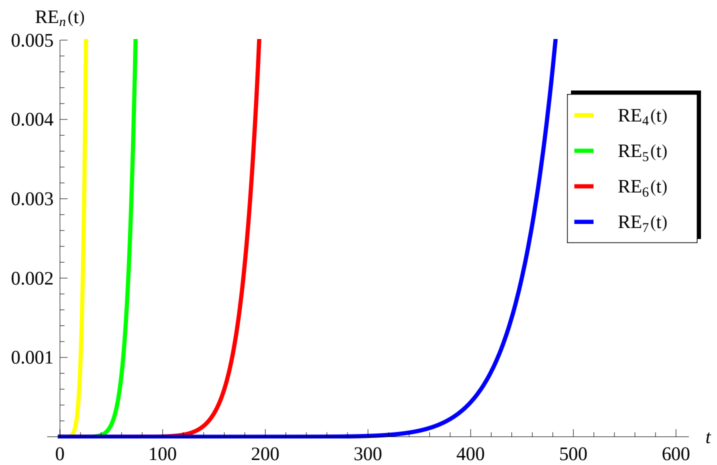

Figure 8. Plots of the residuals $RE_m(t)$ $(m = 4, 5, 6, 7)$ at $\alpha = 2, \beta = -\sqrt{5}$, and $\gamma = -\frac{1}{2}$.

8. Conclusions

A LT-based approach was presented for solving a scalar differential equation with reflective proportional delays. The method yields a recursive representation in the Laplace domain that enables the explicit construction of the solution as an infinite series. The analysis reveals a clear pattern separating even and odd terms, which makes it possible to derive general expressions for the series coefficients. A rigorous convergence proof shows that the solution is globally well defined and analytic in the whole domain. Moreover, the series representation can be summed into a compact closed form involving hyperbolic and polynomial functions under specific assumptions of the model's parameters providing additional analytical insight. The results demonstrate that the LT offers an effective tool for treating scalar equations with symmetry/reflection, producing explicit solutions and guaranteeing convergence. The approach can be extended to solve more complex systems including higher-order equations, which suggests several directions for future research involving PDEs in classical and fractional forms [28, 29].

Author contributions

Laila F. Seddek: Methodology, validation, formal analysis, funding acquisition, investigation, writing-original draft preparation, writing-review and editing; Essam R. El-Zahar: Conceptualization, methodology, validation, formal analysis, investigation, writing-review and editing, visualization; Abdelhalim Ebaid: Conceptualization, methodology, validation, formal analysis, investigation, writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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