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Research article

Construction of hierarchical fuzzy relational models from input-output data

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Abstract: As an important branch of fuzzy systems, hierarchical fuzzy systems (HFSs) are widely used in information science and engineering. Based on the semi-tensor product method, this paper investigates the hierarchical fuzzy relational models (HFRMs) of HFSs, and develops the corresponding construction algorithms. First, the construction methods of fuzzy relation matrices of fuzzy logic units (FLUs) are proposed by means of input-output data, based on which, the algebraic formulation is presented. After that, HFRMs of HFSs are explored via the fuzzy relation matrices based on the input-output data, and some effective algorithms are proposed to construct HFRMs with respect to the different hierarchical structures. Finally, the effectiveness of obtained results is verified by the on-ramp metering of freeway.

Keywords: hierarchical fuzzy systems; hierarchical fuzzy relational models; fuzzy relation matrices; algebraic formulation; semi-tensor product of matrices
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1. Introduction

Fuzzy sets and fuzzy inference systems, introduced by [1], have gone through substantial developments in the past few decades, which have a wide range of applications in control engineering [2], computer science [3], and artificial intelligence [4]. Particularly, due to the ability to deal with uncertainty and noises, fuzzy logic control has become one of the most successful applications [5]. Although fuzzy logic control was originally introduced as a kind of model-free control based on expert experience and knowledge, model-based fuzzy logic control has attracted widespread attention from various research fields because of its stability and robustness [6]. The main challenge of utilizing fuzzy modeling is the "curse of dimensionality", that is, the number of fuzzy rules exponentially increases with the input linguistic variables [7]. As a result, fuzzy modeling is generally difficult to be used in complex and large-scale plants.

To deal with the curse of dimensionality, hierarchical structures were introduced into fuzzy systems by [8], and then hierarchical fuzzy systems (HFSs) were proposed. HFSs are composed of several low-dimensional fuzzy systems, in which the low-dimensional fuzzy systems are called fuzzy logic units (FLUs) [9]. Compared with traditional fuzzy systems, the number of fuzzy rules in HFSs linearly increases with the input variables [10]. By means of this key property, many meaningful works have been performed for HFSs, such as construction [11], universal approximation [12], interpretability [13], and topology structure optimization [14]. Several results were established for hierarchical fuzzy modeling. Especially, hierarchical fuzzy relational models (HFRMs) based on fuzzy relational equations were proposed in [15], based on which, the linguistic interpretation and universal approximation were explored. Hierarchical fuzzy models based on Mamdani-type fuzzy rules have not been well discussed due to the deficiency of mathematical tools.

Recently, the semi-tensor product of matrices has been successfully applied in many research fields [16], including gene regulation [17,18], finite automata [19], information security [20,21], and so on [22,23]. Using the semi-tensor product method, fuzzy relation matrices with multiple input variables were initially proposed in [24], based on which, the algebraic formulation of fuzzy systems was given. By means of the aforementioned fuzzy relation matrices, some significant results were achieved, such as the fuzzy relational equation [25, 26], system identification [27, 28], approximation [29], and fuzzy adaptive control [30]. In addition, basic problems of HFSs were explored, such as universal approximation [31], structure optimization [32, 33], and construction [34, 35]. Most existing HFRMs are constructed based on fuzzy rules, which inherently restricts their direct applicability to situations where such rules are unavailable. However, for many complex practical systems, complete knowledge of the fuzzy rules is often inaccessible, whereas input–output data are typically obtainable [36]. This motivates the investigation of HFRMs that are directly derived from input–output data, which is of both theoretical and practical significance.

By means of the semi-tensor product method, this paper explores HFRMs of HFSs based on input-output data. The main contributions of this paper are summarized below:

(i) We present a kind of HFRM and develop the construction methods for the different hierarchical topological structures. Compared with [32, 34], the proposed HFRM does not rely on fuzzy rules and expert knowledge, and can be obtained from input-output data, which has a wider range of applications.
(ii) The proposed HFRM avoids the processes of defuzzification and fuzzification of FLUs in the middle layer, and thus can reduce the computational complexity and enhance the interpretability [37].
(iii) As an application, we apply the HFRM to deal with traffic congestion of freeways, which not only greatly reduces the computational complexity but also maintains the original performance compared with [33] (see Table 1 and Remark 2).

The rest of this paper is organized as follows: Section 2 recalls the basic preliminaries, and Section 3 formulates the considered problem; and Section 4 explores the construction of HFRMs, which is followed by the applications of freeway ramp metering and a brief conclusion in Sections 5 and 6, respectively.

Notations: δ_n^i is the i-th column of the identity matrix I_n; $Col(M)$ is the set of columns of matrix M; $L \in \mathbb{R}^{m\times n}$ is a logical matrix if $Col(L) \subseteq Col(I_m)$; $\mathcal{L}_{m\times n}$ represents the set of $m \times n$ logical matrices; $\mathcal{F}(U)$ represents the fuzzy set of set U; and for any two scalars a and b, it holds that $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

2. Preliminaries

2.1. Max-min semi-tensor product of matrices

In this subsection, based on the max and min operators, we recall the max-min semi-tensor product which consists of a max-min product and a max-min Kronecker product.

Let $\Psi = (\psi_{i,j}) \in \mathbb{R}^{m\times n}$ and $\Theta = (\theta_{j,k}) \in \mathbb{R}^{n\times p}$ represent two fuzzy relation matrices. The max-min product of Ψ and Θ is denoted by the following:

$$\Psi \circ \Theta := C = (c_{i,k}) \in \mathbb{R}^{m\times p}, \tag{2.1}$$

where $c_{i,k} = \vee_{j=1}^{n} \left(\psi_{i,j} \wedge \theta_{j,k}\right)$, $i = 1, \cdots, m$, $k = 1, \cdots, p$.

For any two fuzzy relation matrices $\Psi = (\psi_{i,j}) \in \mathbb{R}^{m\times n}$ and $\Phi = (\phi_{k,l}) \in \mathbb{R}^{p\times q}$, the max-min Kronecker product of Ψ and Φ is expressed as follows:

$$\Psi \boxtimes \Phi := \begin{bmatrix} \psi_{1,1} \circ \Phi & \cdots & \psi_{1,n} \circ \Phi \\ \vdots & \vdots & \vdots \\ \psi_{m,1} \circ \Phi & \cdots & \psi_{m,n} \circ \Phi \end{bmatrix} \in \mathbb{R}^{mp\times nq}, \tag{2.2}$$

where

$$\psi_{i,j} \circ \Phi = \begin{bmatrix} \psi_{i,j} \wedge \phi_{1,1} & \cdots & \psi_{i,j} \wedge \phi_{1,q} \\ \vdots & \vdots & \vdots \\ \psi_{i,j} \wedge \phi_{p,1} & \cdots & \psi_{i,j} \wedge \phi_{p,q} \end{bmatrix}.$$

Based on (2.1) and (2.2), the max-min semi-tensor product of Ψ and Φ is represented as follows:

$$\Psi \propto \Phi := (\Psi \boxtimes I_{\frac{a}{n}}) \circ (\Phi \boxtimes I_{\frac{a}{p}}) \in \mathbb{R}^{\frac{ma}{n}\times\frac{qa}{p}}, \tag{2.3}$$

where a is the least common multiple of n and p.

There are two main properties of max-min semi-tensor product [34], which will be used in the context.

Lemma 1. *Let the fuzzy vectors $\alpha \in \mathbb{R}^m$, $\beta \in \mathbb{R}^n$ and matrix $W_{[n,m]} = [I_m \boxtimes \delta_n^1 \ \cdots \ I_m \boxtimes \delta_n^n]$ be given. Then, $\alpha \propto \beta = W_{[n,m]} \propto \beta \propto \alpha$.*

Lemma 2. *Let the fuzzy vector $\lambda \in \mathbb{R}^p$ and fuzzy relation matrix $\Psi \in \mathbb{R}^{m\times n}$ be given. Then, $\lambda \propto \Psi = (I_p \boxtimes \Psi) \propto \lambda$.*

2.2. Standard fuzzy systems

Consider a multi-input single-output standard fuzzy system. Let x_i represent the input variable and y denote the output variable, where $i = 1, \cdots, m$. The term sets of x_i and y are denoted by $T_{x_i} = \{A_{i,1}, \cdots, A_{i,k_i}\}$ and $T_y = \{B_1, \cdots, B_\tau\}$, respectively, where $k_i, \tau \in \mathbb{Z}_+$. By means of these term sets, the corresponding fuzzy rule of the considered system can be described as follows:

$$\text{If } x_1 \text{ is } A_{1,\sigma_1} \text{ and } \cdots \text{ and } x_m \text{ is } A_{m,\sigma_m}, \text{then } y \text{ is } B_k, \tag{2.4}$$

where $A_{i,\sigma_i} \in T_{x_i}$, $B_k \in T_y$, $k = 1, \cdots, \tau$, and $\sigma_i = 1, \cdots, k_i$.

Let U_i and V denote the universe of discourse of x_i and y, respectively. The fuzzy rule (2.4) represents a fuzzy relation $\mathcal{R}$ in $U_1 \times \cdots \times U_m \times V$, which is essentially a fuzzy set denoted by $\mathcal{R} \in \mathcal{F}(U_1 \times \cdots \times U_m \times V)$. Here, we use the triangular membership function to describe the fuzzy relation $\mathcal{R}$, whose membership value is expressed as $\mu_{\mathcal{R}}(x_1, \cdots, x_m, y)$, where $x_i \in U_i$ and $y \in V$. Based on the dual fuzzy structure [24], the membership value $\mu_{\mathcal{R}}(x_1, \cdots, x_m, y)$ can be replaced by $\mu_{\mathcal{R}}(A_{1,\sigma_1}, \cdots, A_{m,\sigma_m}, B_k)$, denoted by $\mu_{\sigma_1,\cdots,\sigma_m,k}$. From the above operations, the membership values can be arranged as the following fuzzy relation matrix:

$$M_{\mathcal{R}} = \begin{bmatrix} \mu_{1,1,\cdots,1,1} & \cdots & \mu_{1,1,\cdots,k_m,1} & \cdots & \mu_{k_1,k_2,\cdots,1,1} & \cdots & \mu_{k_1,k_2,\cdots,k_m,1} \\ \mu_{1,1,\cdots,1,2} & \cdots & \mu_{1,1,\cdots,k_m,2} & \cdots & \mu_{k_1,k_2,\cdots,1,2} & \cdots & \mu_{k_1,k_2,\cdots,k_m,2} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ \mu_{1,1,\cdots,1,\tau} & \cdots & \mu_{1,1,\cdots,k_m,\tau} & \cdots & \mu_{k_1,k_2,\cdots,1,\tau} & \cdots & \mu_{k_1,k_2,\cdots,k_m,\tau} \end{bmatrix} \in \mathbb{R}^{\tau\times(k_1\cdots k_m)}. \tag{2.5}$$

According to [24], the vector form of x_i is expressed as follows:

$$X_i = \left[\mu_{A_{i,1}}(x_i) \ \ \mu_{A_{i,2}}(x_i) \ \ \cdots \ \ \mu_{A_{i,k_i}}(x_i)\right]^\top. \tag{2.6}$$

Let Y denote the vector form of variable y. Thus, the algebraic formulation of the considered system is as follows:

$$Y = M_{\mathcal{R}} \propto X_1 \propto \cdots \propto X_m. \tag{2.7}$$

3. Problem formulation

According to the different topological structures, HFSs are typically divided into serial, parallel, and hybrid types. Consider a serial HFS with input variables $x_1, \cdots, x_m$ and output variable y, whose structure diagram is shown in Figure 1. Here, FLU_k represents the k-th layer FLU, p_k is the output variable of FLU_k, where $k = 1, \cdots, m-1$, and $p_{m-1} = y$.

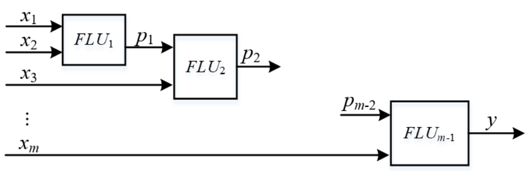

Figure 1. The structure diagram of serial HFSs.

As shown in Figure 1, x_1, x_2, and p_1 denote the input and output variables of FLU_1, respectively. The universes of discourse of x_1, x_2, and p_1 are denoted by $U_1 = [\alpha_1,\beta_1]$, $U_2 = [\alpha_2,\beta_2]$, and $V = [\iota_1, \kappa_1]$, in which the corresponding term sets are expressed as $T_{x_1} = \{A_{1,1}, \cdots, A_{1,\tau_1}\}$, $T_{x_2} = \{A_{2,1}, \cdots, A_{2,\tau_2}\}$, and $T_{p_1} = \{C_{1,1}, \cdots, C_{1,\omega_1}\}$, respectively. Here, the triangular membership functions are used to describe these fuzzy sets [38]. For example, let $\alpha_1 = 0$, $\beta_1 = 1$, and $\tau_1 = 6$; then, the term set T_{x_1} is shown in Figure 2. Notice that b_i is called the center of fuzzy set $A_{1,i}$, where $i = 1, \cdots, 6$.

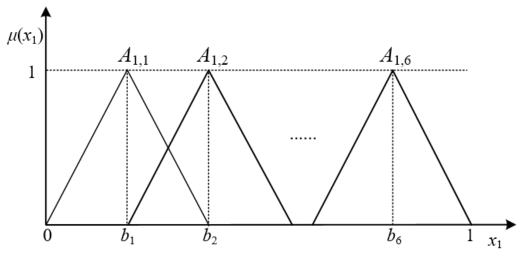

Figure 2. The term set of input variable x_1.

Based on T_{x_1}, T_{x_2}, and T_{p_1}, the fuzzy rule base of FLU_1 is comprised of the following fuzzy rules:

$$\begin{aligned}
&\text{If } x_1 \text{ is } A_{1,1} \text{ and } x_2 \text{ is } A_{2,1}, \text{then } p_1 \text{ is } C_{1,l_1},\\
&\text{If } x_1 \text{ is } A_{1,1} \text{ and } x_2 \text{ is } A_{2,2}, \text{then } p_1 \text{ is } C_{1,l_2},\\
&\qquad\qquad\vdots\\
&\text{If } x_1 \text{ is } A_{1,\tau_1} \text{ and } x_2 \text{ is } A_{2,\tau_2}, \text{then } p_1 \text{ is } C_{1,l_{\tau_1\tau_2}},
\end{aligned} \tag{3.1}$$

where $l_k = 1, \cdots, \omega_1$, and $k = 1, \cdots, \tau_1\tau_2$.

From fuzzy rule base (3.1), one can obtain the fuzzy relation matrix $M_{\mathcal{R}_1} \in \mathcal{L}_{\omega_1 \times \tau_1\tau_2}$. Then, the algebraic formulation of FLU_1 is expressed as follows:

$$P_1 = M_{\mathcal{R}_1} \propto X_1 \propto X_2, \tag{3.2}$$

where X_1, X_2, and P_1 are the vector expressions of x_1, x_2, and p_1, respectively.

The output variable of FLU_1 and variable x_3 are used as the input variables for FLU_2. Based on (3.2), FLU_2 is described as follows:

$$P_2 = M_{\mathcal{R}_2} \propto P_1 \propto X_3. \tag{3.3}$$

Similar to (3.2) and (3.3), by iterating to the final layer FLU, the algebraic formulation of the considered HFS can be represented as follows:

$$\begin{aligned}
Y &= M_{\mathcal{R}_{m-1}} \propto P_{m-2} \propto X_m\\
&= M_{\mathcal{R}_{m-1}} \propto M_{\mathcal{R}_{m-2}} \propto P_{m-3} \propto X_{m-1} \propto X_m\\
&= M_{\mathcal{R}_{m-1}} \propto M_{\mathcal{R}_{m-2}} \propto \cdots \propto M_{\mathcal{R}_1} \propto X_1 \propto \cdots \propto X_{m-1} \propto X_m\\
&= M_{\mathcal{R}} \propto \mathcal{X},
\end{aligned} \tag{3.4}$$

where $M_{\mathcal{R}} = M_{\mathcal{R}_{m-1}} \propto \cdots \propto M_{\mathcal{R}_1}$, and $\mathcal{X} = X_1 \propto \cdots \propto X_m$.

From the algebraic formulation (3.2)–(3.4), it can be observed that the critical procedure in constructing a HFRM is the calculation of fuzzy relation matrices. The fuzzy relation matrices can be directly derived from the fuzzy rules. However, the fuzzy rules are generally unknown for the complex practical systems, but the input-output data are feasible. Therefore, we need consider the following problem: how to construct HFRMs of HFSs based on a given series of n input-output data pairs $(\hat{x}_1^\iota, \cdots, \hat{x}_m^\iota, \hat{p}_1^\iota, \cdots, \hat{p}_{m-2}^\iota, \hat{y}^\iota)$, $\iota = 1, \cdots, n$. In order to deal with this problem, we aim to develop a new theoretical framework to construct HFRMs via the input-output data.

4. Construction of HFRMs

In this section, we explore the different construction methods of HFRMs applicable to serial, parallel, and hybrid hierarchical topological structures, thereby utilizing the input-output data.

First, we recall the derivations of fuzzy relation matrices via the fuzzy rules. Consider the fuzzy rule base (3.1). For the first fuzzy rule, the membership value of premise is calculated as follows:

$$\mu_{A_{1,1}\times A_{2,1}}(x_1, x_2) = \mu_{A_{1,1}}(x_1) \star \mu_{A_{2,1}}(x_2), \tag{4.1}$$

where $\star$ denotes the t-norm operator [28]. Then, one can obtain a variety of membership value expressions with respect to the different interpretations of fuzzy rules. For example, take the Zadeh implication and the Mamdani implication [38], where the membership values are expressed as

$$\begin{aligned} &\mu_{\mathcal{R}_1}(x_1, x_2, p_1) = \max\left[\min(\mu_{A_{1,1}\times A_{2,1}}(x_1, x_2), \mu_{C_{1,l_1}}(p_1)), 1 - \mu_{A_{1,1}\times A_{2,1}}(x_1, x_2)\right], \\ &\mu_{\mathcal{R}_1}(x_1, x_2, p_1) = \mu_{A_{1,1}\times A_{2,1}}(x_1, x_2) \cdot \mu_{C_{1,l_1}}(p_1), \end{aligned} \tag{4.2}$$

respectively, where $\mathcal{R}_1$ denotes a fuzzy relation represented by the first fuzzy rule.

According to (4.1), (4.2), and the composition based inference, we have the following results.

Proposition 1. *Consider the fuzzy rule base (3.1). Based on the Mamdani combination, the fuzzy relation $\mathcal{R}$ is represented as follows:*

$$\mathcal{R} = \bigcup_{\kappa=1}^{\tau_1\tau_2} \mathcal{R}_\kappa.$$

The membership value is calculated as $\mu_{\mathcal{R}}(x_1, x_2, p_1) = \mu_{\mathcal{R}_1}(x_1, x_2, p_1) * \cdots * \mu_{\mathcal{R}_{\tau_1\tau_2}}(x_1, x_2, p_1)$, where $*$ denotes the s-norm operator [25].

Proposition 2. *Consider the fuzzy rule base (3.1). Based on the Gödel combination, the fuzzy relation $\mathcal{R}$ is expressed as follows:*

$$\mathcal{R} = \bigcap_{\kappa=1}^{\tau_1\tau_2} \mathcal{R}_\kappa.$$

The membership value is computed as $\mu_{\mathcal{R}}(x_1, x_2, p_1) = \mu_{\mathcal{R}_1}(x_1, x_2, p_1) \star \cdots \star \mu_{\mathcal{R}_{\tau_1\tau_2}}(x_1, x_2, p_1)$.

From Propositions 1 and 2, we can obtain the fuzzy relation matrix by arranging the membership values. However, the above method cannot be directly applied to derive the fuzzy relation matrix based on the input-output data pairs. Furthermore, the representation of the fuzzy relation matrix is restricted by the number of input variables. Hence, based on Propositions 1 and 2 and the algebraic formulation (2.5), we explore the construction of fuzzy relation matrices by utilizing the input-output data pairs.

Consider a standard multi-input single-output FLU, wherein x_i denotes the input variable and y represents the output variable, where $i = 1, \cdots, m$. Assume that the input-output data pairs are expressed as $(\hat{x}_1^\iota, \cdots, \hat{x}_m^\iota, \hat{y}^\iota)$, $\iota = 1, \cdots, n$. According to the characteristics of the considered system, one can design the appropriate membership functions for $x_1, \cdots, x_m$, and y, such as Gaussian and triangular membership functions. Then, the corresponding term sets can be expressed as follows:

$$T_{x_1} = \{A_{1,1}, \cdots, A_{1,\tau_1}\}, \ \cdots, \ T_{x_m} = \{A_{m,1}, \cdots, A_{m,\tau_m}\}, \ T_y = \{B_1, \cdots, B_\tau\}. \tag{4.3}$$

Based on the above term sets and data pairs, we can derive the membership value vectors of the input-output data pairs, that is,

$$\hat{X}_1^\iota = [\mu_{A_{1,1}}(\hat{x}_1^\iota)\ \cdots\ \mu_{A_{1,\tau_1}}(\hat{x}_1^\iota)]^\top, \cdots, \hat{X}_m^\iota = [\mu_{A_{m,1}}(\hat{x}_m^\iota)\ \cdots\ \mu_{A_{m,\tau_m}}(\hat{x}_m^\iota)]^\top, \hat{Y}^\iota = [\mu_{B_1}(\hat{y}^\iota)\ \cdots\ \mu_{B_\tau}(\hat{y}^\iota)]^\top, \tag{4.4}$$

where $\hat{X}_i^\iota$ and $\hat{Y}^\iota$ denote the vector forms of data $\hat{x}_i^\iota$ and y^ι, respectively.

From (4.3) and (4.4), utilizing the Mamdani implications, the ι-th fuzzy relation matrix is calculated as follows:

$$\begin{aligned} \hat{M}_{\mathcal{R}^\iota} &= \hat{Y}^\iota \propto (\hat{X}_1^\iota \propto \cdots \propto \hat{X}_m^\iota)^\top \\ &= \begin{bmatrix} \mu^\iota_{1,1,\cdots,1,1} & \cdots & \mu^\iota_{1,1,\cdots,\tau_1,1} & \cdots & \mu^\iota_{\tau_1,\tau_2,\cdots,1,1} & \cdots & \mu^\iota_{\tau_1,\tau_2,\cdots,\tau_m,1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ \mu^\iota_{1,1,\cdots,1,\tau} & \cdots & \mu^\iota_{1,1,\cdots,\tau_m,\tau} & \cdots & \mu^\iota_{\tau_1,\tau_2,\cdots,1,\tau} & \cdots & \mu^\iota_{\tau_1,\tau_2,\cdots,\tau_m,\tau} \end{bmatrix}, \end{aligned} \tag{4.5}$$

where $\mu^\iota_{\sigma_1,\cdots,\sigma_m,\kappa} = \mu_{A_{1,\sigma_1}}(\hat{x}_1^\iota)\cdots\mu_{A_{m,\sigma_m}}(\hat{x}_m^\iota)\mu_{B_\kappa}(\hat{y}^\iota)$, $\sigma_i = 1, \cdots, \tau_i$, and $\kappa = 1, \cdots, \tau$.

Based on the Mamdani combination of fuzzy rule base, the complete fuzzy relation matrix is expressed as follows:

$$M_{\mathcal{R}} = \bigvee_{\iota=1}^{n} \hat{M}_{\mathcal{R}^\iota}, \tag{4.6}$$

where $M_{\mathcal{R}} = \hat{M}_{\mathcal{R}^1} \vee \cdots \vee \hat{M}_{\mathcal{R}^n}$.

Similarly, from the Gödel combination, it holds that

$$M_{\mathcal{R}} = \bigwedge_{\iota=1}^{n} \hat{M}_{\mathcal{R}^\iota}, \tag{4.7}$$

where $M_{\mathcal{R}} = \hat{M}_{\mathcal{R}^1} \wedge \cdots \wedge \hat{M}_{\mathcal{R}^n}$.

Based on the above derivations, we have the following result.

Theorem 1. *Based on the Mamdani or Gödel combinations, the fuzzy relation matrix of a multi-input single-output FLU is represented as*

$$M_{\mathcal{R}} = \bigvee_{\iota=1}^{n} \hat{M}_{\mathcal{R}^\iota} \in \mathbb{R}^{\tau\times(\tau_1\cdots\tau_m)}$$

or

$$M_{\mathcal{R}} = \bigwedge_{\iota=1}^{n} \hat{M}_{\mathcal{R}^\iota} \in \mathbb{R}^{\tau\times(\tau_1\cdots\tau_m)},$$

where $\hat{M}_{\mathcal{R}^\iota}$ is shown in (4.5).

Remark 1. *It is worth noting that the Mamdani combination is more commonly used in the engineering field because it can better capture the main features of input-output data, while the Gödel combination is more advantageous in scenarios that emphasize logical rigor and conservative inference.*

In the following, we give a simple example to illustrate the construction procedure of the fuzzy relation matrix.

Example 1. *Consider a standard FLU. Let x_1, x_2, x_3 denote the input variables and y represent the output variable. The universes of discourse of x_1, x_2, x_3, and y are all* $[0, 1]$. *In addition, there are four levels for each term set. Here, the input-output data pairs are represented as* $(0.5, 0.45, 0.35, 0.25)$ *and* $(0.9, 0.3, 0.65, 0.6)$. *The goal is to construct the fuzzy relation matrix of the considered FLU.*

The term sets of x_1, x_2, x_3, and y are defined as $\{A_1, A_2, A_3, A_4\}$. Here, we use the triangular membership functions to describe these fuzzy sets, in which the centers of the fuzzy sets are $\{0.2, 0.4, 0.6, 0.8\}$, *respectively. Taking the input variable x_i as an example, the membership functions are shown in Figure 3, where $i = 1, 2, 3$.*

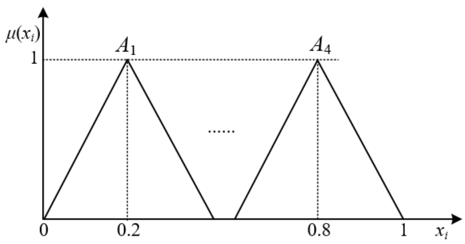

Figure 3. The membership functions of input variable x_i.

By utilizing the input-output data pair $(0.5, 0.45, 0.35, 0.25)$ *and the above term sets, the membership value vectors of x_1, x_2, x_3, and y are expressed as follows:*

$$\hat{X}_1^1 = [0\ 0.5\ 0.5\ 0]^\top,\ \hat{X}_2^1 = [0\ 0.75\ 0.25\ 0]^\top,\ \hat{X}_3^1 = [0.25\ 0.75\ 0\ 0]^\top,\ \hat{Y}^1 = [0.75\ 0.25\ 0\ 0]^\top. \tag{4.8}$$

According to (4.5), the fuzzy relation matrix $\hat{M}_{\mathcal{R}^1}$ is calculated as follows:

$$\begin{aligned}\hat{M}_{\mathcal{R}^1} &= \hat{Y}^1 \propto (\hat{X}_1^1 \propto \hat{X}_2^1 \propto \hat{X}_3^1)^\top \\ &= \begin{bmatrix} 0 & \cdots & 0 & 0.25 & 0.5 & 0 & 0 & 0.25 & 0.5 & 0 & 0 & \cdots & 0 & 0.25 & 0.25 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0.25 & 0.25 & 0 & 0 & 0.25 & 0.25 & 0 & 0 & \cdots & 0 & 0.25 & 0.25 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{4\times 64}.\end{aligned} \tag{4.9}$$

Similar to (4.8), the membership value vectors with respect to data pairs $(0.9, 0.3, 0.65, 0.6)$ *are represented as follows:*

$$\hat{X}_1^2 = [0\ 0\ 0\ 0.5]^\top,\ \hat{X}_2^2 = [0\ 0.5\ 0.5\ 0]^\top,\ \hat{X}_3^2 = [0\ 0\ 0.75\ 0]^\top,\ \hat{Y}^2 = [0\ 0\ 1\ 0]^\top.$$

Then, one can obtain $\hat{M}_{\mathcal{R}^2}$, that is,

$$\hat{M}_{\mathcal{R}^2} = \hat{Y}^2 \propto (\hat{X}_1^2 \propto \hat{X}_2^2 \propto \hat{X}_3^2)^\top = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0.5 & 0.25 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 64}.$$

Based on Theorem 1 and the Mamdani combination, the fuzzy relation matrix $M_{\mathcal{R}}$ is expressed *as follows:*

$$M_{\mathcal{R}} = \hat{M}_{\mathcal{R}^1} \bigvee \hat{M}_{\mathcal{R}^2} = \begin{bmatrix} 0 & \cdots & 0.25 & 0.5 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0.25 & 0.25 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0.5 & 0.25 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{4\times 64}.$$

In what follows, combining Theorem 1 and Figure 1, we explore the construction of serial HFRMs. Without loss of generality, assume that there are τ fuzzy sets for each variable. Hence, the term sets of variables x_i, p_k, and y are denoted by the following:

$$T_{x_i} = \{A_{i,1}, \cdots, A_{i,\tau}\},\ T_{p_k} = \{C_{k,1}, \cdots, C_{k,\tau}\},\ T_y = \{B_1, \cdots, B_\tau\}, \tag{4.10}$$

where $i = 1, \cdots, m$, and $k = 1, \cdots, m-1$. Here, the triangular membership functions are used to describe the above fuzzy sets, which are similar to Figure 2.

Suppose that the input-output data pairs are represented as $(\hat{x}_1^\iota, \hat{x}_2^\iota, \hat{p}_1^\iota, \cdots, \hat{p}_{m-2}^\iota, \hat{x}_m^\iota, \hat{y}^\iota)$, $\iota \in \{1, \cdots, n\}$. Then, we can obtain the membership value vectors of $\hat{x}_i^\iota$, $\hat{p}_k^\iota$ and $\hat{y}^\iota$, that is,

$$\begin{cases} \hat{X}_1^\iota = [\mu_{A_{1,1}}(\hat{x}_1^\iota) \cdots \mu_{A_{1,\tau}}(\hat{x}_1^\iota)]^\top, \ \cdots, \ \hat{X}_m^\iota = [\mu_{A_{m,1}}(\hat{x}_m^\iota) \cdots \mu_{A_{m,\tau}}(\hat{x}_m^\iota)]^\top, \\ \hat{P}_1^\iota = [\mu_{C_{1,1}}(\hat{p}_1^\iota) \cdots \mu_{C_{1,\tau}}(\hat{p}_1^\iota)]^\top, \ \cdots, \hat{P}_{m-2}^\iota = [\mu_{C_{m-2,1}}(\hat{p}_{m-2}^\iota) \cdots \mu_{C_{m-2,\tau}}(\hat{p}_{m-2}^\iota)]^\top, \\ \hat{Y}^\iota = [\mu_{B_1}(\hat{y}^\iota) \cdots \mu_{B_\tau}(\hat{y}^\iota)]^\top. \end{cases}$$

From Theorem 1, the fuzzy relation matrix of FLU_1 is calculated as follows:

$$\begin{cases} \hat{M}_{\mathcal{R}_1^\iota} = \hat{P}_1^\iota \propto (\hat{X}_1^\iota \propto \hat{X}_2^\iota)^\top, \\ M_{\mathcal{R}_1} = \bigvee_{\iota=1}^{n} \hat{M}_{\mathcal{R}_1^\iota}, \end{cases} \tag{4.11}$$

where $\hat{M}_{\mathcal{R}_1^\iota}$ denotes the fuzzy relation matrix corresponding to the ι-th input-output data pair. Based on (2.7) and (4.11), FLU_1 is described as follows:

$$P_1 = M_{\mathcal{R}_1} \propto X_1 \propto X_2, \tag{4.12}$$

where X_1, X_2, and P_1 are the vector forms of x_1, x_2, and p_1, respectively.

Take p_1 and x_3 as the input variables of FLU_2; $M_{\mathcal{R}_2}$ is calculated as follows:

$$\begin{cases} \hat{M}_{\mathcal{R}_2^\iota} = \hat{P}_2^\iota \propto (\hat{P}_1^\iota \propto \hat{X}_3^\iota)^\top, \\ M_{\mathcal{R}_2} = \bigvee_{\iota=1}^{n} \hat{M}_{\mathcal{R}_2^\iota}. \end{cases} \tag{4.13}$$

Then, the algebraic formulation of FLU_2 is expressed as follows:

$$P_2 = M_{\mathcal{R}_2} \propto P_1 \propto X_3. \tag{4.14}$$

Similar to (4.11), (4.13), and (4.14), for FLU_{m-1}, we have the following result:

$$\begin{cases} \hat{M}_{\mathcal{R}_{m-1}^\iota} = \hat{Y}^\iota \propto (\hat{P}_{m-2}^\iota \propto X_m^\iota)^\top, \\ M_{\mathcal{R}_{m-1}} = \bigvee_{\iota=1}^{n} \hat{M}_{\mathcal{R}_{m-1}^\iota}. \end{cases}$$

Then, we can derive that

$$\begin{aligned} Y =& M_{\mathcal{R}_{m-1}} \propto P_{m-2} \propto X_m \\ =& M_{\mathcal{R}_{m-1}} \propto M_{\mathcal{R}_{m-2}} \propto P_{m-3} \propto X_{m-1} \propto X_m \\ =& M_{\mathcal{R}_{m-1}} \propto M_{\mathcal{R}_{m-2}} \propto \cdots \propto M_{\mathcal{R}_2} \propto M_{\mathcal{R}_1} \propto X_1 \propto X_2 \propto \cdots \propto X_{m-1} \propto X_m \\ =& M_{\mathcal{R}} \propto \mathcal{X}, \end{aligned} \tag{4.15}$$

where $M_{\mathcal{R}} = M_{\mathcal{R}_{m-1}} \propto \cdots \propto M_{\mathcal{R}_1}$ *and* $\mathcal{X} = X_1 \propto \cdots \propto X_m$.

Based on (4.12), (4.14), and (4.15), we have the following conclusion.

Theorem 2. *The algebraic formulation of serial HFRMs is represented as follows:*

$$Y = M_{\mathcal{R}} \propto \mathcal{X},$$

where $M_{\mathcal{R}} \in \mathbb{R}^{\tau\times\tau^m}$ *and* $\mathcal{X}$ *are given in (4.15).*

In the following, we explore the construction of parallel HFRMs. Consider a parallel HFS wherein $x_1, \cdots, x_{2m}$ *are input variables and* y *is an output variable, whose structure diagram is shown in Figure 4. Notice that* $FLU_{\kappa,j}$ *is the* j*-th FLU in the* κ*-th layer,* $p_{\kappa,j}$ *is the output variable of* $FLU_{\kappa,j}$*, where* $\kappa = 1, \cdots, l$ *and* $p_l = y$.

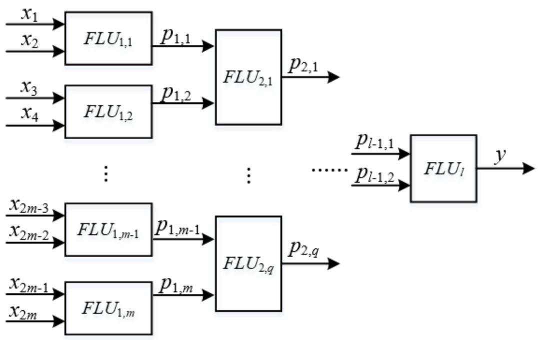

Figure 4. The structure diagram of parallel HFSs.

Similar to (4.10), the term sets of x_i, $p_{\kappa,j}$, *and* y *are denoted by the following:*

$$T_{x_i} = \{A_{i,1}, \cdots, A_{i,\tau}\},\ T_{p_{\kappa,j}} = \{C^1_{\kappa,j}, \cdots, C^\tau_{\kappa,j}\},\ T_y = \{B_1, \cdots, B_\tau\}. \tag{4.16}$$

Given the n *input-output data pairs* $(\hat{x}^\iota_1, \hat{x}^\iota_2, \hat{p}^\iota_{1,1}, \cdots, \hat{p}^\iota_{l-1,1}, \hat{p}^\iota_{l-1,2}, \hat{y}^\iota)$, $\iota = 1, \cdots, n$. *Based on (4.16), the membership value vectors of the aforementioned data pairs are represented as follows:*

$$\begin{cases} \hat{X}^\iota_1 = [\mu_{A_{1,1}}(\hat{x}^\iota_1)\ \cdots\ \mu_{A_{1,\tau}}(\hat{x}^\iota_1)]^\top,\ \cdots,\ \hat{X}^\iota_m = [\mu_{A_{m,1}}(\hat{x}^\iota_m)\ \cdots\ \mu_{A_{m,\tau}}(\hat{x}^\iota_m)]^\top, \\ \hat{P}^\iota_{1,1} = [\mu_{C^1_{1,1}}(\hat{p}^\iota_{1,1})\ \cdots\ \mu_{C^\tau_{1,1}}(\hat{p}^\iota_{1,1})]^\top,\ \cdots,\ \hat{Y}^\iota = [\mu_{B_1}(\hat{y}^\iota)\ \cdots\ \mu_{B_\tau}(\hat{y}^\iota)]^\top. \end{cases} \tag{4.17}$$

Similar to (4.11), we can obtain the algebraic formulation of the first layer FLUs, that is,

$$\begin{cases} \hat{M}_{\mathcal{R}^{\iota}_{1,1}} = \hat{P}^{\iota}_{1,1} \propto (\hat{X}^{\iota}_1 \propto \hat{X}^{\iota}_2)^{\top}, \\ M_{\mathcal{R}_{1,1}} = \bigvee_{\iota=1}^{n} \hat{M}_{\mathcal{R}^{\iota}_{1,1}}; \\ \qquad \vdots \\ \hat{M}_{\mathcal{R}^{\iota}_{1,m}} = \hat{P}^{\iota}_{1,m} \propto (\hat{X}^{\iota}_{2m-1} \propto \hat{X}^{\iota}_{2m})^{\top}, \\ M_{\mathcal{R}_{1,m}} = \bigvee_{\iota=1}^{n} \hat{M}_{\mathcal{R}^{\iota}_{1,m}}. \end{cases} \tag{4.18}$$

Based on the above fuzzy relation matrices, it holds that

$$\begin{cases} P_{1,1} = M_{\mathcal{R}_{1,1}} \propto X_1 \propto X_2, \\ P_{1,2} = M_{\mathcal{R}_{1,2}} \propto X_3 \propto X_4, \\ \qquad \vdots \\ P_{1,m} = M_{\mathcal{R}_{1,m}} \propto X_{2m-1} \propto X_{2m}. \end{cases} \tag{4.19}$$

The input variables of the second layer FLUs are determined by the first layer FLUs. Hence, the corresponding fuzzy relation matrices of the second layer FLUs are calculated as follows:

$$\begin{cases} \hat{M}_{\mathcal{R}^{\iota}_{2,1}} = \hat{P}^{\iota}_{2,1} \propto (\hat{P}^{\iota}_{1,1} \propto \hat{P}^{\iota}_{1,2})^{\top}, \\ M_{\mathcal{R}_{2,1}} = \bigvee_{\iota=1}^{n} \hat{M}_{\mathcal{R}^{\iota}_{2,1}}; \\ \qquad \vdots \\ \hat{M}_{\mathcal{R}^{\iota}_{2,q}} = \hat{P}^{\iota}_{2,q} \propto (\hat{P}^{\iota}_{1,m-1} \propto \hat{P}^{\iota}_{1,m})^{\top}, \\ M_{\mathcal{R}_{2,q}} = \bigvee_{\iota=1}^{n} \hat{M}_{\mathcal{R}^{\iota}_{2,q}}. \end{cases} \tag{4.20}$$

Based on (4.19), (4.20), and Lemma 2, for $FLU_{2,1}$, it holds that

$$\begin{aligned} P_{2,1} &= M_{\mathcal{R}_{2,1}} \propto P_{1,1} \propto P_{1,2} \\ &= M_{\mathcal{R}_{2,1}} \propto M_{\mathcal{R}_{1,1}} \propto X_1 \propto X_2 \propto M_{\mathcal{R}_{1,2}} \propto X_3 \propto X_4 \\ &= M_{\mathcal{R}_{2,1}} \propto M_{\mathcal{R}_{1,1}} \propto (X_1 \propto X_2 \propto M_{\mathcal{R}_{1,2}}) \propto X_3 \propto X_4 \\ &= M_{\mathcal{R}_{2,1}} \propto M_{\mathcal{R}_{1,1}} \propto (I_{\tau^2} \boxtimes M_{\mathcal{R}_{1,2}} \propto X_1 \propto X_2) \propto X_3 \propto X_4 \\ &= M_{\mathcal{R}_{2,1}} \propto M_{\mathcal{R}_{1,1}} \propto (I_{\tau^2} \boxtimes M_{\mathcal{R}_{1,2}}) \propto X_1 \propto \cdots \propto X_4 \\ &= \tilde{M}_{\mathcal{R}_{2,1}} \propto X_1 \propto \cdots \propto X_4, \end{aligned}$$

where $\tilde{M}_{\mathcal{R}_{2,1}} = M_{\mathcal{R}_{2,1}} \propto M_{\mathcal{R}_{1,1}} \propto (I_{\tau^2} \boxtimes M_{\mathcal{R}_{1,2}})$. Similarly, the algebraic formulation of $FLU_{2,j}$ is expressed as follows:

$$P_{2,j} = \tilde{M}_{\mathcal{R}_{2,j}} \propto X_{4j-3} \propto \cdots \propto X_{4j}, \tag{4.21}$$

where $j = 1, \cdots, q$, *and* $\tilde{M}_{\mathcal{R}_{2,j}} = M_{\mathcal{R}_{2,j}} \propto M_{\mathcal{R}_{1,2j-1}} \propto (I_{\tau^2} \boxtimes M_{\mathcal{R}_{1,2j}})$.

Similar to (4.21), by iterating from the second layer, the fuzzy relation matrix of parallel HFRMs $M_{\mathcal{R}_l}$ *is expressed as follows:*

$$\begin{cases} \hat{M}_{\mathcal{R}_l^\iota} = \hat{Y}^\iota \propto (\hat{P}^\iota_{l-1,1} \propto \hat{P}^\iota_{l-1,2})^\top, \\ M_{\mathcal{R}_l} = \bigvee_{\iota=1}^{n} \hat{M}_{\mathcal{R}_l^\iota}. \end{cases} \tag{4.22}$$

According to Lemmas 1 and 2, we can derive that

$$\begin{aligned} Y &= M_{\mathcal{R}_l} \propto P_{l-1,1} \propto P_{l-1,2} \\ &= M_{\mathcal{R}_l} \propto \tilde{M}_{\mathcal{R}_{l-1,1}} \propto X_1 \propto \cdots \propto X_m \propto \tilde{M}_{\mathcal{R}_{l-1,2}} \propto X_{m+1} \propto \cdots \propto X_{2m} \\ &= M_{\mathcal{R}_l} \propto \tilde{M}_{\mathcal{R}_{l-1,1}} \propto (I_{\tau^m} \boxtimes \tilde{M}_{\mathcal{R}_{l-1,2}}) \propto \mathcal{X}, \\ &= M_{\mathcal{R}} \propto \mathcal{X}, \end{aligned} \tag{4.23}$$

where $M_{\mathcal{R}} = M_{\mathcal{R}_l} \propto \tilde{M}_{\mathcal{R}_{l-1,1}} \propto (I_{\tau^m} \boxtimes \tilde{M}_{\mathcal{R}_{l-1,2}})$, *and* $\mathcal{X} = X_1 \propto \cdots \propto X_{2m}$.

Based on (4.19), (4.21), and (4.23), we have the following result.

Theorem 3. *The algebraic formulation of parallel HFRMs is expressed as follows:*

$$Y = M_{\mathcal{R}}\mathcal{X},$$

where $M_{\mathcal{R}} \in \mathbb{R}^{\tau \times \tau^{2m}}$, *and* $\mathcal{X}$ *are shown in (4.23).*

The hybrid HFSs are the combinations of serial and parallel HFSs. Hence, according to Theorems 2 and 3, we can obtain the algebraic formulation of hybrid HFRMs. Here, we give a simple example to illustrate it.

Example 2. *Consider a hybrid HFS wherein* $x_1, \cdots, x_5$ *are input variables,* $p_{1,1}, p_{1,2}, p_2$ *are intermediate variables, and* y *is output variable. The term sets of the aforementioned variables are denoted by* $T_{x_i} = \{A_{i,1}, \cdots, A_{i,\tau}\}$, $T_y = \{B_1, \cdots, B_\tau\}$, $T_{p_{1,1}} = \{C^1_{1,1}, \cdots, C^\tau_{1,1}\}$, $T_{p_{1,2}} = \{C^1_{1,2}, \cdots, C^\tau_{1,2}\}$, *and* $T_{p_2} = \{C^1_2, \cdots, C^\tau_2\}$, *respectively, where* $i = 1, \cdots, 5$. *The structure diagram of the considered HFS is shown in Figure 5.*

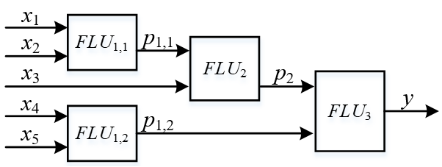

Figure 5. The structure diagram of considered hybrid HFS.

Assume that there are n *input-output data pairs* $(\hat{x}^\iota_1, \hat{x}^\iota_2, \hat{p}^\iota_{1,1}, \cdots, \hat{p}^\iota_2, \hat{y}^\iota)$, $\iota = 1, \cdots, n$. *From (4.18) and Theorem 1, for* $FLU_{1,1}$, *it holds that*

$$\begin{aligned} \hat{M}_{\mathcal{R}^\iota_{1,1}} &= \hat{P}^\iota_{1,1} \propto (\hat{X}^\iota_1 \propto \hat{X}^\iota_2)^\top, \\ M_{\mathcal{R}_{1,1}} &= \bigvee_{\iota=1}^{n} \hat{M}_{\mathcal{R}^\iota_{1,1}}, \end{aligned} \tag{4.24}$$

where

$$\hat{X}_1^\iota = [\mu_{A_{1,1}}(\hat{x}_1^\iota)\ \cdots\ \mu_{A_{1,\tau}}(\hat{x}_1^\iota)]^\top,\ \hat{X}_2^\iota = [\mu_{A_{2,1}}(\hat{x}_2^\iota)\ \cdots\ \mu_{A_{2,\tau}}(\hat{x}_2^\iota)]^\top,\ and \hat{P}^\iota_{1,1} = [\mu_{C^1_{1,1}}(\hat{p}^\iota_{1,1})\ \cdots\ \mu_{C^\tau_{1,1}}(\hat{p}^\iota_{1,1})]^\top.$$

Then, the algebraic formulation of $FLU_{1,1}$ is represented as follows:

$$P_{1,1} = M_{\mathcal{R}_{1,1}} \propto X_1 \propto X_2, \tag{4.25}$$

where $X_1 = [\mu_{A_{1,1}}(x_1)\ \cdots\ \mu_{A_{1,\tau}}(x_1)]^\top$, and $X_2 = [\mu_{A_{2,1}}(x_2)\ \cdots\ \mu_{A_{2,\tau}}(x_2)]^\top$.

Similar to (4.24) and (4.25), $FLU_{1,2}$ can be expressed as follows:

$$\begin{cases} \hat{M}_{\mathcal{R}^\iota_{1,2}} = \hat{P}^\iota_{1,2} \propto (\hat{X}_4^\iota \propto \hat{X}_5^\iota)^\top, \\ M_{\mathcal{R}_{1,2}} = \bigvee_{\iota=1}^{n} \hat{M}_{\mathcal{R}^\iota_{1,2}}, \\ P_{1,2} = M_{\mathcal{R}_{1,2}} \propto X_4 \propto X_5. \end{cases} \tag{4.26}$$

According to the serial structure, FLU_2 can be described as follows:

$$\begin{cases} \hat{M}_{\mathcal{R}^\iota_{2}} = \hat{P}^\iota_{2} \propto (\hat{P}^\iota_{1,1} \propto \hat{X}_3^\iota)^\top, \\ M_{\mathcal{R}_{2}} = \bigvee_{\iota=1}^{n} \hat{M}_{\mathcal{R}^\iota_{2}}, \\ P_{2} = M_{\mathcal{R}_{2}} \propto P_{1,1} \propto X_3. \end{cases} \tag{4.27}$$

From Theorem 3, we can obtain the fuzzy relation matrix of FLU_3, that is,

$$M_{\mathcal{R}_3} = \bigvee_{\iota=1}^{n} \left(\hat{Y}^\iota \propto (\hat{P}^\iota_2 \propto \hat{P}^\iota_{1,2})^\top\right). \tag{4.28}$$

Based on (4.25)–(4.28), for the considered hybrid HFRM, we derive that

$$\begin{aligned} Y &= M_{\mathcal{R}_3} \propto P_2 \propto P_{1,2} \\ &= M_{\mathcal{R}_3} \propto M_{\mathcal{R}_2} \propto P_{1,1} \propto X_3 \propto M_{\mathcal{R}_{1,2}} \propto X_4 \propto X_5 \\ &= M_{\mathcal{R}_3} \propto M_{\mathcal{R}_2} \propto M_{\mathcal{R}_{1,1}} \propto X_1 \propto X_2 \propto X_3 \propto M_{\mathcal{R}_{1,2}} \propto X_4 \propto X_5 \\ &= M_{\mathcal{R}_3} \propto M_{\mathcal{R}_2} \propto M_{\mathcal{R}_{1,1}} \propto (I_{\tau^3} \boxtimes M_{\mathcal{R}_{1,2}}) \propto X_1 \propto X_2 \propto X_3 \propto X_4 \propto X_5 \\ &= M_{\mathcal{R}} \propto \mathcal{X}, \end{aligned}$$

where $M_{\mathcal{R}} = M_{\mathcal{R}_3} \propto M_{\mathcal{R}_2} \propto M_{\mathcal{R}_{1,1}} \propto (I_{\tau^3} \boxtimes M_{\mathcal{R}_{1,2}})$, and $\mathcal{X} = X_1 \propto \cdots \propto X_5$.

5. Hierarchical fuzzy ramp metering strategies

Freeway ramp metering is an effective control strategy to alleviate traffic congestion [33]. The schematic diagram of the freeway model is shown in Figure 6, and the relationship between traffic density and flow is shown in Figure 7. From Figures 6 and 7, it is clear that the ultimate goal of ramp metering is to maximize the traffic flow of segment 4 by adjusting the ramp metering flow u to make the density ρ_4 reach the expected density ρ_d. Notice that the on-ramp queue length l is determined by the on-ramp inflow flow q_{r_2} and u, which affects the traffic efficiency to some extent.

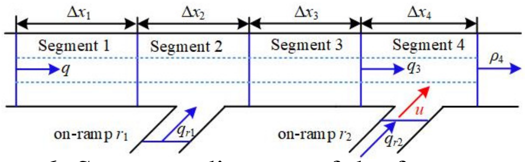

Figure 6. Structure diagram of the freeway model.

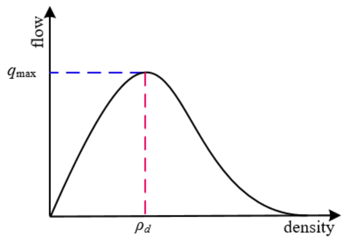

Figure 7. Relationship between traffic density and flow.

The considered freeway model can be described as follows:

$$\begin{cases} \rho_i(k+1) = \rho_i(k) + \dfrac{T}{\lambda_i \Delta x_i}\left[q_{i-1}(k) - q_i(k) + u_i(k)\right], \\ q_i(k) = \rho_i(k) \cdot v_i(k) \cdot \lambda_i, \\ v_i(k+1) = v_i(k) + \dfrac{T}{\tau}\left[V(\rho_i(k)) - v_i(k)\right] + \dfrac{T}{\Delta x_i} v_i(k)\left[v_{i-1}(k) - v_i(k)\right] \dfrac{\mu T}{\tau \Delta x_i} \dfrac{\rho_{i+1}(k) - \rho_i(k)}{\rho_i(k) + \theta}, \\ V\left[\rho_i(k)\right] = v_f\left[1 - \left(\rho_i(k)/\rho_{jam}\right)^{\delta}\right]^m, \end{cases}$$

where ρ_i, q_i, and v_i represent the traffic flow density, speed, and flow of the i-th freeway segment, respectively. The other parameters are described in [33].

The hierarchical fuzzy ramp metering system was proposed in [33], which had 99 FLUs. Based on Theorems 2 and 3, we can calculate the fuzzy relation matrices of FLUs of the aforementioned system. For example, we obtain the 720 input-output data pairs of $FLU_{1,33}$, and select the different data pairs, that is,

$$\begin{gathered} \hat{X}_1^1 = [0\ 0\ 0\ 0\ 1]^\top,\ \hat{X}_2^1 = [0\ 0\ 1\ 0\ 0]^\top,\ \hat{Y}^1 = [0\ 0\ 0\ 0\ 1]^\top; \\ \hat{X}_1^2 = [0\ 0\ 0\ 0\ 1]^\top,\ \hat{X}_2^2 = [0\ 0\ 0\ 1\ 0]^\top,\ \hat{Y}^2 = [0\ 0\ 0\ 1\ 0]^\top; \\ \hat{X}_1^3 = [0\ 0\ 0\ 0\ 1]^\top,\ \hat{X}_2^3 = [0\ 0\ 1\ 0\ 0]^\top,\ \hat{Y}^3 = [1\ 0\ 0\ 0\ 0]^\top; \\ \hat{X}_1^4 = [0\ 0\ 0\ 1\ 0]^\top,\ \hat{X}_2^4 = [0\ 0\ 1\ 0\ 0]^\top,\ \hat{Y}^4 = [0\ 0\ 0\ 1\ 0]^\top; \\ \vdots \\ \hat{X}_1^{10} = [0\ 0\ 1\ 0\ 0]^\top,\ \hat{X}_2^{10} = [0\ 0\ 0\ 1\ 0]^\top,\ \hat{Y}^{10} = [0\ 0\ 1\ 0\ 0]^\top; \\ \hat{X}_1^{11} = [0\ 0\ 0\ 1\ 0]^\top,\ \hat{X}_2^{11} = [0\ 0\ 0\ 1\ 0]^\top,\ \hat{Y}^{11} = [0\ 0\ 0\ 1\ 0]^\top, \end{gathered}$$

where $\hat{X}_1^\iota$ and $\hat{X}_2^\iota$ are the vector forms of data $\hat{x}_1^\iota$ and $\hat{x}_2^\iota$, respectively, and $\iota = 1, \cdots, 11$. Here, x_1 represents the error e between the density ρ_4 and expected density ρ_d, and x_2 denotes the change of error ec.

According to the above data pairs, the fuzzy relation matrix of $FLU_{1,33}$ is calculated as follows:

$$M_{\mathcal{R}_{1,33}} = \delta_5[k_1\ k_2\ 3\ 3\ k_3\ k_4\ k_5\ 3\ 3\ k_6\ k_7\ k_8\ 3\ 3\ k_9\ k_{10}\ k_{11}\ 4\ 4\ k_{12}\ k_{13}\ k_{14}\ 5\ k_{15}\ k_{16}],$$

where $k_1, \cdots, k_{16} \in \{1, 2, 3, 4, 5\}$. Based on the above fuzzy relation matrix, we observe that the fuzzy rules are sparse. According to Theorems 2 and 3, we can calculate all the fuzzy relation matrices of the considered hierarchical fuzzy ramp metering system, and derive all the fuzzy rules.

In what follows, in order to reduce the computational complexity, we only consider the relevant variables of segment 4 and on-ramp r_2 as the input linguistic variables. Therefore, we propose a hierarchical fuzzy on-ramp controller, shown in Figure 8.

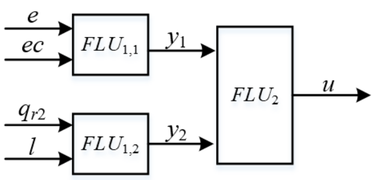

Figure 8. Structure diagram of the hierarchical fuzzy on-ramp controller.

Similar to [33], we use the correction factor method to construct the hierarchical fuzzy controller. For $FLU_{1,1}$, based on the singleton membership function, the term set $T_e = \{NB, NS, ZE, PS, PB\}$ of e is described as follows:

$$NB \sim [e \leq -2],\ NS \sim [-2 < e \leq -1],\ ZE \sim [-1 < e \leq 0],\ PS \sim [0 < e \leq 1],\ PB \sim [e > 1].$$

By replacing e with ec, we can obtain the term set T_{ec}. Then, the fuzzy relation matrix $M_{\mathcal{R}_{1,1}}$ of $FLU_{1,1}$ is represented as follows:

$$M_{\mathcal{R}_{1,1}} = \delta_5[1\ 2\ 3\ 3\ 5\ 2\ 2\ 3\ 3\ 4\ 3\ 2\ 3\ 3\ 3\ 3\ 3\ 4\ 4\ 5\ 2\ 3\ 5\ 4\ 5]. \tag{5.1}$$

For $FLU_{1,2}$, the fuzzification procedures of q_{r_2} and l are expressed as follows:

$$NB \sim [q_{r_2} \leq 380],\ NS \sim [380 < q_{r_2} \leq 415],\ ZE \sim [415 < q_{r_2} \leq 430],\ PS \sim [430 < q_{r_2} \leq 440],$$
$$PB \sim [q_{r_2} > 440],\ NB \sim [l \leq 2],\ NS \sim [2 < l \leq 3],\ ZE\ [3 < l \leq 4],\ PS \sim [4 < l \leq 5],\ PB \sim [l > 5].$$

Then, we derive the fuzzy relation matrix of $FLU_{1,2}$, that is,

$$M_{\mathcal{R}_{1,2}} = \delta_5[1\ 2\ 2\ 3\ 2\ 2\ 2\ 2\ 4\ 3\ 2\ 3\ 3\ 3\ 4\ 2\ 3\ 4\ 4\ 4\ 3\ 2\ 4\ 5\ 5]. \tag{5.2}$$

In addition, for FLU_2, we have the following:

$$M_{\mathcal{R}_2} = \delta_5[1\ 1\ 3\ 1\ 4\ 2\ 2\ 2\ 2\ 1\ 2\ 3\ 3\ 3\ 4\ 3\ 3\ 4\ 4\ 4\ 5\ 3\ 4\ 4\ 5]. \tag{5.3}$$

According to (5.1)–(5.3), the algebraic formulation of the proposed hierarchical on-ramp controller is represented as follows:

$$\begin{cases} Y_1 = M_{\mathcal{R}_{1,1}} \ltimes E \ltimes EC, \\ Y_2 = M_{\mathcal{R}_{1,2}} \ltimes Q_{r_2} \ltimes L, \\ U = M_{\mathcal{R}_2} \ltimes Y_1 \ltimes Y_2, \end{cases} \tag{5.4}$$

where $E, EC, Q_{r_2}, L, Y_1, Y_2,$ and U are the vector expressions of variables $e, ec, q_{r_2}, l, y_1, y_2,$ and u, respectively.

In order to demonstrate the effectiveness of proposed the on-ramp controller, we use the same evaluation criteria as [33], that is, the sum of total travel time (TTT), total waiting time (TWT), and total time spent (TTS). These evaluation criteria are calculated as follows:

$$\begin{cases} TTT = \sum_{i=1}^{4} \sum_{k=1}^{720} \lambda_i \rho_i(k) T, \\ TWT = \sum_{k=1}^{720} l(k) T, \\ TTS = TTT + TWT. \end{cases} \tag{5.5}$$

The traffic density, speed, and on-ramp queue length change curve diagrams under the different control strategies are shown in Figures 9–11, respectively.

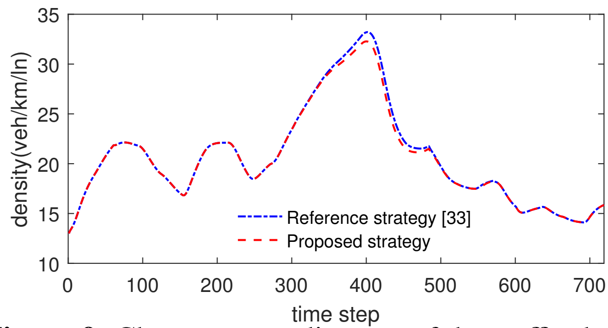

Figure 9. Change curve diagram of the traffic density.

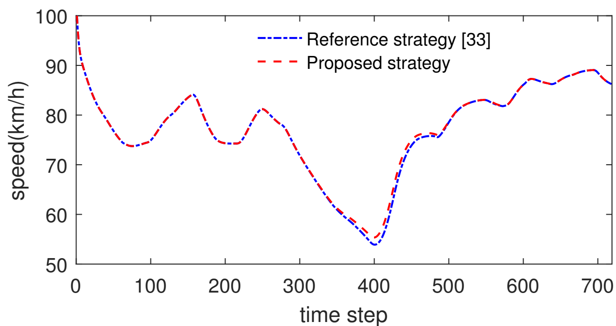

Figure 10. Change curve diagram of the traffic speed.

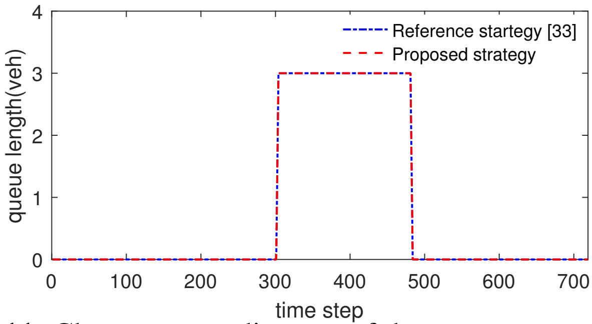

Figure 11. Change curve diagram of the on-ramp queue length.

The comparisons of different control strategies are shown in Table 1. Here, TTT_i denotes the total travel time of segment i, where $i = 1, \cdots, 4$. The evaluation criteria TTT, TWT, and TTS are shown in (5.5).

Table 1. Comparisons of different control strategies.

Criterion\Strategy	reference strategy [33]	proposed strategy
TTT_1(veh·h)	113.55	113.53
TTT_2(veh·h)	98.2	98.14
TTT_3(veh·h)	100.61	100.38
TTT_4(veh·h)	129.89	128.99
TTT (veh·h)	442.25	441.04
TWT (veh·h)	1.5	1.5
TTS (veh·h)	443.75	442.54

From Figures 9–11, within the first 310 time steps and the last 140 time steps, we can observe that when the traffic flow density ρ_4 of segment 4 is much less than the expected density ρ_d, the ramp metering flow is equal to the ramp inflow flow, that is, $u = q_{r_2}$. When the density ρ_4 approaches or exceeds the expected density ρ_d, that is, within the 310–580 time steps, the ramp metering flow u is determined by the density ρ_4 and on-ramp queue length l. Therefore, we take the variables related to the density ρ_4 and the queue length l as input variables. Compared with the 99 FLUs of reference strategy in [33], the proposed control strategy has only 3 FLUs, which greatly reduces the computational complexity. It is clear from Table 1 that the TTT of the proposed control strategy is 441.04 veh· h, which is smaller than the 442.25 veh· h of the reference strategy. Notice that the TWT is the same under both control strategies. Hence, the TTS of the proposed control strategy is smaller than that of the reference control strategy, which can better alleviate the traffic congestion.

Remark 2. *Compared with the control strategy [33],the proposed control strategy simplifies the number of FLUs from 99 to 3 and decreases the computational complexity from* $O(99\times\tau^3)$ *to* $O(3\times\tau^3)$*, where* τ *is the number of fuzzy sets. The proposed strategy not only reduces the computational complexity and inference time, but also maintains the control performance.*

6. Conclusions

In this paper, we discussed the calculation of fuzzy relation matrices of FLUs based on the input-output data pairs, and developed the construction methods of HFRMs with respect to the serial, parallel, and hybrid HFSs. Furthermore, we applied the obtained results to the ramp metering of freeway to alleviate traffic congestion.

Compared with [8], by means of the semi-tensor product method, the proposed HFRM avoids the repeated process of defuzzification and fuzzification in middle layers, thereby improving the interpretability of fuzzy systems. As shown in [39], the interval type-2 fuzzy systems can effectively deal with the multiple uncertainties. The obtained theoretical results can provide a feasible idea for the construction of that in the hierarchical interval type-2 fuzzy systems. Therefore, in the future works, we can will develop a theoretical framework for the hierarchical interval type-2 fuzzy systems through the input-output data pairs, and further explore the interpretability and optimization of such systems.

Author contributions

Changle Sun: Writing-original draft, investigation, formal analysis; Haitao Li: Writing-review & editing, supervision, project administration, funding acquisition; Wenxiu Zhao: Validation; Linlin Guo: Formal analysis; Yalu Li: Writing-review & editing, funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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