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Research article

Szász–Durrmeyer-type operators generated by Euler polynomials of negative order via a Poisson–Gamma representation

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Abstract: We introduced a Szász–Durrmeyer-type family of positive linear operators generated by Euler polynomials of negative order $-k$, where $k \in \mathbb{N}$. The corresponding discrete kernel admits a finite binomial mixture of shifted Poisson probabilities, while the Durrmeyer integral part is described by a Gamma-type kernel. This Poisson–Gamma structure makes positivity and normalization transparent and provides a convenient basis for the computation of algebraic and central moments. Using these moments, we established quantitative estimates, weighted Korovkin-type convergence, and a compact-uniform Voronovskaya-type theorem. The asymptotic formula shows that the classical approximation order is preserved, whereas the negative-order Euler parameter affects only the drift term in the first-order asymptotic profile.

Keywords: Szász–Durrmeyer operators; Euler polynomials of negative order; positive linear operators; Poisson–Gamma representation; weighted Korovkin theorem; Voronovskaya theorem
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1. Introduction

Durrmeyer [1] introduced an integral modification of the classical Bernstein operators on $[0, 1]$, replacing point evaluations by local averages with respect to the Bernstein basis. This integral modification approach was later adapted to the Szász–Mirakyan setting in [2]. In parallel, within the Bernstein–Durrmeyer setting, structural aspects of Bernstein–Durrmeyer operators and approximation properties of Bernstein–Durrmeyer-type operators were studied in [3, 4]. More recent developments include beta-type Szász–Mirakyan operators [5], bivariate GBS-associated Kantorovich-type variants [6], parametric generalized q-Schurer operators in Kantorovich form [7], modified Durrmeyer-type Jakimovski–Leviatan operators [8], general Appell polynomial-based Szász–Beta-type integral operators [9], Durrmeyer-type operators linking adjoint Bernoulli polynomials with the Baskakov basis [10], and composition-type modifications of Szász–Durrmeyer operators leading to Laguerre- and Touchard-type polynomial structures [11].

On the unbounded interval $[0, \infty)$, weighted approximation is a natural framework for positive approximation processes. Korovkin-type theorems and their weighted versions provide effective convergence criteria in such spaces [12–14]. In this setting, moment identities are essential. They are used to prove convergence, to obtain quantitative estimates, and to derive Voronovskaya-type asymptotic formulas.

A classical and well-established approach for constructing positive approximation operators is based on special polynomial families defined by exponential generating functions. The terminology of special polynomials of higher and negative integer order is classical and goes back to Nörlund's treatment of finite differences [15]. The Appell-type construction of Jakimovski and Leviatan [16] is a classical example of this method. Approximation processes involving Bernoulli polynomials of negative order have been studied in [17, 18], while related Euler-type constructions have been considered in [19, 20]. Generating-function-based methods also appear in the study of q-Baskakov-type operators [21] and in combinatorial investigations related to degenerate Stirling numbers [22].

However, a generating function does not by itself guarantee that the associated kernel system is positive and normalized. This observation is particularly important for Szász–Durrmeyer-type operators on $[0, \infty)$, where positivity underlies Korovkin-type arguments, modulus estimates, and moment-based asymptotic analysis. Therefore, when a probabilistic representation of the kernel is available, it is not merely a formal interpretation. It gives a structural explanation of positivity, normalization, and the moment computations used later in the analysis.

Motivated by these considerations, we introduce a Szász–Durrmeyer-type family of positive linear operators generated by Euler polynomials of negative order $-k$, where $k \in \mathbb{N}$. A main feature of the construction is the normalized kernel induced by the negative-order Euler factor. Using the finite representation of these Euler polynomials, the discrete part of the kernel admits a finite binomial mixture of shifted Poisson probabilities. Combined with a Gamma-type Durrmeyer kernel, this yields a Poisson–Gamma representation for the operators.

This representation gives a direct proof of positivity and preservation of constants. It also provides an efficient framework for computing algebraic and central moments. These moments reveal how the Euler parameter k enters the first-order drift term in the Voronovskaya-type formula, while the principal approximation order remains the classical order $1/n$.

Using this structure, we derive explicit moment identities up to order four. We then apply these identities to obtain quantitative estimates in terms of the ordinary modulus of continuity and Lipschitz-type classes. We also establish weighted Korovkin-type convergence in $C^0_\rho[0, \infty)$ for the standard weight $\rho(x) = 1 + x^2$. Finally, we prove a compact-uniform Voronovskaya-type theorem and identify the contribution of the Euler parameter to the limiting drift term.

The paper is organized as follows. In Section 2, we recall the Euler polynomials of negative order $-k$, define the associated Szász–Durrmeyer-type operators, and prove their Poisson–Gamma probabilistic representation. This section also contains the generating-function identities and the algebraic and central moment formulas used throughout the paper. Section 3 presents quantitative estimates based on the ordinary modulus of continuity and Lipschitz-type classes. In Section 4, we prove weighted Korovkin-type convergence in $C^0_\rho[0, \infty)$. Section 5 contains the compact-uniform

Voronovskaya-type theorem and a comparison with the corresponding Bernoulli negative-order drift contribution. Section 6 presents numerical experiments comparing the finite-n behavior of the operators for different values of k. Section 7 concludes the paper.

2. Preliminaries and operator construction

Throughout this paper, we work on the unbounded interval $[0, \infty)$ and fix $k \in \mathbb{N}$.

Definition 2.1. *For a fixed positive $k \in \mathbb{N}$, the Euler polynomials of negative order $-k$, $\{E_m^{(-k)}(y)\}_{m\geq 0}$, are defined by the generating function*

$$\sum_{m=0}^{\infty} E_m^{(-k)}(y)\frac{t^m}{m!} = \left(\frac{\mathrm{e}^t + 1}{2}\right)^k \mathrm{e}^{yt}, \qquad y \geq 0. \tag{2.1}$$

Lemma 2.2. *For $k \in \mathbb{N}$, $m \in \mathbb{N}_0$, and $y \geq 0$, the Euler polynomials of negative order $-k$ satisfy*

$$E_m^{(-k)}(y) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}(y + j)^m. \tag{2.2}$$

Moreover,

$$E_m^{(-k)}(y) \geq 0, \qquad y \geq 0, \quad m \in \mathbb{N}_0.$$

Proof. Expanding the factor $(\mathrm{e}^t + 1)^k$ in (2.1), we obtain

$$\left(\frac{\mathrm{e}^t + 1}{2}\right)^k \mathrm{e}^{yt} = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\mathrm{e}^{(y+j)t}.$$

Comparing the coefficients of $t^m/m!$ gives (2.2). Since the coefficients $2^{-k}\binom{k}{j}$ are nonnegative and sum to one, (2.2) expresses $E_m^{(-k)}(y)$ as a convex combination of the shifted monomials $(y + j)^m$. Hence the asserted positivity follows for $y \geq 0$. □

Definition 2.3. *For $n \in \mathbb{N}$, $m \in \mathbb{N}_0$, and $x \geq 0$, define the Euler-type discrete weights by*

$$p_{n,m}^{(k)}(x) := \left(\frac{2}{\mathrm{e} + 1}\right)^k \mathrm{e}^{-nx}\frac{E_m^{(-k)}(nx)}{m!}. \tag{2.3}$$

We also define the Gamma-type kernel by

$$g_{n,m}(u) = \frac{n^{m+1}}{m!}u^m \mathrm{e}^{-nu}, \qquad u \geq 0. \tag{2.4}$$

By the standard Gamma integral

$$\int_0^{\infty} u^r \mathrm{e}^{-nu}\, du = \frac{\Gamma(r + 1)}{n^{r+1}}, \qquad r > -1,$$

the kernel $g_{n,m}$ satisfies

$$\int_0^{\infty} g_{n,m}(u)\, du = 1,$$

and

$$\int_0^\infty u\, g_{n,m}(u)\, du = \frac{m+1}{n}, \qquad \int_0^\infty u^2\, g_{n,m}(u)\, du = \frac{(m+1)(m+2)}{n^2}.$$

Thus $g_{n,m}$ is a normalized Gamma-type kernel.

We now define the associated Szász–Durrmeyer-type operators generated by Euler polynomials of negative order $-k$. The construction is based on the discrete weight system $\{p_{n,m}^{(k)}(x)\}_{m\geq 0}$ in (2.3) and the Gamma-type kernel $g_{n,m}$ in (2.4).

Definition 2.4. *Let*

$$E := \left\{ f \in C[0,\infty) : |f(u)| \leq M_f(1+u^2) \text{ for some } M_f > 0 \text{ and all } u \geq 0 \right\}.$$

For $f \in E$, $n \in \mathbb{N}$, $k \in \mathbb{N}$, and $x \geq 0$, we define

$$D_n^{(k)}(f;x) := \sum_{m=0}^{\infty} p_{n,m}^{(k)}(x) \int_0^\infty g_{n,m}(u) f(u)\, du. \tag{2.5}$$

Clearly, $C_b[0,\infty) \subset E$.

Throughout the paper, when u^r appears as the argument of an operator, it denotes the monomial function $u \mapsto u^r$.

Remark 2.5. *The restriction $k \in \mathbb{N}$ is essential for the positivity argument. Indeed, Lemma 2.2 provides a finite convex representation of $E_m^{(-k)}(y)$, which guarantees $p_{n,m}^{(k)}(x) \geq 0$. For non-integer negative orders, this finite representation is no longer available, and positivity would require a separate argument.*

Remark 2.6. *For $k = 0$, the generating function* (2.1) *reduces to the standard exponential generating function of the monomials, namely, $E_m^{(0)}(y) = y^m$. Consequently,* (2.3) *becomes*

$$p_{n,m}^{(0)}(x) = e^{-nx} \frac{(nx)^m}{m!},$$

that is, the usual Szász basis. Thus, together with the Gamma-type kernel (2.4)*, the operator $D_n^{(0)}$ coincides with the classical Szász–Durrmeyer operator.*

We now provide a probabilistic interpretation of the discrete Euler weights and of the corresponding Szász–Durrmeyer-type operators. This representation is used to establish positivity and normalization, and to organize the moment computations. In what follows, $\mathbb{P}$ and $\mathbb{E}$ denote probability and expectation with respect to the underlying probability space.

The probability generating function associated with the weights uniquely determines the sequence $\{p_{n,m}^{(k)}(x)\}_{m\geq 0}$. The finite binomial–Poisson representation in Lemma 2.7 is the natural representation induced by the finite Euler expansion and by the normalization of the discrete kernel. No uniqueness is claimed here among all possible abstract mixture representations of the same probability mass function.

Lemma 2.7. *Let $n, k \in \mathbb{N}$ and $x \geq 0$. Set*

$$q := \frac{\mathrm{e}}{\mathrm{e}+1}.$$

Let $J_k \sim \mathrm{Bin}(k, q)$, that is,

$$\mathbb{P}(J_k = j) = \binom{k}{j} q^j (1-q)^{k-j} = \binom{k}{j} \frac{\mathrm{e}^j}{(\mathrm{e}+1)^k}, \qquad j = 0, 1, \ldots, k. \tag{2.6}$$

Conditionally on $J_k = j$, let

$$M_{n,x}^{(k)} \mid (J_k = j) \sim \mathrm{Poisson}(nx + j), \qquad j = 0, 1, \ldots, k. \tag{2.7}$$

Then

$$\mathbb{P}(M_{n,x}^{(k)} = m) = p_{n,m}^{(k)}(x), \qquad m \in \mathbb{N}_0. \tag{2.8}$$

Proof. By (2.2) and (2.3), we have

$$p_{n,m}^{(k)}(x) = \frac{\mathrm{e}^{-nx}}{(\mathrm{e}+1)^k} \sum_{j=0}^{k} \binom{k}{j} \frac{(nx+j)^m}{m!}.$$

Since

$$\mathrm{e}^{-nx} = \mathrm{e}^j \mathrm{e}^{-(nx+j)}, \qquad j = 0, 1, \ldots, k,$$

it follows that

$$p_{n,m}^{(k)}(x) = \sum_{j=0}^{k} \binom{k}{j} \frac{\mathrm{e}^j}{(\mathrm{e}+1)^k} \mathrm{e}^{-(nx+j)} \frac{(nx+j)^m}{m!}.$$

On the other hand, conditioning on J_k and using (2.6)-(2.7), we obtain

$$\mathbb{P}(M_{n,x}^{(k)} = m) = \sum_{j=0}^{k} \binom{k}{j} \frac{\mathrm{e}^j}{(\mathrm{e}+1)^k} \mathrm{e}^{-(nx+j)} \frac{(nx+j)^m}{m!}.$$

Comparing the last two identities gives (2.8). □

Proposition 2.8. *Let $M_{n,x}^{(k)}$ be as in Lemma 2.7. Conditionally on $M_{n,x}^{(k)} = m$, let $U_{n,x}^{(k)}$ have density $g_{n,m}$. Equivalently,*

$$U_{n,x}^{(k)} \mid (M_{n,x}^{(k)} = m) \sim \mathrm{Gamma}(m+1, n), \tag{2.9}$$

where the second parameter denotes the rate. Then, for every $f \in C_b[0, \infty)$,

$$D_n^{(k)}(f; x) = \mathbb{E}[f(U_{n,x}^{(k)})]. \tag{2.10}$$

Proof. Using conditional expectation and Lemma 2.7, we get

$$\mathbb{E}[f(U_{n,x}^{(k)})] = \sum_{m=0}^{\infty} p_{n,m}^{(k)}(x) \int_0^{\infty} f(u) g_{n,m}(u)\, du.$$

By (2.5), the right-hand side is precisely $D_n^{(k)}(f; x)$, which proves (2.10). □

Corollary 2.9. *For every $n \in \mathbb{N}$, $x \geq 0$, and $m \in \mathbb{N}_0$,*

$$p_{n,m}^{(k)}(x) \geq 0, \qquad \sum_{m=0}^{\infty} p_{n,m}^{(k)}(x) = 1.$$

Moreover, $D_n^{(k)}$ is a positive linear operator on $C_b[0,\infty)$ and preserves constants:

$$D_n^{(k)}(1;x) = 1.$$

Proof. By Lemma 2.7, the weights $p_{n,m}^{(k)}(x)$ form a probability mass function. Hence they are nonnegative and sum to one. Linearity follows from the definition of $D_n^{(k)}$, while positivity and preservation of constants follow from Proposition 2.8. □

Lemma 2.10. *Let $k, n \in \mathbb{N}$ and $x \geq 0$. Define the probability generating function associated with the weights $\{p_{n,m}^{(k)}(x)\}_{m\geq 0}$ in (2.3) by*

$$F(z) := \sum_{m=0}^{\infty} p_{n,m}^{(k)}(x) z^m. \tag{2.11}$$

Then

$$F(z) = \frac{(\mathrm{e}^z+1)^k}{(\mathrm{e}+1)^k} \mathrm{e}^{nx(z-1)}, \qquad F(1) = 1. \tag{2.12}$$

Moreover, for $r = 1, 2, 3, 4$,

$$\sum_{m\geq 0} m(m-1)\cdots(m-r+1) p_{n,m}^{(k)}(x) = F^{(r)}(1). \tag{2.13}$$

For brevity, set

$$\delta = \frac{k\mathrm{e}}{\mathrm{e}+1}, \quad \alpha = \frac{k\mathrm{e}}{(\mathrm{e}+1)^2}, \quad \beta = \frac{k\mathrm{e}(1-\mathrm{e})}{(\mathrm{e}+1)^3}, \quad \lambda = \frac{k\mathrm{e}(\mathrm{e}^2-4\mathrm{e}+1)}{(\mathrm{e}+1)^4}. \tag{2.14}$$

Then

$$F'(1) = nx + \delta, \tag{2.15}$$

$$F''(1) = (nx+\delta)^2 + \alpha, \tag{2.16}$$

$$F^{(3)}(1) = (nx+\delta)^3 + 3(nx+\delta)\alpha + \beta, \tag{2.17}$$

$$F^{(4)}(1) = (nx+\delta)^4 + 6(nx+\delta)^2\alpha + 4(nx+\delta)\beta + 3\alpha^2 + \lambda. \tag{2.18}$$

Consequently, the ordinary moments of the weight system are given by

$$\sum_{m\geq 0} m\, p_{n,m}^{(k)}(x) = F'(1), \tag{2.19}$$

$$\sum_{m\geq 0} m^2 p_{n,m}^{(k)}(x) = F''(1) + F'(1), \tag{2.20}$$

$$\sum_{m\geq 0} m^3 p_{n,m}^{(k)}(x) = F^{(3)}(1) + 3F''(1) + F'(1), \tag{2.21}$$

$$\sum_{m\geq 0} m^4 p_{n,m}^{(k)}(x) = F^{(4)}(1) + 6F^{(3)}(1) + 7F''(1) + F'(1). \tag{2.22}$$

Proof. By Lemma 2.7, the weights $\{p^{(k)}_{n,m}(x)\}_{m\geq 0}$ form the probability mass function of $M^{(k)}_{n,x}$. Hence, by the definition of F in (2.11),

$$F(z) = \mathbb{E}\left(z^{M^{(k)}_{n,x}}\right).$$

Using the conditional distribution (2.7), we get

$$\mathbb{E}\left(z^{M^{(k)}_{n,x}} \mid J_k = j\right) = \exp\{(nx + j)(z-1)\}.$$

Therefore, by the law of total expectation,

$$F(z) = \mathbb{E}\left[\mathbb{E}\left(z^{M^{(k)}_{n,x}} \mid J_k\right)\right] = \mathrm{e}^{nx(z-1)}\mathbb{E}\left(\mathrm{e}^{J_k(z-1)}\right). \tag{2.23}$$

Since J_k has the binomial distribution given in (2.6), we have

$$\mathbb{E}\left(\mathrm{e}^{J_k(z-1)}\right) = \left(\frac{1}{\mathrm{e}+1} + \frac{\mathrm{e}}{\mathrm{e}+1}\mathrm{e}^{z-1}\right)^k = \left(\frac{\mathrm{e}^z+1}{\mathrm{e}+1}\right)^k. \tag{2.24}$$

Combining (2.23) and (2.24), we obtain

$$F(z) = \frac{(\mathrm{e}^z+1)^k}{(\mathrm{e}+1)^k}\mathrm{e}^{nx(z-1)},$$

which proves (2.12). In particular, $F(1) = 1$.

Since F is the probability generating function of $M^{(k)}_{n,x}$, its derivatives at $z = 1$ give the factorial moments:

$$F^{(r)}(1) = \sum_{m=0}^{\infty} m(m-1)\cdots(m-r+1)p^{(k)}_{n,m}(x), \qquad r \geq 1.$$

This proves (2.13).

To compute these derivatives, write

$$F(z) = \exp\{H(z)\},$$

where

$$H(z) = nx(z-1) + k\log(\mathrm{e}^z+1) - k\log(\mathrm{e}+1).$$

Using the constants introduced in (2.14), we get

$$H'(1) = nx + \delta, \qquad H''(1) = \alpha, \qquad H^{(3)}(1) = \beta, \qquad H^{(4)}(1) = \lambda. \tag{2.25}$$

Using the standard differentiation formulas for $F = \mathrm{e}^H$, we obtain

$$\begin{aligned}
F'(1) &= H'(1),\\
F''(1) &= H'(1)^2 + H''(1),\\
F^{(3)}(1) &= H'(1)^3 + 3H'(1)H''(1) + H^{(3)}(1),\\
F^{(4)}(1) &= H'(1)^4 + 6H'(1)^2H''(1) + 4H'(1)H^{(3)}(1) + 3H''(1)^2 + H^{(4)}(1).
\end{aligned}$$

Substituting (2.25) into these identities gives (2.15)–(2.18). Finally, the ordinary moments follow from

$$m^2 = m(m-1) + m,$$

$$m^3 = m(m-1)(m-2) + 3m(m-1) + m,$$

and

$$m^4 = m(m-1)(m-2)(m-3) + 6m(m-1)(m-2) + 7m(m-1) + m.$$

Applying (2.13) gives (2.19)–(2.22). The proof is complete. □

Remark 2.11. *The weight sequence* $\{p_{n,m}^{(k)}(x)\}_{m\geq 0}$ *is uniquely determined by the probability generating function* F *in* (2.12)*, since a probability generating function uniquely determines its probability mass function on* $\mathbb{N}_0$*. Hence the discrete kernel is unique as a probability mass function.*

Moreover, within the finite Poisson-mixture form with fixed parameters $nx, nx+1, \ldots, nx+k$*, the coefficients* a_j *in*

$$F(z) = e^{nx(z-1)} \sum_{j=0}^{k} a_j e^{j(z-1)}$$

are uniquely determined by the linear independence of $\{e^{j(z-1)}\}_{j=0}^{k}$*. For the present kernel, the Euler expansion gives*

$$a_j = \binom{k}{j} \frac{e^j}{(e+1)^k}, \qquad j = 0, 1, \ldots, k.$$

Thus, the representation in Lemma 2.7 is canonical within this finite Poisson-mixture structure. No uniqueness is asserted for arbitrary abstract mixture representations of the same probability mass function.

Lemma 2.12. *Let* $k \in \mathbb{N}$*, and let* $D_n^{(k)}$ *be the Szász–Durrmeyer-type operator defined in Definition 2.4. Let* $\delta, \alpha, \beta, \lambda$ *be defined by* (2.14)*. Then, for all* $x \geq 0$ *and* $n \in \mathbb{N}$*, the following identities hold:*

$$D_n^{(k)}(1; x) = 1, \tag{2.26}$$

$$D_n^{(k)}(u; x) = x + \frac{1+\delta}{n}, \tag{2.27}$$

$$D_n^{(k)}(u^2; x) = x^2 + \frac{2\delta+4}{n}x + \frac{\delta^2 + 4\delta + 2 + \alpha}{n^2}, \tag{2.28}$$

$$\begin{aligned} D_n^{(k)}(u^3; x) = x^3 &+ \frac{3(\delta+3)}{n}x^2 + \frac{3(\delta^2 + 6\delta + 6 + \alpha)}{n^2}x \\ &+ \frac{\delta^3 + 9\delta^2 + 18\delta + 6 + 3(\delta+3)\alpha + \beta}{n^3}, \end{aligned} \tag{2.29}$$

$$\begin{aligned} D_n^{(k)}(u^4; x) = x^4 &+ \frac{4(\delta+4)}{n}x^3 + \frac{6\delta^2 + 48\delta + 72 + 6\alpha}{n^2}x^2 \\ &+ \frac{4\delta^3 + 48\delta^2 + 144\delta + 96 + 12\delta\alpha + 48\alpha + 4\beta}{n^3}x \\ &+ \frac{\delta^4 + 16\delta^3 + 72\delta^2 + 96\delta + 24 + 6\delta^2\alpha + 48\delta\alpha + 72\alpha + 4\delta\beta + 16\beta + 3\alpha^2 + \lambda}{n^4}. \end{aligned} \tag{2.30}$$

Consequently, for each fixed $x \geq 0$, as $n \to \infty$,

$$D_n^{(k)}(u^3; x) = x^3 + \frac{3(\delta + 3)}{n} x^2 + O\left(\frac{1}{n^2}\right) \tag{2.31}$$

and

$$D_n^{(k)}(u^4; x) = x^4 + \frac{4(\delta + 4)}{n} x^3 + O\left(\frac{1}{n^2}\right). \tag{2.32}$$

The above asymptotic relations are uniform for x in every compact subinterval of $[0, \infty)$.

Proof. By Proposition 2.8, for $r \in \mathbb{N}_0$,

$$D_n^{(k)}(u^r; x) = \mathbb{E}[(U_{n,x}^{(k)})^r].$$

Moreover, by (2.9),

$$U_{n,x}^{(k)} \mid (M_{n,x}^{(k)} = m) \sim \text{Gamma}(m + 1, n),$$

where the second parameter denotes the rate. Hence, for $r \in \mathbb{N}$,

$$\mathbb{E}[(U_{n,x}^{(k)})^r \mid M_{n,x}^{(k)} = m] = \frac{(m + 1)(m + 2) \cdots (m + r)}{n^r}.$$

Therefore,

$$D_n^{(k)}(u^r; x) = \frac{1}{n^r} \mathbb{E}\Big[(M_{n,x}^{(k)} + 1)(M_{n,x}^{(k)} + 2) \cdots (M_{n,x}^{(k)} + r)\Big], \qquad r \in \mathbb{N}. \tag{2.33}$$

From Lemma 2.10, the factorial moments of $M_{n,x}^{(k)}$ are given by (2.15)–(2.18). In particular, the ordinary moments are

$$\begin{aligned}
\mathbb{E}[M_{n,x}^{(k)}] &= F'(1), \\
\mathbb{E}[(M_{n,x}^{(k)})^2] &= F''(1) + F'(1), \\
\mathbb{E}[(M_{n,x}^{(k)})^3] &= F^{(3)}(1) + 3F''(1) + F'(1),
\end{aligned}$$

and

$$\mathbb{E}[(M_{n,x}^{(k)})^4] = F^{(4)}(1) + 6F^{(3)}(1) + 7F''(1) + F'(1).$$

For $r = 0$, we immediately obtain

$$D_n^{(k)}(1; x) = 1,$$

which proves (2.26). For $r = 1$, using (2.33) and (2.15), we get

$$D_n^{(k)}(u; x) = \frac{1}{n} \mathbb{E}[M_{n,x}^{(k)} + 1] = \frac{nx + \delta + 1}{n} = x + \frac{1 + \delta}{n}.$$

This proves (2.27). Using the identities

$$\begin{aligned}
(M + 1)(M + 2) &= M^2 + 3M + 2, \\
(M + 1)(M + 2)(M + 3) &= M^3 + 6M^2 + 11M + 6,
\end{aligned}$$

and

$$(M + 1)(M + 2)(M + 3)(M + 4) = M^4 + 10M^3 + 35M^2 + 50M + 24,$$

together with (2.15)–(2.18), one obtains (2.28)–(2.30) by direct calculation. The asymptotic relations (2.31) and (2.32) follow directly from (2.29) and (2.30). Since the remaining coefficients are polynomials in x with constants depending only on k, the remainders are uniform for x in every compact subinterval of $[0,\infty)$. The proof is complete. □

Remark 2.13. *The operator $D_n^{(k)}$ is well-defined on the admissible space E, which is introduced in Definition 2.4. Indeed, for $f \in E$, there exists a constant $M_f > 0$ such that*

$$|f(u)| \le M_f(1+u^2), \qquad u \ge 0.$$

Therefore, by the positivity of $p_{n,m}^{(k)}(x)$ and $g_{n,m}(u)$, we have

$$\begin{aligned}\sum_{m=0}^{\infty} p_{n,m}^{(k)}(x) \int_0^{\infty} g_{n,m}(u)|f(u)|\,du &\le M_f \sum_{m=0}^{\infty} p_{n,m}^{(k)}(x) \int_0^{\infty} g_{n,m}(u)(1+u^2)\,du \\ &= M_f\Big[D_n^{(k)}(1;x) + D_n^{(k)}(u^2;x)\Big] < \infty.\end{aligned}$$

Hence, the defining series in (2.5) *is absolutely convergent. The finiteness follows from the moment identities* (2.26) *and* (2.28).

Lemma 2.14. *Let $k \in \mathbb{N}$, and let $D_n^{(k)}$ be the operator defined in Definition 2.4. Let $\delta, \alpha, \beta, \lambda$ be defined by* (2.14). *Then, for all $x \ge 0$ and $n \in \mathbb{N}$,*

$$D_n^{(k)}(u-x;x) = \frac{1+\delta}{n}. \tag{2.34}$$

Moreover,

$$D_n^{(k)}((u-x)^2;x) = \frac{2x}{n} + \frac{\delta^2+4\delta+2+\alpha}{n^2}. \tag{2.35}$$

Furthermore,

$$D_n^{(k)}((u-x)^4;x) = \frac{12x^2}{n^2} + \frac{12(\delta^2+6\delta+6+\alpha)x}{n^3} + \frac{\Theta}{n^4}, \tag{2.36}$$

where

$$\Theta = \delta^4 + 16\delta^3 + 72\delta^2 + 96\delta + 24 + 6\delta^2\alpha + 48\delta\alpha + 72\alpha + 4\delta\beta + 16\beta + 3\alpha^2 + \lambda. \tag{2.37}$$

Consequently, for every compact subset $K \subset [0,\infty)$, as $n \to \infty$, uniformly for $x \in K$,

$$D_n^{(k)}((u-x)^2;x) = \frac{2x}{n} + O\left(\frac{1}{n^2}\right) \tag{2.38}$$

and

$$D_n^{(k)}((u-x)^4;x) = \frac{12x^2}{n^2} + O\left(\frac{1}{n^3}\right). \tag{2.39}$$

In particular,

$$\lim_{n\to\infty} n\,D_n^{(k)}((u-x)^2;x) = 2x \tag{2.40}$$

and

$$\lim_{n\to\infty} n^2\,D_n^{(k)}((u-x)^4;x) = 12x^2, \tag{2.41}$$

uniformly on every compact subset of $[0,\infty)$.

Proof. The first identity follows directly from (2.27):

$$D_n^{(k)}(u-x;x) = D_n^{(k)}(u;x) - x = \frac{1+\delta}{n}.$$

This proves (2.34).

Next, using (2.27) and (2.28), we obtain

$$\begin{aligned} D_n^{(k)}((u-x)^2;x) &= D_n^{(k)}(u^2;x) - 2x\,D_n^{(k)}(u;x) + x^2 \\ &= \left[x^2 + \frac{2\delta+4}{n}x + \frac{\delta^2+4\delta+2+\alpha}{n^2}\right] - 2x\left[x + \frac{1+\delta}{n}\right] + x^2 \\ &= \frac{2x}{n} + \frac{\delta^2+4\delta+2+\alpha}{n^2}. \end{aligned}$$

This proves (2.35). For the fourth central moment, we use the identity

$$(u-x)^4 = u^4 - 4xu^3 + 6x^2u^2 - 4x^3u + x^4. \tag{2.42}$$

Applying $D_n^{(k)}$ to (2.42) and using (2.27)–(2.30), we get

$$D_n^{(k)}((u-x)^4;x) = D_n^{(k)}(u^4;x) - 4x\,D_n^{(k)}(u^3;x) + 6x^2D_n^{(k)}(u^2;x) - 4x^3D_n^{(k)}(u;x) + x^4.$$

Substituting (2.27)–(2.30) into this identity and collecting powers of n^{-1}, the terms of order 1 and n^{-1} cancel, and we obtain

$$D_n^{(k)}((u-x)^4;x) = \frac{12x^2}{n^2} + \frac{12(\delta^2+6\delta+6+\alpha)x}{n^3} + \frac{\Theta}{n^4},$$

where Θ is given by (2.37). This proves (2.36).

Since $\delta, \alpha, \beta, \lambda$ depend only on k, and the remaining coefficients in (2.35) and (2.36) are polynomials in x, the above asymptotic relations are uniform for x in every compact subset of $[0,\infty)$. Hence (2.38) and (2.39) follow. Multiplying these relations by n and n^2, respectively, gives (2.40) and (2.41). The proof is complete. □

Remark 2.15. *The leading term* $12x^2/n^2$ *in* (2.36) *can also be interpreted through the cumulants associated with the Poisson–Gamma representation. Let*

$$M = M_{n,x}^{(k)}, \qquad U = U_{n,x}^{(k)}.$$

Denote by K_M *and* K_U *the cumulant generating functions of* M *and* U*, respectively, and write* $\kappa_r(M)$ *and* $\kappa_r(U)$ *for the corresponding cumulants. Since* $U \mid M = m \sim \Gamma(m+1, n)$*, where* n *is the rate parameter, one has, for* $t < n$*,*

$$K_U(t) = s(t) + K_M(s(t)), \qquad s(t) = \log\frac{n}{n-t}.$$

By (2.15) *and* (2.16)*,*

$$\mathbb{E}[M] = nx + \delta, \qquad \kappa_2(M) = F''(1) + F'(1) - \left(F'(1)\right)^2 = nx + \delta + \alpha.$$

Hence
$$\kappa_2(U) = \frac{\mathbb{E}[M+1] + \kappa_2(M)}{n^2} = \frac{2x}{n} + O(n^{-2}),$$
locally uniformly for x in compact subsets of $[0,\infty)$*. The same cumulant composition gives*
$$\kappa_3(U) = O(n^{-2}), \qquad \kappa_4(U) = O(n^{-3}).$$
Moreover, by (2.27)*,*
$$\mathbb{E}[U] = x + \frac{1+\delta}{n}.$$
Therefore, shifting the center from $\mathbb{E}[U]$ *to x changes the fourth central moment only by* $O(n^{-3})$*, and hence*
$$\mathbb{E}[(U-x)^4] = 3\kappa_2(U)^2 + O(n^{-3}) = \frac{12x^2}{n^2} + O(n^{-3}).$$
Thus the cumulant argument explains the leading term in (2.36)*, whereas the exact Euler-dependent lower-order terms are obtained by the direct computation in Lemma 2.14.*

3. Quantitative estimates

In this section, we obtain pointwise and compact estimates in terms of the ordinary modulus of continuity. Let $C_b[0,\infty)$ denote the space of all bounded continuous functions on $[0,\infty)$, and let $UC_b[0,\infty)$ denote its subspace consisting of bounded uniformly continuous functions.

For $f \in C_b[0,\infty)$ and $\eta > 0$, we define
$$\omega(f;\eta) := \sup_{\substack{u,x\geq 0\\ |u-x|\leq\eta}} |f(u) - f(x)|.$$
Since f is bounded, the quantity $\omega(f;\eta)$ is finite for every $\eta > 0$. However, the condition
$$\lim_{\eta\to 0^+} \omega(f;\eta) = 0$$
is not automatic for arbitrary $f \in C_b[0,\infty)$; it holds, in particular, for $f \in UC_b[0,\infty)$. Therefore, the estimates below are valid for $C_b[0,\infty)$, while the convergence conclusions obtained from them are stated under the additional assumption of uniform continuity.

For $n, k \in \mathbb{N}$ and $x \geq 0$, set
$$\Delta_{n,k}(x) := D_n^{(k)}((u-x)^2; x).$$
By (2.35), we have
$$\Delta_{n,k}(x) = \frac{2x}{n} + \frac{\delta^2 + 4\delta + 2 + \alpha}{n^2}, \tag{3.1}$$
where δ and α are given by (2.14).

Theorem 3.1. *Let* $k \in \mathbb{N}$ *and let* $D_n^{(k)}$ *be the operator defined in Definition 2.4. Then, for every* $f \in C_b[0,\infty)$*,* $x \geq 0$*, and* $n \in \mathbb{N}$*,*
$$\left|D_n^{(k)}(f;x) - f(x)\right| \leq 2\,\omega\left(f;\, \sqrt{\Delta_{n,k}(x)}\right). \tag{3.2}$$

Equivalently,

$$\left|D_n^{(k)}(f;x) - f(x)\right| \le 2\,\omega\left(f;\ \sqrt{\frac{2x}{n} + \frac{\delta^2 + 4\delta + 2 + \alpha}{n^2}}\right).$$

Moreover, if $f \in \mathrm{Lip}_M(\gamma)$*, that is,*

$$|f(u) - f(x)| \le M|u - x|^\gamma, \qquad u, x \ge 0,$$

for some $M > 0$ *and* $0 < \gamma \le 1$*, then*

$$\left|D_n^{(k)}(f;x) - f(x)\right| \le M\left(\Delta_{n,k}(x)\right)^{\gamma/2}. \tag{3.3}$$

Equivalently,

$$\left|D_n^{(k)}(f;x) - f(x)\right| \le M\left(\frac{2x}{n} + \frac{\delta^2 + 4\delta + 2 + \alpha}{n^2}\right)^{\gamma/2}.$$

Consequently, for every $A > 0$ *and every* $f \in C_b[0,\infty)$*,*

$$\sup_{0\le x\le A}\left|D_n^{(k)}(f;x) - f(x)\right| \le 2\,\omega\left(f;\ \sqrt{\frac{2A}{n} + \frac{\delta^2 + 4\delta + 2 + \alpha}{n^2}}\right). \tag{3.4}$$

In particular, if $f \in UC_b[0,\infty)$*, then*

$$\lim_{n\to\infty}\sup_{0\le x\le A}\left|D_n^{(k)}(f;x) - f(x)\right| = 0 \tag{3.5}$$

for every $A > 0$*.*

Proof. Let $f \in C_b[0,\infty)$, $x \ge 0$, and $\eta > 0$. For every $u \ge 0$, the standard property of the modulus of continuity gives

$$|f(u) - f(x)| \le \left(1 + \frac{|u - x|}{\eta}\right)\omega(f;\eta). \tag{3.6}$$

Indeed, if $u = x$, the assertion is immediate. If $u \ne x$, divide the interval between x and u into finitely many subintervals of length not exceeding η and use the definition of $\omega(f;\eta)$.

Since $D_n^{(k)}$ is positive and preserves constants by Corollary 2.9, we have

$$\begin{aligned}\left|D_n^{(k)}(f;x) - f(x)\right| &= \left|D_n^{(k)}(f(u) - f(x);x)\right| \\ &\le D_n^{(k)}(|f(u) - f(x)|;x).\end{aligned}$$

Using (3.6), we obtain

$$\left|D_n^{(k)}(f;x) - f(x)\right| \le \omega(f;\eta)\left[1 + \frac{1}{\eta}D_n^{(k)}(|u - x|;x)\right].$$

By the Cauchy–Schwarz inequality for the positive normalized functional $h \mapsto D_n^{(k)}(h;x)$, we have

$$D_n^{(k)}(|u - x|;x) \le \left(D_n^{(k)}((u - x)^2;x)\right)^{1/2} = \sqrt{\Delta_{n,k}(x)}.$$

Therefore,

$$\left|D_n^{(k)}(f;x) - f(x)\right| \le \omega(f;\eta)\left(1 + \frac{\sqrt{\Delta_{n,k}(x)}}{\eta}\right).$$

Taking $\eta = \sqrt{\Delta_{n,k}(x)}$ gives

$$\left|D_n^{(k)}(f;x) - f(x)\right| \le 2\,\omega\left(f;\ \sqrt{\Delta_{n,k}(x)}\right),$$

which proves (3.2). The displayed explicit form follows immediately from (3.1).

Now suppose that $f \in \mathrm{Lip}_M(\gamma)$, where $0 < \gamma \le 1$. By positivity and the Lipschitz condition,

$$\begin{aligned}\left|D_n^{(k)}(f;x) - f(x)\right| &\le D_n^{(k)}(|f(u) - f(x)|;x)\\ &\le MD_n^{(k)}(|u-x|^\gamma;x).\end{aligned}$$

Using Hölder's inequality for the positive normalized functional $h \mapsto D_n^{(k)}(h;x)$, we get

$$D_n^{(k)}(|u-x|^\gamma;x) \le \left(D_n^{(k)}((u-x)^2;x)\right)^{\gamma/2}.$$

Thus

$$\left|D_n^{(k)}(f;x) - f(x)\right| \le M\,(\Delta_{n,k}(x))^{\gamma/2},$$

which proves (3.3). The displayed explicit estimate follows again from (3.1).

Finally, if $0 \le x \le A$, then by (3.1),

$$\Delta_{n,k}(x) \le \frac{2A}{n} + \frac{\delta^2 + 4\delta + 2 + \alpha}{n^2}.$$

Since the modulus of continuity is nondecreasing with respect to its second argument, (3.4) follows from the displayed explicit form of the modulus estimate.

If $f \in UC_b[0,\infty)$, then

$$\omega(f;\eta) \to 0 \qquad (\eta \to 0^+).$$

Since

$$\sqrt{\frac{2A}{n} + \frac{\delta^2 + 4\delta + 2 + \alpha}{n^2}} \to 0 \qquad (n \to \infty),$$

the right-hand side of (3.4) tends to zero. This proves (3.5). The proof is complete. □

Remark 3.2. *The expectation representation also gives a simple probabilistic counterpart to the modulus estimate. Let $U_{n,x}^{(k)}$ be the random variable in Proposition 2.8. By Markov's inequality and Lemma 2.14, for every $\eta > 0$,*

$$\mathbb{P}(|U_{n,x}^{(k)} - x| \ge \eta) \le \frac{D_n^{(k)}((u-x)^4;x)}{\eta^4}.$$

Consequently, for $f \in C_b[0,\infty)$,

$$|D_n^{(k)}(f;x) - f(x)| \le \omega(f;\eta) + \frac{2\|f\|_\infty}{\eta^4} D_n^{(k)}((u-x)^4;x).$$

This estimate is included as a probabilistic complement to Theorem 3.1, showing how the fourth central moment controls the tail of $U_{n,x}^{(k)}$. It is not claimed to improve the modulus estimate. Sharper concentration or coupling bounds lie beyond the scope of the present paper.

4. Global approximation in a weighted space

We consider the standard weight

$$\rho(x) := 1 + x^2, \qquad x \geq 0. \tag{4.1}$$

Let

$$C_\rho[0,\infty) := \left\{ f \in C[0,\infty) : \|f\|_\rho := \sup_{x\geq 0} \frac{|f(x)|}{\rho(x)} < \infty \right\}, \tag{4.2}$$

and define the weighted Korovkin-type subspace

$$C_\rho^0[0,\infty) := \left\{ f \in C_\rho[0,\infty) : \lim_{x\to\infty} \frac{f(x)}{\rho(x)} = 0 \right\}.$$

Lemma 4.1. *There exists a constant $M > 0$, depending only on k, such that*

$$D_n^{(k)}(\rho; x) \leq M\rho(x), \qquad x \geq 0, \quad n \in \mathbb{N}. \tag{4.3}$$

Consequently, the same series-integral formula defines $D_n^{(k)} f$ for every $f \in C_\rho[0,\infty)$, and

$$\|D_n^{(k)} f\|_\rho \leq M\|f\|_\rho, \qquad f \in C_\rho[0,\infty). \tag{4.4}$$

Proof. For the polynomial test functions appearing below, the action of $D_n^{(k)}$ is understood through the defining series-integral formula, as justified by the moment identities in Lemma 2.12. Using linearity and the definition of ρ in (4.1), we have

$$D_n^{(k)}(\rho; x) = D_n^{(k)}(1; x) + D_n^{(k)}(u^2; x).$$

By (2.26) and (2.28),

$$D_n^{(k)}(\rho; x) = 1 + x^2 + \frac{2\delta + 4}{n} x + \frac{\delta^2 + 4\delta + 2 + \alpha}{n^2}.$$

Therefore,

$$\frac{D_n^{(k)}(\rho; x)}{\rho(x)} = 1 + \frac{2\delta + 4}{n} \frac{x}{1 + x^2} + \frac{\delta^2 + 4\delta + 2 + \alpha}{n^2} \frac{1}{1 + x^2}.$$

Since

$$\sup_{x\geq 0} \frac{x}{1 + x^2} = \frac{1}{2}, \qquad \sup_{x\geq 0} \frac{1}{1 + x^2} = 1,$$

and $n \geq 1$, we obtain

$$\sup_{x\geq 0} \frac{D_n^{(k)}(\rho; x)}{\rho(x)} \leq 1 + \frac{2\delta + 4}{2} + \delta^2 + 4\delta + 2 + \alpha =: M.$$

Thus

$$D_n^{(k)}(\rho; x) \leq M\rho(x), \qquad x \geq 0, \quad n \in \mathbb{N},$$

which proves (4.3).

Now let $f \in C_\rho[0,\infty)$. By (4.2),

$$|f(u)| \leq \|f\|_\rho \rho(u), \qquad u \geq 0.$$

Hence the defining series-integral expression for $D_n^{(k)}(f;x)$ is absolutely controlled by $D_n^{(k)}(\rho;x)$. Using the positivity established in Corollary 2.9, we get

$$|D_n^{(k)}(f;x)| \leq D_n^{(k)}(|f|;x) \leq \|f\|_\rho D_n^{(k)}(\rho;x) \leq M\|f\|_\rho \rho(x).$$

Taking the supremum over $x \geq 0$, we obtain

$$\|D_n^{(k)} f\|_\rho \leq M\|f\|_\rho,$$

which proves (4.4). The proof is complete. □

Theorem 4.2. *For every $f \in C_\rho^0[0,\infty)$,*

$$\lim_{n\to\infty} \|D_n^{(k)} f - f\|_\rho = 0. \tag{4.5}$$

Proof. By Lemma 4.1, the operators $D_n^{(k)}$ are well-defined on $C_\rho[0,\infty)$ and satisfy

$$\|D_n^{(k)} f\|_\rho \leq M\|f\|_\rho, \qquad f \in C_\rho[0,\infty),$$

as stated in (4.4), where $M > 0$ is independent of n. Hence the weighted Korovkin theorem for the weight ρ defined in (4.1) applies once the convergence is verified for the test functions $1, u, u^2$.

From (2.26), we have

$$\|D_n^{(k)}(1) - 1\|_\rho = 0.$$

By (2.27),

$$D_n^{(k)}(u;x) - x = \frac{1+\delta}{n}.$$

Therefore,

$$\|D_n^{(k)}(u) - u\|_\rho = \sup_{x\geq 0} \frac{1+\delta}{n(1+x^2)} \leq \frac{1+\delta}{n} \to 0.$$

For the second test function, (2.28) gives

$$D_n^{(k)}(u^2;x) - x^2 = \frac{2\delta+4}{n}x + \frac{\delta^2+4\delta+2+\alpha}{n^2}.$$

Thus

$$\|D_n^{(k)}(u^2) - u^2\|_\rho \leq \frac{2\delta+4}{n} \sup_{x\geq 0} \frac{x}{1+x^2} + \frac{\delta^2+4\delta+2+\alpha}{n^2} \sup_{x\geq 0} \frac{1}{1+x^2}.$$

Since

$$\sup_{x\geq 0} \frac{x}{1+x^2} = \frac{1}{2}, \qquad \sup_{x\geq 0} \frac{1}{1+x^2} = 1,$$

we obtain

$$\|D_n^{(k)}(u^2) - u^2\|_\rho \leq \frac{2\delta+4}{2n} + \frac{\delta^2+4\delta+2+\alpha}{n^2} \to 0.$$

Thus

$$\lim_{n\to\infty} \|D_n^{(k)}(u^i) - u^i\|_\rho = 0, \qquad i = 0, 1, 2.$$

By the weighted Korovkin theorem, it follows that

$$\lim_{n\to\infty} \|D_n^{(k)} f - f\|_\rho = 0$$

for every $f \in C_\rho^0[0,\infty)$. This proves (4.5). The proof is complete. □

5. Asymptotic analysis

We now establish a Voronovskaya-type theorem describing the first-order asymptotic behavior of the operators. We denote by $C_b^2[0,\infty)$ the class of all functions $f \in C_b[0,\infty)$ such that f' and f'' exist and belong to $C_b[0,\infty)$.

Theorem 5.1. *Let $k \in \mathbb{N}$ and let $f \in C_b^2[0,\infty)$. Then, for every $x \geq 0$,*

$$\lim_{n\to\infty} n(D_n^{(k)}(f;x) - f(x)) = (1+\delta)f'(x) + xf''(x), \tag{5.1}$$

where $\delta = k\mathrm{e}/(\mathrm{e}+1)$ is defined in (2.14). *Moreover, the convergence is uniform on every compact subset of* $[0,\infty)$.

Proof. Let $K \subset [0,\infty)$ be compact. Whenever $D_n^{(k)}$ is applied below to functions not necessarily in $C_b[0,\infty)$, the expression is understood through the defining series-integral formula. The required finiteness follows from Lemma 2.14. For $x \in K$ and $u \geq 0$, Taylor's formula gives

$$f(u) = f(x) + f'(x)(u-x) + \frac{1}{2}f''(x)(u-x)^2 + R_x(u)(u-x)^2, \tag{5.2}$$

where

$$R_x(u) = \int_0^1 (1-s)[f''(x+s(u-x)) - f''(x)]\,ds.$$

Since $f'' \in C_b[0,\infty)$, we have

$$|R_x(u)| \leq \|f''\|_\infty, \qquad u \geq 0, \quad x \in K.$$

Applying $D_n^{(k)}(\cdot;x)$ to (5.2), we obtain

$$\begin{aligned} D_n^{(k)}(f;x) - f(x) = f'(x)D_n^{(k)}(u-x;x) + \frac{1}{2}f''(x)D_n^{(k)}((u-x)^2;x) \\ + D_n^{(k)}(R_x(u)(u-x)^2;x). \end{aligned} \tag{5.3}$$

By (2.34) and (2.40),

$$n\,D_n^{(k)}(u-x;x) = 1+\delta$$

and

$$n\,D_n^{(k)}((u-x)^2;x) \to 2x$$

uniformly for $x \in K$. Hence the first two terms in (5.3), after multiplication by n, converge uniformly on K to

$$(1+\delta)f'(x) + xf''(x).$$

It remains to show that

$$\sup_{x\in K} n\left|D_n^{(k)}(R_x(u)(u-x)^2;x)\right| \to 0. \tag{5.4}$$

Let $\varepsilon > 0$. Choose $A > 0$ such that $K \subset [0, A]$. Since f'' is continuous on $[0,\infty)$, it is uniformly continuous on the compact interval $[0, A+1]$. Hence there exists $\eta \in (0,1)$ such that

$$|f''(y) - f''(z)| < 2\varepsilon \qquad \text{whenever } y, z \in [0, A+1],\ |y-z| < \eta.$$

If $x \in K$ and $|u-x| < \eta$, then $x + s(u-x) \in [0, A+1]$ for every $s \in [0,1]$. Therefore,

$$\begin{aligned} |R_x(u)| &\le \int_0^1 (1-s)\,|f''(x+s(u-x)) - f''(x)|\, ds \\ &\le 2\varepsilon \int_0^1 (1-s)\, ds = \varepsilon. \end{aligned}$$

Thus,

$$\sup_{\substack{x\in K\\ |u-x|<\eta}} |R_x(u)| \le \varepsilon.$$

Using positivity, we get

$$n\left|D_n^{(k)}(R_x(u)(u-x)^2;x)\right| \le \varepsilon n D_n^{(k)}((u-x)^2;x) + nD_n^{(k)}(|R_x(u)|(u-x)^2\mathbf{1}_{\{|u-x|\ge\eta\}};x).$$

On $\{|u-x| \ge \eta\}$, we have

$$(u-x)^2 \le \eta^{-2}(u-x)^4.$$

Therefore,

$$nD_n^{(k)}(|R_x(u)|(u-x)^2\mathbf{1}_{\{|u-x|\ge\eta\}};x) \le \frac{\|f''\|_\infty}{\eta^2} nD_n^{(k)}((u-x)^4;x).$$

Taking the supremum over $x \in K$, and using (2.39) and (2.40), we get

$$\limsup_{n\to\infty} \sup_{x\in K} n\left|D_n^{(k)}(R_x(u)(u-x)^2;x)\right| \le 2\varepsilon \sup_{x\in K} x.$$

Since $\varepsilon > 0$ is arbitrary, (5.4) follows. Combining this with (5.3) proves (5.1) uniformly on K. The proof is complete. □

Remark 5.2. *By Proposition 2.8, $D_n^{(k)}(f;x) = \mathbb{E}[f(U_{n,x}^{(k)})]$, where the conditional Gamma law is understood with rate parameter n. Using the laws of total expectation and total variance, one obtains*

$$\mathbb{E}(U_{n,x}^{(k)}) - x = \frac{1+\delta}{n}$$

and

$$D_n^{(k)}((u-x)^2;x) = \mathbb{E}[(U_{n,x}^{(k)} - x)^2] = \frac{2x}{n} + \frac{\delta^2 + 4\delta + 2 + \alpha}{n^2}.$$

Consequently,

$$nD_n^{(k)}(u-x;x) \to 1+\delta, \qquad nD_n^{(k)}((u-x)^2;x) \to 2x$$

locally uniformly in x. Thus the Euler parameter modifies the drift coefficient, while the leading second-central-moment coefficient remains $2x$*.*

We also briefly compare the drift contribution obtained in the present construction generated by Euler polynomials of negative order with the corresponding one arising from the construction generated by Bernoulli polynomials of negative order. For the corresponding generating factors

$$A_E(t) = \left(\frac{e^t+1}{2}\right)^k, \qquad A_B(t) = \left(\frac{e^t-1}{t}\right)^k,$$

the generating-factor contribution to the first-order drift is determined by the logarithmic derivative $A'(1)/A(1)$. Hence

$$\delta_E = \frac{A'_E(1)}{A_E(1)} = \frac{ke}{e+1}, \qquad \delta_B = \frac{A'_B(1)}{A_B(1)} = \frac{k}{e-1}.$$

Here, for the Bernoulli factor,

$$\frac{A'_B(t)}{A_B(t)} = k\left(\frac{e^t}{e^t-1} - \frac{1}{t}\right).$$

Thus, for fixed $k > 0$, one has $\delta_E > \delta_B$. The two constructions differ only in the magnitude of the drift coefficient, not in the approximation order. Since these drift contributions enter the corresponding asymptotic formulas at order $1/n$, this comparison does not indicate a difference in the order of convergence. Hence the distinction between the two constructions is confined, at this asymptotic level, to the size of the drift contribution.

6. Numerical experiments

In this section, we present numerical experiments illustrating the approximation behavior of the operators introduced above. All computations were performed in double-precision arithmetic. The operator $D_n^{(k)}(f;x)$ was evaluated through the Poisson–Gamma representation in Proposition 2.8. For each grid point x, the discrete sum was truncated at

$$m_{\max} = \left\lfloor \lambda_{\text{mean}} + 10\sqrt{\lambda_{\text{mean}} + k} + 40 \right\rfloor, \qquad \lambda_{\text{mean}} = nx + k\frac{\mathrm{e}}{\mathrm{e}+1}.$$

For the Gamma part, the integral with respect to $g_{n,m}$ was approximated by the composite trapezoidal rule on the interval

$$[\max\{0, \mu - 7\sigma\}, \mu + 9\sigma], \qquad \mu = \frac{m+1}{n}, \qquad \sigma = \frac{\sqrt{m+1}}{n}.$$

The lower endpoint is necessary because $g_{n,m}$ is supported on $[0,\infty)$.

6.1. Test functions

We use four test functions representing complementary behaviors: a smooth oscillatory function, a bounded rational function, a nonsmooth Hölder-type function, and a smooth decaying rational function:

$$f_0(x) = e^{-x}\sin(\pi x), \qquad f_1(x) = \frac{x}{1+x^2}, \qquad f_2(x) = |x-1|^{0.7}, \qquad f_3(x) = \frac{1}{1+x^2}.$$

Here $f_0, f_1, f_3 \in C^0_\rho[0,\infty)$ and $f_0, f_3 \in C^2_b[0,\infty)$. The function f_2 is not bounded on $[0,\infty)$; nevertheless,

$$\frac{|x-1|^{0.7}}{1+x^2} \to 0 \qquad (x \to \infty),$$

so that $f_2 \in C^0_\rho[0,\infty)$ for $\rho(x) = 1 + x^2$. Moreover, f_2 is Hölder continuous of order 0.7. Thus all four test functions are admissible in the weighted setting, while f_2 additionally illustrates the behavior of the operators on a nonsmooth function.

6.2. Approximation and pointwise error

Figures 1 and 2 show, respectively, $D_n^{(k)}(f_i; x)$ and the pointwise error $|D_n^{(k)}(f_i; x) - f_i(x)|$ for $n = 50$, $k = 2$, and $x \in [0,4]$. The compact estimate in (3.4) explains the expected decrease of the error as n increases for bounded uniformly continuous functions; for the Lipschitz-type function f_2, the relevant estimate is obtained from the same positivity–Hölder argument as above.

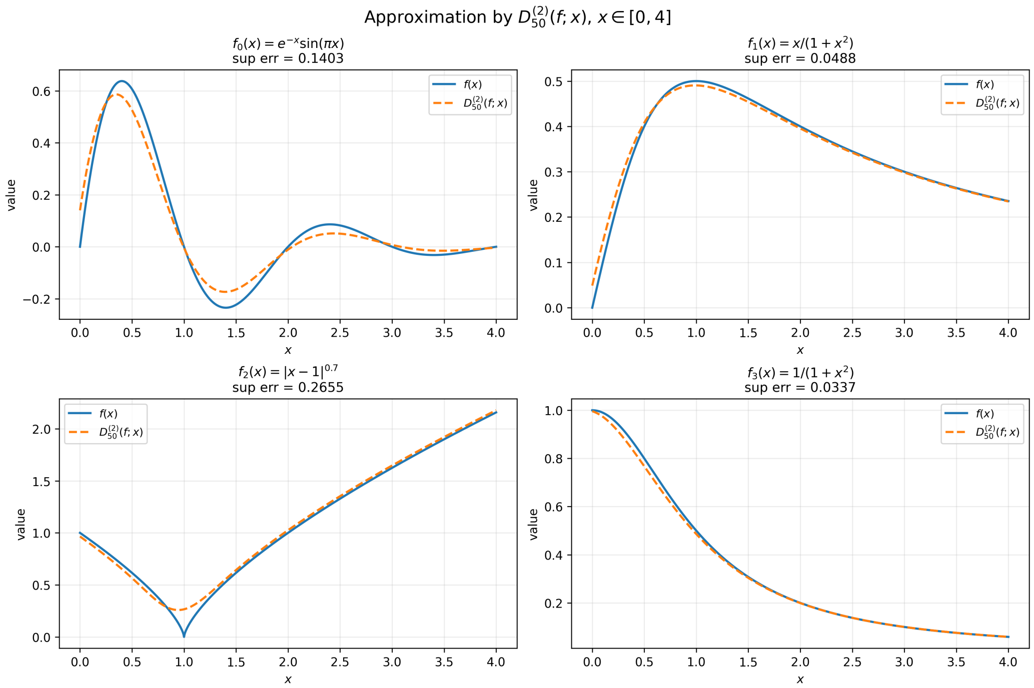

Figure 1. Approximation by $D_n^{(k)}(f; x)$ for the four test functions, with $n = 50$ and $k = 2$. Solid line: target function $f(x)$; dashed line: $D_n^{(k)}(f; x)$. The supremum error on $[0,4]$ is recorded in each panel.

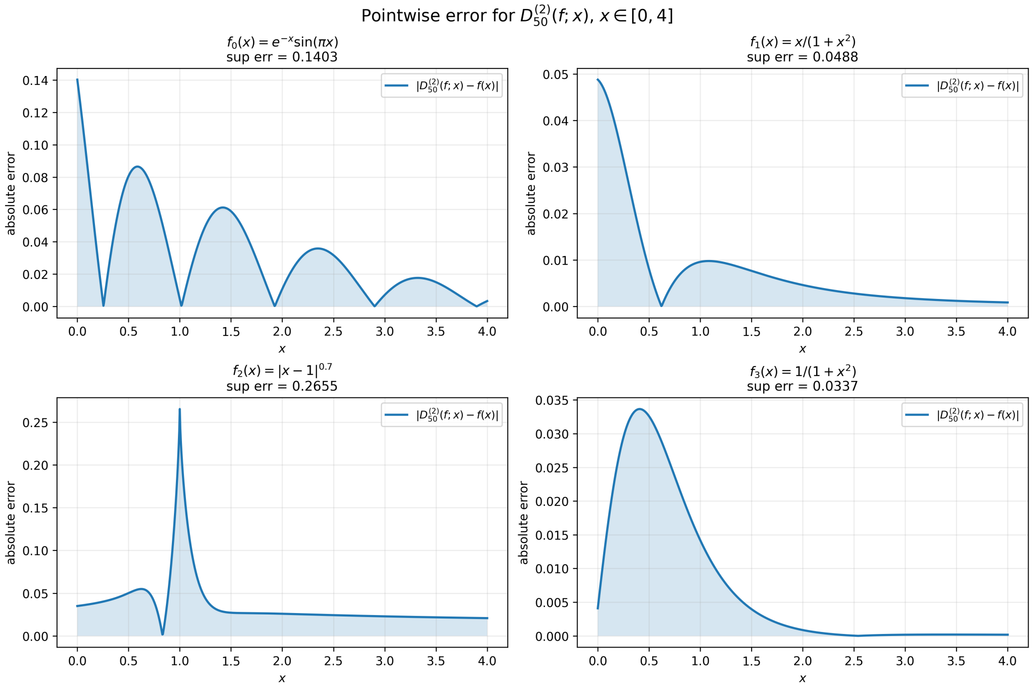

Figure 2. Pointwise error $|D_n^{(k)}(f;x) - f(x)|$ for $n = 50$ and $k = 2$. The shaded region indicates the area under the error curve on $[0, 4]$.

6.3. *Convergence rate and Euler drift*

Figure 3 displays the supremum error

$$\sup_{x\in[0,4]} \left| D_n^{(k)}(f_0; x) - f_0(x) \right|$$

against n on a log–log scale for $k = 0, 1, 2, 4$. The case $k = 0$ corresponds to the classical Szász–Durrmeyer operator, as explained in Remark 2.6. The curves are compared with a reference line of slope -1. For the smooth function f_0, the observed $O(n^{-1})$ behavior is consistent with the Voronovskaya-type formula (5.1), which gives the first-order asymptotic profile on compact intervals.

The exact first-moment identity (2.27) gives

$$n\,[D_n^{(k)}(u; x) - x] = 1 + \delta, \qquad \delta = \frac{k\mathrm{e}}{\mathrm{e}+1},$$

for every n and $x \geq 0$. Thus the Euler parameter modifies the first-order drift coefficient from the classical value 1 to $1 + \delta$, while preserving the order of convergence in n.

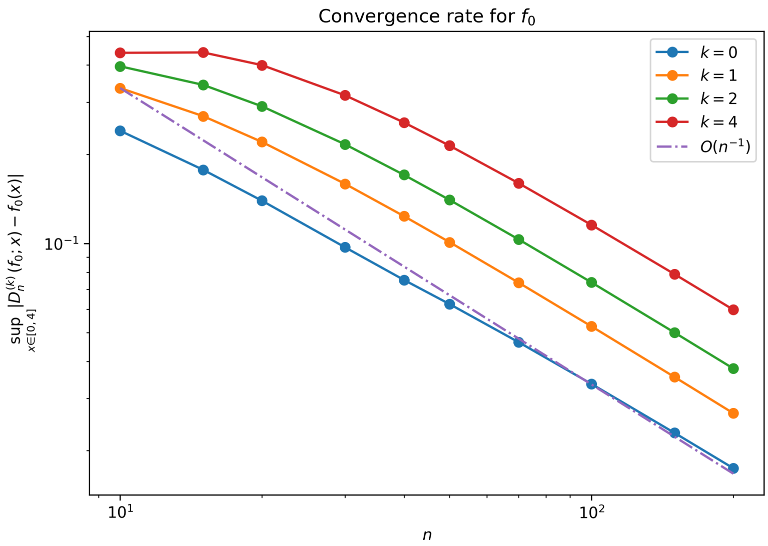

Figure 3. Convergence rate of $D_n^{(k)}$ applied to f_0 for $k = 0, 1, 2, 4$. The case $k = 0$ represents the classical Szász–Durrmeyer operator. The graph is plotted on a log–log scale; the dash-dotted reference line has slope -1, corresponding to $O(n^{-1})$.

6.4. *Voronovskaya verification*

Figure 4 illustrates numerically Theorem 5.1. For $k = 2$, the quantities $n\,[D_n^{(k)}(f_0; x) - f_0(x)]$ are plotted for $n = 30, 50, 100$ and compared with the theoretical limit $(1 + \delta)f_0'(x) + x f_0''(x)$, which is the right-hand side of (5.1). The numerical curves move toward this limit on the compact interval $[0, 4]$, illustrating the compact-uniform asymptotic behavior in Theorem 5.1.

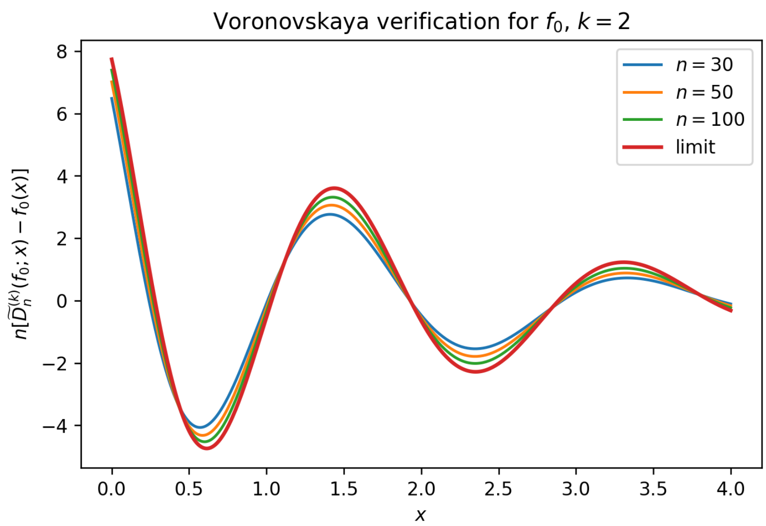

Figure 4. Voronovskaya-type asymptotic behavior for f_0 and $k = 2$. Numerical curves $n[D_n^{(k)}(f_0; x) - f_0(x)]$ for $n = 30, 50, 100$ are compared with the theoretical limit $(1 + \delta)f_0'(x) + x f_0''(x)$ from (5.1).

6.5. Supremum error table

The second central moment is denoted by

$$\Delta_{n,k}(x) = D_n^{(k)}((u-x)^2; x),$$

and its explicit expression is given in (2.35). Hence the asymptotic relation (2.40) shows that $n\Delta_{n,k}(x) \to 2x$ uniformly on compact subsets of $[0,\infty)$. This quantity enters both the modulus estimate (3.2) and the proof of the Voronovskaya formula.

The values in Table 1 are consistent with the $O(n^{-1})$ behavior predicted by the Voronovskaya expansion for the smooth test function f_0. The observed increase of the error with respect to k is compatible with the dependence of the first-order drift coefficient $1+\delta$ on the Euler parameter in (5.1); however, the actual error also depends on the derivatives of the test function and on higher-order terms.

Table 1. Supremum error $\sup_{x\in[0,4]} |D_n^{(k)}(f_0; x) - f_0(x)|$ for selected values of n and k.

n	$k=0$	$k=1$	$k=2$	$k=4$
5	0.3423	0.3725	0.4281	0.5274
10	0.2400	0.3344	0.3961	0.4397
20	0.1393	0.2200	0.2901	0.3996
40	0.0753	0.1236	0.1704	0.2560
80	0.0412	0.0650	0.0912	0.1418
150	0.0230	0.0354	0.0501	0.0788
200	0.0174	0.0267	0.0378	0.0598

7. Conclusions

We have introduced and studied a Szász–Durrmeyer-type family of positive linear operators generated by Euler polynomials of negative order $-k$. The probabilistic representation plays a central role in the analysis: It makes positivity and normalization transparent and provides an effective framework for deriving the algebraic and central moment identities.

A main feature of the construction is the explicit identification of the Euler-generated discrete weights as a finite binomial mixture of shifted Poisson probabilities. Combined with the Gamma-type kernels in the Durrmeyer part, this yields a Poisson–Gamma representation of the operators.

The obtained quantitative estimates, weighted Korovkin-type convergence, and Voronovskaya-type formula show that the Euler structure preserves the principal approximation order and modifies only the drift component of the first-order asymptotic profile. Thus, the proposed operators provide a probabilistically justified Euler-type deformation of the classical Szász–Durrmeyer framework.

Author contributions

Both authors contributed to conceptualization, methodology, formal analysis, investigation, writing–original draft, and writing–review and editing. Both authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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