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Research article

Exact and series solutions for a class of inhomogeneous delay equations with variable coefficients and two proportional delays

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Abstract: This study addresses a class of inhomogeneous delay differential equations characterized by variable coefficients and the presence of two proportional delays. An exponential-type transformation is employed to convert the inhomogeneous formulation into an equivalent homogeneous pantograph equation. This reformulation enables the derivation of an explicit analytical solution represented by a superposition of an exponential function and a power series with coefficients given in closed product form. A detailed investigation of the convergence properties of the associated infinite product and series is conducted, yielding clear criteria for global convergence. Furthermore, specific parameter configurations are identified under which the series solution reduces to a finite sum, leading to exact analytical expressions. As an important application, the inhomogeneous Ambartsumian equation involving two proportional delays is examined, and new closed-form and exact solutions are established. Numerical experiments are included to support the theoretical analysis and to demonstrate the rapid convergence and the practical effectiveness of the proposed solution approach.

Keywords: ordinary differential equation; delay differential equation; initial value problem; series; closed-form
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1. Introduction

Delay differential equations (DDEs) play a fundamental role in modeling phenomena where the present rate of change depends on past states [1–3]. Such equations arise in numerous fields, including, population dynamics [4], biology [5–7], control theory [8–10], physics, engineering, and economics [11–13]. In particular, models with proportional delays, often referred to as pantograph-type equations—have attracted increasing attention due to their ability to capture scale-dependent memory [14–16]. Although significant progress has been made in the analysis of homogeneous pantograph equations (HPEs) [17–19], considerably fewer results are available for inhomogeneous models, especially those involving multiple proportional delays and variable coefficients. The presence of inhomogeneous terms and more than one delay substantially complicates the analytical treatment and limits the availability of closed-form solutions. Consequently, the development of reliable analytical methods for such equations remains an active and important research direction in addition to the stability analysis [20,21]. A special case of the pantograph equation (PE) is known as the Ambartsumian equation (AE) [22–24].

The present paper focuses on a class of inhomogeneous pantograph equations (IPEs) with two proportional delays and exponential forcing terms, expressed as

$$\dot{y}(t) = \alpha y(t) + \beta_1 y(c_1 t) + \beta_2 y(c_2 t) + \beta_3 e^{\delta_1 t} + \beta_4 e^{\delta_2 t}, \quad y(0) = \lambda, \tag{1}$$

where λ, α, β_i $(i = 1, 2, \ldots, 4)$, and δ_j $(j = 1, 2)$ are real constants while $|c_j| < 1$ are two proportional delays. In the literature, numerous numerical methods [25–28] and analytical approaches [29–31] have been developed in addition to the Laplace technique [32]. In [33,34], some variable versions of the PE and the AE have been analytically solved.

The main aim of this work is to construct analytical solutions by transforming the original inhomogeneous equation into a related homogeneous problem. This technique leads to a representation of the solution in terms of an infinite series whose coefficients are explicitly determined through finite products involving the system parameters. In addition to constructing the solution, the paper provides a rigorous analysis of the convergence behavior of the resulting series. Special parameter regimes are identified for which the infinite series truncates/terminates, yielding exact solutions with finite polynomial–exponential structure. The developed approach is further implemented to produce new exact and approximate solutions for the inhomogeneous Ambartsumian equation with two proportional delays, extending the existing results. The remainder of the paper is organized as follows. Section 2 presents the transformation technique and the derivation of the closed-form solution. Section 3 is devoted to convergence analysis. Exact solutions for special classes of equations are discussed in Sections 4 and 5, including the inhomogeneous Ambartsumian equation. Numerical illustrations validating the theoretical findings are provided in Section 6. Concluding remarks are given in the final section.

2. Closed-form solution

We introduce the transformation

$$y(t) = z(t) + ae^{bt}, \tag{2}$$

where a and b are constants to be determined. Substituting Eq (2) into Eq (1) yields

$$\dot{z}(t) + abe^{bt} = \alpha(z(t) + ae^{bt}) + \beta_1(z(c_1 t) + ae^{bc_1 t}) + \beta_2(z(c_2 t) + ae^{bc_2 t}) + \beta_3 e^{\delta_1 t} + \beta_4 e^{\delta_2 t}. \tag{3}$$

Rearranging terms leads to

$$\dot{z}(t) = \alpha z(t) + \beta_1 z(c_1 t) + \beta_2 z(c_2 t) + a(b - \alpha)e^{bt} + \beta_1 ae^{bc_1 t} + \beta_2 ae^{bc_2 t} + \beta_3 e^{\delta_1 t} + \beta_4 e^{\delta_2 t}. \tag{4}$$

Setting

$$b = \alpha, \quad \delta_1 = bc_1, \quad \delta_2 = bc_2, \quad \beta_3 = -\beta_1 a, \quad \beta_4 = -\beta_2 a, \tag{5}$$

then equation Eq (4) reduces to the homogeneous pantograph equation (HPE) with two proportional delays

$$\dot{z}(t) = \alpha z(t) + \beta_1 z(c_1 t) + \beta_2 z(c_2 t), \quad z(0) = \lambda - a. \tag{6}$$

Equation (1) can therefore be written as

$$\dot{y}(t) = \alpha y(t) + \beta_1 y(c_1 t) + \beta_2 y(c_2 t) - \beta_1 a e^{\alpha c_1 t} - \beta_2 a e^{\alpha c_2 t}, \quad y(0) = \lambda, \tag{7}$$

with the solution given by

$$y(t) = z(t) + a e^{\alpha t}, \tag{8}$$

where $z(t)$ is any solution of the HPE (6). In [35], the equation $\dot{z}(t) = \alpha z(t) + \beta_1 z(c_1 t) + \beta_2 z(c_2 t)$ has been solved in series form under the initial condition $z(0) = 1$. Following the same analysis [35], one can obtain the solution of the HPE (6) in the form

$$z(t) = (\lambda - a) \sum_{n=0}^{\infty} \xi_n \frac{t^n}{n!}, \quad \xi_n = \prod_{k=0}^{n-1} \left(\alpha + \beta_1 c_1^k + \beta_2 c_2^k\right). \tag{9}$$

Hence

$$y(t) = a e^{\alpha t} + (\lambda - a) \sum_{n=0}^{\infty} \xi_n \frac{t^n}{n!}, \tag{10}$$

represents a closed-form solution for the IPE (7).

Remark 1. *If $a = 0$ and $\lambda = 1$, the solution (10) reduces to the same one in [35] (see pages 64 and 65), given as*

$$y(t) = 1 + \sum_{n=1}^{\infty} \xi_n \frac{t^n}{n!}, \tag{11}$$

for the following HPE with two proportional delays:

$$\dot{y}(t) = \alpha y(t) + \beta_1 y(c_1 t) + \beta_2 y(c_2 t), \quad y(0) = 1. \tag{12}$$

Moreover, if $a = \lambda$, then $y(t) = \lambda e^{\alpha t}$ is an exact solution for the following IPE with two proportional delays:

$$\dot{y}(t) = \alpha y(t) + \beta_1 y(c_1 t) + \beta_2 y(c_2 t) - \beta_1 \lambda e^{\alpha c_1 t} - \beta_2 \lambda e^{\alpha c_2 t}. \tag{13}$$

This exact solution is independent of the parameters β_j and c_j, $j = 1, 2$. Other forms of exact solutions could be derived from the closed-form solution (10) under some assumptions of the parameters involved in the product ξ_n, mainly α, β_j and c_j. This point will be addressed in a subsequent section.

3. Convergence analysis

This section discusses the convergence of the product ξ_n as $n \to \infty$ in addition to the convergence of the series part in Eq (10). For easier discussion, we rewrite the closed-form solution (10) as

$$y(t) = ae^{\alpha t} + (\lambda - a)\sum_{n=0}^{\infty} \xi_n \frac{t^n}{n!}, \quad \xi_n = \prod_{k=0}^{n-1} \gamma_k, \quad \gamma_k = \alpha + \beta_1 c_1^k + \beta_2 c_2^k. \tag{14}$$

It has been shown in [36] that the product ξ_n converges to a finite nonzero limit as $n \to \infty$ if $\lim_{k\to\infty} \gamma_k = 1$. Since $|c_j| < 1$ $(j = 1, 2)$, then $\lim_{k\to\infty} \gamma_k = \alpha$, this implies the necessary condition $\alpha = 1$.

If $\alpha \neq 1$, then $\xi_n \sim \alpha^n$ as $n \to \infty$. The asymptotic behavior of the solution reveals that $z(t) = (\lambda - a)\sum_{n=0}^{\infty} \xi_n \frac{t^n}{n!} \sim (\lambda - a)e^{\alpha t}$ and consequently $y(t) \sim ae^{\alpha t} + (\lambda - a)e^{\alpha t}$, i.e., $y(t) \sim \lambda e^{\alpha t}$.

Moreover, the condition $\alpha = 1$ implies $\xi_n = \prod_{k=0}^{n-1}\left(1 + \beta_1 c_1^k + \beta_2 c_2^k\right)$ which can be written as $\xi_n = \prod_{k=0}^{n-1}(1 + \theta_k)$, where $\theta_k = \beta_1 c_1^k + \beta_2 c_2^k$. Accordingly, the infinite product $\prod_{k=0}^{\infty}(1 + \theta_k)$ converges to a finite nonzero limit if and only if the series $\sum_{k=0}^{\infty} \theta_k$ converges absolutely. Since $|c_i| < 1$ $(i = 1, 2)$, then the geometric series $\sum_{k=0}^{\infty} |c_1|^k$ and $\sum_{k=0}^{\infty} |c_2|^k$ converge and one can find that $\sum_{k=0}^{\infty} |\theta_k| \leq |\beta_1| \sum_{k=0}^{\infty} |c_1|^k + |\beta_2| \sum_{k=0}^{\infty} |c_2|^k < \infty$. Therefore, the infinite product converges to a finite nonzero limit. The convergence of the series $\sum_{n=0}^{\infty} \xi_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} v_n(t)$ can be checked through applying the ratio test as described below

$$\lim_{n\to\infty} \left| \frac{v_{n+1}(t)}{v_n(t)} \right| = \lim_{n\to\infty} \left| \frac{t\, \xi_{n+1}}{(n+1)\xi_n} \right| = |t| \lim_{n\to\infty} \left| \frac{\alpha + \beta_1 c_1^n + \beta_2 c_2^n}{n+1} \right| = 0. \tag{15}$$

Thus, the series $\sum_{n=0}^{\infty} \xi_n \frac{t^n}{n!}$ converges for every $t > 0$.

Remark 2. *It may be important to mention that the current analytical approach breaks down when transitioning to non-smooth or discontinuous forcing profiles. This is because our exponential transformation mechanism relies on the smooth differentiability of the basis functions, the presence of jump discontinuities or merely measurable forcing profiles invalidating the step-by-step reduction process as pointed out by Difonzo et al. [37].*

4. Exact solutions for some classes of the IPE

As mentioned in the previous section, the product ξ_n $(n \geq 1)$ converges to a finite nonzero limit as $n \to \infty$ if $\alpha = 1$. However, the values of ξ_n become zero at some values of $n \geq N$, where N depends on prescribed conditions of the model's parameters. Consequently, the infinite series transforms to a truncated series which leads exact solutions in different forms if $\alpha \neq 1$. This point is addressed in the next section.

4.1. Class subject to $\alpha = -\beta_1 c_1^p - \beta_2 c_2^p$

Under the conditions $\alpha = -\beta_1 c_1^p - \beta_2 c_2^p$, $p = 1, 2, 3, \ldots$, the IPE (7) takes the form

$$\dot{y}(t) = -\left(\beta_1 c_1^p + \beta_2 c_2^p\right) y(t) + \beta_1 y(c_1 t) + \beta_2 y(c_2 t) - \beta_1 a e^{\alpha c_1 t} - \beta_2 a e^{\alpha c_2 t}, \quad y(0) = \lambda, \tag{16}$$

with the closed-form solution

$$\begin{cases} y(t) = ae^{\alpha t} + (\lambda - a)\sum_{n=0}^{\infty} \xi_n \frac{t^n}{n!}, \\ \xi_n = \prod_{k=0}^{n-1} \gamma_k, \quad \gamma_k = \beta_1 \left(c_1^k - c_1^p\right) + \beta_2 \left(c_2^k - c_2^p\right). \end{cases} \tag{17}$$

Equation (17) leads to

$$\begin{cases} \xi_0 = 1, \\ \xi_1 = \gamma_0 = \beta_1 \left(1 - c_1^p\right) + \beta_2 \left(1 - c_2^p\right), \\ \xi_2 = \gamma_0\gamma_1 = \gamma_0 \left[\beta_1 \left(c_1 - c_1^p\right) + \beta_2 \left(c_2 - c_2^p\right)\right], \\ \xi_3 = \gamma_0\gamma_1\gamma_2 = \gamma_0\gamma_1 \left[\beta_1 \left(c_1^2 - c_1^p\right) + \beta_2 \left(c_2^2 - c_2^p\right)\right], \\ \xi_4 = \gamma_0\gamma_1\gamma_2\gamma_3 = \gamma_0\gamma_1\gamma_2 \left[\beta_1 \left(c_1^3 - c_1^p\right) + \beta_2 \left(c_2^3 - c_2^p\right)\right], \\ \vdots \\ \xi_n = \gamma_0\gamma_1\gamma_2\gamma_3 \ldots \gamma_{n-2}\gamma_{n-1} = \gamma_0\gamma_1\gamma_2\gamma_3 \ldots \gamma_{n-2} \left[\beta_1 \left(c_1^{n-1} - c_1^p\right) + \beta_2 \left(c_2^{n-1} - c_2^p\right)\right], n \geq 1. \end{cases} \tag{18}$$

If $p = 1$, then $\xi_0 = 1$, $\xi_1 = \gamma_0$ and $\xi_n = 0 \; \forall \; n \geq 2$. Accordingly, we obtain from the first equation in (17) the following exact solution:

$$y(t) = ae^{\alpha_1 t} + (\lambda - a)(1 + \gamma_0 t), \; \gamma_0 = \beta_1 \left(1 - c_1\right) + \beta_2 \left(1 - c_2\right), \; \alpha_1 = -\left(\beta_1 c_1 + \beta_2 c_2\right), \tag{19}$$

for the IPE

$$\dot{y}(t) = \alpha_1 y(t) + \beta_1 y(c_1 t) + \beta_2 y(c_2 t) - \beta_1 a e^{\alpha_1 c_1 t} - \beta_2 a e^{\alpha_1 c_2 t}, \quad y(0) = \lambda. \tag{20}$$

If $p = 2$, then $\xi_0 = 1$, $\xi_1 = \gamma_0$, $\xi_2 = \gamma_0\gamma_1$ and $\xi_n = 0 \; \forall \; n \geq 3$. Thus, an exact solution can be established in the form

$$\begin{cases} y(t) = ae^{\alpha_2 t} + (\lambda - a)\left(1 + \gamma_0 t + \gamma_0\gamma_1 \dfrac{t^2}{2!}\right), \\ \gamma_0 = \beta_1 \left(1 - c_1^2\right) + \beta_2 \left(1 - c_2^2\right), \\ \gamma_1 = \beta_1 \left(c_1 - c_1^2\right) + \beta_2 \left(c_2 - c_2^2\right), \\ \alpha_2 = \left(\beta_1 c_1^2 + \beta_2 c_2^2\right), \end{cases} \tag{21}$$

for the IPE

$$\dot{y}(t) = \alpha_2 y(t) + \beta_1 y(c_1 t) + \beta_2 y(c_2 t) - \beta_1 a e^{\alpha_2 c_1 t} - \beta_2 a e^{\alpha_2 c_2 t}, \quad y(0) = \lambda. \tag{22}$$

Similarly, if p is any positive integer then the general exact solution is given by

$$y(t) = ae^{\alpha_p t} + (\lambda - a)\sum_{n=0}^{p} \xi_n \frac{t^n}{n!}, \quad \alpha_p = -\left(\beta_1 c_1^p + \beta_2 c_2^p\right), \tag{23}$$

for the generalized IPE

$$\dot{y}(t) = \alpha_p y(t) + \beta_1 y(c_1 t) + \beta_2 y(c_2 t) - \beta_1 a e^{\alpha_p c_1 t} - \beta_2 a e^{\alpha_p c_2 t}, \quad y(0) = \lambda, \tag{24}$$

where ξ_n can be evaluated from the second equation in (17) once p is specified.

4.2. Class subject to $c_2 = c_1^2$

Suppose that $c_1 = c$ and $c_2 = c_1^2 = c^2$, then Eq (7) becomes

$$\dot{y}(t) = \alpha y(t) + \beta_1 y(ct) + \beta_2 y(c^2 t) - \beta_1 a e^{\alpha c t} - \beta_2 a e^{\alpha c^2 t}, \quad y(0) = \lambda. \tag{25}$$

The closed-form solution of the problem (25) is obtained from (14) as

$$\begin{cases} y(t) = ae^{\alpha t} + (\lambda - a)\sum_{n=0}^{\infty} \zeta_n \frac{t^n}{n!}, \\ \zeta_n = \prod_{k=0}^{n-1} w_k, \quad w_k = \alpha + \beta_1 c^k + \beta_2 c^{2k}. \end{cases} \tag{26}$$

Under the restrictions $\alpha_r = -\left(\beta_1 c^r + \beta_2 c^{2r}\right)$ for $r = 1, 2, 3, \ldots$, the IPE (25) becomes

$$\dot{y}(t) = \alpha_r y(t) + \beta_1 y(ct) + \beta_2 y(c^2 t) - \beta_1 a e^{\alpha_r c t} - \beta_2 a e^{\alpha_r c^2 t}, \quad y(0) = \lambda. \tag{27}$$

On repeating the process in the previous section, one can obtain the exact solution

$$y(t) = ae^{\alpha_r t} + (\lambda - a)\sum_{n=0}^{r} \zeta_n \frac{t^n}{n!}, \quad \alpha_r = -\left(\beta_1 c^r + \beta_2 c^{2r}\right), \tag{28}$$

where ζ_n can be calculated from the formula

$$\zeta_n = \prod_{k=0}^{n-1} \left[\beta_1 \left(c^k - c^r\right) + \beta_2 \left(c^{2k} - c^{2r}\right)\right], \quad 1 \le n \le r, \tag{29}$$

once r is specified while $\zeta_n = 0 \;\forall\; n \ge r + 1$, n is a positive integer.

Hence, this section in addition to the previous one indicate that the infinite series (14) transforms to a truncated series with a finite number of terms p in Eq (23) or r in Eq (28) based on the conditions $\alpha_p = -\left(\beta_1 c_1^p + \beta_2 c_2^p\right)$ and $\alpha_r = -\left(\beta_1 c^r + \beta_2 c^{2r}\right)$, respectively.

5. Solution of the inhomogeneous Ambartsumian equation (IAE)

A closed-form solution for a particular class named as the inhomogeneous Ambartsumian equation (IAE) is introduced in this section as a special case. Additionally, exact solutions in different forms are to be determined subject to some conditions on the model's parameters.

5.1. Closed-form solution for the IAE

Basically, the Ambartsumian equation (AE) is known as [23]

$$\dot{y}(t) = -y(t) + \frac{1}{q} y\left(\frac{t}{q}\right), \quad y(0) = \lambda,\ q > 1, \tag{30}$$

a proportional delay parameter $1/q$. If two proportional delays are involved, then Eq (30) is extended to the homogeneous Ambartsumian equation (HAE)

$$\dot{y}(t) = -y(t) + \frac{1}{q_1} y\left(\frac{t}{q_1}\right) + \frac{1}{q_2} y\left(\frac{t}{q_2}\right), \quad y(0) = \lambda,\ q_1, q_2 > 1. \tag{31}$$

This section solves the inhomogeneous Ambartsumian equation (IAE):

$$\dot{y}(t) = -y(t) + \frac{1}{q_1}y(t/q_1) + \frac{1}{q_2}y(t/q_2) - \frac{a}{q_1}e^{-t/q_1} - \frac{a}{q_2}e^{-t/q_2}, \quad y(0) = \lambda. \tag{32}$$

Comparing Eq (32) with Eq (7) yields $\alpha = -1$, $\beta_1 = c_1 = \frac{1}{q_1}$ and $\beta_2 = c_2 = \frac{1}{q_2}$. The solution of the IAE (32) can be directly obtained via inserting the above values into Eq (14) to give

$$y(t) = ae^{-t} + (\lambda - a)\sum_{n=0}^{\infty} v_n \frac{t^n}{n!}, \quad v_n = \prod_{k=0}^{n-1} \mu_k, \quad \mu_k = q_1^{-(k+1)} + q_2^{-(k+1)} - 1. \tag{33}$$

The infinite series in the solution (33) reduces to a truncated series under some conditions governed by the parameters q_1 and q_2 as indicated in the next section.

5.2. Exact solutions for the IAE

From the formula (33) for μ_n, we have

$$\begin{cases} \mu_0 = 1, \\ \mu_1 = \left(q_1^{-1} + q_2^{-1} - 1\right), \\ \mu_2 = \mu_1\left(q_1^{-2} + q_2^{-2} - 1\right), \\ \mu_3 = \mu_2\left(q_1^{-3} + q_2^{-3} - 1\right), \\ \mu_4 = \mu_3\left(q_1^{-4} + q_2^{-4} - 1\right), \\ \vdots \\ \mu_n = \mu_{n-1}\left(q_1^{-n} + q_2^{-n} - 1\right), \quad n \geq 1. \end{cases} \tag{34}$$

Let $\mu_1 = q_1^{-1} + q_2^{-1} - 1 = 0$, then Eq (34) implies $\mu_0 = 1$ and $\mu_n = 0\ \forall\ n \geq 1$. Inserting these values in Eq (33) yields the exact solution

$$y(t) = \lambda + a\left(e^{-t} - 1\right), \tag{35}$$

for the IAE (32) subject to $q_1^{-1} + q_2^{-1} = 1$. Similarly, if $\mu_2 = q_1^{-2} + q_2^{-2} - 1 = 0$, then $\mu_0 = 1$, $\mu_1 = q_1^{-1} + q_2^{-1} - 1$ and $\mu_n = 0\ \forall\ n \geq 2$. Thus, we obtain

$$y(t) = ae^{-t} + (\lambda - a)(1 + \mu_1 t), \quad \mu_1 = q_1^{-1} + q_2^{-1} - 1, \tag{36}$$

as an exact solution for the IAE (32) $q_1^{-2} + q_2^{-2} = 1$. Generally, if $\mu_l = q_1^{-l} + q_2^{-l} - 1$, $l = 1, 2, 3, \ldots$, we find the exact solution

$$y(t) = ae^{-t} + (\lambda - a)\sum_{n=0}^{l-1} v_n \frac{t^n}{n!}, \quad v_n = \prod_{k=0}^{n-1} \mu_k, \quad \mu_k = q_1^{-k} + q_2^{-k} - 1, \tag{37}$$

for the IAE (32) subject to $q_1^{-l} + q_2^{-l} = 1$, $l \geq 1$, where the coefficients of the l-term truncated series in (37) can be found from the formula for v_n for $0 \leq n < l$.

Remark 3. *The classical Ambartsumian equation models the absorption of light in a cosmic medium. For the present inhomogeneous version of this equation, the dual delays q_1 and q_2 represent distinct physical scaling factors of fluctuations (e.g., turbulent vs. thermal density scaling in the Milky Way), while the inhomogeneous term represents a non-uniform background isotropic emission source. Such type of equations can also be analyzed under history functions as pointed out by Shah et al. [38].*

6. Numerical results

6.1. Approximations

The objective to obtain the exact solutions of some classes of the IPE and the IAE is achieved in Sections 4 and 5 under specific restrictions of the involved parameters. However, if such restrictions are not satisfied then the exact solutions are not available. In this case, the approximate series solutions are only allowed for these models. Additionally, the convergence of the series solution was theoretically addressed in Section 3. To confirm this point through numerical results, a number of plots are produced in this section for the approximate solutions under the conditions $\alpha \neq -(\beta_1 c_1^p + \beta_2 c_2^p)$ for the IPE and $q_1^{-l} + q_2^{-l} \neq 1$ for the IAE, where p and l are positive integers. To achieve this target, we define the m-term approximate solution for the IPE by

$$\chi_m(t) = ae^{\alpha t} + (\lambda - a)\sum_{n=0}^{m-1} \xi_n \frac{t^n}{n!}, \tag{38}$$

and

$$\Theta_m(t) = ae^{-t} + (\lambda - a)\sum_{n=0}^{m-1} v_n \frac{t^n}{n!}, \tag{39}$$

for the IAE, where ξ_n and v_n are defined in Eqs (14) and (33), respectively.

6.2. Truncation error estimate

The closed-form solution of Eq (7) is obtained as

$$y(t) = ae^{\alpha t} + (\lambda - a)\sum_{n=0}^{\infty} \frac{\xi_n t^n}{n!}, \qquad \xi_n = \prod_{k=0}^{n-1}\left(\alpha + \beta_1 c_1^k + \beta_2 c_2^k\right), \tag{40}$$

whereas the m-term approximation is introduced in Eq (38). Therefore, the truncation error $R_m(t)$ is

$$R_m(t) = y(t) - \chi_m(t) = (\lambda - a)\sum_{n=m}^{\infty} \frac{\xi_n t^n}{n!}. \tag{41}$$

Hence

$$|R_m(t)| \leq |\lambda - a| \sum_{n=m}^{\infty} \frac{|\xi_n|\,|t|^n}{n!}. \tag{42}$$

Since $|c_1| < 1$ and $|c_2| < 1$, one has $|c_1|^k \leq 1$ and $|c_2|^k \leq 1$, which implies

$$|\xi_n| \leq \prod_{k=0}^{n-1}\left(|\alpha| + |\beta_1||c_1|^k + |\beta_2||c_2|^k\right) \leq \prod_{k=0}^{n-1}\left(|\alpha| + |\beta_1| + |\beta_2|\right). \tag{43}$$

Thus

$$|\xi_n| \leq M^n, \qquad M = |\alpha| + |\beta_1| + |\beta_2|. \tag{44}$$

Substituting into (42) gives

$$|R_m(t)| \leq |\lambda - a| \sum_{n=m}^{\infty} \frac{(M|t|)^n}{n!}. \tag{45}$$

Let $K = M|t|$ and applying the standard exponential-tail inequality, then

$$\sum_{n=m}^{\infty} \frac{K^n}{n!} \leq e^K \frac{K^m}{m!}. \tag{46}$$

This yields

$$|y(t) - \chi_m(t)| \leq |\lambda - a| e^K \frac{K^m}{m!}, \tag{47}$$

or equivalently,

$$|y(t) - \chi_m(t)| \leq |\lambda - a| \exp\Big((|\alpha| + |\beta_1| + |\beta_2|)|t|\Big) \frac{\Big[(|\alpha| + |\beta_1| + |\beta_2|)|t|\Big]^m}{m!}. \tag{48}$$

Moreover, retaining the delay factors gives the sharper estimate

$$|\xi_n| \leq \prod_{k=0}^{n-1} \Big(|\alpha| + |\beta_1||c_1|^k + |\beta_2||c_2|^k\Big), \tag{49}$$

which explicitly demonstrates the influence of the scaling factors $|c_1|$ and $|c_2|$ on the convergence rate. Since $|c_1|, |c_2| < 1$, the quantities $|\beta_1||c_1|^k$ and $|\beta_2||c_2|^k$ decay geometrically as $k \to \infty$, resulting in a faster decay of the series coefficients and therefore a smaller truncation error. It was previously stated that $\alpha = 1$ is a necessary condition for the convergence of the product ξ_n as $n \to \infty$ for the IPE. Considering $\alpha = 1$, the approximations χ_m are depicted in Figures 1 and 2 at $\alpha = 1$, $a = 2$, $\beta_1 = 1$, $\beta_2 = 2$, $c_1 = -\frac{1}{4}$, $c_2 = -\frac{1}{2}$ for different $m = 2, 3, 4, 5$ in Figure 1 and $m = 6, 7, 8, 9$ in Figure 2 while $\lambda = 1$ is fixed. These figures show that the convergence of the approximations $\chi_m(t)$ is achieved using a few number of terms. Beside, the approximations converge rapidly toward a certain curve by increasing m. Moreover, the domain of convergence of the approximate solution enlarges as m increases. It can be also shown that the convergence of the approximations $\chi_m(t)$ is not affected by choosing $\alpha \neq 1$.

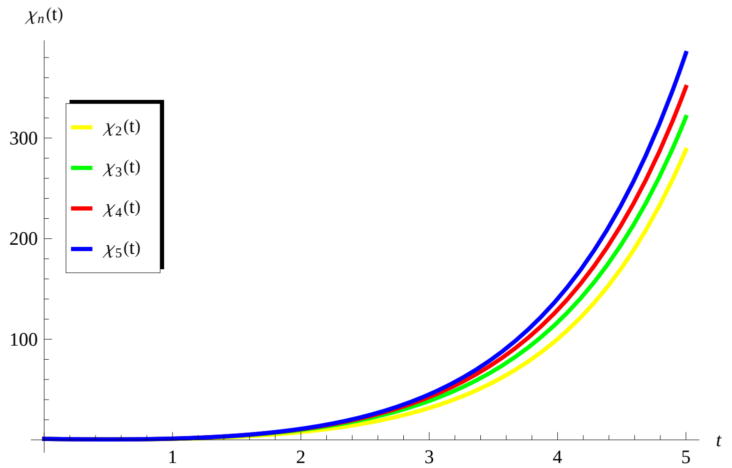

Figure 1. Plots of the approximations $\chi_m(t)$ ($m = 2, 3, 4, 5$) at $\alpha = 1$, $a = 2$, $\beta_1 = 1$, $\beta_2 = 2$, $c_1 = -\frac{1}{4}$, and $c_2 = -\frac{1}{2}$.

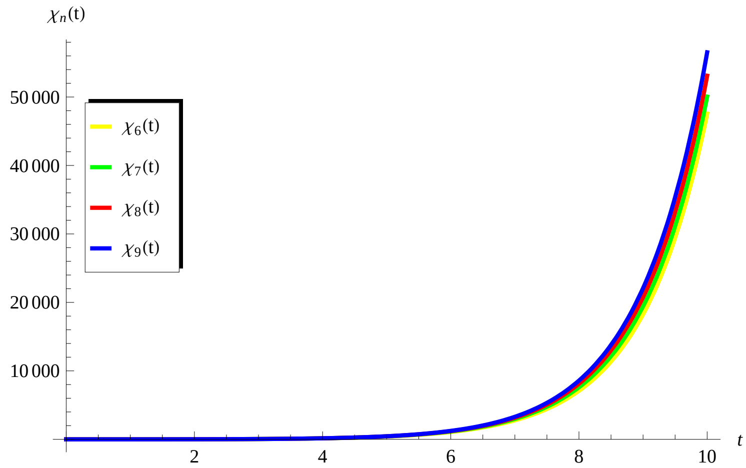

Figure 2. Plots of the approximations $\chi_m(t)$ $(m = 6, 7, 8, 9)$ at $\alpha = 1$, $a = 2$, $\beta_1 = 1$, $\beta_2 = 2$, $c_1 = -\frac{1}{4}$, and $c_2 = -\frac{1}{2}$.

For illustration, Figures 3 and 4 display the approximations $\chi_m(t)$ at $\alpha = -1$, $a = 2$, $\beta_1 = 1$, $\beta_2 = 2$, $c_1 = -\frac{1}{4}$, $c_2 = -\frac{1}{2}$ for $m = 2, 3, 4, 5$ in Figure 3 and for $m = 6, 7, 8, 9$ in Figure 4. It is seen form these figures that the convergence of the approximations $\chi_m(t)$ is also achieved and this conclusion is confirmed through the plots in Figures 5 and 6 with $\alpha = -\frac{1}{2}$ for $m = 2, 3, 4, 5$ in Figure 5 and for $m = 6, 7, 8, 9$ in Figure 6, respectively. The numerical results in Figures 7 and 8 prove the convergence of the approximate solutions $\Theta_m(t)$ for the IAE. In view of the above discussion, one can conclude that the proposed analysis was effective and efficient to deal with the IPE and IAE in both analytical and numerical sense.

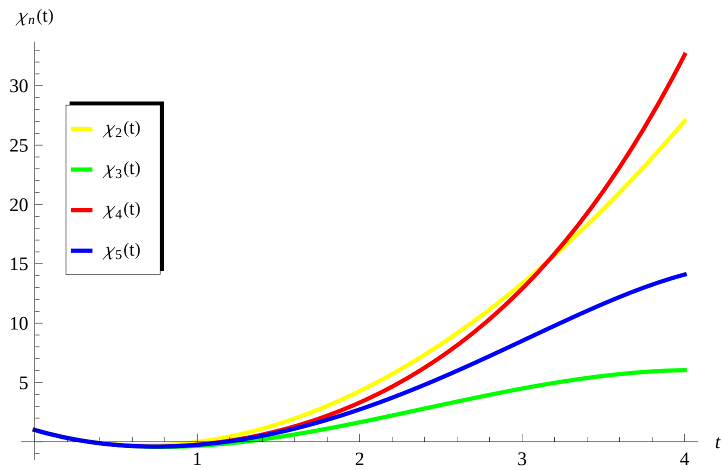

Figure 3. Plots of the approximations $\chi_m(t)$ $(m = 2, 3, 4, 5)$ at $\alpha = -1$, $a = 2$, $\beta_1 = 1$, $\beta_2 = 2$, $c_1 = -\frac{1}{4}$, and $c_2 = -\frac{1}{2}$.

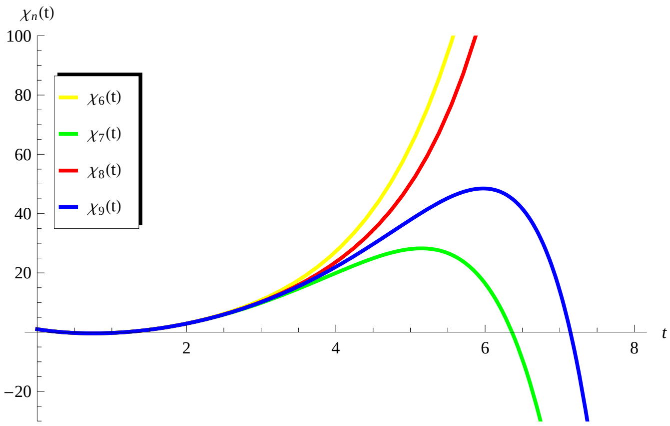

Figure 4. Plots of the approximations $\chi_m(t)$ $(m = 6, 7, 8, 9)$ at $\alpha = -1$, $a = 2$, $\beta_1 = 1$, $\beta_2 = 2$, $c_1 = -\frac{1}{4}$, and $c_2 = -\frac{1}{2}$.

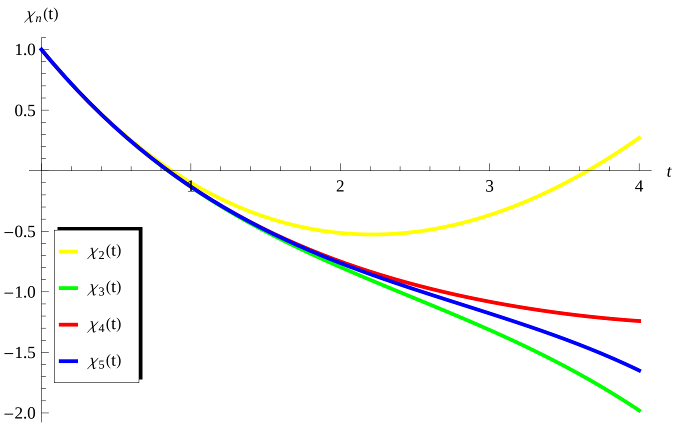

Figure 5. Plots of the approximations $\chi_m(t)$ $(m = 2, 3, 4, 5)$ at $\alpha = -\frac{1}{2}$, $a = 2$, $\beta_1 = -1$, $\beta_2 = 2$, $c_1 = -\frac{3}{4}$, and $c_2 = -\frac{1}{2}$.

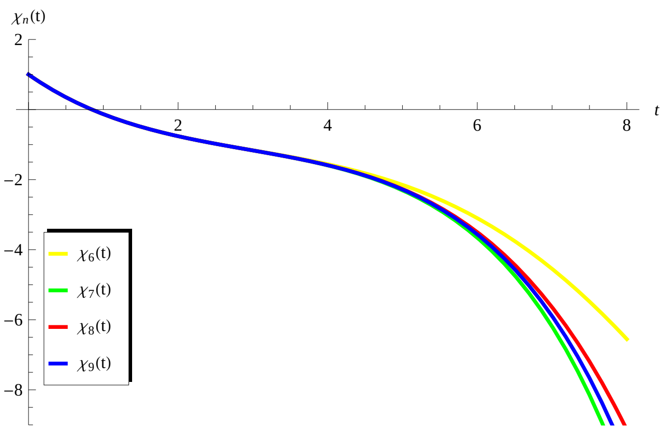

Figure 6. Plots of the approximations $\chi_m(t)$ $(m = 6, 7, 8, 9)$ at $\alpha = -\frac{1}{2}$, $a = 2$, $\beta_1 = -1$, $\beta_2 = 2$, $c_1 = -\frac{3}{4}$, and $c_2 = -\frac{1}{2}$.

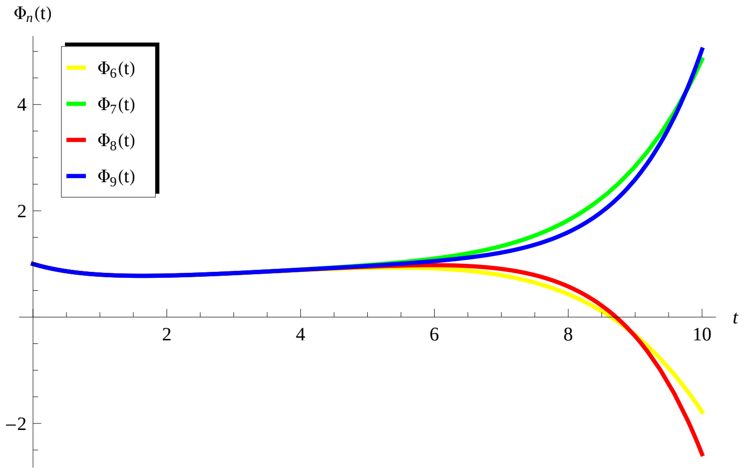

Figure 7. Plots of the approximations $\Theta_m(t)$ ($m = 6, 7, 8, 9$) at $a = \frac{1}{2}$, $q_1 = \frac{4}{3}$, and $q_2 = 2$.

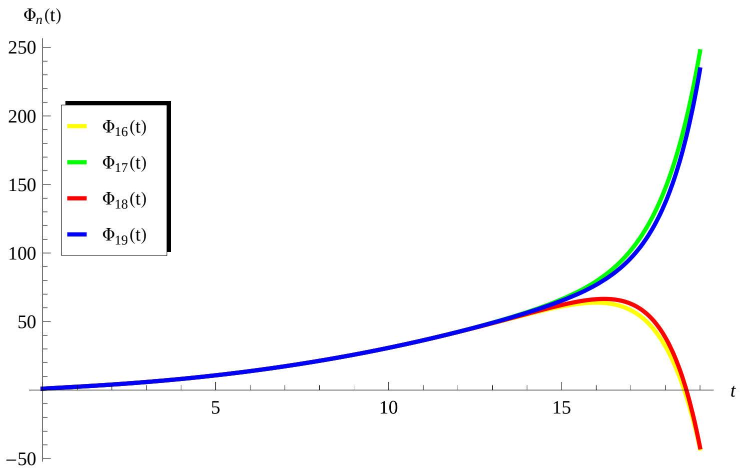

Figure 8. Plots of the approximations $\Theta_m(t)$ ($m = 16, 17, 18, 19$) at $a = -\frac{1}{2}$, $q_1 = \frac{5}{4}$, and $q_2 = \frac{6}{5}$.

7. Conclusions

This paper developed an analytical approach for solving a class of inhomogeneous delay differential equations with variable coefficients and two proportional delays of the pantograph-type. By applying an appropriate exponential transformation, the original IPE was reduced to a HPE, allowing the solution to be expressed as a combination of an exponential term and a power series with explicitly computable coefficients. A thorough convergence analysis was performed, leading to explicit conditions that guarantee the convergence of both the infinite product defining the coefficients and the associated series solution. Moreover, it was shown that for certain parameter choices, the series solution becomes truncated, resulting in exact closed-form solutions. Several such classes were identified and analyzed in detail. The obtained analytic solution for the IPE was invested to derive a new analytic solution for the IAE with two proportional delays. Numerical simulations confirmed the theoretical results and demonstrated that accurate approximations can be achieved using only a small number of the series terms. Overall, the results presented in this work provide a systematic and effective approach for analyzing inhomogeneous pantograph-type equations with multiple proportional delays. The techniques developed here may be extended to more complex models, including fractional-order delay equations, delayed convection-diffusion-reaction equations with constant delay [39] and vanishing delay [40], or equations involving more general forms of time-dependent delays.

Author contributions

Essam R. El-Zahar: Conceptualization, methodology, validation, formal analysis, investigation, writing-review and editing, visualization; Abdelhalim Ebaid: Conceptualization, methodology, validation, formal analysis, investigation, writing-review and editing; Laila F. Seddek: Methodology, validation, formal analysis, investigation, writing-original draft preparation; A. M. Altamimi: Conceptualization, methodology, software, validation, formal analysis, investigation, data curation, writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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