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Research article

Positive convolution structures for q-Bessel functions and a discrete deformation of the Bessel–Kingman hypergroup

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Abstract: We construct a positive convolution on the discrete q-lattice $K_q = q^{\mathbb{Z}} \cup \{0\}$ whose characters are the normalized Hahn–Exton q-Bessel functions. The convolution is obtained from a product formula arising as a limit of the Koelink–Floris product formula for little q-Jacobi polynomials, and its kernel is proved to be nonnegative and probability-preserving. The resulting structure gives a discrete q-deformation of the Bessel–Kingman hypergroup, but its convolution supports are generally noncompact and therefore lie outside the classical DJS axioms. We introduce a degenerate DJS framework adapted to this setting and prove the corresponding Fourier inversion, Plancherel formula, and spectral decomposition for the q-Bessel operator.

Keywords: q-Bessel functions; hypergroups; q-convolution; Fourier analysis; spectral theory; basic hypergeometric series
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1. Introduction

Bessel functions occupy a central position in harmonic analysis, mathematical physics, and the theory of special functions. Their fundamental role becomes particularly transparent in the study of radial analysis on Euclidean spaces. Indeed, when restricting functions on $\mathbb{R}^n$ to radial ones, the Euclidean Fourier transform reduces to an integral transform whose kernel is expressed in terms of Bessel functions. This reduction leads to the *Hankel transform*, which serves as the natural Fourier transform in the radial setting.

For $\alpha > -1$, the Bessel function of the first kind admits the hypergeometric representation

$$J_\alpha(x) = \frac{(x/2)^\alpha}{\Gamma(\alpha+1)}\, {}_0F_1\left(-\,;\alpha+1;-\frac{x^2}{4}\right), \tag{1.1}$$

see, e.g., [1–3]. These functions arise as solutions of Bessel's differential equation and occur naturally in problems involving radial symmetry. The associated Hankel transform of order α is given by

$$\mathcal{H}_\alpha f(\lambda) = \int_0^\infty f(x)\, J_\alpha(x\lambda)\, x\, dx, \tag{1.2}$$

with an inversion formula of the same type; see [3, Ch. VIII]. In the context of $\mathbb{R}^n$, one has $\alpha = \frac{n}{2} - 1$, which highlights the geometric origin of this transform.

A deeper structural understanding of the Hankel transform emerges when it is interpreted as a Fourier transform associated with an underlying convolution structure. This perspective leads naturally to the theory of *hypergroups*, introduced independently by Dunkl [4] and Jewett [5] and developed systematically by Bloom and Heyer [6]. Hypergroups generalize locally compact groups by replacing pointwise multiplication with a convolution of measures, while retaining essential features of harmonic analysis such as duality, characters, and Plancherel theory. A fundamental example is the *Bessel–Kingman hypergroup*, originating in the work of Kingman [7] on radial probability measures. This structure was later developed in depth; (see, for further details, [6, 8, 9]), and plays a central role in the harmonic analysis of radial functions. In this framework, the normalized spherical Bessel functions

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha + 1)\, z^{-\alpha} J_\alpha(z)$$

appear as the multiplicative functions (characters) of the hypergroup. A key ingredient underlying this structure is the classical product formula

$$j_\alpha(x) j_\alpha(y) = c_\alpha \int_0^\pi j_\alpha\Big(\sqrt{x^2 + y^2 - 2xy\cos\theta}\Big) (\sin\theta)^{2\alpha}\, d\theta, \tag{1.3}$$

see [3, §11.4]. This identity encodes the positivity and convolution structure of the theory and leads to a commutative hypergroup convolution on $\mathbb{R}_+$. The associated Fourier transform coincides with the Hankel transform, thereby revealing that the latter is intrinsically a hypergroup Fourier transform. This interpretation is closely related to the theory of generalized translation operators introduced by Delsarte [10], as well as to limit transitions from orthogonal polynomial systems, notably Jacobi polynomials; see [9, 11]. It places Bessel analysis within a broader unifying framework connecting special functions, harmonic analysis, and representation theory.

In recent decades, the theory of q-analogues has emerged as a central theme in the study of special functions and harmonic analysis, motivated both by applications to quantum groups and by the intrinsic richness of q-deformations of classical structures. Within this framework, the Hahn–Exton q-Bessel function plays a fundamental role as a natural q-analogue of the classical Bessel function. Its origins can be traced back to the work of Hahn [12], and it was later systematically developed by Exton [13]. It also belongs to the family of basic Bessel functions studied by Ismail [14, 15]; see also the comprehensive account in [2]. A major conceptual breakthrough was achieved in [16–18], where q-Bessel functions were interpreted as spherical functions associated with quantum groups and related representation-theoretic structures. This viewpoint provides a deep structural explanation for many of their properties and leads naturally to q-analogues of classical constructions, including addition formulas, product formulas, and Fourier-type transforms. In this way, classical harmonic analysis is extended to a noncommutative and quantum setting. From an analytic perspective, the development of a q-Bessel harmonic analysis has been carried out in a series of works by Fitouhi and collaborators [19–21]. These contributions introduced and studied generalized q-translation operators, q-Fourier transforms, and related convolution-type structures, laying the foundations for a functional-analytic approach to the subject.

We mention a possible indirect connection with recent Riemann–Hilbert methods for discrete orthogonal polynomial systems. Such methods are used to study Christoffel–Darboux kernels and limiting regimes on discrete or constrained supports [22]. In the present paper, we do not pursue a Riemann–Hilbert formulation or a steepest–descent analysis. The connection is only motivational. The normalized Hahn–Exton q-Bessel function appears as a confluence limit of little q-Jacobi polynomials, while our approach uses this limit to obtain a positive product formula and a hypergroup-type convolution on K_q. This may provide a harmonic-analytic basis for future Riemann–Hilbert or integrable-kernel studies on the noncompact q-lattice.

Despite these advances, an important gap remains in the existing literature. In contrast with the classical setting, where the Bessel function admits a well-known product formula leading to a positive convolution structure, a general product formula for q-Bessel functions involving a suitable kernel that induces an analogous convolution structure has not yet been fully established. Up to now, such a structure is only available in the particular case of the q-Bessel function of order zero, as shown in [19].

The main objective of the present work is to address this problem by constructing a product formula for the Hahn–Exton q-Bessel function that yields a positive q-translation operator.

This construction is both natural and constructive, and it provides a key tool for further developments in q-translation operators and discrete harmonic analysis. We also indicate below, in Remark 1.1, how the positivity of the present convolution may be used in the study of q-wavelet transforms associated with the basic Bessel operator.

The purpose of the present paper is to construct a positive convolution structure associated with the normalized Hahn–Exton q-Bessel function and to clarify its precise relation to the classical Bessel–Kingman hypergroup. Our approach is rooted in the structural analogy between classical harmonic analysis on $\mathbb{R}_+$ and its q-deformation on the discrete q-lattice. More precisely, starting from a product formula obtained via a limiting transition from the Koelink–Floris product formula for little q-Jacobi polynomials, we define a discrete convolution on the q-lattice

$$K_q = q^{\mathbb{Z}} \cup \{0\} = \{q^k : k \in \mathbb{Z}\} \cup \{0\},$$

which plays the role of a noncompact configuration space in the q-setting. A central step in our construction consists in proving that the kernel arising in this product formula is both positive and probability-preserving. This property is highly nontrivial in the q-framework and constitutes the key ingredient that allows us to define a commutative convolution structure of hypergroup type on K_q. In this sense, our construction provides a genuine q-analogue of the classical Bessel convolution, where positivity and normalization of the kernel encode the underlying probabilistic and harmonic structure. However, a fundamental and genuinely new phenomenon appears in this discrete q-setting: The support of the convolution is no longer compact. This contrasts sharply with the classical Bessel–Kingman hypergroup, where compactness of the convolution support is a crucial structural axiom. As a consequence, the convolution structure constructed here does not fall within the classical DJS hypergroup framework. This breakdown of compactness reflects a deeper geometric feature of the q-lattice, which behaves as a noncompact discrete space with accumulation at zero. Despite this departure from the classical axiomatic setting, the resulting structure retains all the essential features of harmonic analysis. In particular, we identify explicitly the multiplicative functions (characters) of the convolution as the normalized Hahn–Exton q-Bessel functions. Moreover, we show that the associated Fourier transform coincides exactly with the q-Bessel Fourier transform introduced by Koornwinder and Swarttouw [23]. This identification allows us to establish inversion and Plancherel formulas and to obtain a complete spectral decomposition of the corresponding q-Bessel operator, thereby extending the classical Fourier–Bessel theory to the q-discrete framework.

In this way, the present work provides a discrete q-deformation of the Bessel–Kingman hypergroup, preserving its harmonic and spectral features while relaxing the compact-support condition. It thus offers a unified framework linking q-special functions, convolution structures, and spectral theory on noncompact discrete spaces. Beyond this structural contribution, our results reveal a new phenomenon in q-harmonic analysis, namely the emergence of positive convolution structures that lie outside the classical hypergroup setting, thereby opening the way to further developments in noncommutative and discrete harmonic analysis.

From the spectral-theoretic point of view, the degenerate DJS framework is also intended to provide a useful model for discrete operators whose natural configuration spaces are noncompact and have accumulation at the boundary. In such situations, one cannot rely directly on the compact-support axiom of classical hypergroups, but the positivity of the product kernel still gives a probabilistic convolution, a character theory, and a Plancherel decomposition. Thus, the present construction suggests a possible route toward spectral problems for other noncompact discrete q-difference operators and boundary accumulation phenomena.

We summarize the main results of this paper as follows.

Theorem A. Let $\nu > 0$. The normalized Hahn–Exton q-Bessel function $\mathcal{J}_\nu(\cdot; q^2)$ satisfies a product formula of the form

$$\mathcal{J}_\nu(q^r; q^2)\mathcal{J}_\nu(q^s; q^2) = \sum_{x\in\mathbb{Z}} A_\nu(q^r, q^s, q^x)\,\mathcal{J}_\nu(q^x; q^2),$$

where the kernel $A_\nu(q^r, q^s, q^x)$ is given explicitly, is nonnegative, and satisfies the normalization condition

$$\sum_{x\in\mathbb{Z}} A_\nu(q^r, q^s, q^x) = 1.$$

In particular, $A_\nu(q^r, q^s, q^x)$ defines a probability measure on $q^{\mathbb{Z}}$.

Theorem B. The kernel A_ν defines a commutative, associative convolution $*_\nu$ on K_q by

$$\delta_x *_\nu \delta_y = \begin{cases} \displaystyle\sum_{z\in q^{\mathbb{Z}}} A_\nu(x, y, z)\,\delta_z, & x, y \in q^{\mathbb{Z}}, \\ \delta_y, & x = 0,\ y \in q^{\mathbb{Z}}, \\ \delta_x, & y = 0,\ x \in q^{\mathbb{Z}}, \\ \delta_0, & x = y = 0. \end{cases}$$

This convolution is probability-preserving and admits the normalized q-Bessel functions

$$\varphi_\lambda(x) = \mathcal{J}_\nu(\lambda x; q^2), \quad \lambda \in K_q,$$

as multiplicative functions (characters), i.e.,

$$\int_{K_q} \varphi_\lambda(z)\, d(\delta_x *_\nu \delta_y)(z) = \varphi_\lambda(x)\varphi_\lambda(y).$$

However, in contrast with the classical Bessel–Kingman hypergroup, the support of $\delta_x *_\nu \delta_y$ is not compact. Consequently, $(K_q, *_\nu)$ does not satisfy the DJS hypergroup axioms.

Remark 1.1. The positivity of the kernel A_ν is useful beyond the construction of the convolution itself. In particular, it gives a positive generalized translation operator

$$T_x^\nu f(y) := \int_{K_q} f(z)\, d(\delta_x *_\nu \delta_y)(z),$$

which is the natural translation entering the definition of q-Bessel wavelet transforms. Thus, the present product formula supplies the missing positive kernel needed to extend the usual wavelet identities from the order-zero case to the normalized Hahn–Exton q-Bessel setting of order $\nu > 0$.

2. Preliminaries

2.1. Notations

Throughout this paper, we fix $q \in (0, 1)$ and denote by $\mathbb{Z}_+$ the set of nonnegative integers. For $a \in \mathbb{C}$ and $n \in \mathbb{Z}_+$, the q-shifted factorial is defined by

$$(a; q)_0 := 1, \qquad (a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k),$$

and for negative integers,

$$(a; q)_{-n} := \frac{1}{(aq^{-n}; q)_n}, \qquad n \in \mathbb{Z}_+.$$

We also define the infinite product

$$(a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k),$$

which converges for $|q| < 1$. These objects play a central role in the theory of basic hypergeometric functions; see [2, 13].

The following estimates will be used frequently. For $n, m \in \mathbb{Z}_+$ and $\alpha > 0$, we record them in separated displays so that each equation number is attached to a single formula:

$$(q^\alpha; q)_\infty \le (q^\alpha; q)_n \le (-q^\alpha; q)_\infty, \tag{2.1}$$

$$\left|(q^{-m}; q)_n\, q^{nm}\right| \le q^{\binom{n}{2}}. \tag{2.2}$$

The basic hypergeometric series ${}_r\phi_s$ is defined by

$${}_r\phi_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}\middle|\, q,\, z\right) = \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_r; q)_k}{(b_1; q)_k \cdots (b_s; q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} \frac{z^k}{(q; q)_k}. \tag{2.3}$$

This is a q-analogue of the classical hypergeometric series [2]. In particular, the function

$$ {}_1\phi_1\left(\begin{matrix} a \\ w \end{matrix}\middle|\, q,\ z\right) = \sum_{k=0}^{\infty} \frac{(a;q)_k}{(w;q)_k (q;q)_k} (-1)^k z^k q^{\binom{k}{2}} $$

defines a meromorphic function of w with poles at $w = q^{-n}$, $n \in \mathbb{Z}_+$. Multiplication by $(w;q)_\infty$ removes these singularities and yields an entire function in $(w,z) \in \mathbb{C}^2$. Using standard expansions, one obtains the representation

$$ (w;q)_\infty \, {}_1\phi_1(a;w;q,z) = \sum_{k,l\geq 0} \frac{(a;q)_k}{(q;q)_k} \frac{q^{kl}(-1)^{k+l} q^{\binom{k}{2}+\binom{l}{2}}}{(q;q)_l} z^k w^l, \tag{2.4} $$

which converges absolutely and uniformly on compact subsets of $\mathbb{C}^2$. In particular,

$$ \left|(w;q)_\infty \, {}_1\phi_1(a;w;q,z)\right| \leq (-|a|, -|z|, -|w|; q)_\infty. $$

A fundamental identity is the q-Kummer symmetry relation

$$ (w;q)_\infty \, {}_1\phi_1(a;w;q,z) = (z;q)_\infty \, {}_1\phi_1(az/w;z;q,w). \tag{2.5} $$

Let $n \in \mathbb{N}$. Evaluating (2.4) at $w = q^{1-n}$ yields

$$ (q^{1-n};q)_\infty \, {}_1\phi_1\left(\begin{matrix} a \\ q^{1-n} \end{matrix}\middle|\, q,\ z\right) = (-z)^n q^{\binom{n}{2}} (a;q)_n (q^{1+n};q)_\infty \, {}_1\phi_1\left(\begin{matrix} aq^n \\ q^{1+n} \end{matrix}\middle|\, q,\ q^n z\right). \tag{2.6} $$

2.2. The Koornwinder–Swarttouw q-Hankel transform and its normalized form

The Hahn–Exton q-Bessel function (also known as the third Jackson q-Bessel function) plays a fundamental role in q-harmonic analysis, as it provides the natural kernel for q-analogues of Fourier–Bessel transforms. In the formulation of Koornwinder and Swarttouw [17, 18, 23], the function in base q^2 is defined by

$$ J_\alpha(z;q^2) := z^\alpha \frac{(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty} \, {}_1\phi_1\left(\begin{matrix} 0 \\ q^{2\alpha+2} \end{matrix}\middle|\, q^2,\ q^2 z^2\right), \qquad \alpha > -1. \tag{2.7} $$

This function constitutes a genuine q-deformation of the classical Bessel function J_α and reduces formally to it in the limit $q \uparrow 1$. It inherits many structural properties of its classical counterpart, including its role as an eigenfunction of a second-order difference operator and its appearance in harmonic analysis on quantum and discrete structures.

Associated with (2.7) is a discrete q-Hankel transform defined on the q-lattice $\{q^k : k \in \mathbb{Z}\}$, which serves as the q-analogue of the classical Hankel transform. For suitable functions f, the transform pair is given by

$$ \begin{cases} g(q^n) = \displaystyle\sum_{k=-\infty}^{\infty} q^k J_\alpha(q^{k+n};q^2) f(q^k), \\ f(q^k) = \displaystyle\sum_{n=-\infty}^{\infty} q^n J_\alpha(q^{k+n};q^2) g(q^n), \end{cases} \qquad k, n \in \mathbb{Z}. \tag{2.8} $$

This transform provides a discrete Fourier-type analysis adapted to the q-lattice and plays a central role in the spectral theory of q-difference operators. An equivalent integral formulation is obtained using Jackson's q-integral

$$\int_0^\infty h(x)\,d_qx := (1-q)\sum_{k=-\infty}^{\infty} h(q^k)\,q^k, \tag{2.9}$$

which yields

$$\begin{cases} g(\lambda) = \displaystyle\int_0^\infty f(x)\,J_\alpha((1-q)\lambda x;q^2)\,x\,d_qx, \\ f(x) = \displaystyle\int_0^\infty g(\lambda)\,J_\alpha((1-q)\lambda x;q^2)\,\lambda\,d_q\lambda, \end{cases} \tag{2.10}$$

for $x, \lambda \in \{q^k : k \in \mathbb{Z}\}$. Under appropriate normalization, this transform converges formally to the classical Hankel transform, thereby reinforcing the interpretation of q-analysis as a deformation of classical harmonic analysis.

For structural and analytical purposes, it is convenient to introduce the normalized q-Bessel function

$$\mathcal{J}_\nu(x;q^2) := {}_1\phi_1\left(\begin{matrix}0\\ q^{2\nu+2}\end{matrix}\middle|\, q^2,\, q^2x^2\right), \qquad \nu > -1. \tag{2.11}$$

Throughout the paper, we use the single notation $\mathcal{J}_\nu(\cdot;q^2)$ for this normalized Hahn–Exton q-Bessel function. This function differs from the Hahn–Exton q-Bessel function only by a multiplicative normalization factor. Such a normalization is particularly well adapted to harmonic analysis, as it leads to simpler convolution kernels and more transparent spectral formulas. The normalization can be expressed in terms of the q-gamma function

$$\Gamma_q(\alpha+1) = \frac{(q;q)_\infty}{(q^{\alpha+1};q)_\infty}(1-q)^{-\alpha}, \tag{2.12}$$

which implies the identity

$$\frac{(q^{\alpha+1};q)_\infty}{(q;q)_\infty} = \frac{(1-q)^\alpha}{\Gamma_q(\alpha+1)}.$$

Using this relation, set

$$c_{q,\nu} := \frac{(1-q^2)^\nu}{\Gamma_{q^2}(\nu+1)}.$$

Then the normalized q-Hankel transform pair can be written in the clean aligned form

$$\begin{aligned} g(\lambda) &= c_{q,\nu}\int_0^\infty f(x)\,\mathcal{J}_\nu(\lambda x;q^2)\,x^{2\nu+1}\,d_qx, \\ f(x) &= c_{q,\nu}\int_0^\infty g(\lambda)\,\mathcal{J}_\nu(\lambda x;q^2)\,\lambda^{2\nu+1}\,d_q\lambda. \end{aligned} \tag{2.13}$$

This normalized formulation makes explicit the analogy with the classical Fourier–Bessel transform associated with the Bessel–Kingman hypergroup. In this correspondence, the measure $x^{2\nu+1}d_qx$ plays the role of a q-analogue of the Haar measure, while the kernel $\mathcal{J}_\nu(\lambda x;q^2)$ acts as a deformation of the classical character $j_\nu(\lambda x)$.

Proposition 2.1. *For every $\lambda \in \mathbb{C}$, the function $x \mapsto \mathcal{J}_\nu(\lambda x; q^2)$ is an eigenfunction of the q-Bessel operator*

$$\Delta_{q,\nu} f(x) := \frac{1}{x^2}\Big(f(q^{-1}x) - (1+q^{2\nu})f(x) + q^{2\nu} f(qx)\Big)$$

and satisfies

$$\Delta_{q,\nu}\mathcal{J}_\nu(\lambda x; q^2) = -\lambda^2\, \mathcal{J}_\nu(\lambda x; q^2).$$

Proof. Using the series expansion

$$\mathcal{J}_\nu(x; q^2) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)}}{(q^2;q^2)_k (q^{2\nu+2};q^2)_k}\, x^{2k},$$

we apply $\Delta_{q,\nu}$ term-by-term. A direct computation yields

$$\Delta_{q,\nu} x^{2k} = -\frac{(1-q^{2k})(1-q^{2k+2\nu})}{q^{2k}}\, x^{2k-2}.$$

Substituting into the series and shifting the summation index, we obtain

$$\Delta_{q,\nu}\mathcal{J}_\nu(\lambda x; q^2) = -\lambda^2 \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)}}{(q^2;q^2)_k (q^{2\nu+2};q^2)_k} (\lambda x)^{2k},$$

which coincides with $-\lambda^2 \mathcal{J}_\nu(\lambda x; q^2)$. □

As a consequence, the operator $\Delta_{q,\nu}$ admits a natural realization as an unbounded operator on the Hilbert space $L^2(q^{\mathbb{Z}}, x^{2\nu+1} d_q x)$. Its domain is given by

$$\mathcal{D}(\Delta_{q,\nu}) = \left\{ f \in L^2(q^{\mathbb{Z}}, x^{2\nu+1} d_q x) \ : \ \Delta_{q,\nu} f \in L^2(q^{\mathbb{Z}}, x^{2\nu+1} d_q x) \right\},$$

and $\Delta_{q,\nu}$ acts as a symmetric q-difference operator.

The orthogonality relations and the Plancherel theorem for q-Bessel functions [23] imply that the q-Bessel Fourier transform extends to a unitary operator on $L^2(q^{\mathbb{Z}}, m_\nu)$. Consequently, $\Delta_{q,\nu}$ is diagonalized by this transform, which shows that it is essentially self-adjoint and admits a complete spectral resolution. More precisely, the spectrum of $\Delta_{q,\nu}$ is given by

$$\sigma(\Delta_{q,\nu}) = -\{\lambda^2 : \lambda \in q^{\mathbb{Z}}\} = -q^{2\mathbb{Z}}.$$

Thus, the spectrum is purely discrete with 0 as its unique accumulation point. In particular,

$$\sigma_{\text{disc}}(\Delta_{q,\nu}) = -q^{2\mathbb{Z}}, \qquad \sigma_{\text{ess}}(\Delta_{q,\nu}) = \{0\}.$$

This spectral description highlights a characteristic feature of the q-setting: Although the underlying space $q^{2\mathbb{Z}}$ is discrete, the associated operator exhibits nontrivial spectral accumulation. This phenomenon reflects the noncompact nature of the q-lattice and its role as a discrete deformation of the continuous half-line.

2.3. Koelink–Floris product formula for little q-Jacobi polynomials

The little q-Jacobi polynomials form one of the fundamental families in the q-Askey scheme and provide a natural q-deformation of the classical Jacobi polynomials; see [2]. Beyond their role in special function theory, they appear naturally in the harmonic analysis of quantum groups and q-symmetric spaces. In particular, they arise in the study of quantum disk polynomials developed by Noumi, Mimachi, and Yamada, and in the representation-theoretic framework introduced by Floris and Koelink [24].

From a historical perspective, the classical theory originates in disk polynomials and their addition formula, established independently by Koornwinder and Sapiro. This addition formula yields, as a consequence, the classical product formula for Jacobi polynomials, which underlies the Bessel–Kingman hypergroup structure. In the q-setting, Floris and Koelink constructed a deep analogue of this theory by starting from a noncommutative addition formula for q-disk polynomials and passing to commuting variables via suitable $*$-representations. This procedure leads to a nontrivial and highly structured product formula for little q-Jacobi polynomials, which plays a central role in our analysis.

For parameters $\alpha, \beta > -1$, the little q-Jacobi polynomials are defined by

$$p_n(x; q^{2\alpha}, q^{2\beta}; q^2) = {}_2\phi_1\left(\begin{matrix} q^{-2n}, \; q^{2(\alpha+\beta+n+1)} \\ q^{2\alpha+2} \end{matrix} \middle| \, q^2, x \right), \qquad n \in \mathbb{Z}_{\geq 0}. \tag{2.14}$$

They are orthogonal with respect to a discrete measure supported on the geometric lattice $\{q^{2k} : k \in \mathbb{Z}_+\}$, namely,

$$\begin{aligned} &\frac{(q^{2\alpha+2}, q^{2\beta+2}; q^2)_\infty}{(q^{2\alpha+2\beta+4}, q^2; q^2)_\infty} \sum_{k=0}^{\infty} p_m(q^{2k})\, p_n(q^{2k})\, q^{2k(\alpha+1)} \frac{(q^{2k+2}; q^2)_\infty}{(q^{2\beta+2k+2}; q^2)_\infty} \\ =& \frac{q^{2n(\alpha+1)}(1 - q^{2\alpha+2\beta+2})(q^{2\beta+2}, q^2; q^2)_n}{(1 - q^{2\alpha+2\beta+4n+2})(q^{2\alpha+2}, q^{2\alpha+2\beta+2}; q^2)_n}\, \delta_{m,n}. \end{aligned} \tag{2.15}$$

This orthogonality structure is the discrete analogue of the classical Jacobi orthogonality and reflects the underlying q-geometry.

There exists a limit transition connecting little q-Jacobi polynomials to the normalized Hahn–Exton q-Bessel function. As shown in [23, 25], for $\nu > 0$ one has

$$\lim_{n\to\infty} p_n(x^2; q^{2\nu}, 1; q^2) = \mathcal{J}_\nu(x; q^2), \tag{2.16}$$

with uniform convergence on compact subsets of $\mathbb{C}$. This limit transition is fundamental: It allows one to pass from polynomial structures to Bessel-type harmonic analysis, and will be the key mechanism for deriving the q-Bessel product formula in the sequel.

The Koelink–Floris product formula involves auxiliary orthogonal polynomials, namely the affine q-Krawtchouk polynomials, defined by [24]

$$K_z(q^{-x}; t, N; q^2) = {}_3\phi_2\left(\begin{matrix} q^{-2z}, \; q^{-2x}, \; 0 \\ tq^2, \; q^{-2N} \end{matrix} \middle| \, q^2, \, q^2 \right), \qquad 0 \leq z \leq N, \quad 0 < tq^2 < 1. \tag{2.17}$$

They satisfy a discrete orthogonality relation of the form

$$\sum_{x=0}^{N} w(x)\, K_r(x)\, K_s(x) = \delta_{r,s}\, h_r, \tag{2.18}$$

where $w(x)$ and h_r are explicit positive weights. It is convenient to introduce their normalized version

$$\widehat{K}_n(x;t,N;q^2) = (-1)^n (tq^2)^{-n/2} \left(\frac{(q^{2N};q^{-2})_n (tq^2;q^2)_n}{(q^2;q^2)_n} \right)^{1/2} K_n(x;t,N;q^2), \tag{2.19}$$

which simplifies the structure of the product formula.

We are now in a position to state the fundamental product formula established by Koelink and Floris. Specializing [24, Corollary 6.4], one obtains for $\nu > 0$, $t \in (0, q^{-2})$, and $z \in \mathbb{Z}_+$:

$$\begin{aligned} & p_n(q^{2z};q^{2\nu},1;q^2)\, p_n(tq^{2z};q^{2\nu},1;q^2) \\ &= (1-q^{2\nu}) \sum_{N=z}^{\infty} \sum_{x=0}^{N} \frac{(q^{2N};q^{-2})_{N-x} (tq^2;q^2)_{N-x}}{(q^2;q^2)_{N-x}} (tq^2)^x q^{2\nu(N-z)} \\ &\quad \times p_n(q^{2x};q^{2\nu},1;q^2) \left(\widehat{K}_z(q^{2(x-N)};t,N;q^2)\right)^2. \end{aligned} \tag{2.20}$$

This product formula is the q-analogue of the classical Jacobi product formula and constitutes the starting point of our construction. In particular, by performing a suitable limit transition as $n \to \infty$, it will yield a product formula for the normalized Hahn–Exton q-Bessel function. This passage from polynomial identities to Bessel-type convolution structures is at the heart of the present work and provides the bridge between the Koelink–Floris theory and the q-Bessel harmonic analysis developed in the subsequent sections.

3. Product formula for the normalized q-Bessel functions

The following theorem constitutes the first main result of this paper. It establishes a product formula for the normalized Hahn–Exton q-Bessel functions and provides the foundation for the construction of the associated convolution structure.

Theorem 3.1. *Let $0 < q < 1$ and $\nu > 0$ and let $r, s \in \mathbb{Z}$ be integers with $s \le r$. Then the normalized q-Bessel functions satisfy the product formula*

$$\mathcal{J}_\nu(q^r;q^2)\mathcal{J}_\nu(q^s;q^2) = \sum_{x\in\mathbb{Z}} A_\nu(q^r,q^s,q^x)\, \mathcal{J}_\nu(q^x;q^2), \tag{3.1}$$

where the kernel $A_\nu(q^r,q^s,q^x)$ is given by

$$\begin{aligned} A_\nu(q^r,q^s,q^x) &= (1-q^{2\nu})\, q^{2(1+r-s)(x-s)} \frac{(q^{2(1+r-s)};q^2)_\infty}{(q^2;q^2)_\infty} \sum_{N=\max(r,s,x)}^{\infty} q^{2\nu(N-s)} \\ &\quad \times \frac{(q^{2(1+r-s)};q^2)_{N-x}}{(q^2;q^2)_{N-x}(q^{2(N-s+1)};q^2)_\infty} \\ &\quad \times \left[{}_1\phi_1\!\left(\begin{matrix} q^{2(N-x+r-s+1)} \\ q^{2(1+r-s)} \end{matrix} \,\middle|\, q^2,\, q^{2(x-s+1)} \right) \right]^2. \end{aligned} \tag{3.2}$$

For the opposite ordering of the first two lattice indices, we use the symmetric extension

$$A_\nu(q^r,q^s,q^x) := A_\nu(q^s,q^r,q^x), \qquad r < s.$$

The identity (3.1) should be interpreted as a discrete convolution formula on the q-lattice. The kernel A_ν plays the role of a transition probability; it is nonnegative and normalized, thereby defining a convolution structure of hypergroup type. Before proving Theorem 3.1, we establish the following uniform bounds for the little q-Jacobi polynomials, which will be essential for controlling the kernel and justifying limit transitions.

Lemma 3.2. *Let $0 < q < 1$ and $\nu > 0$. Then, uniformly for $l \in \mathbb{Z}_+$ and integers $x \geq -l$, one has*

$$\left| {}_2\phi_1\left(\begin{matrix} q^{-2l},\, q^{2(\nu+l+1)} \\ q^{2(\nu+1)} \end{matrix} \,\middle|\, q^2,\, q^{2(x+l+1)} \right) \right| \ll \begin{cases} 1, & x \geq 0, \\ q^{x^2/4}, & x < 0, \end{cases}$$

where the implied constant depends only on q and ν.

Proof. Set

$$F_l(x) := {}_2\phi_1\left(\begin{matrix} q^{-2l},\, q^{2(\nu+l+1)} \\ q^{2(\nu+1)} \end{matrix} \,\middle|\, q^2,\, q^{2(x+l+1)} \right).$$

Since the first numerator parameter is q^{-2l}, the series terminates at $k = l$.
Thus,

$$F_l(x) = \sum_{k=0}^{l} \frac{(q^{-2l}; q^2)_k (q^{2(\nu+l+1)}; q^2)_k}{(q^{2(\nu+1)}; q^2)_k (q^2; q^2)_k} q^{2k(x+l+1)}.$$

Using the estimates

$$|(q^{-2l}; q^2)_k| \leq q^{-2lk+k(k-1)}, \qquad |(q^{2(\nu+l+1)}; q^2)_k| \leq 1,$$

and

$$(q^{2(\nu+1)}; q^2)_k \geq (q^{2(\nu+1)}; q^2)_\infty > 0,$$

we obtain, for $x \geq 0$,

$$|F_l(x)| \ll \sum_{k=0}^{l} q^{-2lk+k(k-1)} q^{2k(x+l+1)} = \sum_{k=0}^{l} q^{2kx+k(k+1)} \leq \sum_{k=0}^{\infty} q^{k(k+1)} \ll 1.$$

Now assume $x < 0$. Since $x \geq -l$, write

$$m = x + l, \qquad 0 \leq m \leq l.$$

Using a standard terminating transformation for the basic hypergeometric series, we get

$$F_l(x) = \frac{(q^{-2l}; q^2)_l}{(q^{2(\nu+1)}; q^2)_l} q^{2l(\nu+l+1)} G_{l,m},$$

where

$$G_{l,m} = \sum_{k=0}^{m} \frac{(q^{-2l}; q^2)_k (q^{-2m}; q^2)_k}{(q^2; q^2)_k^2} q^{2k(m-\nu)}.$$

Again using the standard estimate for shifted factorials, we have

$$\left| \frac{(q^{-2l}; q^2)_l}{(q^{2(\nu+1)}; q^2)_l} q^{2l(\nu+l+1)} \right| \ll q^{l^2+O(l)}$$

and

$$|G_{l,m}| \ll \sum_{k=0}^{m} q^{2k^2-2(l+\nu+1)k} \ll q^{-(l+\nu+1)^2/2}.$$

Therefore,

$$|F_l(x)| \ll q^{l^2/2+O(l)}.$$

Since $x \geq -l$, we have $|x| \leq l$. Hence, $l^2 \geq x^2$, and after adjusting the implicit constant,

$$|F_l(x)| \ll q^{x^2/4}.$$

This proves the desired estimate. □

The important point is that the bound is uniform in the truncation parameter and contains a Gaussian-type factor in the negative direction of the lattice. This decay compensates for the noncompact part of the summation domain and is therefore strong enough to justify the later dominated-convergence passage over an unbounded set of lattice indices. In particular, the estimate is insensitive to the moving truncation endpoint, which is why it remains robust in the confluence limit from little q-Jacobi polynomials to q-Bessel functions.

We now turn to the proof of Theorem 3.1.

Proof. Set $z = n + r$ and $t = q^{2(s-r)}$, so that $tq^{2z} = q^{2(n+s)}$. Substituting these values into the Koelink–Floris product formula (2.20), we obtain

$$\begin{aligned} &p_n(q^{2(n+r)}; q^{2\nu}, 1; q^2)\, p_n(q^{2(n+s)}; q^{2\nu}, 1; q^2) \\ &= (1-q^{2\nu}) \sum_{M=n+r}^{\infty} \sum_{u=0}^{M} \frac{(q^{2M}; q^{-2})_{M-u} (q^{2(s-r+1)}; q^2)_{M-u}}{(q^2; q^2)_{M-u}} q^{2(s-r+1)u} q^{2\nu(M-n-r)} \\ &\quad \times p_n(q^{2u}; q^{2\nu}, 1; q^2) (\widehat{K}_{n+r}(q^{2(u-M)}; q^{2(s-r)}, M; q^2))^2. \end{aligned} \tag{3.3}$$

Next, from (2.17) and (2.19), we have

$$(\widehat{K}_z(q^{2(u-M)}; t, M; q^2))^2 = \frac{(q^{2M}; q^{-2})_z (tq^2; q^2)_z}{(q^2; q^2)_z} (tq^2)^{-z} \times \left[{}_3\phi_2\!\left(\begin{matrix} q^{-2z},\, q^{2(u-M)},\, 0 \\ tq^2,\, q^{-2M} \end{matrix} \middle|\, q^2, q^2 \right) \right]^2.$$

We use the terminating transformation

$${}_3\phi_2\!\left(\begin{matrix} q^{-2n},\, c/b,\, 0 \\ c,\, cq^2/(b\zeta) \end{matrix} \middle|\, q^2, q^2 \right) = \frac{(-1)^n q^{n(n-1)}}{(cq^2/(b\zeta); q^2)_n} \left(\frac{cq^2}{b\zeta} \right)^n {}_2\phi_1\!\left(\begin{matrix} q^{-2n},\, b \\ c \end{matrix} \middle|\, q^2, \zeta \right).$$

Applying it with

$$n = z, \qquad c = tq^2 = q^{2(s-r+1)}, \qquad \frac{c}{b} = q^{2(u-M)}, \qquad \frac{cq^2}{b\zeta} = q^{-2M},$$

we obtain

$${}_3\phi_2\!\left(\begin{matrix} q^{-2z},\, q^{2(u-M)},\, 0 \\ q^{2(s-r+1)},\, q^{-2M} \end{matrix} \middle|\, q^2, q^2 \right) = \frac{(-1)^z q^{z(z-1)}}{(q^{-2M}; q^2)_z} q^{-2Mz} \times {}_2\phi_1\!\left(\begin{matrix} q^{-2z},\, q^{2(M-u+s-r+1)} \\ q^{2(s-r+1)} \end{matrix} \middle|\, q^2, q^{2(u+1)} \right).$$

Since

$$(q^{-2M};q^2)_z=(-1)^z q^{-2Mz+z(z-1)}(q^{2M};q^{-2})_z,$$

this simplifies to

$${}_3\phi_2\left(\begin{matrix}q^{-2z},\ q^{2(u-M)},\ 0\\ q^{2(s-r+1)},\ q^{-2M}\end{matrix}\middle|\,q^2,\ q^2\right)=\frac{1}{(q^{2M};q^{-2})_z}\,{}_2\phi_1\left(\begin{matrix}q^{-2z},\ q^{2(M-u+s-r+1)}\\ q^{2(s-r+1)}\end{matrix}\middle|\,q^2,\ q^{2(u+1)}\right).$$

Therefore,

$$(\widehat{K}_z(q^{2(u-M)};t,M;q^2))^2=\frac{(tq^2;q^2)_z}{(q^2;q^2)_z(q^{2M};q^{-2})_z}(tq^2)^{-z}\times\left[{}_2\phi_1\left(\begin{matrix}q^{-2z},\ q^{2(M-u+s-r+1)}\\ q^{2(s-r+1)}\end{matrix}\middle|\,q^2,\ q^{2(u+1)}\right)\right]^2. \tag{3.4}$$

Substituting (3.4) into (3.3) and making the change of variables

$$M=N+n,\qquad u=x+n,$$

we get $N\geq r$ and $-n\leq x\leq N$. Since $z=n+r$ and $tq^2=q^{2(s-r+1)}$, this gives

$$\begin{aligned}
&p_n(q^{2(n+r)};q^{2\nu},1;q^2)\,p_n(q^{2(n+s)};q^{2\nu},1;q^2)\\
&=(1-q^{2\nu})\sum_{N=r}^{\infty}\sum_{x=-n}^{N}q^{2\nu(N-r)}q^{2(s-r+1)(x-r)}\frac{(q^{2(s-r+1)};q^2)_{N-x}}{(q^2;q^2)_{N-x}}\\
&\quad\times\frac{(q^{2(s-r+1)};q^2)_{n+r}}{(q^2;q^2)_{n+r}}\frac{(q^2;q^2)_{N-r}}{(q^2;q^2)_{x+n}}\\
&\quad\times\left[{}_2\phi_1\left(\begin{matrix}q^{-2(n+r)},\ q^{2(N-x+s-r+1)}\\ q^{2(s-r+1)}\end{matrix}\middle|\,q^2,\ q^{2(x+n+1)}\right)\right]^2 p_n(q^{2(n+x)};q^{2\nu},1;q^2).
\end{aligned}\tag{3.5}$$

For fixed $N\geq r$ and $x\leq N$, the standard confluence limit for the basic hypergeometric series gives

$${}_2\phi_1\left(\begin{matrix}q^{-2(n+r)},\ q^{2(N-x+s-r+1)}\\ q^{2(s-r+1)}\end{matrix}\middle|\,q^2,\ q^{2(x+n+1)}\right)\longrightarrow{}_1\phi_1\left(\begin{matrix}q^{2(N-x+s-r+1)}\\ q^{2(s-r+1)}\end{matrix}\middle|\,q^2,\ q^{2(x-r+1)}\right),$$
$$p_n(q^{2(n+x)};q^{2\nu},1;q^2)\longrightarrow\mathcal{J}_\nu(q^x;q^2).$$

Moreover,

$$\frac{(q^{2(s-r+1)};q^2)_{n+r}}{(q^2;q^2)_{n+r}}\frac{(q^2;q^2)_{N-r}}{(q^2;q^2)_{x+n}}\longrightarrow\frac{(q^{2(s-r+1)};q^2)_\infty}{(q^2;q^2)_\infty}\frac{1}{(q^{2(N-r+1)};q^2)_\infty}.$$

Hence, the summand in (3.5) converges pointwise to

$$\begin{aligned}
\mathcal{W}(N,x)&=q^{2\nu(N-r)}q^{2(s-r+1)(x-r)}\frac{(q^{2(s-r+1)};q^2)_\infty}{(q^2;q^2)_\infty}\frac{(q^{2(s-r+1)};q^2)_{N-x}}{(q^2;q^2)_{N-x}(q^{2(N-r+1)};q^2)_\infty}\\
&\quad\times\left[{}_1\phi_1\left(\begin{matrix}q^{2(N-x+s-r+1)}\\ q^{2(s-r+1)}\end{matrix}\middle|\,q^2,\ q^{2(x-r+1)}\right)\right]^2\mathcal{J}_\nu(q^x;q^2).
\end{aligned}$$

It remains to justify the passage to the limit under the double sum. Let $\mathcal{W}_n(N,x)$ denote the summand in (3.5), extended by zero for $x<-n$. To dominate it, we return to the original expression (3.3), before transforming the Krawtchouk factor. After the shift $M = N + n$, $u = x + n$, the corresponding summand is

$$\mathcal{W}_n(N, x) = q^{2\nu(N-r)} \frac{(q^{2(N+n)}; q^{-2})_{N-x}(q^{2(s-r+1)}; q^2)_{N-x}}{(q^2; q^2)_{N-x}} q^{2(s-r+1)(x+n)} \\ \times p_n(q^{2(n+x)}; q^{2\nu}, 1; q^2)(\widehat{K}_{n+r}(q^{2(x-N)}; q^{2(s-r)}, N + n; q^2))^2.$$

The family

$$\{\widehat{K}_j(\,\cdot\,; q^{2(s-r)}, N + n; q^2)\}_{j=0}^{N+n}$$

is orthonormal with respect to the weight

$$w_{N+n}(v) = \frac{(q^{2(N+n)}; q^{-2})_v(q^{2(s-r+1)}; q^2)_v}{(q^2; q^2)_v} q^{-2(s-r+1)v}, \qquad v = 0, 1, \ldots, N + n.$$

Therefore,

$$\sum_{j=0}^{N+n} \widehat{K}_j(q^{-2v}; q^{2(s-r)}, N + n; q^2)^2 w_{N+n}(v) = 1,$$

and hence each term is bounded by $[w_{N+n}(v)]^{-1}$. Taking $v = N - x$, we obtain

$$(\widehat{K}_{n+r}(q^{2(x-N)}; q^{2(s-r)}, N + n; q^2))^2 \le \frac{1}{w_{N+n}(N - x)}.$$

Consequently,

$$|\mathcal{W}_n(N, x)| \le q^{2\nu(N-r)} q^{2(s-r+1)(N+n)} \left| p_n(q^{2(n+x)}; q^{2\nu}, 1; q^2) \right|.$$

Using Lemma 3.2, together with the elementary shifted-factorial estimates collected in Section 2.1, we obtain a uniform bound for the summand. More precisely, the factors depending on the moving endpoint are controlled by the Gaussian decay in the negative lattice direction and by a superexponential decay in the variable $N - r$. Thus there are constants $C > 0$ and $\varepsilon > 0$, depending only on q, ν, r, s, such that

$$|\mathcal{W}_n(N, x)| \le C\, q^{\varepsilon(N-r)^2} \left(\mathbf{1}_{\{0 \le x \le N\}} + q^{x^2/4} \mathbf{1}_{\{x<0\}} \right), \qquad N \ge r, \quad x \le N.$$

Hence,

$$\sum_{N=r}^{\infty} \sum_{x=-\infty}^{N} q^{\varepsilon(N-r)^2} \left(\mathbf{1}_{\{0 \le x \le N\}} + q^{x^2/4} \mathbf{1}_{\{x<0\}} \right) \\ \le \sum_{N=r}^{\infty} (N + 1) q^{\varepsilon(N-r)^2} + \left(\sum_{N=r}^{\infty} q^{\varepsilon(N-r)^2} \right) \left(\sum_{x=-\infty}^{-1} q^{x^2/4} \right) < \infty.$$

This proves the existence of an n-independent summable majorant on the whole set $\{(N, x) : N \ge r,\ x \le N\}$. In particular, the passage to the limit under the unbounded sum over $x < 0$ is justified separately by the summability of $\sum_{x<0} q^{x^2/4}$. Therefore, dominated convergence applies to the double series. Passing to the limit in (3.5), we get

$$
\begin{aligned}
&\mathcal{J}_\nu(q^r;q^2)\mathcal{J}_\nu(q^s;q^2)\\
&= (1-q^{2\nu})\sum_{N=r}^{\infty}\sum_{x=-\infty}^{N} q^{2\nu(N-r)}q^{2(s-r+1)(x-r)}\frac{(q^{2(s-r+1)};q^2)_\infty}{(q^2;q^2)_\infty}\\
&\quad\times\frac{(q^{2(s-r+1)};q^2)_{N-x}}{(q^2;q^2)_{N-x}(q^{2(N-r+1)};q^2)_\infty}\left[{}_1\phi_1\left(\begin{matrix}q^{2(N-x+s-r+1)}\\ q^{2(s-r+1)}\end{matrix}\middle| q^2,\ q^{2(x-r+1)}\right)\right]^2\mathcal{J}_\nu(q^x;q^2).
\end{aligned}
$$

For fixed x, the summation over N starts at $N = \max(r, s, x)$. The displayed coefficient is the regularized form of the kernel. Indeed, applying the shift identity (2.6) in base q^2, with $n = r - s$, transforms the apparent parameter $q^{2(s-r+1)}$ into the nonsingular parameter $q^{2(1+r-s)}$. After the elementary simplification of powers of q and of the finite and infinite q-shifted factorials, this coefficient is exactly the kernel in (3.2). Consequently,

$$\mathcal{J}_\nu(q^r;q^2)\mathcal{J}_\nu(q^s;q^2) = \sum_{x\in\mathbb{Z}} A_\nu(q^r,q^s,q^x)\,\mathcal{J}_\nu(q^x;q^2),$$

which proves the product formula. □

The following proposition collects the fundamental structural properties of the kernel A_ν, which are essential for the construction of the q-convolution structure associated with the normalized q-Bessel functions.

Proposition 3.3. *Let* $0 < q < 1$*,* $\nu > 0$*, and* $r, s, x, l \in \mathbb{Z}$*. Then the kernel* A_ν *satisfies the following properties:*

(i) $A_\nu(q^r, q^s, q^x) \ge 0$.
(ii) $A_\nu(q^r, q^s, q^x) = A_\nu(q^s, q^r, q^x)$.
(iii) $A_\nu(q^{r+l}, q^{s+l}, q^{x+l}) = A_\nu(q^r, q^s, q^x)$.
(iv) $q^{2(\nu+1)(s-x)} A_\nu(q^r, q^s, q^x) = A_\nu(q^r, q^x, q^s)$.
(v) $\sum_{x\in\mathbb{Z}} A_\nu(q^r, q^s, q^x) = 1$.
(vi) $\lim_{\nu\to 0^+} A_\nu(q^r, q^s, q^x) = [J_{r-x}(q^{s-x}; q^2)]^2$.

Proof. (i) The proof of *(i)* is immediate. For $0 < q < 1$ and $\nu > 0$, all factors in the summand of (3.2) are nonnegative: The q-Pochhammer symbols and infinite products lie in $(0, 1)$, the powers of q are positive, and the squared basic hypergeometric term is nonnegative. Hence, each summand is nonnegative, and therefore

$$A_\nu(q^r, q^s, q^x) \ge 0.$$

(ii) For $r < s$, the assertion is part of the symmetric extension of the kernel. For $s \le r$, the same symmetry may also be checked directly from the regularization identity (2.6). Apply it with

$$n = r - s, \qquad a = q^{2(N-x+s-r+1)}, \qquad z = q^{2(x-r+1)}.$$

Then

$$
\begin{aligned}
&(q^{2(s-r+1)};q^2)_\infty\, {}_1\phi_1\left(\begin{matrix} q^{2(N-x+s-r+1)} \\ q^{2(s-r+1)} \end{matrix}\,\middle|\, q^2,\ q^{2(x-r+1)}\right) \\
&\quad = (-q^{2(x-r+1)})^{r-s} q^{(r-s)(r-s-1)} (q^{2(N-x+s-r+1)};q^2)_{r-s} (q^{2(r-s+1)};q^2)_\infty \\
&\qquad \times {}_1\phi_1\left(\begin{matrix} q^{2(N-x+1)} \\ q^{2(r-s+1)} \end{matrix}\,\middle|\, q^2,\ q^{2(x-s+1)}\right).
\end{aligned}
$$

Substituting this identity into the kernel and simplifying the powers of q together with the q-shifted factorials, one obtains exactly the expression with r and s interchanged. Hence,

$$
A_\nu(q^r, q^s, q^x) = A_\nu(q^s, q^r, q^x).
$$

(iii) Starting from the definition,

$$
\begin{aligned}
A_\nu(q^{r+l}, q^{s+l}, q^{x+l}) &= (1-q^{2\nu}) q^{2(1+r-s)(x-s)} \frac{(q^{2(1+r-s)};q^2)_\infty}{(q^2;q^2)_\infty} \\
&\quad \times \sum_{N=\max(r,s,x)+l}^{\infty} q^{2\nu(N-s-l)} \frac{(q^{2(1+r-s)};q^2)_{N-x-l}}{(q^2;q^2)_{N-x-l}(q^{2(N-s-l+1)};q^2)_\infty} \\
&\quad \times \left[{}_1\phi_1\left(\begin{matrix} q^{2(N-x+r-s+1-l)} \\ q^{2(1+r-s)} \end{matrix}\,\middle|\, q^2,\ q^{2(x-s+1)}\right)\right]^2.
\end{aligned}
$$

Now replace N by $N+l$. Since

$$
\max(r+l, s+l, x+l) = \max(r,s,x) + l,
$$

this becomes

$$
\begin{aligned}
A_\nu(q^{r+l}, q^{s+l}, q^{x+l}) &= (1-q^{2\nu}) q^{2(1+r-s)(x-s)} \frac{(q^{2(1+r-s)};q^2)_\infty}{(q^2;q^2)_\infty} \\
&\quad \times \sum_{N=\max(r,s,x)}^{\infty} q^{2\nu(N-s)} \frac{(q^{2(1+r-s)};q^2)_{N-x}}{(q^2;q^2)_{N-x}(q^{2(N-s+1)};q^2)_\infty} \\
&\quad \times \left[{}_1\phi_1\left(\begin{matrix} q^{2(N-x+r-s+1)} \\ q^{2(1+r-s)} \end{matrix}\,\middle|\, q^2,\ q^{2(x-s+1)}\right)\right]^2,
\end{aligned}
$$

which is exactly $A_\nu(q^r, q^s, q^x)$.

(iv) By symmetry, it suffices to assume $r \leq s$.

Case 1: $x \geq s$**.** Then

$$
\begin{aligned}
A_\nu(q^r, q^s, q^x) &= (1-q^{2\nu}) \frac{(q^{2(s-r+1)};q^2)_\infty}{(q^2;q^2)_\infty} \sum_{N=x}^{\infty} q^{2\nu(N-r)} q^{2(s-r+1)(x-r)} \\
&\quad \times \frac{(q^{2(s-r+1)};q^2)_{N-x}}{(q^2;q^2)_{N-x}(q^{2(N-r+1)};q^2)_\infty} \left[{}_1\phi_1\left(\begin{matrix} q^{2(N-x+s-r+1)} \\ q^{2(s-r+1)} \end{matrix}\,\middle|\, q^2,\ q^{2(x-r+1)}\right)\right]^2.
\end{aligned}
$$

Apply the q-Kummer identity with

$$a = q^{2(N-x+s-r+1)}, \quad w = q^{2(s-r+1)}, \quad z = q^{2(x-r+1)},$$

so that $aw^{-1}z = q^{2(N-r+1)}$. Substituting the resulting identity and setting $M = N + s - x$, a direct simplification gives

$$q^{2(\nu+1)(s-x)} A_\nu(q^r, q^s, q^x) = A_\nu(q^r, q^x, q^s).$$

Case 2: $x \le r$**.** Set $n = r - x \ge 0$. Then $q^{2(x-r+1)} = q^{2(1-n)}$. Applying first the q-Kummer identity and then the shift identity in base q^2,

$$(q^{2(1-n)}; q^2)_\infty\, {}_1\phi_1 = (-z)^n q^{n(n-1)} (a; q^2)_n (q^{2(1+n)}; q^2)_\infty\, {}_1\phi_1,$$

with $a = q^{2(N-r+1)}$ and $z = q^{2(s-r+1)}$. We obtain a regularized form of the kernel. Substituting into A_ν and using

$$(q^{2(N-r+1)}; q^2)_\infty = (q^{2(N-r+1)}; q^2)_{r-x} (q^{2(N-x+1)}; q^2)_\infty,$$

one finds

$$\begin{aligned} A_\nu(q^r, q^s, q^x) = (1 - q^{2\nu}) &\frac{(q^{2(r-x+1)}; q^2)_\infty^2}{(q^2; q^2)_\infty (q^{2(s-r+1)}; q^2)_\infty} q^{2(r-x)(s-x)} \sum_{N=r}^{\infty} q^{2\nu(N-r)} \\ &\times \frac{(q^{2(s-r+1)}; q^2)_{N-x} (q^{2(N-r+1)}; q^2)_{r-x}}{(q^2; q^2)_{N-x} (q^{2(N-x+1)}; q^2)_\infty} \left[{}_1\phi_1\left(\left. \begin{matrix} q^{2(N-x+1)} \\ q^{2(r-x+1)} \end{matrix} \right| q^2,\, q^{2(s-x+1)} \right) \right]^2. \end{aligned}$$

Multiplying by $q^{2(\nu+1)(s-x)}$ and setting $M = N + s - r$, a direct simplification yields

$$q^{2(\nu+1)(s-x)} A_\nu(q^r, q^s, q^x) = A_\nu(q^x, q^r, q^s).$$

By *(ii)*,

$$A_\nu(q^x, q^r, q^s) = A_\nu(q^r, q^x, q^s),$$

and therefore

$$q^{2(\nu+1)(s-x)} A_\nu(q^r, q^s, q^x) = A_\nu(q^r, q^x, q^s).$$

(v) Starting from the product formula

$$\mathcal{J}_\nu(q^r; q^2) \mathcal{J}_\nu(q^s; q^2) = \sum_{x \in \mathbb{Z}} A_\nu(q^r, q^s, q^x)\, \mathcal{J}_\nu(q^x; q^2),$$

replace r, s, x by $r + l, s + l, x + l$. Using *(iii)*, we obtain

$$\mathcal{J}_\nu(q^{r+l}; q^2) \mathcal{J}_\nu(q^{s+l}; q^2) = \sum_{x \in \mathbb{Z}} A_\nu(q^r, q^s, q^x)\, \mathcal{J}_\nu(q^{x+l}; q^2).$$

Letting $l \to \infty$, we have $q^{2(x+l)} \to 0$, hence

$$\mathcal{J}_\nu(q^{r+l}; q^2) \to 1, \qquad \mathcal{J}_\nu(q^{s+l}; q^2) \to 1, \qquad \mathcal{J}_\nu(q^{x+l}; q^2) \to 1.$$

By dominated convergence,

$$1 = \sum_{x \in \mathbb{Z}} A_\nu(q^r, q^s, q^x).$$

(vi) Assume $r \geq s$, and set $t := q^{2\nu} \in (0, 1)$, so that $t \to 1^-$ as $\nu \to 0^+$. From the definition of the kernel,

$$\begin{aligned} A_\nu(q^r, q^s, q^x) &= (1-t)\, q^{2(1+r-s)(x-s)} \frac{(q^{2(1+r-s)}; q^2)_\infty}{(q^2; q^2)_\infty} \\ &\quad \times \sum_{N=\max(r,s,x)}^{\infty} t^{N-s} \frac{(q^{2(1+r-s)}; q^2)_{N-x}}{(q^2; q^2)_{N-x} (q^{2(N-s+1)}; q^2)_\infty} \\ &\quad \times \left[{}_1\phi_1\!\left(\begin{matrix} q^{2(N-x+r-s+1)} \\ q^{2(1+r-s)} \end{matrix} \,\middle|\, q^2,\, q^{2(x-s+1)} \right) \right]^2 . \end{aligned}$$

Write

$$A_\nu(q^r, q^s, q^x) = (1-t) \sum_{N=\max(r,s,x)}^{\infty} B_N\, t^{N-s},$$

where

$$\begin{aligned} B_N &:= q^{2(1+r-s)(x-s)} \frac{(q^{2(1+r-s)}; q^2)_\infty}{(q^2; q^2)_\infty} \frac{(q^{2(1+r-s)}; q^2)_{N-x}}{(q^2; q^2)_{N-x} (q^{2(N-s+1)}; q^2)_\infty} \\ &\quad \times \left[{}_1\phi_1\!\left(\begin{matrix} q^{2(N-x+r-s+1)} \\ q^{2(1+r-s)} \end{matrix} \,\middle|\, q^2,\, q^{2(x-s+1)} \right) \right]^2 . \end{aligned}$$

Let $N_0 := \max(r, s, x)$ and write $N = N_0 + k$. Then

$$A_\nu(q^r, q^s, q^x) = t^{N_0-s}(1-t) \sum_{k=0}^{\infty} B_{N_0+k}\, t^k.$$

Since $t^{N_0-s} \to 1$, Abel's theorem yields

$$\lim_{\nu \to 0^+} A_\nu(q^r, q^s, q^x) = \lim_{N \to \infty} B_N.$$

Now

$$(q^{2(1+r-s)}; q^2)_{N-x} \to (q^{2(1+r-s)}; q^2)_\infty, \qquad (q^{2(N-s+1)}; q^2)_\infty \to 1,$$

and hence

$${}_1\phi_1\!\left(\begin{matrix} q^{2(N-x+r-s+1)} \\ q^{2(1+r-s)} \end{matrix} \,\middle|\, q^2,\, q^{2(x-s+1)} \right) \longrightarrow {}_1\phi_1\!\left(\begin{matrix} 0 \\ q^{2(1+r-s)} \end{matrix} \,\middle|\, q^2,\, q^{2(x-s+1)} \right).$$

Therefore,

$$\lim_{N \to \infty} B_N = q^{2(1+r-s)(x-s)} \frac{(q^{2(1+r-s)}; q^2)_\infty^2}{(q^2; q^2)_\infty^2} \left[{}_1\phi_1\!\left(\begin{matrix} 0 \\ q^{2(1+r-s)} \end{matrix} \,\middle|\, q^2,\, q^{2(x-s+1)} \right) \right]^2 .$$

Finally, using (2.7) with $\alpha = r - s$ and $z = q^{x-s}$, we obtain

$$J_{r-s}(q^{x-s}; q^2) = q^{(r-s)(x-s)} \frac{(q^{2(1+r-s)}; q^2)_\infty}{(q^2; q^2)_\infty}\, {}_1\phi_1\!\left(\begin{matrix} 0 \\ q^{2(1+r-s)} \end{matrix} \,\middle|\, q^2,\, q^{2(x-s+1)} \right),$$

and hence

$$\lim_{\nu\to 0^+} A_\nu(q^r, q^s, q^x) = q^{2(x-s)} \left[J_{r-s}(q^{x-s}; q^2)\right]^2.$$

We use here the following symmetry identity for the third Jackson q-Bessel function:

$$q^{2(x-s)}[J_{r-s}(q^{x-s}; q^2)]^2 = [J_{r-x}(q^{s-x}; q^2)]^2. \tag{3.6}$$

This is formula (18) in [19]. Applying (3.6) to the last displayed expression gives

$$\lim_{\nu\to 0^+} A_\nu(q^r, q^s, q^x) = [J_{r-x}(q^{s-x}; q^2)]^2.$$

□

Remark 3.4. When the lattice index tends to $x \to -\infty$, the point q^x tends to $+\infty$ in K_q. Thus, the values $A_\nu(q^r, q^s, q^x)$ describe the mass transported toward the noncompact end of the q-lattice. The normalization in Proposition 3.3 shows that no mass is lost at this end: The kernel is a probability distribution on the whole lattice, although its support is generally infinite and noncompact. This is the geometric reason why the resulting convolution behaves like a hypergroup convolution analytically, but not in the compact-support sense of the classical DJS axioms.

Remark 3.5. Property (vi) shows that, as $\nu \to 0^+$, the kernel A_ν converges pointwise to

$$A_0(q^r, q^s, q^x) = [J_{r-x}(q^{s-x}; q^2)]^2.$$

Consequently, the convolution constructed in this paper reduces, in the order-zero limit, to the positive convolution structure previously obtained for the third basic zero-order Bessel function [19]. Thus, the present construction extends the order-zero case to the full family of normalized Hahn–Exton q-Bessel functions with $\nu > 0$.

4. DJS-hypergroups and their degenerate variant

The notion of a hypergroup was developed in the late 1960s and early 1970s by Jewett, Dunkl, and Spector as a natural extension of locally compact groups, in which the product of two points is replaced by a probability measure. This framework provides a flexible setting for harmonic analysis beyond group structures and is particularly well suited for describing product formulas and convolution structures associated with special functions; see [4, 5, 34]. A key insight, due to Koornwinder [9], is that positive linearization formulas for orthogonal polynomials give rise to hypergroup structures. In particular, classical disk polynomials and their addition formulas generate commutative hypergroups. A q-analogue of this construction was later developed via quantum group methods and q-disk polynomials, notably by Floris and Koelink [24], where the convolution arises from representation-theoretic considerations.

Let K be a locally compact Hausdorff space. Denote by $M(K)$ the space of complex regular Borel measures on K, and by $M_1(K) \subset M(K)$ the set of probability measures. For $x \in K$, let δ_x denote the Dirac measure at x. Assume that the following structures are given:

(a) A convolution mapping

$$K \times K \longrightarrow M_1(K), \qquad (x, y) \longmapsto \delta_x * \delta_y,$$

which is weakly continuous.

(b) A continuous involution $x \mapsto \bar{x}$ on K.

(c) A distinguished identity element $e \in K$.

The convolution extends uniquely to a bilinear, weakly continuous map on $M(K)$, still denoted by $*$, and the involution extends to measures by

$$\mu^*(E) := \mu(\bar{E}), \qquad E \subset K \text{ Borel}.$$

Definition 4.1. The quadruple $(K, *, \bar{\cdot}, e)$ is called a *(DJS) hypergroup* if, for all $x, y, z \in K$, the following axioms hold:

(1) **Associativity:**

$$\delta_x * (\delta_y * \delta_z) = (\delta_x * \delta_y) * \delta_z.$$

(2) **Commutativity:**

$$\delta_x * \delta_y = \delta_y * \delta_x.$$

(3) **Compact support:**

$$\operatorname{supp}(\delta_x * \delta_y) \text{ is compact.}$$

(4) **Involution:**

$$(\delta_x * \delta_y)^* = \delta_y * \delta_x.$$

(5) **Identity:**

$$\delta_e * \delta_x = \delta_x = \delta_x * \delta_e.$$

(6) **Separation:**

$$e \in \operatorname{supp}(\delta_x * \delta_y) \quad \Longleftrightarrow \quad x = \bar{y}.$$

(7) **Continuity of support:**

$$(x, y) \longmapsto \operatorname{supp}(\delta_x * \delta_y)$$

is continuous with respect to the Vietoris topology.

The convolution structure associated with the q-Bessel functions provides a natural example in which the compact-support axiom fails. This motivates the following weaker notion.

Definition 4.2. A quadruple $(K, *, \bar{\cdot}, e)$ is called a *degenerate DJS hypergroup* (or *hypergroup of noncompact support*) if it satisfies the algebraic DJS axioms of bilinearity, associativity, commutativity, involution, and identity, while the compact-support, support-continuity, and DJS separation axioms are not imposed. They are replaced by the following conditions:

(3′) **Probability preservation:**

$$\delta_x * \delta_y \in M_1(K), \qquad \forall x, y \in K.$$

(6′) **Identity-atom condition:**

$$(\delta_x * \delta_y)(\{e\}) > 0 \quad \Longleftrightarrow \quad x = y = e.$$

This condition is weaker than the usual DJS separation axiom $e \in \operatorname{supp}(\delta_x * \delta_y) \Longleftrightarrow x = \bar{y}$.

(7′) **Weak continuity:**

$$(x, y) \longmapsto \delta_x * \delta_y$$

is continuous from $K \times K$ into $M_1(K)$ endowed with the weak topology.

We introduce the space

$$K_q := q^{\mathbb{Z}} \cup \{0\} = \{q^n : n \in \mathbb{Z}\} \cup \{0\},$$

which will serve as the underlying locally compact space throughout this work. When convenient, we use the purely notational convention $q^\infty := 0$; however, the space itself will always be denoted by the consistent notation $K_q = q^{\mathbb{Z}} \cup \{0\}$.

We endow K_q with the topology induced from $[0, \infty)$. Then K_q is a locally compact Hausdorff space. Indeed, each point q^n, $n \in \mathbb{Z}$ is isolated, while 0 is the unique accumulation point. Moreover, 0 admits a compact neighbourhood of the form

$$\{0\} \cup \{q^n : n \geq N\}, \qquad N \in \mathbb{Z}.$$

Thus, K_q is locally compact but not compact since it is unbounded, with $q^n \to +\infty$ as $n \to -\infty$.

Equivalently, in the index space $\mathbb{Z} \cup \{\infty\}$, the point ∞ compactifies only the direction $n \to +\infty$ (i.e., $q^n \to 0$), while the direction $n \to -\infty$ remains noncompact (since $q^n \to +\infty$).

We extend the Jackson measure to K_q by defining

$$m_\nu(\{q^n\}) = c_{q,\nu}(1 - q)\, q^{n(2\nu+2)}, \quad n \in \mathbb{Z}, \qquad m_\nu(\{0\}) = 0,$$

where

$$c_{q,\nu} := \frac{(1 - q^2)^\nu}{\Gamma_{q^2}(\nu + 1)}.$$

Equivalently, for any function $f : K_q \to \mathbb{C}$,

$$\int_{K_q} f(t)\, dm_\nu(t) = c_{q,\nu} \int_0^\infty f(t) t^{2\nu+1}\, d_q t.$$

In this sense, m_ν is the natural extension of the Jackson-type measure

$$dm_\nu(t) = c_{q,\nu}\, t^{2\nu+1}\, d_q t.$$

In particular, m_ν is a positive Radon measure on K_q since it is finite on compact subsets.

Definition 4.3. Let $f \in L^1(K_q, m_\nu)$. The q-Bessel Fourier transform of f is defined by

$$\mathcal{F}_{q,\nu} f(\lambda) := \int_{K_q} f(t)\, \mathcal{J}_\nu(\lambda t; q^2)\, dm_\nu(t), \qquad \lambda \in K_q.$$

Proposition 4.4. *Let $f \in L^1(K_q, m_\nu)$ and assume that $\mathcal{F}_{q,\nu} f \in L^1(K_q, m_\nu)$ and $f(0) := \lim_{k\to+\infty} f(q^k)$. Then, for every $t \in K_q$,*

$$f(t) = \int_{K_q} \mathcal{F}_{q,\nu} f(\lambda) \mathcal{J}_\nu(\lambda t; q^2)\, dm_\nu(\lambda).$$

Furthermore, the transform $\mathcal{F}_{q,\nu}$ extends uniquely to a unitary operator on $L^2(K_q, m_\nu)$. In particular, the Plancherel identity holds:

$$\|\mathcal{F}_{q,\nu} f\|_{L^2(K_q,m_\nu)} = \|f\|_{L^2(K_q,m_\nu)}, \qquad f \in L^2(K_q, m_\nu).$$

Proof. Since $m_\nu(\{0\}) = 0$, integration over K_q coincides with the Jackson integral on $q^{\mathbb{Z}}$. More precisely,

$$\int_{K_q} h(\lambda)\, dm_\nu(\lambda) = c_{q,\nu} \int_0^\infty h(\lambda) \lambda^{2\nu+1}\, d_q\lambda.$$

Hence, for $k \in \mathbb{Z}$,

$$\mathcal{F}_{q,\nu} f(q^k) = c_{q,\nu} \int_0^\infty f(t) \mathcal{J}_\nu(q^k t; q^2) t^{2\nu+1}\, d_q t,$$

which is precisely the normalized q-Bessel Fourier transform on $q^{\mathbb{Z}}$. By the inversion theorem for the normalized q-Bessel transform, if $f, \mathcal{F}_{q,\nu} f \in L^1(K_q, m_\nu)$, then

$$f(q^k) = c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu} f(\lambda) \mathcal{J}_\nu(\lambda q^k; q^2) \lambda^{2\nu+1}\, d_q\lambda.$$

Using the definition of m_ν, this yields

$$f(q^k) = \int_{K_q} \mathcal{F}_{q,\nu} f(\lambda) \mathcal{J}_\nu(\lambda q^k; q^2)\, dm_\nu(\lambda), \qquad k \in \mathbb{Z}.$$

To handle $t = 0$, note that

$$\mathcal{J}_\nu(\lambda q^k; q^2) \longrightarrow 1 \quad (k \to +\infty).$$

If $\mathcal{F}_{q,\nu} f \in L^1(K_q, m_\nu)$, then by dominated convergence,

$$\lim_{k\to+\infty} \int_{K_q} \mathcal{F}_{q,\nu} f(\lambda) \mathcal{J}_\nu(\lambda q^k; q^2)\, dm_\nu(\lambda) = \int_{K_q} \mathcal{F}_{q,\nu} f(\lambda)\, dm_\nu(\lambda),$$

which proves the formula at $t = 0$. Finally, the Plancherel theorem for the normalized q-Bessel transform on $q^{\mathbb{Z}}$ gives

$$c_{q,\nu} \int_0^\infty |\mathcal{F}_{q,\nu} f(\lambda)|^2 \lambda^{2\nu+1}\, d_q\lambda = c_{q,\nu} \int_0^\infty |f(t)|^2 t^{2\nu+1}\, d_q t.$$

Equivalently,

$$\int_{K_q} |\mathcal{F}_{q,\nu} f(\lambda)|^2\, dm_\nu(\lambda) = \int_{K_q} |f(t)|^2\, dm_\nu(t).$$

Thus, $\mathcal{F}_{q,\nu}$ is an isometry on $L^1 \cap L^2(K_q, m_\nu)$, and by density, it extends uniquely to a unitary operator on $L^2(K_q, m_\nu)$. $\square$

Proposition 4.5 (A q-Bessel Lévy continuity criterion)**.** *Let $(\mu_n)_{n\geq 1} \subset M_1(K_q)$ and $\mu \in M_1(K_q)$. Assume that*

$$\widehat{\mu_n}(\lambda) \longrightarrow \widehat{\mu}(\lambda), \qquad \lambda \in K_q,$$

and that the sequence (μ_n) is tight on K_q. Then $\mu_n \rightharpoonup \mu$ weakly on K_q. In particular, if $a_n \to 0$ in K_q and $b_n \to b \in K_q$, then

$$\delta_{a_n} *_\nu \delta_{b_n} \rightharpoonup \delta_b.$$

Proof. The proof is the usual Lévy argument, but it is applied only to the present q-Bessel transform. By the inversion and Plancherel formulas above, the linear span of the functions

$$x \longmapsto \mathcal{J}_\nu(\lambda x; q^2), \qquad \lambda \in K_q,$$

is convergent determining on K_q. Hence, convergence of the transforms implies vague convergence. Since all measures involved are probability measures and the sequence is tight, vague convergence is equivalent to weak convergence.

For the last assertion, the product formula gives

$$\widehat{\delta_{a_n} *_\nu \delta_{b_n}}(\lambda) = \mathcal{J}_\nu(\lambda a_n; q^2)\, \mathcal{J}_\nu(\lambda b_n; q^2) \longrightarrow \mathcal{J}_\nu(\lambda b; q^2) = \widehat{\delta_b}(\lambda).$$

The probability preservation of the kernel gives tightness in this case, and the first part of the proposition proves the desired weak convergence. $\square$

We define the convolution on point masses of K_q by

$$\delta_x *_\nu \delta_y = \begin{cases} \displaystyle\sum_{z\in q^{\mathbb{Z}}} A_\nu(x,y,z)\, \delta_z, & x, y \in q^{\mathbb{Z}}, \\ \delta_y, & x = 0,\ y \in q^{\mathbb{Z}}, \\ \delta_x, & y = 0,\ x \in q^{\mathbb{Z}}, \\ \delta_0, & x = y = 0. \end{cases} \tag{4.1}$$

The convolution (4.1), initially defined on point masses, is extended bilinearly to $M_b(K_q)$ by

$$\mu *_\nu \eta := \int_{K_q} \int_{K_q} (\delta_x *_\nu \delta_y)\, d\mu(x)\, d\eta(y), \qquad \mu, \eta \in M_b(K_q),$$

that is, for every bounded Borel function f on K_q,

$$\int_{K_q} f(z)\, d(\mu *_\nu \eta)(z) = \int_{K_q} \int_{K_q} \left(\int_{K_q} f(z)\, d(\delta_x *_\nu \delta_y)(z) \right) d\mu(x)\, d\eta(y).$$

For $\mu \in M_b(K_q)$, define its q-Fourier–Bessel–Stieltjes transform by

$$\widehat{\mu}(\lambda) := \int_{K_q} \mathcal{J}_\nu(\lambda t; q^2)\, d\mu(t), \qquad \lambda \in K_q.$$

In particular,

$$\widehat{\delta_{q^r}}(\lambda) = \mathcal{J}_\nu(\lambda q^r; q^2).$$

For $r, s \in \mathbb{Z}$, using the product formula,

$$\widehat{\delta_{q^r} *_\nu \delta_{q^s}}(\lambda) = \sum_{x\in\mathbb{Z}} A_\nu(q^r, q^s, q^x)\mathcal{J}_\nu(\lambda q^x; q^2) = \mathcal{J}_\nu(\lambda q^r; q^2)\mathcal{J}_\nu(\lambda q^s; q^2).$$

Hence,

$$\widehat{\delta_{q^r} *_\nu \delta_{q^s}}(\lambda) = \widehat{\delta_{q^r}}(\lambda)\,\widehat{\delta_{q^s}}(\lambda).$$

By bilinearity and Fubini's theorem, for all $\mu, \eta \in M_b(K_q)$,

$$\widehat{\mu *_\nu \eta}(\lambda) = \widehat{\mu}(\lambda)\,\widehat{\eta}(\lambda), \qquad \lambda \in K_q.$$

Indeed,

$$\begin{aligned}
\widehat{\mu *_\nu \eta}(\lambda) &= \int_{K_q} \mathcal{J}_\nu(\lambda z; q^2)\, d(\mu *_\nu \eta)(z)\\
&= \int_{K_q}\int_{K_q}\left(\int_{K_q} \mathcal{J}_\nu(\lambda z; q^2)\, d(\delta_x *_\nu \delta_y)(z)\right) d\mu(x)\, d\eta(y)\\
&= \int_{K_q}\int_{K_q} \mathcal{J}_\nu(\lambda x; q^2)\mathcal{J}_\nu(\lambda y; q^2)\, d\mu(x)\, d\eta(y)\\
&= \widehat{\mu}(\lambda)\widehat{\eta}(\lambda).
\end{aligned}$$

Theorem 4.6. *Let $0 < q < 1$ and $\nu > 0$. Endow K_q with the convolution* (4.1)*, the involution $\bar{x} = x$, and the identity element $e = 0$. Then $(K_q, *, \bar{\cdot}, 0)$ is a commutative degenerate DJS hypergroup.*

Proof. We verify the axioms.

(1) **Probability preservation.** By Proposition 3.3 (i) and (v),

$$A_\nu(q^r, q^s, q^x) \geq 0, \qquad \sum_{x\in\mathbb{Z}} A_\nu(q^r, q^s, q^x) = 1.$$

Hence $\delta_{q^r} * \delta_{q^s} \in M_1(K_q)$. The cases involving 0 follow directly from (4.1).

(2) **Commutativity.** By Proposition 3.3 (ii),

$$A_\nu(q^r, q^s, q^x) = A_\nu(q^s, q^r, q^x),$$

and therefore

$$\delta_{q^r} * \delta_{q^s} = \delta_{q^s} * \delta_{q^r}.$$

(3) **Identity.** We show that δ_0 is the identity. For $r, s \in \mathbb{Z}$,

$$\delta_{q^r} *_\nu \delta_{q^s} = \sum_{x\in\mathbb{Z}} A_\nu(q^r, q^s, q^x)\,\delta_{q^x}.$$

The definition is extended to 0 by weak continuity. Moreover,

$$\widehat{\delta_a}(\lambda) = \mathcal{J}_\nu(\lambda a; q^2).$$

Thus,

$$\widehat{\delta_{q^r} *_\nu \delta_{q^s}}(\lambda) = \sum_{x\in\mathbb{Z}} A_\nu(q^r, q^s, q^x)\, \mathcal{J}_\nu(\lambda q^x; q^2).$$

By the product formula,

$$\widehat{\delta_{q^r} *_\nu \delta_{q^s}}(\lambda) = \mathcal{J}_\nu(\lambda q^r; q^2)\mathcal{J}_\nu(\lambda q^s; q^2).$$

Letting $r \to \infty$, we have $q^r \to 0$ and $\mathcal{J}_\nu(0; q^2) = 1$. Hence,

$$\widehat{\delta_{q^r} *_\nu \delta_{q^s}}(\lambda) \longrightarrow \mathcal{J}_\nu(\lambda q^s; q^2) = \widehat{\delta_{q^s}}(\lambda).$$

By Proposition 4.5,

$$\delta_{q^r} *_\nu \delta_{q^s} \rightharpoonup \delta_{q^s}.$$

Since also $\delta_{q^r} \rightharpoonup \delta_0$, the definition of the extension at 0 gives

$$\delta_0 *_\nu \delta_{q^s} = \delta_{q^s}.$$

Similarly,

$$\delta_0 *_\nu \delta_0 = \delta_0.$$

(4) **Involution.** Since $\bar{x} = x$, we have $\delta_x^* = \delta_x$. Therefore, by commutativity,

$$(\delta_x * \delta_y)^* = \delta_y * \delta_x.$$

(5) **Associativity.** Associativity follows from the product formula and the fact that the q-Bessel Fourier transform converts convolution into pointwise multiplication. Indeed, for every $\lambda \in K_q$,

$$\widehat{(\delta_x * \delta_y) * \delta_z}(\lambda) = \mathcal{J}_\nu(\lambda x; q^2)\mathcal{J}_\nu(\lambda y; q^2)\mathcal{J}_\nu(\lambda z; q^2) = \widehat{\delta_x * (\delta_y * \delta_z)}(\lambda).$$

Since the q-Bessel transform separates measures, we obtain

$$(\delta_x * \delta_y) * \delta_z = \delta_x * (\delta_y * \delta_z).$$

(6) **Weak continuity.** Let $(x_n, y_n) \to (x, y)$ in $K_q \times K_q$. If $x, y \in q^{\mathbb{Z}}$, then the assertion is immediate, because all nonzero points of K_q are isolated. If $x_n \to 0$, then Proposition 4.5 gives

$$\delta_{x_n} *_\nu \delta_{y_n} \rightharpoonup \delta_y.$$

The case $y_n \to 0$ is analogous, and if both coordinates tend to 0, then $\delta_{x_n} *_\nu \delta_{y_n} \rightharpoonup \delta_0$. The only non-isolated limit point of K_q is therefore harmless: The extension at 0 is fixed by the identity-atom rule and by $\mathcal{J}_\nu(0; q^2) = 1$, so no singular mass is created at the boundary. Hence,

$$(x, y) \longmapsto \delta_x *_\nu \delta_y$$

is weakly continuous.

(7) **The usual DJS separation axiom is not part of Definition 4.2.** What is retained is the weaker identity-atom condition. For $r, s \in \mathbb{Z}$, the convolution

$$\delta_{q^r} *_\nu \delta_{q^s} = \sum_{x\in\mathbb{Z}} A_\nu(q^r, q^s, q^x)\, \delta_{q^x}$$

has no atom at 0, because the sum is over $q^{\mathbb{Z}}$. If exactly one of the variables is 0, then $\delta_0 *_\nu \delta_y = \delta_y$ or $\delta_x *_\nu \delta_0 = \delta_x$, again with no atom at 0 unless the other variable is also 0. Finally,

$$\delta_0 *_\nu \delta_0 = \delta_0.$$

Therefore,

$$(\delta_x *_\nu \delta_y)(\{0\}) > 0 \quad \Longleftrightarrow \quad x = y = 0.$$

Thus all axioms of a commutative degenerate DJS hypergroup, in the sense of Definition 4.2, are satisfied. □

Definition 4.7. A bounded continuous function $\varphi \in C_b(K_q)$, $\varphi \not\equiv 0$, is called a character of $(K_q, *_\nu)$ if

$$\int_{K_q} \varphi(z)\, d(\delta_x *_\nu \delta_y)(z) = \varphi(x)\varphi(y), \qquad x, y \in K_q.$$

We denote the character space by

$$\mathcal{X}_b(K_q) := \left\{\varphi \in C_b(K_q) : \varphi \not\equiv 0,\ (\delta_x *_\nu \delta_y)(\varphi) = \varphi(x)\varphi(y),\ x, y \in K_q\right\}.$$

The symmetric character space is

$$\mathcal{X}(K_q) := \left\{\varphi \in \mathcal{X}_b(K_q) : \varphi(x) = \overline{\varphi(x)},\ x \in K_q\right\}.$$

Proposition 4.8. *For every $\lambda \in K_q$, define*

$$\varphi_\lambda(x) := \mathcal{J}_\nu(\lambda x; q^2), \qquad x \in K_q.$$

Then $\varphi_\lambda \in \mathcal{X}_b(K_q)$. More precisely,

$$\int_{K_q} \varphi_\lambda(z)\, d(\delta_x *_\nu \delta_y)(z) = \varphi_\lambda(x)\varphi_\lambda(y), \qquad x, y \in K_q.$$

Hence,

$$\widehat{K_q} := \{\varphi_\lambda : \lambda \in K_q\} \subset \mathcal{X}_b(K_q).$$

Moreover, since $\mathcal{J}_\nu(\lambda x; q^2)$ is real-valued for $\lambda, x \in K_q$, one has

$$\widehat{K_q} \subset \mathcal{X}(K_q).$$

Thus, the q-Bessel dual is naturally identified with K_q through

$$K_q \ni \lambda \longmapsto \varphi_\lambda \in \widehat{K_q}.$$

Proof. Let $x = q^r$, $y = q^s$, and $\lambda \in K_q$. By the product formula,

$$\mathcal{J}_\nu(\lambda q^r; q^2)\mathcal{J}_\nu(\lambda q^s; q^2) = \sum_{k\in\mathbb{Z}} A_\nu(q^r, q^s, q^k)\mathcal{J}_\nu(\lambda q^k; q^2).$$

Using the definition of the convolution,

$$\delta_{q^r} *_\nu \delta_{q^s} = \sum_{k\in\mathbb{Z}} A_\nu(q^r, q^s, q^k)\delta_{q^k},$$

we obtain

$$(\delta_{q^r} *_\nu \delta_{q^s})(\varphi_\lambda) = \varphi_\lambda(q^r)\varphi_\lambda(q^s).$$

The cases $x = 0$ or $y = 0$ follow from

$$\mathcal{J}_\nu(0; q^2) = 1$$

and from the identity property

$$\delta_0 *_\nu \delta_x = \delta_x.$$

Therefore, φ_λ is a character. □

Remark 4.9. In the classical theory of commutative hypergroups, the character space $\mathcal{X}_b(X)$ is canonically homeomorphic to the Gelfand spectrum $\Delta(L^1(X, m))$, while the symmetric character space $\mathcal{X}(X)$ corresponds to the symmetric spectrum $\Delta_S(L^1(X, m))$. The Fourier transform is then the Gelfand transform, and the abstract Bochner and Plancherel theorems provide a Plancherel measure π satisfying

$$\int_X |f(x)|^2\, dm(x) = \int_{\mathcal{X}(X)} |\widehat{f}(\varphi)|^2\, d\pi(\varphi), \qquad f \in C_c(X).$$

In the present q-Bessel setting, however, the compact-support axiom of DJS hypergroups fails. Therefore, the abstract Gelfand and Plancherel theory of commutative hypergroups cannot be invoked directly. Nevertheless, the functions

$$\varphi_\lambda(x) = \mathcal{J}_\nu(\lambda x; q^2), \qquad \lambda, x \in K_q,$$

are genuine multiplicative functions for the q-Bessel convolution and play the role of characters. The corresponding Fourier transform is defined by

$$\mathcal{F}_{q,\nu} f(\lambda) = \int_{K_q} f(x)\varphi_\lambda(x)\, dm_\nu(x) = \int_{K_q} f(x)\mathcal{J}_\nu(\lambda x; q^2)\, dm_\nu(x).$$

Its Plancherel theorem is obtained from the intrinsic q-Bessel Fourier theory:

$$\int_{K_q} |f(x)|^2\, dm_\nu(x) = \int_{K_q} |\mathcal{F}_{q,\nu} f(\lambda)|^2\, dm_\nu(\lambda), \qquad f \in C_c(K_q).$$

Thus, under the identification $\lambda \leftrightarrow \varphi_\lambda$, the Plancherel measure is m_ν, but this conclusion is justified directly by the q-Bessel Plancherel theorem rather than by the abstract hypergroup Gelfand theory.

Remark 4.10. The preceding construction shows that the loss of compact support does not prevent a full Fourier theory when the product kernel remains positive, normalized, and weakly continuous. In this sense, the degenerate DJS structure on K_q may serve as a prototype for other noncompact discrete harmonic spaces, especially those governed by q-difference or discrete spectral operators whose natural convolution kernels have infinite support. The Plancherel and inversion formulas are therefore not merely consequences of a formal analogy with classical hypergroups; they indicate that a robust harmonic analysis can persist beyond the standard DJS compact-support framework.

5. Conclusions

We have constructed a positive convolution structure on the discrete q-lattice $K_q = q^{\mathbb{Z}} \cup \{0\}$ whose characters are the normalized Hahn–Exton q-Bessel functions. The construction is obtained from the Koelink–Floris product formula by a confluence limit from little q-Jacobi polynomials. The resulting kernel is nonnegative, normalized, and compatible with the q-Bessel Fourier transform. This gives Fourier inversion, Plancherel theory, and a spectral decomposition for the corresponding q-Bessel operator.

The main structural feature is that the convolution has, in general, noncompact support. Thus the space does not satisfy the classical DJS compact-support axiom, although it retains the essential analytic features needed for harmonic analysis. The degenerate DJS framework introduced here therefore provides a natural setting for noncompact discrete q-lattice convolution structures and suggests further applications to q-translation operators, q-wavelets, and spectral problems for noncompact q-difference operators.

Use of Generative-AI tools declaration

The author declares that generative-AI tools were used only for language polishing and formatting assistance. The mathematical content, proofs, and final text were checked and approved by the author.

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Conflict of interest

The author declares that there is no conflict of interest.

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