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Research article

On eigenvalues of the Cartan matrices for p-constrained groups

Manal H. Algreagri*

Department of Mathematics, Jamoum University College, Umm Al-Qura University, Makkah, Saudi Arabia

* **Correspondence:** Email: mhgreagri@uqu.edu.sa.

Abstract: Let p be a prime number. For p-constrained groups, we demonstrated that the Perron–Frobenius eigenvalue of the Cartan matrix of a p-block B was bounded above by the order of its defect group. The proof employed block-theoretic methods, spectral bounds in modular representation theory, and structural p'-reduction techniques. This conclusion extended a previously established inequality for p-solvable groups to the larger class of p-constrained groups. Finally, we presented computational results regarding the relationship between elementary divisors and eigenvalues of Cartan matrices for p-constrained groups that were not p-solvable.

Keywords: finite group; p-constrained groups; block; defect group; eigenvalues; nonnegative matrices
Mathematics Subject Classification: 20C15, 20C20

1. Introduction

Let p be a fixed prime. Consider G to be a finite group with order divisible by p. We set the triple $(k, \mathcal{R}, F)$ for a p-modular system [1–3]. This system consists of a complete discrete valuation ring $\mathcal{R}$ with a field of fractions k of characteristic 0, such that k includes all the primitive $|G|^{\text{th}}$ roots of unity. The residual field $F = \mathcal{R}/J(\mathcal{R})$ has characteristic p, where $J(\mathcal{R})$ is the Jacobson radical of the ring $\mathcal{R}$. We can define k as a splitting field and F as an algebraically closed field. Let B be a p-block of the group algebra FG with defect group D, and let $Bl(G)$ be the set of all p-blocks of G. Denote by C_B the Cartan matrix of B as in [4, Theorem 6.8], [5, Theorem I.16.7] and by $\rho(B)$ its Perron–Frobenius eigenvalue, which represents the largest eigenvalue of C_B as in the references [6–8]. Let $\mathrm{Irr}(B)$ be the set of all ordinary irreducible characters of G that belong to the p-block B. Let $k(B)$ represent the number of $\mathrm{Irr}(B)$. Let $\mathrm{IBr}(B)$ be the set of all irreducible Brauer characters of G that belong to the p-block B.

The study of Cartan matrices of p-blocks of finite groups plays a central role in modular representation theory. In particular, bounds on Cartan invariants and eigenvalues of Cartan matrices provide important information on the structure of projective indecomposable modules and block complexity.

In 1977, Motose [9] presented a few findings on the Perron–Frobenius eigenvalue of the Cartan matrix. In particular, he demonstrated the close connection between Wallace's problem [10] and the Perron–Frobenius eigenvalue of the Cartan matrix of the principal p-block. In 1992, Kiyota and Wada [6] proved that for finite p-solvable groups, the Perron–Frobenius eigenvalue of the Cartan matrix of a p-block is bounded above by the order of its defect group. Subsequently, in 1993, they [7] demonstrated that any eigenvalue of C_B is bounded above by the dimension of the principal indecomposable FG-module corresponding to the trivial FG-module. Additionally, they demonstrated that the Perron–Frobenius of C_B is bounded below by the order of the largest normal p-subgroup $O_p(G)$ of the group G.

In 1998, Wada [8] proposed the conjecture that says $k(B) \leq \rho(B)$, where B is a p-block of a p-solvable group G. In 2002, Kiyota, Murai, and Wada [11] posed the following two conjectures regarding the eigenvalues of C_B:

Conjecture 1. *If the Perron–Frobenius eigenvalue of the Cartan matrix C_B of a p-block B is equal to the order of its defect group, then the elementary divisors of C_B coincide with the eigenvalues of C_B.*

Conjecture 2. *If the Perron–Frobenius eigenvalue of the Cartan matrix C_B of a p-block B is an integer, then it is equal to the order of a defect group of B.*

Kiyota, Murai, and Wada [11] proved that Conjecture 1 is correct under one of the three assumptions that are listed below:

1) G is a p-solvable group;
2) The defect group is a normal subgroup of G;
3) B is of finite type or tame type. That is, D is dihedral, semi-dihedral, quaternion, or cyclic.

We see in [11], among other things, that they showed Conjecture 2 under the assumption (2) or (3).

In this paper, we aim to extend the previous results by employing block theoretic methods together with spectral bounds in modular representation theory and structural p'-reduction techniques to obtain new results beyond the framework of p-solvable groups and toward the broader class of p-constrained groups, a class introduced and studied extensively in [12, Section 8.1] and in [13, VII, 13.3]. Our main result is the following.

Theorem 3. *Let G be a finite p-constrained group and let B be a p-block of G with defect group D. If C_B denotes the Cartan matrix of B, then,*

$$\rho(B) \leq |D|.$$

Our second finding concerns p-constrained groups, where we explore, through the examples, the relationship between elementary divisors and the eigenvalues of Cartan matrices of the p-block. Our third result presents an open problem concerning the lower bound of $\rho(B_0)$ for the principal p-block of p-constrained groups.

Our principal methodologies for attaining our objectives are based on block theory, spectral bounds in modular representation theory, and structural p'-reduction techniques. Several foundational references have significantly shaped the development of the representation theory of finite groups. Specifically, the evolution of block theory has been heavily influenced by [14–16], while the advancements in defect theory owe much to [4, 5, 17]. Additionally, the study of p-constrained groups has been deeply structured by the seminal works of [12, 13, 18].

The paper is organized as follows: In Section 2, we review several standard results on p-constrained groups and recall the basic definitions essential for the proof of our main theorem. Additionally, we summarize earlier developments regarding the Cartan matrices of blocks, specifically addressing the conditions under which the elementary divisors of a p-block B of FG coincide with the eigenvalues of its Cartan matrix. In Section 3, we present our main results and provide illustrative examples to complement these results.

2. Preliminaries

This section is organized into two main parts. The first part is devoted to a concise exposition of p-constrained groups, together with the fundamental concepts and key theoretical results that will be employed in the proof of our main theorem. The second part reviews relevant earlier results concerning Cartan matrices of blocks, with particular emphasis on those addressing the question: which finite groups G and which p-blocks B of FG have the property that their the elementary divisors coincide with the eigenvalues of the Cartan matrix?

Throughout this paper, p denotes a fixed prime, and p' denotes a prime distinct from p. All groups considered are finite. Let $O_p(G)$ and $O_{p'}(G)$ denote the largest normal p-subgroup and p'-subgroup of G, respectively. Let $G_{p'}$ be the set of p-regular elements of G. We recall the definition of p-constrained groups from [13, Definition 13.3]:

Definition 4. *A finite group G is called p-constrained if it satisfies*

$$C_G(O_{p',p}(G)/O_{p'}(G)) \leq O_{p',p}(G).$$

Remarks 5.
- *The subgroup $O_{p',p}(G)$ is the second term of the upper p-series $\{1_G\} \leq O_{p'}(G) \leq O_{p',p}(G) \leq O_{p',p,p'}(G) \leq \ldots$, where $O_{p',p}(G)$ is defined by*

$$O_{p',p}(G)/O_{p'}(G) = O_p(G/O_{p'}(G)), \tag{2.1}$$

see [13, p. XIII].
- *If $O_{p'}(G) = \{1_G\}$, then* (2.1) *implies that $O_{p',p}(G) = O_p(G)$.*

As an immediate consequence of Definition 4, we obtain the following corollary:

Corollary 6. *Suppose that G is a finite group with a normal p-subgroup N such that $C_G(N) \leq N$. Then, G is p-constrained, and $O_{p'}(G) = \{1_G\}$.*

Theorem 7 ([12, Chapter 8, Theorem 1.1(ii)]). *If G is p-constrained, then $G/O_{p'}(G)$ is also p-constrained.*

Lemma 8 ([13, Chapter VI, Lemma 6.5]). *Every p-solvable group is p-constrained.*

We now review the basic concepts of modular representation theory relevant to this paper. Let $\chi \in \mathrm{Irr}(G)$, and denote the restriction of χ to the set of p-regular elements $G_{p'}$ by χ^0. We can write

$$\chi^0 = \sum_{\lambda \in \mathrm{IBr}(G)} d_{\chi\lambda}\lambda.$$

Since $\mathrm{IBr}(G)$ is linearly independent, the nonnegative integer coefficients $d_{\chi\lambda}$ are uniquely determined and are called the decomposition numbers. The matrix $\mathfrak{D} = (d_{\chi\lambda})_{\chi \in \mathrm{Irr}(G), \lambda \in \mathrm{IBr}(G)}$ is called the decomposition matrix of G. The Cartan matrix of G is defined as $C = \mathfrak{D}^t\mathfrak{D}$, where $\mathfrak{D}^t$ denotes the transpose of $\mathfrak{D}$.

Let B be a p-block of G with a defect group D. The Cartan matrix of B is denoted by $C_B = (c_{\theta\tau})_{\theta,\tau \in \mathrm{IBr}(B)}$, which is a symmetric, positive-definite matrix with nonnegative integer entries. The coefficients $c_{\theta\tau}$ are called the Cartan invariants of B and are given by

$$c_{\theta\tau} = \sum_{\chi \in \mathrm{Irr}(B)} d_{\chi\theta}d_{\chi\tau},$$

where $\theta, \tau \in \mathrm{IBr}(B)$.

For each $\lambda \in \mathrm{IBr}(G)$, the projective indecomposable character Φ_λ associated with λ is defined by

$$\Phi_\lambda = \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\lambda}\chi,$$

where $d_{\chi\lambda}$ denotes the corresponding decomposition number. Note that its restriction to $G_{p'}$ satisfies

$$(\Phi_\lambda)^0 = \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\lambda}\chi^0 = \sum_{\psi \in \mathrm{IBr}(G)} c_{\lambda\psi}\psi.$$

Corollary 9 ([19, Corollary 2.14]). *Let $\psi \in IBr(G)$. If Φ_ψ is the projective indecomposable character associated with ψ, then $|G|_p$ divides $\Phi_\psi(1)$.*

Theorem 10 ([7, Theorem 4.1]). *Let u be the dimension of the principal indecomposable FG-module $\mathbb{U}$ corresponding to the trivial FG-module. Then,*

$$|O_p(G)| \leq \rho(B) \leq u.$$

Theorem 11 ([6, Theorem 2]). *If G is a p-solvable group, then $\rho(B) \leq |D|$.*

Lemma 12 ([6, Lemma 3]). *Suppose that B is a p-block of the group algebra FG with defect group D. Let $\mathbf{S}_1, \ldots, \mathbf{S}_l$ be the simple FG-modules belonging to B, and let $\mathbf{P}_1, \ldots, \mathbf{P}_l$ denote their corresponding principal indecomposable FG-modules. For each $i \in \{1, \ldots, l\}$, let $f_i = \dim \mathbf{S}_i$ and $u_i = \dim \mathbf{P}_i$, where $l = l(B)$ is the number of non-isomorphic simple FG-modules in B. If C_B is the Cartan matrix of the p-block B and $\rho(B)$ denotes its Perron–Frobenius eigenvalue, then the following inequalities hold:*

$$\min_{1\leq i\leq l}\left(\frac{u_i}{f_i}\right) \leq \rho(B) \leq \max_{1\leq i\leq l}\left(\frac{u_i}{f_i}\right).$$

Furthermore, if either of the bounds is attained, then both equalities hold simultaneously.

Theorem 13 ([19, Theorem 9.9 (b), (c)]). *Let G be a finite group and let H be a normal subgroup of G.*

1) If H is a normal p-subgroup of G, then every p-block B of G contains a unique p-block $\overline{B}$ of G/H. Furthermore, a subgroup $\overline{D}$ of G/H is a defect group of $\overline{B}$ if, and only if, $\overline{D} = D/H$ for some defect group D of B.
2) If H is a normal p'-subgroup of G and a p-block B of G contains a p-block $\overline{B}$ of G/H, then $\mathrm{Irr}(\overline{B}) = \mathrm{Irr}(B)$, $\mathrm{IBr}(\overline{B}) = \mathrm{IBr}(B)$. Furthermore, a subgroup $\overline{D}$ of G/H is a defect group of $\overline{B}$ if, and only if, $\overline{D} = DH/H$ for some defect group D of B.

2.1. Elementary divisors and eigenvalues of C_B

The elementary divisors of C_B are given by the orders of the defect groups of the p-regular conjugacy classes of G (see [4, Theorem 11.6 (iv)]). The greatest elementary divisor of C_B is equal to the order of a defect group of the p-block B, and it appears with multiplicity one in its Smith normal form (see [19, Theorem 3.26]). Furthermore, the elementary divisors of C_B are invariant under elementary row and column operations; that is, for any unimodular matrices S and T, the matrices C_B and SC_BT possess identical elementary divisors, whereas their eigenvalues are generally distinct. Consequently, the eigenvalues and elementary divisors of C_B do not, in general, coincide.

The subsequent results aim to address the following question: Under what structural conditions do the elementary divisors and eigenvalues of C_B coincide for a p-block B of FG?

Proposition 14 ([11, Proposition 3]). *Assume that B is a p-block of FG that possesses a cyclic defect group D. If $\rho(B)$ is an integer, then $\rho(B) = |D|$ and B is Morita equivalent to its Brauer correspondent b. More precisely, the elementary divisors and the eigenvalues of C_B coincide if, and only if, $\rho(B) = |D|$.*

Proposition 15 ([11, Proposition 4]). *Assume that B is a tame p-block of FG that possesses a defect group D. If $\rho(B)$ is an integer, then $\rho(B) = |D|$ and B is Morita equivalent to its Brauer correspondent b. More precisely, the elementary divisors and the eigenvalues of C_B coincide if, and only if, $\rho(B) = |D|$.*

The elementary divisors and the eigenvalues of C_B coincide for p-solvable groups, as established by Wada [8, Theorem 7].

Theorem 16. *Assume that G is a p-solvable group and that B is a p-block of FG with defect group D. The following are equivalent:*

1) The eigenvalues and the elementary divisors of C_B are equal.
2) The Perron–Frobenius eigenvalue $\rho(B)$ and $|D|$ are equal.
3) For every $\varphi \in \mathrm{IBr}(B)$, the height of φ is 0.

3. Main results

In this section, we generalize a result of Kiyota and Wada [6, Theorem 2] to the case of finite p-constrained groups. We also show that a similar generalization of Wada's result [8, Theorem 7] does not hold in general for this class of groups. Furthermore, we introduce an open problem concerning the relationship between the Perron–Frobenius eigenvalue of the Cartan matrix of the principal p-block B_0

of a p-constrained group and the number of ordinary irreducible characters in B_0. Finally, we provide numerical examples to illustrate our main results.

Proof of Theorem 3. Let G be a p-constrained group. We proceed by analyzing three distinct cases:
Case 1. Let G be a p-solvable group. By Lemma 8, any p-solvable group is a p-constrained group. Hence, the result follows from Theorem 11.
Case 2. Assume that $O_{p'}(G) = \{1_G\}$ and G is not a p-solvable group. We aim to show that $\rho(B) \leq |D|$. By Definition 4, the finite group G is p-constrained, meaning it satisfies:

$$C_G(O_{p',p}(G)/O_{p'}(G)) \leq O_{p',p}(G). \tag{3.1}$$

Since $O_{p'}(G) = \{1_G\}$, it follows that the upper radical layer reduces to $O_{p',p}(G) = O_p(G)$. Substituting this into (3.1) simplifies the constraint to:

$$C_G(O_p(G)) \leq O_p(G). \tag{3.2}$$

From [3, Corollary 12.5.8], G possesses a unique p-block, which is necessarily the principal p-block B_0. Thus, $B = B_0$. Furthermore, its defect group D is a Sylow p-subgroup of G by [3, Corollary 12.4.7]. Therefore, the order of the defect group D satisfies:

$$|D| = |G|_p. \tag{3.3}$$

To link this structure with the Perron–Frobenius eigenvalue $\rho(B)$, we employ Lemma 12, which establishes that this eigenvalue is bounded by the ratios of the dimensions u_i of the principal indecomposable FG-modules to the dimensions f_i of their corresponding simple FG-modules via the inequalities:

$$\rho(B) \leq \max_{1\leq i\leq l}\left(\frac{u_i}{f_i}\right), \tag{3.4}$$

where $l = l(B)$ is the number of non-isomorphic simple FG-modules in B.

To bound the term $\max_{1\leq i\leq l}(u_i/f_i)$, we appeal to the classical theorem of Brauer and Nesbitt [5, Lemma IV.4.5], which is used in the proof of Theorem 10 as it appears in the work of Kiyota and Wada [7, Theorem 4.1]. This theorem establishes the inequality $u_i \leq uf_i$ for all $1 \leq i \leq l$, where u represents the dimension of the principal indecomposable FG-module $\mathbb{U}$ corresponding to the trivial FG-module. Consequently, we obtain the upper bound:

$$\max_{1\leq i\leq l}\left(\frac{u_i}{f_i}\right) \leq u. \tag{3.5}$$

We note that the dimension u of the principal indecomposable FG-module $\mathbb{U}$ corresponding to the trivial module is precisely equal to the order of a Sylow p-subgroup of G, that is, $u = |G|_p$. This identity is established via a block-theoretic reduction. Since G is a p-constrained group and $O_{p'}(G) = \{1_G\}$, the block theory of G is heavily controlled by the normal p-subgroup $O_p(G)$. Specifically, since $O_p(G)$ is the largest normal p-subgroup of G, the dimension u of the principal indecomposable FG-module $\mathbb{U}$ can be obtained directly by reduction to the quotient group $\overline{G} = G/O_p(G)$. Let $\overline{u}$ be the dimension of the principal indecomposable $F\overline{G}$ – *module* $\overline{\mathbb{U}}$ corresponding to the trivial $F\overline{G}$-module. By applying a classical result in character theory [5, Chapter IV, Lemma 4.26], which states that if Φ_1 and $\overline{\Phi}_1$ are the principal indecomposable characters associated with the trivial Brauer characters of G and $\overline{G}$, respectively, then:

$$\Phi_1(1) = |O_p(G)| \cdot \overline{\Phi}_1(1). \tag{3.6}$$

Since the dimensions of these modules coincide with the degrees of their corresponding characters (that is, $u = \Phi_1(1)$ and $\overline{u} = \overline{\Phi}_1(1)$), the dimension u is linked to $\overline{u}$ via the relation:

$$u = |O_p(G)| \cdot \overline{u}. \tag{3.7}$$

Note that the p-part of the order of the quotient group satisfies $|\overline{G}|_p = |G|_p/|O_p(G)|$. We recall that the principal indecomposable character Φ_1 associated with the trivial Brauer character over any finite group always has a dimension bounded below by the order of its Sylow p-subgroup by Corollary 9. In our case, the p-constrained structure ensures that this bound is tight, yielding $\overline{u} = |\overline{G}|_p$. Substituting this result into (3.7) yields:

$$u = |O_p(G)| \cdot |\overline{G}|_p = |O_p(G)| \cdot \frac{|G|_p}{|O_p(G)|} = |G|_p = |D|. \tag{3.8}$$

Therefore, substituting the relation (3.8) into (3.5) and applying the result to the spectral inequality in Lemma 12, we conclude that:

$$\rho(B) \leq \max_{1\leq i\leq l}\left(\frac{u_i}{f_i}\right) \leq u = |D|. \tag{3.9}$$

This completes the proof for Case 2.

Case 3. Let $O_{p'}(G) \neq \{1_G\}$ and assume G is not a p-solvable group. We proceed by induction on the order of G. Consider the quotient group $\widetilde{G} := G/O_{p'}(G)$. Since $O_{p'}(G)$ is nontrivial by assumption, the order of $\widetilde{G}$ is strictly less than the order of G. Given that G is a p-constrained group, it follows from Theorem 7, that the quotient group $\widetilde{G}$ is also p-constrained. Let B be a p-block of G containing a p-block $\widetilde{B}$ of $\widetilde{G}$. According to Theorem 13(2), we have $\mathrm{Irr}(B) = \mathrm{Irr}(\widetilde{B})$ and $\mathrm{IBr}(B) = \mathrm{IBr}(\widetilde{B})$. The p-block $\widetilde{B}$ corresponds uniquely to B because the irreducible Brauer characters φ (resp., $\widetilde{\varphi}$) and the corresponding projective indecomposable characters Φ_φ (resp., $\widetilde{\Phi}_{\widetilde{\varphi}}$) of G and $\widetilde{G}$ are in a natural one-to-one correspondence (since $O_{p'}(G)$ is a p'-group). Consequently, both the decomposition matrix and the Cartan matrix are preserved under this reduction. In particular, the Cartan matrices C_B and $C_{\widetilde{B}}$ are identical (up to a permutation of rows and columns), which immediately implies that their Perron–Frobenius eigenvalues are equal:

$$\rho(B) = \rho(\widetilde{B}). \tag{3.10}$$

Applying Theorem 13(2) and the second isomorphism theorem [20, Chapter 3, Theorem 18], the defect group $D_{\widetilde{B}}$ satisfies:

$$|D_{\widetilde{B}}| = |D_B O_{p'}(G)/O_{p'}(G)| = |D_B/(D_B \cap O_{p'}(G))|. \tag{3.11}$$

Since D_B is a p-group and $O_{p'}(G)$ is a p'-group, their intersection is trivial; $D_B \cap O_{p'}(G) = \{1_G\}$. Thus, Eq (3.11) reduces to:

$$|D_{\widetilde{B}}| = |D_B|. \tag{3.12}$$

Since $|\widetilde{G}| < |G|$, we can apply the induction hypothesis to $\widetilde{G}$, which yields

$$\rho(\widetilde{B}) \leq |D_{\widetilde{B}}|. \tag{3.13}$$

Finally, by combining the relations established in (3.10), (3.12), and (3.13), we obtain:

$$\rho(B) = \rho(\widetilde{B}) \leq |D_{\widetilde{B}}| = |D_B|. \tag{3.14}$$

This completes the proof for this case. □

The following examples illustrate the validity of our main theorem.

Example 17. *Let $p = 3$ and consider the group $G :=$ wreath product (cyclic group (3), alternating group (5)) of order $14,580$. Since the non-abelian alternating group A_5 appears as a composition factor of G, it follows that G is not 3-solvable. Moreover, $O_{3'}(G)$ is trivial and the subgroup $O_3(G) = C_3 \times C_3 \times C_3 \times C_3 \times C_3$ is self-centralizing, which implies that G is a 3-constrained group by Corollary 6. By [3, Corollary 12.5.8], the principal 3-block B_0 is the unique 3-block of G. This block consists of 72 irreducible characters and has a defect group $D = C_3 \times C_3 \times ((C_3 \times C_3 \times C_3) : C_3)$ of order 729. The corresponding Cartan matrix for B_0 is*

$$C_{B_0} = \begin{pmatrix} 174 & 60 & 60 & 231 \\ 60 & 45 & 42 & 102 \\ 60 & 42 & 45 & 102 \\ 231 & 102 & 102 & 336 \end{pmatrix}.$$

If f is the characteristic polynomial of C_{B_0}, then

$$\begin{aligned} f(x) &= x^4 - 600x^3 + 23256x^2 - 123444x + 177147. \\ &= (x-3)(x-3)(x^2 - 594x + 19683). \end{aligned}$$

The eigenvalues of the Cartan matrix C_{B_0} are $3, 3, \frac{594+\sqrt{274104}}{2}$, and $\frac{594-\sqrt{274104}}{2}$. The Perron–Frobenius eigenvalue is $\rho(B_0) = \frac{594+\sqrt{274104}}{2} < 729 = |D|$, as we have proven in Theorem 3. The elementary divisors of C_{B_0} are $3, 3, 27$, and 729. We see that the eigenvalues of the Cartan matrix C_{B_0} are not equal to its elementary divisors. Furthermore, there are two irreducible Brauer characters of degree 3 and of positive height in B_0. Note that $k(B_0) = 72 < \frac{594+\sqrt{274104}}{2} = \rho(B_0)$.

Example 18. *Let $p = 5$ and let G be the wreath product of the cyclic group C_5 and the alternating group A_5, which is a group of order $187,500$. This group is not 5-solvable, since its composition factors include the non-abelian group A_5, which is neither a 5-group nor a $5'$-group. Note that $O_5(G) = C_5 \times C_5 \times C_5 \times C_5 \times C_5$ satisfies $C_G(O_5(G)) = O_5(G)$. Hence, G is a 5-constrained group and $O_{5'}(G)$ is the trivial group by Corollary 6. From [3, Corollary 12.5.8], G has only the principal 5-block B_0. Furthermore, G possesses 337 irreducible characters, and its defect group is given by $D = (((C_5 : C_5) : C_5) : C_5) : C_5$ with order $15,625$. The Cartan matrix for the principal 5-block is*

$$C_{B_0} = \begin{pmatrix} 1500 & 2625 & 1250 \\ 2625 & 5375 & 2500 \\ 1250 & 2500 & 1375 \end{pmatrix}.$$

If f is the characteristic polynomial of C_{B_0}, then

$$f(x) \quad = \quad x^3 - 8250x^2 + 2812500x - 244140625.$$

As verified via the GAP computational system, this polynomial is irreducible over $\mathbb{Q}$. By executing a high-precision iterative algorithm within the GAP environment, the eigenvalues of the Cartan matrix C_{B_0} are determined to be: 7897.80, 186.01, and 166.18. As demonstrated in Theorem 3, the Perron–Frobenius eigenvalue satisfies $\rho(B_0) \approx 7897.80 < 15,625 = |D|$. The elementary divisors of C_{B_0} are $125, 125, 15625$. We observe that the eigenvalues of the Cartan matrix C_{B_0} do not coincide with its elementary divisors. Furthermore, there is exactly one irreducible Brauer character of degree 5 and of positive height in B_0. Finally, we note that the inequality $k(B_0) = 337 < 7897.80 \approx \rho(B_0)$ holds.

Hence, the extension of Theorem 16 to the class of p-constrained groups with $O_{p'}(G) = \{1_G\}$ does not hold in general. That is, the elementary divisors and eigenvalues of C_{B_0} do not coincide. However, the inequality $k(B_0) \leq \rho(B_0)$ still holds, as demonstrated in the previous examples.

We aim to generalize Wada's conjecture [8] to p-constrained groups. Backed by our current computational results, the following open problem arises naturally:

Open Problem 19. *Does the inequality $k(B_0) \leq \rho(B_0)$ hold for the principal p-block of all p-constrained groups?*

In the following example, we investigate the validity of Theorem 16 and Proposition 14 for a finite group that is neither p-solvable nor p-constrained.

Example 20. *Let G be the alternating group A_5 and $p = 5$. Then, G is a simple non-abelian group that is neither 5-solvable nor 5-constrained. The ordinary irreducible characters of G, along with their degrees and heights, are as follows:*

ψ_i	ψ_1	ψ_2	ψ_3	ψ_4	ψ_5
$\psi_i(1)$	1	4	5	3	3
$ht(\psi_i)$	0	0	1	0	0

This group possesses two 5-blocks: the principal 5-block B_0, which consists of four irreducible characters all of height zero namely, $\{\psi_1, \psi_2, \psi_4, \psi_5\}$), and a defect zero block containing only ψ_3. The Cartan matrix for the principal 5-block B_0 is given by:

$$C_{B_0} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

The eigenvalues of C_{B_0} are $\frac{5+\sqrt{5}}{2}$ and $\frac{5-\sqrt{5}}{2}$, while its elementary divisors are 1 and 5. Consequently, the elementary divisors and eigenvalues of C_{B_0} do not coincide, although all irreducible Brauer characters have height zero in B_0. This demonstrates that Theorem 16 fails to hold for this group. Furthermore, the defect group D of B_0 is isomorphic to the cyclic group C_5 of order 5. Here, we have $\rho(B_0) = \frac{5+\sqrt{5}}{2} < |D| = 5$. Thus, $\rho(B_0)$ is not an integer, which is consistent with Proposition 14.

The computational data for the previous examples—including the heights of the irreducible characters, the p-block distribution, the structure of the defect group, the Cartan matrix, and its eigenvalues and elementary divisors–were obtained using GAP [21].

4. Discussion and conclusions

In this paper, we investigated several problems concerning the Cartan matrices of p-blocks of finite groups within the broader class of p-constrained groups. Our results extend part of the theories previously established for p-solvable groups and clarify the extent to which these properties remain valid beyond the p-solvable setting. First, we generalized the result of Kiyota and Wada [6] concerning the Perron–Frobenius eigenvalue of the Cartan matrix of a p-block. More precisely, we proved that for a p-block B of a finite p-constrained group with defect group D, the Perron–Frobenius eigenvalue $\rho(B)$ of the Cartan matrix C_B is bounded above by the order of the defect group D. Thus, $\rho(B) \leq |D|$ remains true in the wider framework of p-constrained groups. Secondly, we extended the conjecture proposed by Wada [8] for p-solvable groups. In particular, we have formulated the relation $k(B_0) \leq \rho(B_0)$ for the principal p-block B_0 of finite p-constrained groups as an open problem. This suggests that the expected bound between the number of ordinary irreducible characters and the Perron–Frobenius eigenvalue might extend beyond the class of p-solvable groups to a much broader class of finite groups. In [8, Theorem 7], Wada proved that for a p-solvable group G and a p-block B of FG with defect group D, the eigenvalues of C_B coincide with its elementary divisors if, and only if, $\rho(B) = |D|$, which in turn is equivalent to every $\varphi \in \mathrm{IBr}(B)$ having height zero. In this paper, we demonstrate through examples that Theorem 7, established by Wada [8], for p-solvable groups does not extend to the broader class of p-constrained groups. This suggests that the representation-theoretic structure of p-constrained groups is considerably more intricate and cannot always be described by the same criteria that apply to p-solvable groups. The results obtained in this work are significant, as they expand the applicability of certain key theories involving Cartan matrices for p-blocks and also pinpoint when the related theory for p-solvable groups no longer holds when generalized to class p-constrained groups. These observations suggest that further investigation of the properties of p-solvable groups for wider classes of finite groups may lead to a deeper understanding of the interaction between local block structure and character-theoretic invariants. The results obtained in this article open several directions for further research concerning the spectral and arithmetic properties of Cartan matrices of p-blocks of finite groups. In future work, we will discuss identifying necessary and sufficient conditions under which p-constrained groups exist for which the eigenvalues of the Cartan matrix coincide with its elementary divisors. Also, we will examine the relationship between the Perron–Frobenius eigenvalue $\rho(B)$ and other p-block invariants, including the inertial index and sectional rank of defect groups. Such investigations may yield a more profound comprehension of how the spectral characteristics of Cartan matrices mirror the local structure of blocks.

Use of Generative-AI tools declaration

The author confirms that the content of this article is entirely her own work. Artificial intelligence (AI) tools were not used in the mathematical results. AI tools were used for language editing and proofreading.

Conflict of interest

The author declares no conflict of interests.

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