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Research article

On t-Cayley hypergraphs of cyclic groups with vertex transitive property on the set of non-injective endomorphisms

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Abstract: Hypergraphs provide a powerful framework for modeling polyadic relationships, generalizing classical graph structures. t-Cayley hypergraphs, a natural extension of Cayley graphs, are inherently vertex-transitive under the automorphism group (Aut). However, this vertex-transitivity is not necessarily preserved when replacing Aut with non-injective endomorphisms (End′). This work investigates the precise conditions under which t-Cayley hypergraphs of finite cyclic groups retain this property. Our main result establishes that a t-Cayley hypergraph H over the cyclic group $\mathbb{Z}_n$ is End′-vertex-transitive if t divides n.

Keywords: Cayley graph; hypergraph; t-Cayley hypergraph; vertex-transitive graph; endomorphism; automorphism
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1. Introduction

As generalizations of graphs where edges can be incident to any number of vertices, hypergraphs offer a powerful way to represent complex relationships between sets of objects. In its simplest form, a standard graph $G = (V(G), E(G))$ is defined as a pair consisting of a vertex set $V(G)$ and an edge set $E(G)$, where edges join pairs of vertices. In contrast, a hypergraph $H = (V(H), E(H))$ also has a vertex set $V(H)$, which is a nonempty set, but its hyperedge set $E(H)$ is a family of subsets of $V(H)$.

The study of symmetry in graphs and their generalizations has long been a central topic in algebraic graph theory, particularly regarding vertex-transitive graphs, which carry significant practical value and potential applications. A graph G is said to be vertex-transitive if its automorphism group, Aut(G), acts transitively on its vertex set. In recent years, hypergraphs have emerged as an essential mathematical framework for modeling complex systems where interactions are inherently polyadic, such as biological pathways, social networks, and modern communication infrastructures [4, 11, 14–16]. A fundamental family within this class is the Cayley graph, first introduced by Arthur Cayley in 1878 to represent groups graphically. For a group Γ and a connection set $S \subseteq \Gamma$, a Cayley graph represents group elements as vertices, with edges defined by the relation (g, ga) for each $a \in S$. Although all Cayley graphs are vertex-transitive, the converse is generally false [12, 13]. Notable constructions of non-Cayley vertex-transitive graphs include those of order $2pq$ [6], illustrating the structural complexity of these objects. Furthermore, the study of such symmetry has been extended beyond group-based structures; for instance, Kelarev and Praeger [9] characterized the Aut-vertex-transitivity of Cayley graphs of semigroups. In particular, Cayley graphs are valuable due to their highly symmetric and well-structured nature. These properties make them excellent candidates for designing interconnection networks in parallel computing, distributed systems, and data centers [1,5,17]. In such architectures, symmetry ensures optimal routing, efficient load balancing, and high fault tolerance, which are critical for the performance and reliability of modern network topologies.

Over time, several approaches have been proposed to generalize the Cayley graph construction to the hypergraph setting [7,8]. In particular, Lee and Kwon [10] introduced a formal definition of Cayley hypergraphs, proving that every Cayley hypergraph of a group is vertex-transitive, thereby extending the classical property of Cayley graphs to this more general framework.

Within this context, the t-Cayley hypergraph emerges as a significant structure. Originally introduced by Buratti in 1994 [3], it provides a natural generalization of the Cayley graph characterized by a parameter t, which determines the number of elements used to generate each hyperedge. Although this construction allows for the specification of t-uniform hypergraphs, the actual uniformity depends on the specific properties of the connection set. Due to these characteristics, the t-Cayley hypergraph has gained attention as a notable generalization of Cayley graphs, as seen in [2]. Furthermore, as a specialized case of the general Cayley hypergraph, every t-Cayley hypergraph is inherently vertex-transitive.

However, focusing on the set of all non-injective endomorphisms, denoted by End′, reveals a more complex landscape. Unlike the case of automorphisms, t-Cayley hypergraphs exhibit varying behavior regarding End′-vertex-transitivity, with some instances possessing this property and others lacking it. This inconsistency highlights End′-vertex-transitivity as a compelling subject for further investigation. Moreover, the investigation of endomorphisms, specifically non-injective endomorphisms, provides deeper insights into the homomorphic mappings and structural retractions of these networks. In this paper, we identify the precise conditions under which t-Cayley hypergraphs of cyclic groups achieve End′-vertex-transitivity.

2. Preliminaries

A hypergraph $H = (V, E)$ is often defined where E is a family of subsets of V, allowing repeated hyperedges. In this work, multiplicities are irrelevant, so we treat E as a set of subsets of V, ensuring that each hyperedge appears at most once. Two vertices $x, y \in V$ (or two hyperedges) are *adjacent* if some hyperedge contains both of them (or if the hyperedges intersect nontrivially, respectively), and a vertex x is *incident* with a hyperedge e when $x \in e$. For any vertex x, the star $H(x)$ is the set of hyperedges containing x, and its *degree* is $d(x) = |H(x)|$. Additionally, the hypergraph is t-uniform if every hyperedge has exactly t vertices and is r-regular if every vertex has degree r.

Let $H = (V, E)$ be a hypergraph. A subhypergraph of H, denoted by $H' \subseteq H$, is a hypergraph $H' = (V', E')$ such that $V' \subseteq V$, and $E' \subseteq E$. For $V' \subseteq V$, the induced subhypergraph $H(V') = (V', E')$ is formed by all intersections $e \cap V' \neq \emptyset$ with $e \in E$ such that $|e| = 1$ or $|e \cap V'| \geq 2$. A (u, v)-walk of length k is a sequence $v_0, e_1, v_1, \ldots, e_k, v_k$ with $v_0 = u$ and $v_k = v$, where each pair of consecutive vertices is adjacent through the corresponding hyperedge; if all vertices and hyperedges (except possibly $v_0 = v_k$) are distinct, the sequence is a *path* of length k. The *distance* $d(x, y)$ is the minimum length of a path between x and y, or ∞ if no such path exists. A connected component is an induced subhypergraph $H(A)$, where $A \subseteq V$ is maximal with the property that $d(x, y) \neq \infty$ for all $x, y \in A$; thus, the hypergraph is connected precisely when it has exactly one connected component.

In the context of hypergraph theory, a morphism is a structure-preserving function between hypergraphs. Specifically, a homomorphism $f : V(H) \to V(H')$ preserves hyperedges: If $e = \{u_1, u_2, \ldots, u_n\} \in E(H)$, then $f(e) = \{f(u_1), f(u_2), \ldots, f(u_n)\} \in E(H')$. A homomorphism f is an isomorphism if it is a bijection that preserves the hyperedge structure in both directions. An endomorphism is a homomorphism from a hypergraph H to itself, and a bijective endomorphism is an automorphism. Crucially, in the study of finite graphs and hypergraphs, the concepts of injectivity, surjectivity, and bijectivity are equivalent for endomorphisms. This means that an endomorphism on a finite structure is either an automorphism or a non-injective map. The families of automorphisms and endomorphisms of a hypergraph H are denoted by $\mathrm{Aut}(H)$ and $\mathrm{End}(H)$, respectively. We note that $\mathrm{Aut}(H)$ together with the composition operation is a group, and $\mathrm{End}(H)$ is a monoid which contains $\mathrm{Aut}(H)$. We will denote by $\mathrm{End}'(H)$ the set of all non-injective endomorphisms on H. Moreover, when just one hypergraph is under discussion, we often write Aut for $\mathrm{Aut}(H)$, End for $\mathrm{End}(H)$, and End$'$ for $\mathrm{End}'(H)$.

Vertex transitivity, denoted as $\mathrm{Aut}(H)$-vertex-transitive for a hypergraph H, means that for any two vertices $x, y \in V(H)$, there is an automorphism $\phi \in \mathrm{Aut}(H)$ such that $\phi(x) = y$. This concept can be generalized to $\mathcal{U}$-vertex-transitive by requiring the set $\mathcal{U}$ (a subset of the endomorphism monoid $\mathrm{End}(H)$) to act transitively on the vertices. Because $\mathrm{Aut}(H) \subseteq \mathrm{End}(H)$, any $\mathrm{Aut}(H)$-vertex-transitive structure is automatically $\mathrm{End}(H)$-vertex-transitive.

The t-Cayley hypergraph was first introduced by Buratti in 1994 [3] as a natural extension of the classical Cayley graph to hypergraphs. When we set the parameter t to 2, the t-Cayley hypergraph reduces to the well-known Cayley graph.

Definition 2.1. Let Γ be a group with the identity e_Γ, $A \subseteq \Gamma \setminus \{e_\Gamma\}$, and let t be an integer such that $2 \leq t \leq max\{o(\omega) : \omega \in A\}$, where $o(\omega)$ denotes the order of the element of Γ. The t-Cayley hypergraph t-$CH(\Gamma, A)$ of Γ relative to A is defined as $V(t - CH(\Gamma, A)) = \Gamma$ and $E(t - CH(\Gamma, A)) = \{\{g, g\omega, g\omega^2, \ldots, g\omega^{t-1}\} : g \in \Gamma, \omega \in A\}$. We also call A the connection set of $t - CH(\Gamma, A)$.

Because a t-Cayley hypergraph is a special case of a Cayley hypergraph, it is well-known that every t-Cayley hypergraph is *Aut*(H)-vertex-transitive (see [10]).

We see that each hyperedge admits a natural algebraic description. Therefore, in the proofs, we represent a hyperedge by $e_{g,a} = \{\, g,\ ga,\ \ldots,\ ga^{t-1}\,\} \in E(t\text{-}CH(\Gamma, A))$, where $g \in \Gamma$, and $a \in A$, and we refer to a hyperedge $e_{g,a}$ as a hyperedge of type a.

Remark 2.2. *In the t-Cayley hypergraph t-*$\mathrm{CH}(\Gamma, A)$*, if both a and* a^{-1} *belong to A, then for any* $g \in \Gamma$*, we have*

$$\begin{aligned} e_{g,a} &= \{\, g,\ ga,\ \ldots,\ ga^{t-1}\,\} \\ &= \{\, ga^{t-1},\ ga^{t-1}a^{-1},\ \ldots,\ ga^{t-1}a^{-(t-1)} = g\,\} \\ &= e_{ga^{t-1},\, a^{-1}}. \end{aligned}$$

In particular, for a t-Cayley hypergraph $H = t\text{-CH}(\Gamma, A)$ where A contains both a and a^{-1}, the collections of hyperedges generated by a and by a^{-1} consist of exactly the same sets, that is, $\{ e_{h,a} \mid h \in \Gamma \} = \{ e_{h,a^{-1}} \mid h \in \Gamma \}$.

Throughout this work, all groups under consideration are assumed to be finite. Recall that a group is called *cyclic* if it is generated by a single element, and it is well-known that every finite cyclic group is isomorphic to the additive group of integers modulo n. For the remainder of this paper, any finite cyclic group is identified with the additive group of integers modulo n, denoted by $(\mathbb{Z}_n, +_n)$. Under this convention, the group operation is $a +_n b = a + b \pmod{n}$, and the integer multiple ka is used in place of the multiplicative notation a^k. Furthermore, we let $k \pmod{n}$ denote the remainder of k divided by n.

3. Vertex transitivity in hypergraphs

In the context of an endomorphism f of the hypergraph $H = (V, E)$, we can denote the image of f as $f(H)$, representing the hypergraph $H' = (V', E')$, where

$$V' = \{f(v) \mid v \in V\},$$

and

$$E' = \{\{f(e) \mid e \in E\} \text{ with } f(e) = \{f(u) \mid u \in e\}.$$

Because f is an endomorphism, it follows that $V' \subseteq V$, and $E' \subseteq E$. This implies that $f(H)$ is a subhypergraph of H. Previous studies have established many transitivity results for graphs. In this work, we show that several of these results also hold in the hypergraph environment. As these lemmas can be proven directly via a construction analogous to the Cayley graph, the corresponding proofs are provided in this paper.

Lemma 3.1. *Let G be a disconnected finite hypergraph, $f \in \mathrm{Aut}(G)$, and C_1, C_2 are components of G. If $f(x) = y$ for some $x \in C_1$, and $y \in C_2$, then $f(C_1) = C_2$.*

Proof. Let G be a disconnected finite hypergraph, and let $f \in \mathrm{Aut}(G)$ be an automorphism on G. Suppose C_1 and C_2 are distinct connected components of G, and there exist vertices $x \in V(C_1)$ and $y \in V(C_2)$ such that $f(x) = y$. Moreover, because C_1 is a connected component, for any vertex $u \in V(C_1)$, there exists a (x, u)-walk $w : x = v_0, e_1, v_1, e_2, v_2 \ldots, e_k, v_k = u$. Because f is an automorphism, each hyperedge e_i is mapped to $f(e_i)$, which remains a hyperedge in G for all $i \in \{1, 2, \ldots, k\}$, and $f(v_{i-1})$ and $f(v_i)$ are adjacent through the hyperedge $f(e_i)$, forming a walk $f(v_0), f(e_1), f(v_1) f(e_2), f(v_2), \ldots, f(e_k), f(v_k)$. Because C_2 is a connected component, the sequence of hyperedges mapped by f remains entirely within C_2, which means that $f(C_1)$ is a subhypergraph of C_2. Because f is a bijection on $V(G)$, its inverse f^{-1} is also an automorphism on G. Applying the same argument to f^{-1}, we conclude that $f^{-1}(C_2) \subseteq C_1$, which implies $f(C_1) = C_2$. □

Lemma 3.2. *Let G be a finite hypergraph, and let $C_1, C_2, \ldots, C_k$ be the components of G. Then G is* $\mathrm{Aut}(G)$*-vertex-transitive if and only if the following conditions hold:*

(a) *Components $C_1, C_2, \ldots, C_k$ are isomorphic.*
(b) *Each component C_i is* $\mathrm{Aut}(C_i)$*-vertex transitive.*

Proof. (⇒) To prove Condition (a), consider components C_i and C_j of the hypergraph G, and let $u_i \in V(C_i)$, and $v_j \in V(C_j)$. Because G is an Aut(G)-vertex-transitive hypergraph, there exists an automorphism $f \in \mathrm{Aut}(G)$ such that $f(u_i) = v_j$. By Lemma 3.1, $f(C_i) = C_j$. Hence, $f|_{V(C_i)} : V(C_i) \to V(C_j)$ is an isomorphism. Next, we show that C_i is Aut(C_i)-vertex-transitive. Consider vertices u, v in the same component C_i. There is an automorphism $g \in \mathrm{Aut}(G)$ such that $g(u) = v$. By Lemma 3.1, we have $g(C_i) = C_i$. Hence, the restriction $g|_{V(C_i)} : V(C_i) \to V(C_i)$ is an isomorphism such that $g|_{V(C_i)}(u) = v$. Thus, C_i is Aut(C_i)-vertex-transitive.
(⇐) Suppose the conditions (a) and (b) are true. Let $u, v \in V(G)$, and assume $u \in V(C_i)$ and $v \in V(C_j)$ for some $i, j \in \{1, 2, \ldots, k\}$. This can be considered in two scenarios.

Case I: $i = j$. Because C_i is Aut(C_i)-vertex-transitive, there is an automorphism $f \in \mathrm{Aut}(C_i)$ such that $f(u) = v$. Let $\varphi : V(G) \to V(G)$ be defined by

$$\varphi(x) = \begin{cases} f(x) & x \in V(C_i), \\ x & \textit{otherwise}. \end{cases}$$

It is clear that $\varphi \in \mathrm{Aut}(G)$, and $\varphi(u) = v$.

Case II: $i \neq j$. By Assumption (a), C_i and C_j are isomorphic. Consider f as a hypergaph isomorphism from C_i onto C_j. Thus, $u, f^{-1}(v) \in V(C_i)$, and $f(u), v \in V(C_j)$. Because C_i is Aut(C_i)-vertex-transitive, and C_j is Aut(C_j)-vertex-transitive, there is an automorphism $\varphi_i \in \mathrm{Aut}(C_i)$ and $\varphi_j \in \mathrm{Aut}(C_j)$ such that $\varphi_i(f^{-1}(v)) = u$, and $\varphi_j(f(u)) = v$. Let $\varphi : V(G) \to V(G)$ be defined by

$$\varphi(x) = \begin{cases} \varphi_j(f(x)) & x \in V(C_i), \\ \varphi_i(f^{-1}(x)) & x \in V(C_j), \\ x & \textit{otherwise}. \end{cases}$$

Hence, φ is an automorphism on G such that $\varphi(u) = v$.

From the two cases above, we conclude that G is Aut(G)-vertex-transitive. □

The primary objective of this work is to investigate the End′-vertex-transitivity of t-Cayley hypergraphs. Our study distinguishes between two fundamental cases: connected t-Cayley hypergraphs and those that are disconnected. To this end, we establish certain underlying conditions for these structures. The following lemma establishes the necessary and sufficient conditions for these hypergraphs to be connected.

Lemma 3.3. *Let Γ be a finite group with cardinality n. Define $H = t\text{-}Cay(\Gamma, A)$. A pair of vertices $u, v \in V(H)$ are in the same component if and only if $u\langle A\rangle = v\langle A\rangle$.*

Proof. (⇒) Let u, v be in the same component. Then there is a walk w from u to v of length m where $W : u = v_0, e_1, v_1, e_2, \ldots, e_m, v_m = v$. Because v_0 and v_1 are adjacent, there exists $\omega_1 \in A$ such that $v_0 = v_1\omega_1^{j_1}$ for some $j_1 \in \mathbb{Z}$. Similarly, we obtain $v_1 = v_2\omega_2^{j_2}$, $v_2 = v_3\omega_3^{j_3}$, …, $v_{m-1} = v_m\omega_m^{j_m}$, where $\omega_2, \omega_3, \ldots, \omega_m \in A$, and $j_2, j_3, \ldots, j_m \in \mathbb{Z}$. Then, $u = v_1{\omega_1}^{j_1} = v_2{\omega_2}^{j_2}{\omega_1}^{j_1} = \cdots = v_m{\omega_m}^{j_m}{\omega_{m-1}}^{j_{m-1}} \ldots {\omega_1}^{j_1} = v{\omega_m}^{j_m}{\omega_{m-1}}^{j_{m-1}} \ldots {\omega_1}^{j_1}$. Because ${\omega_m}^{j_m}{\omega_{m-1}}^{j_{m-1}} \ldots {\omega_1}^{j_1} \in \langle A\rangle$, it follows that $v^{-1}u = {\omega_m}^{j_m}{\omega_{m-1}}^{j_{m-1}} \ldots {\omega_1}^{j_1} \in \langle A\rangle$. Thus, $u\langle A\rangle = v\langle A\rangle$.

($\Leftarrow$) Let $u\langle A\rangle = v\langle A\rangle$. Then $v^{-1}u = a$ for some $a \in \langle A\rangle$. Moreover, $a = \omega_1\omega_2\dots\omega_n$, where $\omega_i \in A$ for all $1 \geq i \geq n$. Let $u = v_0, v_1 = v_0\omega_1, v_2 = v_1\omega_2, v_2 = v_1\omega_2, \dots, v_n = v_{n-1}\omega_n$. We see that v_0 is adjacent to v_1, v_1 is adjacent to v_2, and so on. Then there exists a walk from u to v. Hence, u, v are in the same component. □

Suppose G is an $\mathrm{Aut}(G)$-vertex-transitive hypergraph that is disconnected. Let the components of G be $C_1, \dots, C_k$. By Lemma 3.2, these components are all mutually isomorphic, and each C_i is $\mathrm{Aut}(C_i)$-vertex-transitive. Consequently, any two vertices $u, v \in V(G)$ can be mapped to each other by a suitable endomorphism. Specifically, if u and v lie in the same component, we extend an automorphism of that component to an endomorphism on G; if they lie in different components, we compose an isomorphism between components with an automorphism of the target component and extend it to all of G. In either case, we can ensure the resulting mapping is a proper endomorphism (non-automorphism) sending u to v. This foundational idea leads directly to the following result.

Lemma 3.4. *Let G be an $\mathrm{Aut}(G)$-vertex-transitive hypergraph. If G is disconnected, then G is $\mathrm{End}'(G)$-vertex-transitive.*

4. End′-vertex transitivity of t-Cayley hypergraphs of cyclic groups

The hypergraph $H = t\text{-}CH(\Gamma, A)$ is either connected or disconnected, depending on the connection set A. By Lemma 3.3, it is a connected hypergraph when $\langle A\rangle = \Gamma$.

Theorem 4.1. *Let $H = t\text{-}\mathrm{CH}(\Gamma, A)$ be a t-Cayley hypergraph of a finite group Γ. If H is disconnected, then H is $\mathrm{End}'(H)$-vertex-transitive.*

Proof. It is known that every t-Cayley hypergraph is $\mathrm{Aut}(H)$-vertex-transitive (see [10]). By Lemma 3.4, any disconnected $\mathrm{Aut}(H)$-vertex-transitive hypergraph is necessarily $\mathrm{End}'(H)$-vertex-transitive. Therefore, the result follows immediately for disconnected t-Cayley hypergraphs. □

Consequently, we focus our attention on the more substantial problem of determining when connected t-Cayley hypergraphs admit non-injective endomorphisms and, in particular, when they are End′-vertex-transitive.

Because varying the orders of elements in the connection set may produce hyperedges of different sizes, the structural analysis can become considerably more complex. However, for cyclic groups, by choosing the connection set $A \subseteq \mathcal{A}$, where $\mathcal{A}$ is the set of all generators of the group Γ, the resulting hypergraph remains t-uniform for all $2 \leq t \leq n$. Therefore, from this point onward, we focus exclusively on t-Cayley hypergraphs of cyclic groups with connection sets $A \subseteq \mathcal{A}$. To establish conditions for these hypergraphs to admit non-injective endomorphisms, we rely solely on elementary number theory, basic combinatorics, and several structural properties of t-Cayley hypergraphs.

Remark 4.2. *Let t and m be integers. For any distinct integers i and j with $0 \leq i, j < t$, the remainders when $m + i$ and $m + j$ are divided by t are distinct.*

Remark 4.3. *Let w and t be integers such that $\gcd(w, t) = 1$. Then for any integer k and any distinct integers i, j with $0 \leq i, j < t$, the remainders when $w(k + i)$ and $w(k + j)$ are divided by t are distinct.*

Remark 4.4. *Let $a, k, t \in \mathbb{Z}$ with $k > t > 0$. If k is a multiple of t, then the remainder obtained by first dividing a by k and then dividing that remainder by t is equal to the remainder obtained by dividing a directly by t; that is,*

$$((a \bmod k) \bmod t) = a \bmod t.$$

The following lemma demonstrates that for the t-Cayley hypergraphs under consideration, the existence of a non-injective endomorphism is sufficient to conclude that the hypergraph is End′-vertex-transitive.

Lemma 4.5. *Let $\mathbb{Z}_n$ be the additive group of order n, and let $2 \leq t < n$ be an integer. Consider the t-Cayley hypergraph t-$CH(\mathbb{Z}_n, A)$, where $A \subseteq \mathcal{A}$. If $\mathrm{End}'(t\text{-}CH(\mathbb{Z}_n, A)) \neq \emptyset$, then t-$CH(\mathbb{Z}_n, A)$ is an* End′*-vertex-transitive hypergraph.*

Proof. Let $H = t\text{-}CH(\mathbb{Z}_n, A)$. Take any non-injective endomorphism $f \in \mathrm{End}'(H)$. Let $u, v \in V(H) = \mathbb{Z}_n$ be arbitrary. Define a map $\phi_{u,v} : V(H) \to V(H)$ by

$$\phi_{u,v}(x) = x - f(u) + v \qquad \text{for all } x \in \mathbb{Z}_n.$$

Note that $\phi_{u,v}$ is the translation by the element $c = v - f(u)$. We claim that $\phi_{u,v}$ is well-defined. To show that $\phi_{u,v}$ is an endomorphism, consider $e \in E(H)$; that is, $e = \{q, q + a, q + 2 \cdot a, \ldots, q + (t-1) \cdot a\}$ for some $q \in \mathbb{Z}_n$ and $a \in A$.
Now, we evaluate $\phi_{u,v}(e)$:

$$\begin{aligned}\phi_{u,v}(e) &= \{\phi_{u,v}(q), \phi_{u,v}(q + a), \phi_{u,v}(q + 2 \cdot a), \ldots, \phi_{u,v}(q + (t-1) \cdot a)\} \\ &= \{q - f(u) + v, q + a - f(u) + v, q + 2 \cdot a - f(u) + v, \ldots, q + (t-1) \cdot a - f(u) + v\}.\end{aligned}$$

Let $q' = q - f(u) + v \in \mathbb{Z}_n$. We then have

$$\phi_{u,v}(e) = \{q', q' + a, q' + 2 \cdot a, \ldots, q' + (t-1) \cdot a\} \in E(H).$$

Thus, $\phi_{u,v}$ maps edges to edges, and therefore, $\phi_{u,v}$ is an endomorphism.

Now, consider the composition $g := \phi_{u,v} \circ f$. As a composition of endomorphisms, g is an endomorphism of H. Because $\phi_{u,v}$ is bijective, g is non-injective exactly when f is non-injective; hence, $g \in \mathrm{End}'(H)$. Moreover,

$$(g)(u) = (\phi_{u,v} \circ f)(u) = \phi_{u,v}(f(u)) = f(u) - f(u) + v = v.$$

Thus, for any arbitrary $u, v \in V(H)$, we have produced an element $g \in \mathrm{End}'(H)$ with $g(u) = v$. This shows that H is $\mathrm{End}'(H)$-vertex-transitive. □

To prove that n being divisible by t is necessary for the existence of a non-injective endomorphism, we show that the t-Cayley hypergraph can be iteratively compressed into a single hyperedge. The following lemmas ensure that each step of this compression remains valid.

Lemma 4.6. *Let $H = t\text{-}CH(\mathbb{Z}_n, A)$, where $A \subseteq \mathcal{A}$. Let $f : \mathbb{Z}_n \to \mathbb{Z}_n$ be an endomorphism of H, and assume $|f(V(H))| > t$. Then there exist two distinct hyperedges $e_1, e_2 \in f(E(H))$ with $|e_1 \cap e_2| = t - 1$.*

Proof. Fix any $a \in A$ (which is a generator) and any $g \in \mathbb{Z}_n$. Write $e_{g,a} = \{g,\ g+a,\ \ldots,\ g+(t-1)a\}$. Because f is an endomorphism, $f(e_{g,a}) = e_{g',a'} \in f(E(H))$ for some $g' \in \mathbb{Z}_n$ and $a' \in A$.

Consider the sequence of consecutive hyperedges in the domain

$$e_{g,a},\ e_{g+a,a},\ e_{g+2a,a},\ \ldots,\ e_{g+(n-1)a,a},$$

where, because a is a generator, we have $\{g + ka : 0 \leq k \leq n-1\} = \mathbb{Z}_n$. We consider two cases based on whether all these consecutive hyperedges map to the same image hyperedge.

Case 1: There exists $k \in \{0, 1, 2, \ldots, n-1\}$ with $f(e_{g+ka,a}) \neq e_{g',a'}$.

Let $m \geq 1$ be the minimal such index. By minimality, we have

$$f(e_{g+ja,a}) = e_{g',a'} \quad \text{for } 0 \leq j \leq m-1 \qquad \text{and} \qquad f(e_{g+ma,a}) = e_{h,b} \neq e_{g',a'}.$$

Now, the two domain edges $e_{g+(m-1)a,a}$ and $e_{g+ma,a}$ intersect in exactly $t-1$ vertices:

$$e_{g+(m-1)a,a} \cap e_{g+ma,a} = \{\, g+ma,\ g+(m+1)a, \ldots, g+(m+t-2)a \,\},$$

and by the assumption $f(e_{g+(m-1)a,a}) = e_{g',a'}$, all those $t-1$ domain vertices map into $e_{g',a'}$. Because $f(e_{g+ma,a}) = e_{h,b}$, the images of these same $t-1$ vertices also belong to $e_{h,b}$. Hence,

$$|e_{g',a'} \cap e_{h,b}| \geq t-1.$$

Both $e_{g',a'}$ and $e_{h,b}$ have size t, so their intersection cardinality is either t (equal edges) or $t-1$. It cannot be t because we assume that $e_{h,b} \neq e_{g',a'}$. Thus,

$$|e_{g',a'} \cap e_{h,b}| = t-1,$$

which proves the claim in this case.

Case 2: For all $k \in \{0, 1, 2, \ldots, n-1\}$, we have $f(e_{g+ka,a}) = e_{g',a'}$. Because a is a generator, the union of the domain vertices of the edges $e_{g+ka,a}$ (as k runs) equals the whole vertex set $\mathbb{Z}_n$. Therefore, every vertex of $\mathbb{Z}_n$ is mapped into $e_{g',a'}$, so

$$f(V(H)) \subseteq e_{g',a'}.$$

Hence, $|f(V(H))| \leq |e_{g',a'}| = t$, contradicting the hypothesis $|f(V(H))| > t$. Thus, this alternative cannot occur.

Combining the two cases yields that some pair of distinct hyperedges in $f(E(H))$ intersects in exactly $t-1$ vertices. □

Lemma 4.7. *Let $H = t\text{-}CH(\mathbb{Z}_n, A)$ where $A \subseteq \mathcal{A}$. Let $f : \mathbb{Z}_n \to \mathbb{Z}_n$ be a non-injective endomorphism of H, and write $Y = Im(f)$ and*

$$\mathcal{D}_f = \{x_1 - x_2 \mid x_1 \neq x_2,\ f(x_1) = f(x_2)\}, \mathcal{D}_Y = \{y_1 - y_2 \mid y_1, y_2 \in Y\}.$$

Assume, furthermore, that there exists a fixed $a \in A$ such that every image hyperedge under f is of type a. If $|Y| > t$, then $\mathcal{D}_f \cap \mathcal{D}_Y \neq \emptyset$.

Proof. First, we show that $at \in \mathcal{D}_Y$. By the hypothesis $|Y| > t$ and the preceding Lemma 4.6, there exist two distinct image hyperedges whose intersection is of size $t-1$. Under the added assumption that every image hyperedge is of type a, any two distinct type-a hyperedges that meet in $t-1$ vertices must be consecutive; hence, they are of the form $e_{g,a}$ and $e_{g+a,a}$ for some $g \in \mathbb{Z}_n$. Therefore, both g and $g+ta$ belong to Y, and so $(g+ta)-g = at \in \mathcal{D}_Y$.

Next, we show that $at \in \mathcal{D}_f$. Because f is non-injective, there exist distinct elements $x_1, x_2 \in \mathbb{Z}_n$ with $f(x_1) = f(x_2)$. We then have $|Y| < n$. Choose the same $a \in A$ from above, and consider the n hyperedges $e_{h,a} \in E(H)$ for all $h \in \mathbb{Z}_n$, whose union is $\mathbb{Z}_n$. If every hyperedge of type a appears among the image hyperedges; that is, if $e_{h,a} \in E(f(H))$ for all $h \in \mathbb{Z}_n$, then the union of the image hyperedges would be $\mathbb{Z}_n$, forcing $|Y| = n$, contradicting non-injectivity. Hence, there exists some $h \in \mathbb{Z}_n$ with $e_{h-a,a} \in E(f(H))$ but $e_{h,a} \notin E(f(H))$.

Let hyperedges $e_{x,a}, e_{x+a,a} \in E(H)$ be such that $f(e_{x,a}) = e_{h-a,a}$. Because every hyperedge in $E(f(H))$ is assumed to be of type a, and $e_{h,a} \notin E(f(H))$, this forces $f(x) = f(x+ta)$. That is, we obtain $x + ta - x = at \in \mathcal{D}_f$.

From above, we obtain $at \in \mathcal{D}_Y \cap \mathcal{D}_f$, proving the lemma. □

The following lemma shows that the hypergraph can be collapsed onto a single hyperedge via a non-injective endomorphism.

Lemma 4.8. *Let $H = t\text{-CH}(\mathbb{Z}_n, A)$ be the t-Cayley hypergraph of the group $\mathbb{Z}_n$, where $t \geq 2$, and $a \in A \subseteq \mathcal{A}$. Let $f \in \mathrm{End}'(H)$ with the additional condition that every image hyperedge under f is of type a. Define the translation $\sigma^{\ell}(x) = x + \ell$ for all $x \in \mathbb{Z}_n$, and recursively set*

$$f_0 = f, \qquad f_{k+1} = f \circ \sigma^{\ell_k} \circ f_k \quad (k \geq 0),$$

where at each step, the shift $\ell_k \in \mathbb{Z}$ is chosen suitably depending on f and $\mathrm{Im}(f_k)$. Then there exists $K \in \mathbb{N}$ such that $\mathrm{Im}(f_K)$ is contained in a single hyperedge of H.

Proof. A non-injective endomorphism f of the hypergraph H maps each hyperedge of H into some hyperedge of H, and it composes with the already defined translation $\sigma^{\ell}(x)$. For any hyperedge $e_{g,a'} = \{g, g+a', \ldots, g+(t-1)a'\} \in E(H)$, applying σ^{ℓ} gives

$$\sigma^{\ell}(e_{g,a'}) = \{g+\ell, g+a'+\ell, \ldots, g+(t-1)a'+\ell\} = e_{g+\ell,a'} \in E(H).$$

Therefore, σ^{ℓ} is a hypergraph endomorphism. Consequently, the composition $f_k = f \circ \sigma^{\ell_{k-1}} \circ f_{k-1}$ is an endomorphism of H for every $k \geq 1$, with $f_0 = f$, as compositions of endomorphisms yield endomorphisms. Moreover, f_k is non-injective. Denote $S_k = \mathrm{Im}(f_k)$ and $s_k = |S_k|$. Because each step involves applying f, which is non-injective, it is clear that $s_{k+1} \leq s_k < n$. It follows that the sequence $\{s_k\}_{k\geq 0}$ is nonincreasing.

Because every hyperedge in H consists of exactly t vertices, and any nontrivial endomorphism must map some hyperedge to another, we deduce that $s_k \geq t$ for all k. Thus, the sequence $\{s_k\}$ is bounded below by t and must eventually stabilize. That is, there exists $k_0 \geq 0$ such that $s_k = s_{k_0}$ for all $k \geq k_0$. Let us consider $S_{k_0} = \mathrm{Im}(f_{k_0})$.

Suppose, for contradiction, that $s_{k_0} > t$. We define $\mathcal{D}_f = \{y - x \mid x \neq y,\ f(x) = f(y)\}$. Because f is non-injective, $\mathcal{D}_f \neq \emptyset$. For a $S_{k_0} = \mathrm{Im}(f_{k_0}) \subseteq \mathbb{Z}_n$, define the $\mathcal{D}_{S_{k_0}} = \{v-u \in \mathbb{Z}_n : u, v \in S_{k_0},\ u \neq v\}$. That is, $\mathcal{D}_{S_{k_0}}$ does not contain 0. By Lemma 4.7, we can find that $\mathcal{D}_{f_{k_0}} \cap \mathcal{D}_{S_{k_0}} \neq \emptyset$. Take $d \in \mathcal{D}_{f_{k_0}} \cap \mathcal{D}_{S_{k_0}}$.

Then there exists $d = v - u$ for some $u, v \in S_{k_0}$, and $x, y \in \mathbb{Z}_n$ with $f_{k_0}(x) = f_{k_0}(y)$ and $y - x = d$. Setting $\ell_{k_0} = x - u$, we obtain

$$f(\sigma^{\ell_{k_0}}(u)) = f(u + \ell_{k_0}) \quad = f(x) = f(y) \quad = f(x + d) = f(\sigma^{\ell_{k_0}}(v)).$$

Thus, two distinct vertices in S_{k_0} are identified under $f \circ \sigma^{\ell_{k_0}}$. That is, the restriction $f \circ \sigma^{\ell_{k_0}}|_{S_{k_0}}$ is not injective. Hence, $|S_{k_0+1}| < |S_{k_0}|$; that is, $s_{k_0+1} < s_{k_0}$, which contradicts $s_{k_0+1} = s_{k_0}$. Therefore, $s_{k_0} \leq t$. Combining this with the lower bound of the sequence $\{s_k\}$, we obtain $s_{k_0} = t$.

We already know that $\mathrm{Im}(f_{k_0})$ has size exactly t and that f_{k_0} is an endomorphism of H. Because every hyperedge of H has size t, it follows immediately that $\mathrm{Im}(f_{k_0})$ must coincide with a single hyperedge of H. Hence, the image of f_k is eventually contained entirely in one hyperedge. □

Because the underlying groups are finite cyclic, we can analyze hypergraph endomorphisms through finite arrangements. The following lemma applies combinatorial techniques to characterize these endomorphism properties.

Lemma 4.9. *Consider a total of n objects, where an integer $2 \leq t \leq n$ is given. The objects are divided into two types: The first type consists of more than n/t objects, and the remaining objects belong to the second type. If these n objects are arranged in a circular arrangement, then there exists at least one pair of objects of the first type that are at most $t - 1$ positions apart.*

Proof. Assume for contradiction that there are at least $t - 1$ objects of the second type between every consecutive pair of objects of the first type.

Let k be the number of objects of the first type, where $k > n/t$. Because there are k objects of the first type arranged in a circle, there are k gaps between them. Each gap must contain at least $t - 1$ positions occupied by objects of the second type, leading to a total of at least $k(t-1)$ positions occupied by objects of the second type.

Because the total number of positions is n, we obtain the inequality

$$k(t - 1) \leq n - k.$$

Rearranging this inequality, we get

$$kt - k \leq n - k.$$

Adding k to both sides, we obtain

$$kt \leq n.$$

However, as $k > n/t$, it follows that

$$kt > n,$$

which is a contradiction. Hence, the assumption is false, and there must exist at least one pair of objects of the first type that are at most $t - 1$ positions apart. □

This is a key part of the paper. Although our results focus on (End′)-vertex transitivity, the core contribution is characterizing the necessary and sufficient conditions for the existence of non-injective endomorphisms on t-Cayley hypergraphs.

Theorem 4.10. *Let t-$CH(\mathbb{Z}_n, A)$ be a t-Cayley hypergraph of a cyclic group $\mathbb{Z}_n$ with $2 \leq t < n$ and $A \subseteq \mathcal{A}$. For $a \in A$, there exists a non-injective endomorphism $f \in \mathrm{End}'(t\text{-}CH(\mathbb{Z}_n, A))$ such that every image hyperedge under f is of type a, if and only if $t \mid n$.*

Proof. Let $a \in A$.
($\Rightarrow$) Assume there exists a non-injective endomorphism $f \in \mathrm{End}'(t\text{-}CH(\mathbb{Z}_n, A))$ such that every image hyperedge under f is of type a. By Lemma 4.8, there exists $\Phi \in \mathrm{End}'(H)$ such that $\Phi(H) = e$ for some $e \in E(H)$. Let $e = \{v_1, v_2, \ldots, v_t\}$. Then $V(H)$ is divided into t partitions, denoted as $\Phi^{-1}(v_1), \Phi^{-1}(v_2), \ldots, \Phi^{-1}(v_t)$. Suppose that $t \nmid n$. By the pigeonhole principle, there is at least one partition that has order greater than n/t. Let us call this partition $\Phi^{-1}(v_x)$ for some $x \in \{1, 2, \ldots, t\}$. Consider the circular arrangement of the vertices of $t\text{-}CH(\mathbb{Z}_n, A)$ by ordering $a, 2a, 3a, \ldots, na$. Because a is a generator of $\mathbb{Z}_n$, all elements $a, 2a, 3a, \ldots, na$ are distinct.

By Lemma 4.9, we see that there exists one pair of vertices $u, v \in \Phi^{-1}(v_x)$ that are at most $t - 1$ positions apart. That is, there must be a hyperedge

$$e = \{ja, (j +_n 1)a, \ldots, (j +_n t)a\} \in E(H)$$

for some $j \in \{1, 2, \ldots, n\}$ such that $u, v \in e$. Thus, $\Phi(e)$ has an order less than t, which is impossible because H is a t-uniform hypergraph. Hence, $t \mid n$.
($\Leftarrow$) Assume n is divisible by t. Define a map

$$f_a : \mathbb{Z}_n \to \mathbb{Z}_n$$

by

$$f_a(x) = (k \bmod t)a, \text{ where } x = ka, \text{ and } 0 \le k \le n - 1.$$

The map f_a is well-defined because a is a generator of $\mathbb{Z}_n$.
Next, consider a hyperedge $e \in E(t\text{-}CH(\mathbb{Z}_n, A))$. We analyze the following two cases:
Case 1: $e = \{g, g +_n a, g +_n 2a, \ldots, g +_n (t-1)a\}$, where $g \in \mathbb{Z}_n$. Let $g = ma$ for some integer $0 \le m < n$. Then:

$$\begin{aligned}
f_a(e) &= f_a(\{g, g +_n a, g +_n 2a, \ldots, g +_n (t-1)a\}) \\
&= f_a(\{ma, ma +_n a, ma +_n 2a, \ldots, ma +_n (t-1)a\}) \\
&= f_a(\{ma, (m +_n 1)a, (m +_n 2)a, \ldots, (m +_n (t-1))a\}) \\
&= f_a(\{(m \bmod n)a, ((m+1) \bmod n)a, \ldots, ((m+(t-1)) \bmod n)a\}) \\
&= \{((m \bmod n) \bmod t)a, (((m+1) \bmod n) \bmod t)a, \ldots, (((m+(t-1)) \bmod n) \bmod t)a\}.
\end{aligned}$$

Because $t \mid n$, Remark 4.4 implies that

$$f_a(e) = \{(m \bmod t)a, ((m+1) \bmod t)a, \ldots, ((m+(t-1)) \bmod t)a\}.$$

By Remark 4.2, the values $(m + i) \bmod t$ for $0 \le i < t$ are distinct. Because a is a generator of $\mathbb{Z}_n$, we have

$$f_a(e) = \{0a, 1a, 2a, \ldots, (t-1)a\} \in E(t\text{-}CH(\mathbb{Z}_n, A)).$$

Case 2: $e' = \{g, g + a', g + 2a', \ldots, g + (t-1)a'\}$, where $g \in \mathbb{Z}_n$, and $a' \in A$ with $a' \neq a$. Because a' is a generator of the additive group $\mathbb{Z}_n$, it follows that $g = ka'$ for some integer $0 \le k < n$. Thus,

$$\begin{aligned}
e' &= \{g, g + a', g + 2a', \ldots, g + (t-1)a'\} \\
&= \{ka', (k+1)a', (k+2)a', \ldots, (k+t-1)a'\}.
\end{aligned}$$

As a is a generator of $\mathbb{Z}_n$, it follows that $a' = wa$ for some integer $1 \le w < n$. Then,

$$e' = \{kwa, (k+1)wa, \dots, (k+t-1)wa\}.$$

Similarly to the previous case, it can be observed that, by Remark 4.4,

$$\begin{aligned} f_a(e') &= f_a(\{wka, w(k+1)a, \dots, w(k+t-1)a\}) \\ &= \{(wk \bmod n) \bmod ta,\ (w(k+1) \bmod n) \bmod ta,\ \dots,\ (w(k+t-1) \bmod n) \bmod ta\} \\ &= \{(wk \bmod t)a,\ (w(k+1) \bmod t)a,\ \dots,\ (w(k+t-1) \bmod t)a\}. \end{aligned}$$

Because wa is a generator of the cyclic group of order n, by the properties of finite cyclic groups, it immediately follows that $\gcd(w,n) = 1$. Given the hypothesis that $t \mid n$ (i.e., $n = qt$ for some $q \in \mathbb{Z}$), we conclude that $t \nmid w$. By Remark 4.3, the elements $\{0a, 1a, 2a, \dots, (t-1)a\}$ modulo n are mapped to distinct residues modulo t. Thus, the map is as follows:

$$f_a(e') = \{0a, 1a, 2a, \dots, (t-1)a\} \in E(t\text{-}CH(\mathbb{Z}_n, A)).$$

From the above case, we conclude that f_a is a non-injective endomorphism. □

Theorem 4.11. *Let $H = t\text{-}CH(\mathbb{Z}_n, A)$ be a t-Cayley hypergraph with $A \subseteq \mathcal{A}$ and $2 \le t < n$. If $t \mid n$, then H is* $\mathrm{End}'(H)$*-vertex-transitive.*

Proof. Assume that n is divisible by t. By Theorem 4.10, we know that $\mathrm{End}'(t\text{-}CH(\mathbb{Z}_n, A)) \neq \emptyset$, which guarantees the existence of an non-injective endomorphism for the hypergraph $t\text{-}CH(\mathbb{Z}_n, A)$. Additionally, by Theorem 4.5, it follows that the hypergraph $H = t\text{-}CH(\mathbb{Z}_n, A)$ possesses the $\mathrm{End}'(H)$-vertex-transitive property. Thus, $t\text{-}CH(\mathbb{Z}_n, A)$ is proven to be End'-vertex-transitive when $t \mid n$. □

5. Discussion and conclusions

In this paper, we investigated non-injective endomorphisms of t-Cayley hypergraphs over finite cyclic groups. We established structural conditions under which such endomorphisms exist and clarified their role in determining End′-vertex-transitivity. In particular, we showed that disconnected t-Cayley hypergraphs are always End′-vertex-transitive, whereas connected cases may fail to admit any non-injective endomorphism. Moreover, we characterized the existence of non-injective endomorphisms whose image hyperedges are of a fixed type, providing a condition in terms of the divisibility of n by t.

In particular, in Lemma 4.8, we imposed an additional condition requiring that the image hyperedges of the initial function be of a fixed type. This condition ensures that the iteration of composite endomorphisms proceeds in a controlled and well-behaved manner.

Nevertheless, by examining small instances step by step, we observed that even when the initial function does not satisfy such additional constraints, the iteration process can often be resolved in a flexible manner. This observation suggests that the extra conditions imposed in our current framework may not be essential.

We conjecture that these additional assumptions can likely be relaxed.

Example 5.1. Consider the additive group $\mathbb{Z}_9$ and a generating set $A = \{1_9, 2_9\}$. Let $H = 3\text{-}CH(Z_9, A)$ denote the 3-Cayley hypergraph. Suppose $f : \mathbb{Z}_9 \to \mathbb{Z}_9$ is a non-injective endomorphism defined by

$$\begin{aligned} &f(1_9) = 1_9, \quad f(2_9) = 2_9, \quad f(3_9) = 3_9, \\ &f(4_9) = 4_9, \quad f(5_9) = 5_9, \quad f(6_9) = 3_9, \\ &f(7_9) = 1_9, \quad f(8_9) = 2_9, \quad f(0_9) = 3_9. \end{aligned}$$

Furthermore, consider the hyperedges $e_{3_9,1_9}$ and $e_{3_9,2_9}$. Their hyperedge images under f, specifically $f(e_{3_9,1_9}) = e_{3_9,1_9}$ and $f(e_{3_9,2_9}) = e_{3_9,2_9}$, are of distinct types (type 1_9 and type 2_9, respectively).

Next, consider the composite mapping $(f \circ \sigma^1 \circ f \circ \sigma^1 \circ f)(H)$, which yields a single hyperedge $e_{3_9,1_9}$. This demonstrates that although the images of the hyperedges are not initially of the same type, there exists an appropriate iteration that collapses the hypergraph into a single hyperedge.

Conjecture 5.2. *Let $H = t\text{-CH}(\mathbb{Z}_n, A)$ be the t-Cayley hypergraph of the group $\mathbb{Z}_n$, where $t \geq 2$, and A is a generating set. Let $f \in \text{End}'(H)$. Define the translation $\sigma^{\ell}(x) = x + \ell$ for all $x \in \mathbb{Z}_n$, and define a sequence of mappings recursively by*

$$f_0 = f, \qquad f_{k+1} = f \circ \sigma^{\ell_k} \circ f_k \quad (k \geq 0),$$

where at each step, the shift $\ell_k \in \mathbb{Z}_n$ is chosen suitably depending on f and $\text{Im}(f_k)$. Then there exists $K \in \mathbb{N}$ such that $\text{Im}(f_K)$ is contained in a single hyperedge of H.

This conjecture would lead to a complete characterization of the necessary and sufficient conditions for a t-Cayley hypergraph of a finite cyclic group with a given generating set to be End′-vertex-transitive.

Author contributions

Tanapat Chalarux: Conceptualization, Writing–original draft preparation; Sayan Panma: Writing–review and editing, Validation, Supervision, Project administration. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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