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Research article

On the flexibility of monophonic paths and connectedness property: application to the human circulatory system

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Abstract: This paper explores the interplay between graph theory and topology by introducing a novel framework based on flexible paths in directed graphs. Extending the classical notion of monophonic paths, we define flexible graphical topologies and characterize classes of directed graphs whose associated flexible graphical topologies are discrete or indiscrete. Fundamental topological properties, including connectedness, compactness, and homeomorphism, are investigated within this framework. Based on these properties, the notions of flexible connectedness and flexible discreteness are introduced and studied. To demonstrate the applicability of the proposed framework, we considered graph representations derived from the human circulatory system. The obtained flexible graphical topological structures are analyzed to illustrate their ability to capture connectedness, flexible connectedness, and flexible discreteness in biologically inspired networks. The results showed that flexible graphical topology provides an extended mathematical framework for studying structural properties of directed networks and offers potential for future applications in biological and complex systems.

Keywords: digraph; monophonic path; topology; connectedness; continuity
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1. Introduction

The intersection of graph theory and topology has attracted significant attention in mathematics, with numerous researchers constructing topological structures on the vertex and edge sets of graphs, whether undirected or directed. Early contributions largely focused on undirected simple graphs. For instance, in [18], some topologies on the vertex set of an undirected simple graph $\zeta = (\theta(\zeta), \omega(\zeta))$ without isolated vertices, using the subbase $S_{IG} = \{ends(\xi) : \xi \in \omega(\zeta)\}$ were presented. Later, in [1] this line of work was extended to directed graphs by defining two edge-based constructions. The paper [3] developed the notion of topologies of a graph ζ, expressed as $(\theta(\zeta), T_{\eta\theta})$, where $T_{\eta\theta}$ is a topology on $\theta(\zeta)$ generated by the family $\eta\theta$ of vertex neighborhoods. Recent studies have demonstrated the growing role of approximation and topological frameworks in modeling complex systems and practical applications. Hosny et al. [17] developed generalized approximation spaces generated from $\mathbb{I}_j$-neighborhoods and ideals and illustrated their applicability in disease-related modeling. El-Bably [16] introduced primal approximation spaces based on κ-neighborhood structures to extend approximation mechanisms in applied settings. Medical-oriented rough topological representations were further investigated by El-Bably et al. [12] through β-basic rough sets and their applications in medicine. Moreover, El-Gayar and Abu-Gdairi [13] extended topological structures using lattice and rough-set constructions. These studies highlight the increasing importance of topology- and neighborhood-based frameworks in applied modeling and provide additional motivation for developing flexible graphical topologies and investigating their applications in biological networks. Building on this direction, Damag et al. [8] introduced the concept of flexible graphical topologies for directed graphs and applied them to investigate connectedness in network models of the human nervous system. In subsequent work [9], they defined new classes of topological spaces using monophonic eccentric neighborhoods, applying the framework to study connectedness in networks describing COVID-19 diffusion. In [19] the authors proposed relations on graphs that induce further types of topological structures. Additional contributions examined products of graphs and neighborhood-based systems. The paper [24] generated topologies on the vertex sets of undirected graphs without isolated vertices, analyzing openness and continuity of functions in these settings. In [20,21], this approach was expanded by constructing vertex topologies via open hop neighborhoods. Gamorez et al. [15] defined a topology $\mathcal{T}_\zeta$ on undirected simple graphs and explored topological spaces induced by tensor products and edge corona. The work [5] presented some adjacencies of finite digraphs without loops or multiple arcs. In [14], the authors constructed topologies using monophonic eccentric neighborhoods to characterize graphs that produce point, discrete, or indiscrete topologies, while the paper [29] studied conditions under which graph topologies satisfy the Alexandroff property. In the study of directed graphs. in [2] the authors developed the 1-neighborhood system (1-NS), integrating it with rough set generalizations to construct the heart topological graph model, highlighting medical applications through decision-making algorithms and visualization. Damag et al. [7] proposed rough directed topological spaces and applied them to models of the human nervous and circulatory systems. In [6], the int-graphic topology τ_G^{int} on the set of vertices of a directed graph G was constructed and several of its topological properties are obtained. Othman et al. [23] introduced pathless directed topological spaces and investigated their connections to relative topologies and E-generated subdirected graphs, with applications to blood circulation. In a related extension, in [22], the notion of C-set was defined in order to construct L_2-directed topological spaces. The paper [11] further developed the rough set approach on adjacency matrices of digraphs, analyzing classical notions such as approximations, indiscernibility, and discernibility within the graph-theoretical framework. Finally, Shokry and Aly [27] introduced neighborhood-based topologies in graphs and demonstrated their applicability to models of the human heart.

The study is motivated by the inadequacy of classical graph-theoretical and topological approaches in capturing the complexity of directed networks. Traditional concepts, such as monophonic paths, often fail to represent essential structural and functional aspects, including connectedness, compactness, and functional equivalence. These limitations are particularly evident in biological systems, where directionality plays a crucial role in information transfer and network organization. To address this challenge, the present work introduces the concepts of flexible paths and flexible graphical topologies, offering a broader theoretical framework with direct applications to biological models such as the human circulatory system.

Recent developments in approximation-based topological frameworks have further expanded the applicability of graph-related mathematical models. In particular, rough approximation spaces, primal approximation structures, and neighborhood-based constructions have demonstrated their effectiveness in modeling uncertainty and supporting structural analysis in complex systems. Related studies have also emphasized medical and decision-oriented applications through generalized approximation spaces, rough topological representations, and extended topological structures. These developments provide additional motivation for introducing more flexible topological mechanisms capable of capturing local structural interactions in directed networks.

The paper is organized as follows: Section 1 presents the introduction, background, and motivation for developing new topological structures in directed graphs. In Section 2, we give basic information about directed graphs that will be used in the next sections. Section 3 defines flexible paths as an extension of monophonic paths and uses them to construct flexible graphical topologies, also characterizing the directed graphs that induce discrete flexible topologies or not. Section 4 examines homeomorphism within flexible graphical topologies and establishes its relation to graph isomorphism. Section 5 investigates fundamental topological properties, particularly connectedness and compactness, and introduces the notions of flexible connectedness and flexible discreteness. Section 6 demonstrates the applicability of the framework through examples, focusing on diagrams of the human circulatory system to illustrate flexible graphical topological properties in biological networks. Finally, Section 7 concludes with a summary of the main results and outlines potential directions for future research.

Unlike classical monophonic paths, which depend only on chordless connectivity, the proposed flexible path introduces an additional structural condition based on the balance between in-degree and out-degree of vertices along the path. This distinguishes the present framework from pathless directed topologies, rough directed topological spaces, and neighborhood-based graphical topologies, which primarily rely on neighborhood relations or approximation mechanisms. The proposed approach therefore extends existing graphical-topological constructions by incorporating flexible path behavior into the definition of topology and the associated notions of connectedness and discreteness.

2. Preliminaries

A directed graph (or digraph) ζ is defined as the ordered pair $(\theta(\zeta), \omega(\zeta))$, where $\theta(\zeta)$ denotes a non-empty set of vertices, and $\omega(\zeta)$ denotes the set of directed edges. An edge $\xi \in \omega(\zeta)$ with initial point τ and terminal point ρ is represented by $\xi_{\tau\rho}$, and the corresponding point set is $\theta(\xi_{\tau\rho}) = \{\tau, \rho\}$. Two vertices in a digraph are adjacent if there is a directed edge connecting them. Two edges $\xi_{\tau\rho}$ and $\xi'_{\tau'\rho'}$ are said to be adjacent if they have a common vertex. For each point $\tau \in \theta(\zeta)$, ∂_τ denotes the degree of τ which is the number of vertices adjacent with τ, ∂_τ^{po} is the number of edges entering τ, and ∂_τ^{ne} is the number of edges leaving τ. A digraph is called *finite* if both $\theta(\zeta)$ and $\omega(\zeta)$ are finite, and locally finite if ∂_τ is finite for all $\tau \in \theta(\zeta)$. An edge of the form $\xi_{\tau\tau}$ is called a loop, while two edges with the same initial and terminal vertices are called parallel edges. A simple digraph is one without loops or parallel edges, and a directed edge from τ to ρ is denoted by $\zeta_{\tau\rho}$. A directed path P is an alternating sequence of distinct vertices and distinct edges, denoted as $\{\xi^1_{1'2'}, \xi^2_{2'3'}, \ldots\}$. If the first and last vertices of P coincide, then P is called a closed directed path. A digraph is said to be weakly connected if its underlying undirected graph $Un(\zeta)$ is connected, connected if for any distinct $\tau, \rho \in \theta(\zeta)$ there exists a directed path from one to the other, and strongly connected if every point can be reached from every other point by a directed path.

Special classes of digraphs include the cycle digraph $\mathbb{C}_n$ ($n > 2$), which consists of n vertices and n edges, where $\partial^{po}_\tau = \partial^{ne}_\tau = 1$ for all $\tau \in \theta(C_n)$. A complete digraph $\mathbb{K}_n$ has n vertices, each satisfying $\partial^{po}_\tau = \partial^{ne}_\tau = n$. A complete bipartite digraph $\mathbb{K}_{n\to m}$, with $n, m > 0$, partitions its point set into two disjoint subsets V_n and V_m such that every point $v \in V_n$ is connected by directed edges to all vertices $u \in V_m$, and no edges within either partition. The digraph $\mathbb{K}_{n,m}$ represents the complete bipartite digraph that contains all possible directed edges between the two partitions.

For any point $\tau \in \theta(\zeta)$, the open in-neighborhood is defined as $\mathcal{R}^{po}(\tau) = \{\rho \in \theta(\zeta) : \xi_{\rho\tau} \in \omega(\zeta)\}$, while the open out-neighborhood is $\mathcal{R}^{ne}(\tau) = \{\rho \in \theta(\zeta) : \xi_{\tau\rho} \in \omega(\zeta)\}$. The corresponding closed neighborhoods are $\mathcal{R}^{po}[\tau] = \mathcal{R}^{po}(\tau) \cup \{\tau\}$ and $\mathcal{R}^{ne}[\tau] = \mathcal{R}^{ne}(\tau) \cup \{\tau\}$. The open neighborhood of τ is $\mathcal{R}(\tau) = \mathcal{R}^{po}(\tau) \cup \mathcal{R}^{ne}(\tau)$, and the closed neighborhood is $\mathcal{R}[\tau] = \mathcal{R}(\tau) \cup \{\tau\}$. A chord of a directed path P is an edge connecting two non-adjacent vertices of P. A chordless directed path of length greater than one is called a monophonic directed path, and its length is denoted by $\partial_\zeta P$.

3. The flexible graphical topology

In this section, we introduce the flexible graphical topology on digraphs and consider its basic properties.

Let $\zeta = (\theta(\zeta), \omega(\zeta))$ be any simple digraph. A monophonic directed path P is called a flexible path if $\partial^{ne}_\tau = \partial^{po}_\tau$ for some $\tau \in \theta(P)$, where $\theta(P)$ denotes the set of all vertices of P. A point $\tau \in \theta(\zeta)$ is called a flexible point with respect to another point $\rho \in \theta(\zeta)$ if there exists a flexible path connecting them. For a point $\tau \in \theta(\zeta)$, the flexible open neighborhood $\Theta(\tau)$ of τ is defined as the set of all vertices flexible with respect to τ, and the flexible closed neighborhood Θ_τ of τ is given by $\Theta_\tau = \Theta(\tau) \cup \{\tau\}$. The collection of all flexible closed neighborhoods of vertices in ζ is denoted by $\Theta(\zeta)$. For any simple digraph $\zeta = (\theta(\zeta), \omega(\zeta))$, the flexible graphical topology on $\theta(\zeta)$ is denoted by $T_{\Theta(\zeta)}$ and is defined as the topology generated by the subbase $\Theta(\zeta)$.

Example 3.1. *The digraph $\zeta = (\theta(\zeta), \omega(\zeta))$ in Figure 1(a) has the vertices $1'$ and $13'$ that satisfy the property $\partial^{ne}_\tau \neq \partial^{po}_\tau$.*

So, the subbase $\Theta(\zeta)$ is given by $\Theta(\zeta) = \{\Theta_{k'} : k = 1, 2, \cdots, 13\}$, where

$$\Theta_{1'} = \theta(\zeta) - \{2', 3'\},\ \ \Theta_{2'} = \theta(\zeta) - \{1', 3', 4', 5'\},\ \ \Theta_{3'} = \theta(\zeta) - \{1', 2', 4', 5'\},$$

$$\Theta_{4'} = \theta(\zeta) - \{2', 3', 5', 6'\},\ \ \Theta_{5'} = \theta(\zeta) - \{2', 3', 4', 6'\},\ \ \Theta_{6'} = \theta(\zeta) - \{4', 5', 7', 8'\},$$

$$\Theta_{7'} = \theta(\zeta) - \{6', 8', 10', 11'\},\ \ \Theta_{8'} = \theta(\zeta) - \{6', 7', 10', 11'\},$$

$$\Theta_{9'} = \theta(\zeta) - \{10', 11', 12', 13'\},\ \ \Theta_{10'} = \theta(\zeta) - \{7', 8', 9', 11'\},\ \ \Theta_{11'} = \theta(\zeta) - \{7', 8', 9', 10'\},$$

$$\Theta_{12'} = \theta(\zeta) - \{9', 13'\},\ \ \Theta_{13'} = \theta(\zeta) - \{9', 12'\}.$$

(See more details in Table 1.)

Note that in Figure 1(b), all vertices satisfy the property $\partial_\tau^{ne} = \partial_\tau^{po}$. The subbase $\Theta(\zeta)$ is given by

$$\Theta(\zeta) = \{\Theta_{k'} : k = 0, 1, 2, \cdots, n\},$$

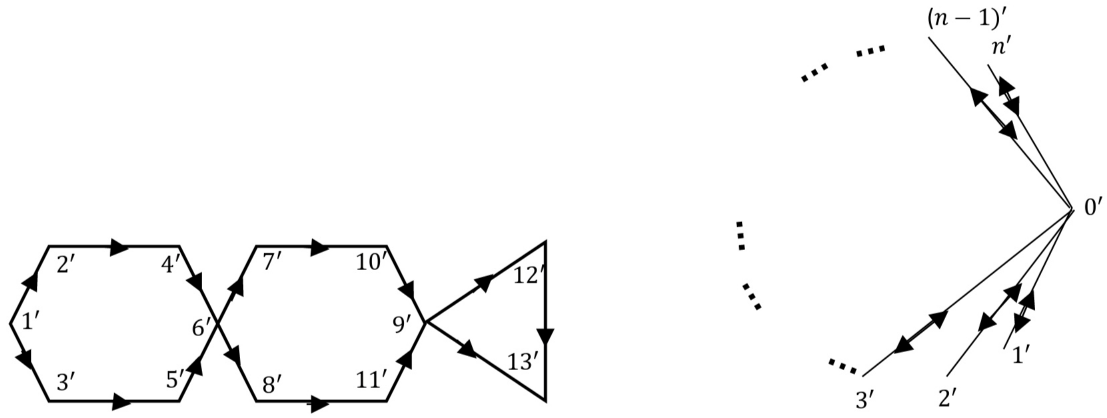
where

$$\Theta_{0'} = \{0'\} \text{ and } \Theta_{k'} = \theta(\zeta) - \{0'\}$$

for all $k = 1, 2, \cdots, n$. Therefore, the topology $T_{\Delta(\zeta)}$ is given by

$$T_{\Delta(\zeta)} = \{\emptyset, \theta(\zeta), \theta(\zeta) - \{0'\}, \{0'\}\}.$$

(See more details in Table 2.)



(a) Flexible topology 1 (b) Flexible topology 2

Figure 1. Flexible graphical topologies.

Table 1. The flexible graphical topological space in Figure 1(a).

τ	∂_τ^{ne}	∂_τ^{po}	$\Theta(\tau)$	Θ_τ
$1'$	2	0	$\theta(\zeta) - \{1', 2', 3'\}$	$\theta(\zeta) - \{2', 3'\}$
$2'$	1	1	$\theta(\zeta) - \{1', 2', 3', 4', 5'\}$	$\theta(\zeta) - \{1', 3', 4', 5'\}$
$3'$	1	1	$\theta(\zeta) - \{1', 2', 3', 4', 5'\}$	$\theta(\zeta) - \{1', 2', 4', 5'\}$
$4'$	1	1	$\theta(\zeta) - \{2', 3', 4', 5', 6'\}$	$\theta(\zeta) - \{2', 3', 5', 6'\}$
$5'$	1	1	$\theta(\zeta) - \{2', 3', 4', 5', 6'\}$	$\theta(\zeta) - \{2', 3', 4', 6'\}$
$6'$	2	2	$\theta(\zeta) - \{4', 5', 6', 7', 8'\}$	$\theta(\zeta) - \{4', 5', 7', 8'\}$
$7'$	1	1	$\theta(\zeta) - \{6', 7', 8', 10', 11'\}$	$\theta(\zeta) - \{6', 8', 10', 11'\}$
$8'$	1	1	$\theta(\zeta) - \{6', 7', 8', 10', 11'\}$	$\theta(\zeta) - \{6', 7', 10', 11'\}$
$9'$	2	2	$\theta(\zeta) - \{9', 10', 11', 12', 13'\}$	$\theta(\zeta) - \{10', 11', 12', 13'\}$
$10'$	1	1	$\theta(\zeta) - \{7', 8', 9', 10', 11'\}$	$\theta(\zeta) - \{7', 8', 9', 11'\}$
$11'$	1	1	$\theta(\zeta) - \{7', 8', 9', 10', 11'\}$	$\theta(\zeta) - \{7', 8', 9', 10'\}$
$12'$	1	1	$\theta(\zeta) - \{9', 12', 13'\}$	$\theta(\zeta) - \{9', 13'\}$
$13'$	0	2	$\theta(\zeta) - \{9', 12', 13'\}$	$\theta(\zeta) - \{9', 12'\}$

Table 2. The flexible graphical topological space in Figure 1(b).

τ	∂_τ^{ne}	∂_τ^{po}	$\Theta(\tau)$	Θ_τ
$0'$	n	n	$\emptyset$	$\{0'\}$
$1'$	2	2	$\theta(\zeta)-\{0',1'\}$	$\theta(\zeta)-\{0'\}$
$2'$	2	2	$\theta(\zeta)-\{0',2'\}$	$\theta(\zeta)-\{0'\}$
$3'$	2	2	$\theta(\zeta)-\{0',3'\}$	$\theta(\zeta)-\{0'\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(n-1)'$	2	2	$\theta(\zeta)-\{0',(n-1)'\}$	$\theta(\zeta)-\{0'\}$
n'	2	2	$\theta(\zeta)-\{0',n'\}$	$\theta(\zeta)-\{0'\}$

Let $\zeta=(\theta(\zeta),\omega(\zeta))$ be any digraph. It is evident that if $\tau\in\theta(\zeta)$ is an isolated point, then the set $\{\tau\}$ is an open set in the flexible graphical topological space $(\theta(\zeta),T_{\Theta(\zeta)})$. Similarly, if $\xi_{\tau\rho}\in\omega(\zeta)$ is an isolated directed edge, then the sets $\{\tau\}$ and $\{\rho\}$ are open. Furthermore, if there exists an isolated path P in ζ of length two of the form $P=\{\xi_{\tau\rho},\xi_{\rho\nu}\}$, then the sets $\{\tau,\nu\}$ and $\{\rho\}$ are also open.

Theorem 3.2. *The following assertions hold:*

(1) The space $(\theta(\mathbb{K}_n),T_{\Theta(\mathbb{K}_n)})$ is discrete;
(2) The space $(\theta(\mathbb{K}_{n,m}),T_{\Theta(\mathbb{K}_{n,m})})$ is discrete;
(3) The space $(\theta(\mathbb{K}_{n\to m}),T_{\Theta(\mathbb{K}_{n\to m})})$ is discrete;
(4) The space $(\theta(\mathbb{C}_n),T_{\Theta(\mathbb{C}_n)})$ is discrete.

Proof. (1) Note that in $\mathbb{K}_n$ with $n>0$, for any $\tau,\rho\in\theta(\mathbb{K}_n)$, there exist two directed edges $(\mathbb{K}_n)_{\tau\rho}$ and $(\mathbb{K}_n)_{\rho\tau}$. Hence, any monophonic path that connects τ and ρ has length 1. Moreover, for every $\nu\in\theta(\mathbb{K}_n)$, $\partial_\nu^{ne}=\partial_\nu^{po}=n$. In this case, we have $\Theta_\tau=\{\tau\}$ for all $\tau\in\theta(\mathbb{K}_n)$.

(2) Let $\theta(\mathbb{K}_{n,m})=\theta_n\cup\theta_m$ and take any $\tau\in\theta(\mathbb{K}_{n,m})$. Since $\theta_n\cap\theta_m=\emptyset$, we have either $\tau\in\theta_n$ or $\tau\in\theta_m$. Suppose $\tau\in\theta_n$, and let $\rho\in\theta_m$ be any point. By the definition of $\mathbb{K}_{n,m}$, there exist two directed edges $(\mathbb{K}_{n,m})_{\tau\rho}$ and $(\mathbb{K}_{n,m})_{\rho\tau}$. Hence, any monophonic path which connects τ and ρ has length 1, so $\rho\notin\Theta_\tau$. Now, let $\rho\in\theta_n$ be any point . Then, any monophonic path joining τ and ρ has length 2. Since for all $\nu\in\theta_n$, $\partial_\nu^{ne}=\partial_\nu^{po}=m$, and for all $\nu\in\theta_m$, $\partial_\nu^{ne}=\partial_\nu^{po}=n$, we have $\rho\notin\Theta_\tau$. Thus, $\Theta_\tau=\{\tau\}$ for all $\tau\in\theta_n$. Similarly, $\Theta_\tau=\{\tau\}$ for all $\tau\in\theta_m$. Therefore, the flexible graphical topological space of the complete bipartite digraph $\mathbb{K}_{n,m}$ is a discrete space.

(3) In the digraph $\mathbb{K}_{n\to m}$ with $n,m>0$, all directed paths are of length one. Hence, the flexible graphical topological space $(\theta(\mathbb{K}_{n\to m}),T_{\Theta(\mathbb{K}_{n\to m})})$ is a discrete space.

(4) To determine the flexible graphical topological space of a cycle digraph $\mathbb{C}_n$ with $n>2$, it is unnecessary to identify the monophonic directed paths and their lengths, since $\partial_\tau^{po}=\partial_\tau^{ne}=1$ for all $\tau\in\theta(\mathbb{C}_n)$. Therefore, $\Theta_\tau=\{\tau\}$ for all $\tau\in\theta(\mathbf{C}_n)$, and the flexible graphical topological space of $\mathbf{C}_n$ is a discrete space. □

In what follows by an $\mathcal{S}$-digraph, we mean a locally finite, simple directed graph.

Theorem 3.3. *The flexible graphical topological space associated with any $\mathcal{S}$-digraph is an Alexandroff space.*

Proof. Let $\zeta=(\theta(\zeta),\omega(\zeta))$ be a $\mathcal{S}$-digraph, and let C be any nonempty subset of $\theta(\zeta)$. Let C denote the collection of all flexible closed neighborhoods of elements of C. We will show that $\bigcap_{\tau\in C}\Theta_\tau$ is an open set in $(\theta(\zeta), T_{\Theta(\zeta)})$. Take any $\rho \in \bigcap_{\tau\in C}\Theta_\tau$. Then, $\rho \in \Theta_\tau$ for every $\tau \in C$. Consequently, $\tau \in \Theta_\rho$ for all $\tau \in C$, and hence $C \subseteq \Theta_\rho$. Since ζ is locally finite, Θ_ρ is finite, and thus C is finite. Therefore, $\bigcap_{\tau\in C}\Theta_\tau$ is an open set. Hence, the space $(\theta(\zeta), T_{\Theta(\zeta)})$ is an Alexandroff space. □

In any $\mathcal{S}$-digraph $\zeta = (\theta(\zeta), \omega(\zeta))$, by the Alexandroff property of the space $(\theta(\zeta), T_{\Theta(\zeta)})$, the smallest open set containing a point $a \in \theta(\zeta)$ is denoted by $\mathbb{O}^{\min}(a)$. As noted earlier, if $a \in \theta(\zeta)$ is an isolated point, then $\mathbb{O}^{\min}(a) = \{a\}$. If $\zeta_{ab} \in \omega(\zeta)$ is an isolated directed edge, then $\mathbb{O}^{\min}(a) = \{a\}$ and $\mathbb{O}^{\min}(b) = \{b\}$. Moreover, if ζ contains an isolated path $P = \{\zeta_{ab}, \zeta_{bc}\}$ or $P = \{\zeta_{ab}, \zeta_{cb}\}$ of length two, then $\mathbb{O}^{\min}(a) = \{a\}$, $\mathbb{O}^{\min}(b) = \{b\}$, and $\mathbb{O}^{\min}(c) = \{c\}$.

Theorem 3.4. *Let $\zeta = (\theta(\zeta), \omega(\zeta))$ be any $\mathcal{S}$-digraph and $\tau \in \theta(\zeta)$. Then, $\mathbb{O}^{\min}(\tau)$ is the intersection of all flexible closed neighborhoods containing τ.*

Proof. By definition of the subbase $\Theta(\zeta)$, each Θ_τ is open in $(\theta(\zeta), T_{\Theta(\zeta)})$ for all $\tau \in \theta(\zeta)$. From the Alexandroff property of the flexible graphical topology (Theorem 3.3), $\bigcap_{\rho\in\Theta_\tau}\Theta_\rho$ is an open set. If $\rho \in \Theta_\tau$, then clearly $\tau \in \Theta_\rho$, and thus $\bigcap_{\rho\in\Theta_\tau}\Theta_\rho$ is an open set containing τ. Since $\mathbb{O}^{\min}(\tau)$ is the smallest open set containing τ, we have $\mathbb{O}^{\min}(\tau) \subseteq \bigcap_{\rho\in\Theta_\tau}\Theta_\rho$.

Conversely, since $\mathbb{O}^{\min}(\tau)$ is the intersection of all open sets containing τ, we can write $\mathbb{O}^{\min}(\tau) = \bigcap_{\rho\in F}\Theta_\rho$ for some subset $F \subseteq \theta(\zeta)$. Then, $\tau \in \Theta_\rho$ for every $\rho \in F$, which implies $\rho \in \Theta_\tau$ for all $\rho \in F$. Hence $F \subseteq \Theta_\tau$, and therefore $\bigcap_{\rho\in\Theta_\tau}\Theta_\rho \subseteq \bigcap_{\rho\in F}\Theta_\rho = \mathbb{O}^{\min}(\tau)$. Thus, $\mathbb{O}^{\min}(\tau) = \bigcap_{\rho\in\Theta_\tau}\Theta_\rho$, i.e., $\mathbb{O}^{\min}(\tau)$ is the intersection of all flexible closed neighborhoods containing τ. □

Theorem 3.5. *Let $\zeta = (\theta(\zeta), \omega(\zeta))$ be any $\mathcal{S}$-digraph, and let $\tau, \rho \in \theta(\zeta)$. Then, $\Theta_\rho \subseteq \Theta_\tau$ if, and only if, $\tau \in \mathbb{O}^{\min}(\rho)$.*

Proof. Let $\tau, \rho \in \theta(\zeta)$ and suppose $\Theta_\rho \subseteq \Theta_\tau$. By Theorem 3.4, $\tau \in \bigcap_{\nu\in\Theta_\tau}\Theta_\nu$. Since $\Theta_\rho \subseteq \Theta_\tau$, we have $\bigcap_{\nu\in\Theta_\tau}\Theta_\nu \subseteq \bigcap_{\nu\in\Theta_\rho}\Theta_\nu$. Hence, $\tau \in \bigcap_{\nu\in\Theta_\rho}\Theta_\nu = \mathbb{O}^{\min}(\rho)$.

Conversely, suppose $\tau \in \mathbb{O}^{\min}(\rho)$. By Theorem 3.4, $\Theta_\rho = \bigcap_{\nu\in\Theta_\rho}\Theta_\nu$. Thus, $\tau \in \bigcap_{\nu\in\Theta_\rho}\Theta_\nu$, which means $\tau \in \Theta_\nu$ for all $\nu \in \Theta_\rho$. Therefore, $\nu \in \Theta_\tau$ for every $\nu \in \Theta_\rho$, and hence, $\Theta_\rho \subseteq \Theta_\tau$. □

Theorem 3.6. *Let $\zeta = (\theta(\zeta), \omega(\zeta))$ be any $\mathcal{S}$-digraph. Then, for all $\tau, \rho \in \theta(\zeta)$, $\Theta_\rho \nsubseteq \Theta_\tau$ and $\Theta_\tau \nsubseteq \Theta_\rho$ if, and only if, the space $(\theta(\zeta), T_{\Theta(\zeta)})$ is discrete.*

Proof. Let τ be any point in $\theta(\zeta)$. It is clear that $\tau \in \Theta_\tau$. Suppose that $\rho \in \theta(\zeta)$ is any point. If $\rho \in \Theta_\tau$, then by Theorem 3.5, $\Theta_\tau \subseteq \Theta_\rho$, contradicting the hypothesis. Hence, $\mathbb{O}^{\min}(\tau) = \{\tau\}$ for all $\tau \in \theta(\zeta)$. Therefore, $(\theta(\zeta), T_{\Theta(\zeta)})$ is discrete.

The converse is immediate since $\Theta_\tau = \{\tau\}$ for all $\tau \in \theta(\zeta)$. □

Let $\zeta = (\theta(\zeta), \omega(\zeta))$ be any $\mathcal{S}$-digraph and let $H \subseteq \theta(\zeta)$. Following [10], the closure of H is denoted by $Cl(H)$.

Corollary 3.7. *Let $\zeta = (\theta(\zeta), \omega(\zeta))$ be any $\mathcal{S}$-digraph and $\tau \in \theta(\zeta)$. Then, $Cl(\Theta_\tau) \subseteq Cl(\Theta_\rho)$ for all $\rho \in \Theta_\tau$. In particular, $Cl(\{\tau\}) \subseteq Cl(\Theta_\rho)$ for all $\rho \in \Theta_\tau$.*

Corollary 3.8. *For any simple graph $\zeta = (\theta(\zeta), \omega(\zeta))$ and any $\tau, \rho \in \theta(\zeta)$, $\tau \in Cl(\{\rho\})$ if, and only if, $\Theta_\tau \subseteq \Theta_\rho$.*

4. Mappings

Let $\zeta_1 = (\theta(\zeta_1), \omega(\zeta_1))$ and $\zeta_2 = (\theta(\zeta_2), \omega(\zeta_2))$ be two $\mathcal{S}$-digraphs without isolated vertices. If there exists a bijective mapping $\mathcal{J} : \theta(\zeta_1) \to \theta(\zeta_2)$ such that $(\zeta_1)_{\tau\rho} \in \omega(\zeta_1)$ if, and only if, $(\zeta_2)_{\mathcal{J}(\tau)\mathcal{J}(\rho)} \in \omega(\zeta_2)$ for all $\tau, \rho \in \theta(\zeta_1)$, then the two graphs ζ_1 and ζ_2 are said to be *isomorphic*, denoted by $\zeta_1 \cong \zeta_2$. A mapping $F : (A_1, \mathcal{T}_1) \to (A_2, \mathcal{T}_2)$ between topological spaces is said to be *continuous* if $F(Cl(G)) \subseteq Cl(F(G))$ for all $G \subseteq A_1$. A mapping $F : (A_1, \mathcal{T}_1) \to (A_2, \mathcal{T}_2)$ is called a *closed function* if $F(G)$ is closed in A_2 for every closed set $G \subseteq A_1$. A mapping $F : (A_1, \mathcal{T}_1) \to (A_2, \mathcal{T}_2)$ is a *homeomorphism* if it is bijective, continuous, and closed [10].

Theorem 4.1. *Let $\zeta_1 = (\theta(\zeta_1), \omega(\zeta_1))$ and $\zeta_2 = (\theta(\zeta_2), \omega(\zeta_2))$ be two $\mathcal{S}$-digraphs. The following are equivalent:*

(1) A mapping $f : (\theta(\zeta_1), T_{\Theta(\zeta_1)}) \to (\theta(\zeta_2), T_{\Theta(\zeta_2)})$ is continuous;
(2) $\Theta_\tau \subseteq \Theta_\rho$ implies $\Theta_{\mathcal{J}(\tau)} \subseteq \Theta_{\mathcal{J}(\rho)}$ for all $\tau, \rho \in \theta(\zeta_1)$.

Proof. (1) $\Rightarrow$ (2) Let $\tau, \rho \in \theta(\zeta_1)$ be such that $\Theta_\tau \subseteq \Theta_\rho$. By Corollary 3.8, $\tau \in Cl(\{\rho\})$, and by the continuity of f,
$$f(\tau) \in f(Cl(\{\rho\})) \subseteq Cl(f(y)\}).$$
Again, by Corollary 3.8, $\Theta_{f(\tau)} \subseteq \Theta_{f(\rho)}$.

(2) $\Rightarrow$ (1) Suppose that for all $\tau, \rho \in \theta(\zeta_1)$, $\Theta_\tau \subseteq \Theta_\rho$ implies $\Theta_{f(\tau)} \subseteq \Theta_{f(\rho)}$. Let G be any subset of $\theta(\zeta_1)$ and $\tau \in G$. If $\tau \in Cl(G)$, then $\tau \in Cl(\{\rho\})$ for some $\rho \in G$, which implies $\Theta_\tau \subseteq \Theta_\rho$. By the hypothesis, $\Theta_{f(\tau)} \subseteq \Theta_{f(\rho)}$. Hence, $f(\tau) \in Cl(f(\rho)\}) \subseteq Cl(v(G))$, and therefore, f is continuous. □

Theorem 4.2. *Let ζ_1 and ζ_2 be $\mathcal{S}$-digraphs and $f : \theta(\zeta_1) \to \theta(\zeta_2)$ a mapping.*

(1) If f is onto and for all $\tau, \rho \in \theta(\zeta_1)$, $\Theta_{f(\tau)} \subseteq \Theta_{f(\rho)}$ implies $\Theta_\tau \subseteq \Theta_\rho$, then f is a closed mapping;
(2) If f is closed and one-to-one, then for all $\tau, \rho \in \theta(\zeta_1)$, $\Theta_{f(\tau)} \subseteq \Theta_{f(\rho)}$ implies $\Theta_\tau \subseteq \Theta_\rho$.

Proof. (1) Let G be any closed set in $\theta(\zeta_1)$. Since f is onto, there exists a mapping $g : \theta(\zeta_2) \to \theta(\zeta_1)$ such that $f \circ g = id_{\theta(\zeta_2)}$. We show that g is continuous. Let $\tau, \rho \in \theta(\zeta_2)$ be arbitrary vertices such that $\Theta_\tau \subseteq \Theta_\rho$. Then, $\Theta_{f(g(\tau))} \subseteq \Theta_{f(g(\rho))}$. By the hypothesis, it follows that $\Theta_{g(\tau)} \subseteq \Theta_{g(\rho)}$. By Theorem 4.1, g is continuous. Hence, $f(G) = g^{\leftarrow}(G)$ is closed, and therefore f is a closed mapping.

(2) Let $\tau, \rho \in \theta(\zeta_2)$ be such that $\Theta_{f(\tau)} \subseteq \Theta_{f(\rho)}$. Since f is one-to-one, there exists a mapping $\psi : \theta(\zeta_2) \to \theta(\zeta_1)$ such that $\psi \circ f = id_{\theta(\zeta_1)}$. As f is bijective and closed, ψ is continuous. Therefore, $\Theta_{\psi(f(\tau))} \subseteq \Theta_{\psi(f(\rho))}$, i.e., $\Theta_\tau \subseteq \Theta_\rho$. □

Corollary 4.3. *A bijective mapping $F : \theta(\zeta_1) \to \theta(\zeta_2)$ between two $\mathcal{S}$-digraphs $\zeta_1 = (\theta(\zeta_1), \omega(\zeta_1))$ and $\zeta_2 = (\theta(\zeta_2), \omega(\zeta_2))$ is a homeomorphism if, and only if, for all $\tau, \rho \in \theta(\zeta_1)$, and $\Theta_\tau \subseteq \Theta_\rho$ if, and only if, $\Theta_{F(\tau)} \subseteq \Theta_{F(\rho)}$.*

As mentioned earlier, the flexible graphical topological spaces $(\mathbb{C}_n, T_{\Theta(\mathbb{C}_n)})$ and $(\mathbb{K}_n, T_{\Theta(\mathbb{K}_n)})$ of a cycle digraph $\mathbb{C}_n$ and a complete digraph $\mathbb{K}_n$, respectively, are discrete spaces. Since $|\mathbb{C}_n| = |\mathbb{K}_n|$, the spaces $(\mathbb{C}_n, T_{\Theta(\mathbb{C}_n)})$ and $(\mathbb{K}_n, T_{\Theta(\mathbb{K}_n)})$ are homeomorphic, even though the graphs $\mathbb{C}_n$ and $\mathbb{K}_n$ are not isomorphic.

The following theorem establishes that the isomorphism relation between $\mathcal{S}$-digraphs without isolated vertices induces a homeomorphism between their corresponding flexible graphical topological spaces.

Theorem 4.4. *Let ζ_1 and ζ_2 be two $\mathcal{S}$-digraphs without isolated vertices. If $\zeta_1 \cong \zeta_2$, then the spaces $(\theta(\zeta_1), T_{\Theta(\zeta_1)})$ and $(\theta(\zeta_2), T_{\Theta(\zeta_2)})$ are homeomorphic.*

Proof. Let $h : \theta(\zeta_1) \to \theta(\zeta_2)$ be a bijective mapping such that $(\zeta_1)_{\tau\rho} \in \omega(\zeta_1)$ if, and only if, $(\zeta_2)_{h(\tau)h(\rho)} \in \omega(\zeta_2)$ for all $\tau, \rho \in \theta(\zeta_1)$. Let $\tau, \rho \in \theta(\zeta_1)$ be such that ρ is a flexible point of τ. Then, there exists a flexible path

$$P_{\zeta_1} : (\zeta_1)_{\tau\tau_1}(\zeta_1)_{\tau_1\tau_2}(\zeta_1)_{\tau_2\tau_3}\cdots(\zeta_1)_{\tau_{n-1}\tau_n}(\zeta_1)_{\tau_n\rho}$$

with $\partial_{\zeta_1}P_{\zeta_1} > 1$ and $\partial^{ne}_{\tau_k} = \partial^{po}_{\tau_k}$ for some $\tau_k \in \theta(P_{\zeta_1})$.

By the isomorphism relation between ζ_1 and ζ_2, we obtain

$$P_{\zeta_2} : (\zeta_2)_{h(\tau)h(\tau_1)}(\zeta_2)_{h(\tau_1)h(\tau_2)}(\zeta_2)_{h(\tau_2)h(\tau_3)}\cdots(\zeta_2)_{h(\tau_{n-1})h(\tau_n)}(\zeta_2)_{h(\tau_n)h(\rho)}$$

with $\partial_{\zeta_2}P_{\zeta_2} > 1$ and $\partial^{ne}_{h(\tau_k)} = \partial^{po}_{h(\tau_k)}$ for some $h(\tau_k) \in \theta(P_{\zeta_2})$. That is, P_{ζ_2} is a flexible path from $h(\tau)$ to $h(\rho)$. Hence, $h(\rho)$ is a flexible point of $h(\tau)$. Therefore, for all $\tau, \rho \in \theta(\zeta_1)$, $\Theta_\tau \subseteq \Theta_\rho$ if, and only if, $\Theta_{h(\tau)} \subseteq \Theta_{h(\rho)}$. By Corollary 4.3, h is a homeomorphism between spaces $(\theta(\zeta_1), T_{\Theta(\zeta_1)})$ and $(\theta(\zeta_2), T_{\Theta(\zeta_2)})$. □

5. Compactness and connectedness

It is evident that any flexible graphical topological space $(\theta(\zeta), T_{\Theta(\zeta)})$ associated with an $\mathcal{S}$-digraph $\zeta = (\theta(\zeta), \omega(\zeta))$ is compact whenever $\theta(\zeta)$ is finite. However, the converse does not necessarily hold.

Example 5.1. In Figure 2, consider the $\mathcal{S}$-digraph $\zeta = (\theta(\zeta), \omega(\zeta))$ with the infinite point set $\theta(\zeta) = \{0'', 0', 1', 2', 3', \cdots\}$. The subbase $\Theta(\zeta)$ is defined by

$$\Theta(\zeta) = \{\Theta_{0''}, \Theta_{k'} : k = 0, 1, 2, 3, \cdots\},$$

where

$$\Theta_{k'} = \{0', 1', 2', 3', \cdots\} \text{ for all } k = 0, 1, 2, 3, \cdots, \text{ and } \Theta_{0''} = \{0''\}.$$

The flexible graphical topology of ζ is given by

$$T_{\Theta(\zeta)} = \{\emptyset, \theta(\zeta), \Theta_{0'}, \Theta_{0''}\}.$$

Thus, the space $(\theta(\zeta), T_{\Theta(\zeta)})$ is compact even though $\theta(\zeta)$ is infinite.

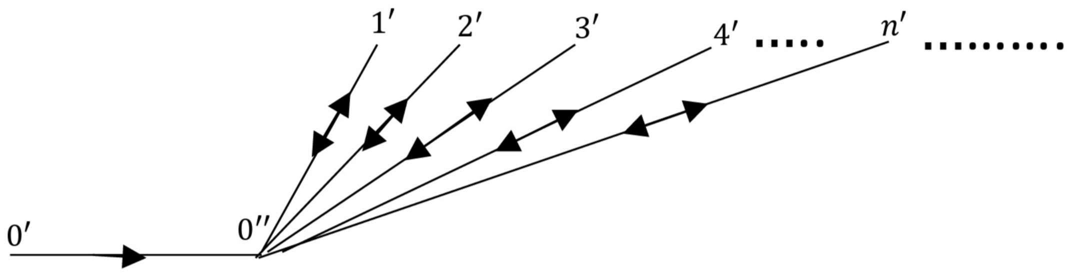


Figure 2. Compactness property.

The following theorems address the connectedness property in the class of $\mathcal{S}$-digraphs and their corresponding flexible graphical topological spaces.

Theorem 5.2. *Let $\zeta = (\theta(\zeta), \omega(\zeta))$ be an $\mathcal{S}$-digraph. If the space $(\theta(\zeta), T_{\Theta(\zeta)})$ is connected and ζ has no isolated vertices, then ζ is weakly connected.*

Proof. Assume that ζ is not weakly connected. Then, its associated undirected graph $Un(\zeta)$ is disconnected. Let $\mathcal{H} := \{\zeta_d : d \in \delta\}$ be the family of all components of $Un(\zeta)$, where $\zeta_d = (\theta(\zeta_d), \omega(\zeta_d))$ for each $d \in \delta$. For each $d \in \delta$, $\theta(\zeta_d) = \bigcup_{\tau \in \theta(\zeta_d)} \Theta_\tau$. Choose $d_0 \in \delta$ and define $M := \theta(\zeta_{d_o})$, which is a proper nonempty open subset of $\theta(\zeta)$. Then, $G := M^c = \bigcup_{d \in \delta \setminus \{do\}} \theta(\zeta_d)$ is also a proper nonempty open subset of $\theta(\zeta)$. Hence, $(\theta(\zeta), T_{\Theta(\zeta)})$ is disconnected, which contradicts the assumption. Therefore, ζ must be weakly connected. □

Theorem above should be understood as a one-way implication. More precisely, connectedness of the flexible graphical topological space implies weak connectivity of the underlying digraph, provided that the digraph has no isolated vertices. However, the converse implication is not valid in general. Thus, weak connectivity of the underlying digraph is a necessary condition for topological connectedness in this setting, but it is not sufficient.

Remark 5.3. *The converse of Theorem 5.2 does not hold in general. That is, a weakly connected digraph may generate a disconnected flexible graphical topological space. For example, the flexible graphical topological space $(\theta(C_n), T_{\Theta(C_n)})$ of a cycle digraph C_n is discrete, and hence disconnected, although C_n is weakly connected. Similarly, the complete digraph K_n is strongly connected, but $(\theta(K_n), T_{\Theta(K_n)})$ is disconnected.*

Let $\zeta = (\theta(\zeta), \omega(\zeta))$ be an $\mathcal{S}$-digraph. Define

$$\mathbb{H}^{\mathrm{com}}(\zeta) = \{a \in \theta(\zeta) : \partial_a^{ne} \neq \partial_a^{po}\}.$$

An $\mathcal{S}$-digraph ζ is said to be *flexible connected* if the induced subgraph $\zeta^{\mathbb{H}^{\mathrm{com}}}$ is weakly connected. The space $(\theta(\zeta), T_{\Theta(\zeta)})$ is called *flexible discrete* if the relative topology $T_{\Theta(\zeta)}|_{\mathbb{H}^{com}(\zeta)}$ is discrete in $\mathbb{H}^{\mathrm{com}}(\zeta)$.

Example 5.4. (1) The cycle digraph $\mathbb{C}_n$ $(n > 2)$ is flexible connected since $\mathbb{H}^{\mathrm{com}}(\mathbb{C}_n) = \theta(\mathbb{C}_n)$, and $(\theta(\mathbb{C}_n), T_{\Theta(\mathbb{C}_n)})$ is flexible discrete. Similarly, the complete digraph $\mathbb{K}_n$ $(n > 2)$ is flexible connected, and $(\theta(\mathbb{K}_n), T_{\Theta(\mathbb{K}_n)})$ is flexible discrete.

(2) In contrast, the complete bipartite digraph $\mathbb{K}_{n \to m}$ $(n, m > 2)$ is not flexible connected since $\mathbb{H}^{\mathrm{com}}(\mathbb{K}_{n \to m}) = \theta_m$ and the induced subgraph $\mathbb{K}_{n \to m}^{\mathbb{H}^{\mathrm{com}}}$ is not weakly connected, although $(\theta(\mathbb{K}_{n,m}), T_{\Theta(\mathbb{K}_{n,m})})$ is flexible discrete.

(3) On the other hand, $\mathbb{K}_{n,m}$ $(n, m > 2)$ is flexible connected since $\mathbb{H}^{\mathrm{com}}(\mathbb{K}_{n,m}) = \theta(\mathbb{K}_{n,m})$, and $(\theta(\mathbb{K}_{n,m}), T_{\Theta(\mathbb{K}_{n,m})})$ is flexible discrete.

(4) In Figure 2, $\mathbb{H}^{\mathrm{com}}(\zeta) = \{0', 1', 2', 3', \cdots\}$; the subgraph $\zeta^{\mathbb{H}^{\mathrm{com}}}$ is not weakly connected, and $(\theta(\zeta), T_{\Theta(\zeta)})$ is not flexible discrete.

Theorem 5.5. *Let ζ be a flexible connected graph. Then, the set $\mathbb{Q}$ of vertices in $\mathbb{H}^{\mathrm{com}}(\zeta)$ having degree greater than one is dense in $(\mathbb{H}^{\mathrm{com}}(\zeta), T_{\eta\mathbb{H}^{\mathrm{com}}(\zeta)})$.*

Proof. Let $\tau \in \mathbb{H}^{\mathrm{com}}(\zeta)$. Since $\mathbb{O}^{\min}(\tau)$ is the smallest open set containing τ, it suffices to show that $\mathbb{Q} \cap \mathbb{O}^{\min}(\tau) \neq \emptyset$ for all $\tau \in \mathbb{H}^{\mathrm{com}}(\zeta) \setminus \mathbb{Q}$. Because τ is not isolated, there exists $\rho \in \mathbb{H}^{\mathrm{com}}(\zeta)$ such that $\rho \in \mathbb{O}^{\min}(\tau)$. Hence, $\mathbb{O}^{\min}(\tau) = \mathbb{O}^{\min}(\rho)$ and $\partial_\rho > 1$. Thus, there exists $\nu \in \mathbb{Q}$ such that $\nu \in \mathbb{O}^{\min}(\rho) = \mathbb{O}^{\min}(\tau)$. Therefore, $\mathbb{Q} \cap \mathbb{O}^{\min}(\tau) \neq \emptyset$ for all $\tau \in \mathbb{H}^{\mathrm{com}}(\zeta) \setminus \mathbb{Q}$, proving that $\mathbb{Q}$ is dense. □

Theorem 5.6. *Let ζ be a flexible connected graph, Γ the family of smallest open sets of all vertices in $\mathbb{H}^{\text{com}}(\zeta)$, and Ω_Γ the set of all minimal elements in Γ.*

(1) If $\mathbb{Q} \subseteq \mathbb{H}^{\text{com}}(\zeta)$ is a minimal dense subset of $(\mathbb{H}^{\text{com}}(\zeta), T_{\eta\mathbb{H}^{\text{com}}(\zeta)})$, then there exists a surjective mapping $\psi : \Omega_\Gamma \to \mathbb{Q}$ such that $\psi(\Psi_\tau) \in \Psi_\tau$ for all $\Psi_\tau \in \Omega_\Gamma$;

(2) If $\psi : \Omega_\Gamma \to \theta(\zeta)$ satisfies $\psi(\Psi_\tau) \in \Psi_\tau$ for all $\Psi_\tau \in \Omega_\Gamma$, then $\psi(\Omega_\Gamma)$ is a minimal dense subset of $(\mathbb{H}^{\text{com}}(\zeta), T_{\eta\mathbb{H}^{\text{com}}(\zeta)})$.

Proof. (1) Since distinct elements of Ω_Γ are disjoint and $Cl(\mathbb{Q}) = \mathbb{H}^{\text{com}}(\zeta)$, each $G \in \Omega_\Gamma$ contains a unique point $\tau \in G \cap \mathbb{Q}$. Define $\psi(G) = \tau$. Then, $\psi : \Omega_\Gamma \to \mathbb{Q}$ is surjective and satisfies $\psi(\Psi_\tau) \in \Psi_\tau$.

(2) For each $\tau \in \mathbb{H}^{\text{com}}(\zeta)$, there exists ρ such that $\Psi_\rho \in \Omega_\Gamma$ and $\Psi_\rho \subseteq \Psi_\tau$. Hence, $\psi(\Psi_\rho) \in \Psi_\tau \cap \psi(\Omega_\Gamma)$, proving density. Minimality follows by contradiction: if $\mathbb{Q} \subset \psi(\Omega_\Gamma)$ with $Cl(\mathbb{Q}) = \mathbb{H}(\zeta)$, then for some Ψ_τ, $\psi(\Psi_\tau) \notin \mathbb{Q}$, which contradicts the closure property. Thus, $\psi(\Omega_\Gamma)$ is a minimal dense set. □

Theorem 5.7. *Flexible connectedness is an isomorphic property in the class of $\mathcal{S}$-digraphs without isolated vertices.*

Proof. Let ζ_1 and ζ_2 be $\mathcal{S}$-digraphs without isolated vertices, with ζ_1 flexible connected and $\zeta_1 \cong \zeta_2$. Then there exists a bijection $\mathcal{J} : \theta(\zeta_1) \to \theta(\zeta_2)$ preserving adjacency. Thus, $\partial_a^{ne} \neq \partial_a^{po}$ if, and only if, $\partial_{\mathcal{J}(a)}^{ne} \neq \partial_{\mathcal{J}(a)}^{po}$. If ζ_2 were not flexible connected, then ζ_1 would not be either, a contradiction. Hence, ζ_2 is flexible connected. □

Theorem 5.8. *Flexible discreteness is a topological property in the class of $\mathcal{S}$-digraphs without isolated vertices.*

Proof. Let ζ_1 and ζ_2 be $\mathcal{S}$-digraphs without isolated vertices. Suppose $(\theta(\zeta_1), T_{\Theta(\zeta_1)})$ is flexible discrete and there exists a homeomorphism $\Gamma : \theta(\zeta_1) \to \theta(\zeta_2)$. For any $a' \in \mathbb{H}^{\text{com}}(\zeta_2)$, there exists $a \in \mathbb{H}^{\text{com}}(\zeta_1)$ such that $a' = \Gamma(a)$. Since $\{a\}$ is open in $(\mathbb{H}^{\text{com}}(\zeta_1), T_{\Theta(\zeta_1)}|_{\mathbb{H}^{\text{com}}(\zeta_1)})$, and Γ is open and bijective, it follows that $\{a'\}$ is open in $(\mathbb{H}^{\text{com}}(\zeta_2), T_{\Theta(\zeta_2)}|_{\mathbb{H}^{\text{com}}(\zeta_2)})$. Thus, $(\theta(\zeta_2), T_{\Theta(\zeta_2)})$ is flexible discrete. □

6. Some applications

It is well known that the cardiovascular system (or the human circulatory system) forms a closed and dynamic network responsible for the transportation of oxygen, nutrients, hormones, and metabolic waste throughout the body. This system comprises the heart, blood vessels, and blood. The heart, a muscular organ with four chambers, functions as a dual pump: the right atrium and ventricle direct deoxygenated blood to the lungs for oxygenation, whereas the left atrium and ventricle propel oxygenated blood to the rest of the body. The vascular system consists of arteries that convey oxygen-rich blood from the heart, veins that return deoxygenated blood back to it, and capillaries that facilitate the exchange of gases, nutrients, and waste at the cellular level. The circulating blood itself includes red blood cells for oxygen transport, white blood cells for immune defense, platelets for coagulation, and plasma that serves as a carrier for nutrients, hormones, and proteins. Moreover, the heart muscle is supplied through the coronary circulation, ensuring its continuous rhythmic activity. Altogether, this interconnected system maintains homeostasis, supports metabolic activity, and sustains life [28]. This section investigates the structural and topological properties of connectedness, flexible connectedness, and flexible discreteness in the graphs constructed from the network representations of the human circulatory system. Figure 3(a) (see [28, Figure 1]) illustrates the overall schematic of the human circulatory system. From an interpretative perspective, the obtained topological structures provide qualitative information about the organization of the circulatory network. In particular, connectedness reflects the existence of continuous circulation pathways, whereas flexible connectedness characterizes regions exhibiting balanced local flow behavior. The absence of flexible discreteness indicates that local functional interactions remain structurally dependent rather than isolated. Although the present analysis is theoretical and not intended for clinical prediction, such topological representations may support future studies in medical decision-making by assisting in the identification of structurally sensitive regions, circulation organization patterns, and possible changes caused by functional perturbations or pathological conditions.

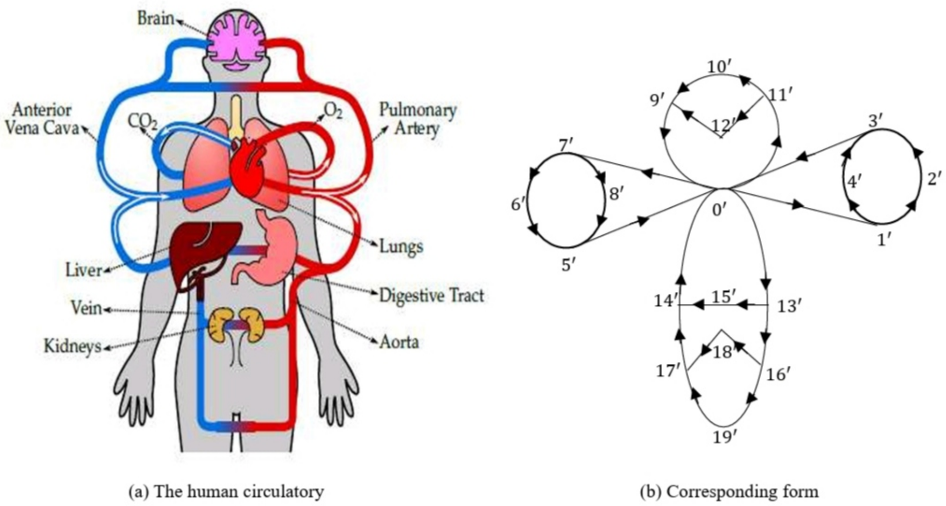


Figure 3. The human circulatory system.

First, we consider the corresponding graph derived from the network diagram that represents the human circulatory system shown in Figure 3(a), [28, Figure 1] and Table 3. Let the associated graph be denoted by $\zeta = (\theta(\zeta), \omega(\zeta))$ as illustrated in Figure 3(b). The point set of ζ is

$$\theta(\zeta) = \{0', 1', 2', \ldots, 19'\},$$

comprising 20 vertices. The graph ζ is connected but not flexible connected, since the induced subgraph $\zeta^{\mathbb{H}^{\mathrm{com}}}$ is not weakly connected and has the point set

$$\theta(\zeta^{\mathbb{H}^{\mathrm{com}}}) = \{3', 5', 9', 11', 13', 14', 16', 17'\}.$$

The corresponding flexible graphical topological space $(\theta(\zeta), T_{\Theta(\zeta)})$ of ζ possesses a subbase $\Theta(\zeta)$ that consists of the family of all flexible open neighborhoods.

$$\Theta_{0'} = \theta(\zeta) - \{1', 3', 5', 7', 9', 11', 13', 14'\},\ \Theta_{1'} = \theta(\zeta) - \{0', 2', 4', 5', 9', 14', 17'\}$$

$$\Theta_{2'} = \theta(\zeta) - \{0', 1', 3', 4', 5', 9', 14', 17'\},\ \Theta_{3'} = \theta(\zeta) - \{0', 2', 4'\},$$

$$\Theta_{4'} = \theta(\zeta) - \{1', 2', 3'\},\ \Theta_{5'} = \theta(\zeta) - \{0', 1', 6', 8'\},$$

$$\Theta_{6'} = \theta(\zeta) - \{5', 7', 8'\},\ \Theta_{7'} = \theta(\zeta) - \{0', 6', 8', 9'\},$$

$$\Theta_{8'} = \theta(\zeta) - \{0', 5', 6', 7'\},\ \Theta_{9'} = \theta(\zeta) - \{0', 10', 12'\},\ \Theta_{10'} = \theta(\zeta) - \{9', 11', 12'\},$$

$$\Theta_{11'} = \theta(\zeta) - \{0', 10', 12'\},\ \Theta_{12'} = \theta(\zeta) - \{9', 10', 11'\},\ \Theta_{13'} = \theta(\zeta) - \{0', 15', 16'\}.$$

$$\Theta_{14'} = \theta(\zeta) - \{0', 15', 17'\},\ \Theta_{15'} = \theta(\zeta) - \{13', 14', 16', 17', 18', 19'\}$$

$$\Theta_{16'} = \theta(\zeta) - \{13', 18', 19'\},\ \Theta_{17'} = \theta(\zeta) - \{14', 15', 18', 19'\},$$

$$\Theta_{18'} = \theta(\zeta) - \{15', 16', 17', 19'\},\ \Theta_{19'} = \theta(\zeta) - \{15', 16', 17', 18'\}.$$

Table 3. Correspondence between vertices in Figure 3(b) and functional descriptions in the human circulatory system.

Vertex	Functional description
$0'$	Main blood distribution and circulation junction
$1'$	Blood transmission toward upper circulation branch
$2'$	Local blood flow continuation
$3'$	Transition node connecting circulation pathways
$4'$	Intermediate blood transport node
$5'$	Functional blood redistribution point
$6'$	Peripheral circulation continuation
$7'$	Blood transfer toward secondary vascular branch
$8'$	Return blood circulation path
$9'$	Pulmonary/systemic circulation transition
$10'$	Oxygenated blood transport path
$11'$	Blood redistribution node
$12'$	Functional circulation continuation
$13'$	Connection toward lower vascular region
$14'$	Blood flow regulation branch
$15'$	Exchange and distribution node
$16'$	Peripheral vascular circulation
$17'$	Return circulation branch
$18'$	Terminal blood distribution region
$19'$	Blood return and circulation completion

More details are seen in Table 4. So, the flexible graphical topological space $(\theta(\zeta), T_{\Theta(\zeta)})$ of ζ is not discrete, not flexible discrete, and disconnected. To facilitate readability, Table 4 summarizes the flexible neighborhood relationships in a compact form. Each row describes the local flexible connectivity pattern associated with a vertex and illustrates how the corresponding flexible graphical topology reflects structural interactions in the circulatory network.

Table 4. The flexible neighborhood system in Figure 3(b).

τ	∂_τ^{ne}	∂_τ^{po}	$\mathbb{H}^{\text{com}}(\zeta)$	$\Theta(\tau)$	Θ_τ
$0'$	4	2	0	$\theta(\zeta)-\{0',1',3',5',7',9',11',13',14'\}$	$\theta(\zeta)-\{1',3',5',7',9',11',13',14'\}$
$1'$	2	2	1	$\theta(\zeta)-\{0',1',2',4',5',9',14',17'\}$	$\theta(\zeta)-\{0',2',4',5',9',14',17'\}$
$2'$	1	1	1	$\theta(\zeta)-\{0',1',2',3',4',5',9',14',17'\}$	$\theta(\zeta)-\{0',1',3',4',5',9',14',17'\}$
$3'$	1	2	0	$\theta(\zeta)-\{0',2',3',4'\}$	$\theta(\zeta)-\{0',2',4'\}$
$4'$	1	1	1	$\theta(\zeta)-\{1',2',3',4'\}$	$\theta(\zeta)-\{1',2',3'\}$
$5'$	1	2	0	$\theta(\zeta)-\{0',1',5',6',8'\}$	$\theta(\zeta)-\{0',1',6',8'\}$
$6'$	1	1	1	$\theta(\zeta)-\{5',6',7',8'\}$	$\theta(\zeta)-\{5',7',8'\}$
$7'$	2	2	1	$\theta(\zeta)-\{0',6',7',8',9'\}$	$\theta(\zeta)-\{0',6',8',9'\}$
$8'$	1	1	1	$\theta(\zeta)-\{0',5',6',7',8'\}$	$\theta(\zeta)-\{0',5',6',7'\}$
$9'$	1	2	0	$\theta(\zeta)-\{0',9',10',12'\}$	$\theta(\zeta)-\{0',10',12'\}$
$10'$	1	1	1	$\theta(\zeta)-\{9',10',11',12'\}$	$\theta(\zeta)-\{9',11',12'\}$
$11'$	2	1	0	$\theta(\zeta)-\{0',10',11',12'\}$	$\theta(\zeta)-\{0',10',12'\}$
$12'$	1	1	1	$\theta(\zeta)-\{9',10',11',12'\}$	$\theta(\zeta)-\{9',10',11'\}$
$13'$	2	1	0	$\theta(\zeta)-\{0',13',15',16'\}$	$\theta(\zeta)-\{0',15',16'\}$
$14'$	1	2	0	$\theta(\zeta)-\{0',14',15',17'\}$	$\theta(\zeta)-\{0',15',17'\}$
$15'$	1	1	1	$\theta(\zeta)-\{13',14',15',16',17',18',19'\}$	$\theta(\zeta)-\{13',14',16',17',18',19'\}$
$16'$	2	1	0	$\theta(\zeta)-\{13',16',18',19'\}$	$\theta(\zeta)-\{13',18',19'\}$
$17'$	1	2	0	$\theta(\zeta)-\{14',15',17',18',19'\}$	$\theta(\zeta)-\{14',15',18',19'\}$
$18'$	1	1	1	$\theta(\zeta)-\{15',16',17',18',19'\}$	$\theta(\zeta)-\{15',16',17',19'\}$
$19'$	1	1	1	$\theta(\zeta)-\{15',16',17',18',19'\}$	$\theta(\zeta)-\{15',16',17',18'\}$

7. Conclusions

This study proposed flexible paths and the corresponding flexible graphical topologies as a new framework for analyzing directed graphs. The work characterized classes of directed graphs that generate discrete or indiscrete flexible graphical topologies, investigated the relationship between homeomorphism and graph isomorphism, and introduced the notions of flexible connectedness and flexible discreteness. To demonstrate its applicability, the framework was applied to graph representations of the human circulatory system. The results highlight the usefulness of flexible graphical topology as both a theoretical advancement and a mathematical tool for modeling biological networks.

Future work may extend this framework to intelligent systems, rough approximation models, and computational topology.

Author contributions

Husniyah Alzubaidi and Hakeem A. Othman: conceptualized the research idea and applications, prepared a draft of the manuscript, and made formal analysis. Ljubiša D.R. Kočinac: supervised the research, verified and refined the theoretical analysis, and proofs and revised the manuscript. All authors have read and approved the final version of the paper for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflict of interests.

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