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Research article

Soliton solutions to the inverse curvature flow on the de Sitter surface dS_2

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Abstract: In geometric flows, soliton solutions evolve by isometries of the ambient surface. In this paper, we study soliton solutions to the inverse curvature flow of regular curves on the de Sitter surface dS_2, a maximally symmetric Lorentzian manifold of constant positive curvature embedded in $\mathbb{R}^3_1$. We show that a regular curve is a soliton solution if and only if its inverse geodesic curvature equals the Lorentzian inner product of its tangent vector field with a fixed nonzero vector $\mathbf{v} \in \mathbb{R}^3_1$. On the basis of this characterization, we prove the existence of a two-parameter family of nontrivial soliton solutions for each $\mathbf{v}$, including both spacelike and timelike curves. Moreover, we completely classify both spacelike and timelike soliton solutions to the inverse curvature flow on the de Sitter surface. The classification is determined by the causal character of $\mathbf{v}$. For spacelike solitons, each solution is either complete or has a finite singular endpoint at which the geodesic curvature blows up, while the other end escapes to infinity. Moreover, complete spacelike solitons exist only when $\mathbf{v}$ is spacelike. By contrast, timelike solitons may be complete for any causal character of $\mathbf{v}$. In the incomplete case, each solution possesses a finite endpoint whose geometry depends on the causal character of $\mathbf{v}$. In all cases, we determine the asymptotic behavior of the curves and their curvatures at every nonsingular end. We also present several explicit examples and visualizations illustrating the different types of soliton solutions.

Keywords: soliton solutions; isometry; inverse curvature flow; de Sitter surface; classification
Mathematics Subject Classification: 53C50, 53E10

1. Introduction

Inverse curvature flow (ICF) has been a central topic in geometric analysis due to its deep connections with problems in general relativity, geometric inequalities, and the study of minimal surfaces. The inverse mean curvature flow (IMCF), rigorously formulated by Huisken and Ilmanen [1], played a decisive role in proving the Riemannian–Penrose inequality, which is a fundamental result relating the Arnowitt–Deser–Misner mass, the total mass of an asymptotically flat manifold as measured at spatial infinity, to the area of its black hole horizons. This deep connection between geometric flows and spacetime invariants has since inspired further work, including applications to quasi-local mass problems [2–4] and the analysis of cosmological spacetimes [5]. Beyond relativity, ICF has become indispensable in establishing geometric inequalities [6, 7] and in the analysis of minimal surfaces via parabolic regularization techniques [8, 9]. In Euclidean space, Gerhardt [10] and Urbas [11] established key results on the long-time behavior and convergence properties of ICFs; later, Gerhardt [12] extended these results to hyperbolic space. Recently, ICF has been applied to more complex ambient spaces, deriving various geometric inequalities [13–15].

The notion of a soliton originated in the study of nonlinear wave equations and solitary wave propagation in mathematical physics, beginning with the pioneering work of Zabusky and Kruskal [16], while its mathematical structure was further developed by Lax [17]. We also refer to [18–20] for some recent developments concerning soliton phenomena in mathematical physics. In geometric analysis, the concept of a soliton is used in a generalized sense to describe special solutions that evolve purely through the symmetries of the ambient space. Such soliton solutions are of fundamental importance, as they often serve as canonical models for singularity formation and asymptotic behavior in geometric flows. For the ICF, only partial classification results are currently known. Chang [21] classified all self-similar solutions to the ICF on the Euclidean plane $\mathbb{R}^2$. More recently, Pyo and the author (submitted for publication) classified soliton solutions to the ICF on the sphere $\mathbb{S}^2$ and the hyperbolic plane $\mathbb{H}^2$. On the other hand, soliton solutions to the curvature flow (CF) have been extensively studied on totally umbilical surfaces in $\mathbb{R}^3$ and $\mathbb{R}^3_1$. Halldorsson [22, 23] analyzed CF solitons on the Euclidean plane $\mathbb{R}^2$ and the Minkowski plane $\mathbb{L}^2$. Santos dos Reis and Tenenblat [24] classified CF solitons on the sphere $\mathbb{S}^2$, while da Silva and Tenenblat [25], independently of Woolgar and Xie [26], classified CF solitons on the hyperbolic plane $\mathbb{H}^2$. Pyo and the author (submitted for publication) further classified CF solitons on the de Sitter surface dS_2. Additionally, da Silva and Tenenblat [27] investigated both CF and ICF solitons on the lightlike cone.

The de Sitter surface dS_2 is the Lorentzian analog of the Euclidean sphere, realized as a totally umbilical quadric with constant positive sectional curvature in $\mathbb{R}^3_1$. While previous studies have classified soliton solutions for the CF as well as the ICF in various two-dimensional Riemannian and Lorentzian space forms, the investigation of the ICF on the de Sitter surface has remained a necessary gap to fill. By exploring soliton solutions to the ICF on dS_2, we complete the classification of such solitons for both CF and ICF within the framework of two-dimensional Riemanninan and Lorentzian space forms, which are realized as totally umbilical surfaces in the Euclidean 3-space $\mathbb{R}^3$ and the Minkowski 3-space $\mathbb{R}^3_1$. More precisely, we prove that for any nonzero vector in the Minkowski 3-space, a spacelike or timelike soliton solution to the ICF on dS_2 exists. Furthermore, we analyze the qualitative behavior of these solutions according to the causal character of the vector. Finally, we provide several visualizations illustrating both the geometry of the soliton solutions and their short-time evolution under the flow.

The paper is organized as follows. In Section 2, we provide some fundamental preliminaries. In Section 3, we prove the equivalent formulation and the existence of the ICF solitons, as stated in Theorems 2.1 and 2.3. Sections 4 and 5 are devoted to the study of spacelike and timelike soliton solutions, respectively. After establishing the corresponding qualitative analysis of the associated ordinary differential equation (ODE) systems through the asymptotic behavior of the relevant functions, we present the main classification results, in Theorems 4.14 and 5.3, and then prove them.

Section 6 describes the geometric conditions satisfied by the illustrated solitons and explains the notation used in the figures.

2. Preliminaries

The three-dimensional Minkowski space, $\mathbb{R}^3_1$, is defined as $\mathbb{R}^3$ equipped with the Lorentzian metric

$$\langle\cdot,\cdot\rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

with respect to the standard coordinates (x_1, x_2, x_3). Within this ambient space, the de Sitter surface is realized as the two-dimensional Lorentzian space form of constant positive sectional curvature, given by the quadric

$$dS_2 = \{x \in \mathbb{R}^3_1 : \langle x, x\rangle = 1\}.$$

Let X: $I \subset \mathbb{R} \to dS_2$ be a regular curve parametrized by the arc length s. We use $T(s) = X'(s)$ to denote its unit tangent vector field and use $N(s) = X(s) \wedge T(s)$ to denote the corresponding unit normal vector field, where $\wedge$ denotes the Lorentzian cross–product in $\mathbb{R}^3_1$. The geodesic curvature of X is then defined by $\kappa(s) = \langle T'(s), N(s)\rangle$.

A one-parameter family of curves $\widehat{X}$: $I \times J \to dS_2$ is said to evolve by the ICF if it satisfies

$$\left\langle \frac{\partial}{\partial t}\widehat{X}(s,t), \widehat{N}(s,t)\right\rangle = \widehat{\kappa}^{-1}(s,t), \tag{2.1}$$

with the initial data $\widehat{X}(s, 0) = X(s)$, where $\widehat{N}$ and $\widehat{\kappa}$ denote the unit normal vector field and the geodesic curvature of $\widehat{X}(s, t)$, respectively.

We call the initial curve X a soliton solution of the ICF if the evolution is generated by a one-parameter group of isometries of dS_2; namely, if a family $\{M(t)\}_{t\in J} \subset O_1(3)$ with $M(0) = \mathrm{Id}$ exists such that

$$\widehat{X}(s,t) = M(t)X(s),\ t \in J.$$

Here, $O_1(3)$ denotes the Lorentz group preserving the Minkowski metric and leaving dS_2 invariant.

Null and pseudo-null curves are excluded from our discussion. Indeed, geodesic curvature is not defined for null curves, since the Frenet construction degenerates in the null setting, and no natural curvature invariant exists in the pseudo-null setting [28].

The first result provides an equivalent condition of soliton solutions to the ICF, which allows us to reduce the problem to the analysis of curves satisfying (2.2).

Theorem 2.1. *A regular curve X: $I \to dS_2$, parametrized by the arc length s, is a soliton solution to the ICF on the de Sitter surface if and only if a nonzero vector $\mathbf{v} \in \mathbb{R}^3_1$ exists such that*

$$\langle T(s), \mathbf{v}\rangle = \kappa^{-1}(s) \quad \textit{for all } s \in I, \tag{2.2}$$

where $T(s)$ and $\kappa(s)$ denote the unit tangent vector field and the geodesic curvature of X, respectively.

Remark 2.2. *From Eq (2.2), if $\langle T(s), \mathbf{v}\rangle$ vanishes at some point, the geodesic curvature $\kappa(s)$ becomes unbounded; consequently, X develops singularities.*

Hereafter, all soliton solutions are assumed to be defined on their maximal interval of existence $I = (w_-, w_+)$. In particular, the arc length parametrization $X\colon I \to dS_2$ cannot be prolonged past either endpoint of I as a smooth solution.

Regarding the existence of soliton solutions to the ICF, there is a two-parameter family of such solutions.

Theorem 2.3. *Let $\mathbf{v} \in \mathbb{R}^3_1$ be an arbitrary nonzero vector. Then there is a two-parameter family of nontrivial spacelike (respectively (resp.) timelike) soliton solutions $X\colon I \to dS_2$ to the ICF on the de Sitter surface.*

3. Proof of Theorems 2.1 and 2.3

Proof of Theorem 2.1. Let $X\colon I \to dS_2$ be a soliton solution to the ICF, then $\widehat{X}(s,t) = M(t)X(s)$ is a solution to the ICF Eq (2.1). Since $M(t)$ is an isometry for each $t \in J$, in particular, at $t = 0$, $M'(0)$ is an element of the Lie algebra $\mathfrak{o}_1(3)$, which can be expressed as

$$M'(0) = \begin{pmatrix} 0 & c_1 & c_2 \\ c_1 & 0 & c_3 \\ c_2 & -c_3 & 0 \end{pmatrix}.$$

By direct computation, we have

$$\kappa^{-1}(s) = \widehat{\kappa}^{-1}(s,0) = \left\langle \frac{\partial}{\partial t}\widehat{X}(s,0), \widehat{N}(s,0) \right\rangle = \langle M'(0)X(s), N(s)\rangle = \langle T(s), (-c_3, -c_2, c_1)\rangle.$$

Therefore, there is a nonzero vector $\mathbf{v} = (-c_3, -c_2, c_1)$ such that $\langle T(s), \mathbf{v}\rangle = \kappa^{-1}(s)$.

Conversely, we consider vectors of the form $\mathbf{v}_1 = a(-1, 0, 0)$ (timelike), $\mathbf{v}_2 = a(1, 1, 0)$ (lightlike), and $\mathbf{v}_3 = a(0, 0, -1)$ (spacelike) for some $a > 0$.

Assume a regular curve $X\colon I \to dS_2$ whose inverse geodesic curvature satisfies $\kappa_i^{-1}(s) = \langle T(s), \mathbf{v}_i\rangle$, $i = 1, 2, 3$. Define the evolution of X in dS_2 to be $\widehat{X_i}(s,t) = M_i(t)X(s)$, $i = 1, 2, 3$, where $M_i(t)$ represents the isometries of dS_2 given by one of the following:

$$M_1(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1(t) & -\sin\theta_1(t) \\ 0 & \sin\theta_1(t) & \cos\theta_1(t) \end{pmatrix},$$

$$M_2(t) = \begin{pmatrix} 1 + \frac{\theta_2^2(t)}{2} & -\frac{\theta_2^2(t)}{2} & \theta_2(t) \\ \frac{\theta_2^2(t)}{2} & 1 - \frac{\theta_2^2(t)}{2} & \theta_2(t) \\ \theta_2(t) & -\theta_2(t) & 1 \end{pmatrix},$$

$$M_3(t) = \begin{pmatrix} \cosh\theta_3(t) & \sinh\theta_3(t) & 0 \\ \sinh\theta_3(t) & \cosh\theta_3(t) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with $\theta_i(t) = at$ for each $i = 1, 2, 3$.

Write

$$X(s) = (x_1, x_2, x_3)(s), T(s) = (T_1, T_2, T_3)(s), N(s) = (N_1, N_2, N_3)(s).$$

Given

$$T(s) = N(s) \wedge X(s) = (-(x_3N_2 - x_2N_3), x_1N_3 - x_3N_1, x_2N_1 - x_1N_2)(s).$$

By direct computation, we can check that

$$\begin{aligned}\left\langle \frac{\partial}{\partial t}\widehat{X}_i(s,t), \widehat{N}_i(s,t) \right\rangle &= \langle M_i'(t)X(s), M_i(t)N(s)\rangle \\ &= \begin{cases} \theta_1'(t)T_1(s) \\ \theta_2'(t)(-T_1(s) + T_2(s)) \\ \theta_3'(t)(-T_3(s)) \end{cases} \\ &= \langle T(s), \mathbf{v}_i\rangle = \kappa_i^{-1}(s) = \widehat{\kappa_i}^{-1}(s,t).\end{aligned}$$

Therefore, $\widehat{X}_i(s,t)$ solves (2.1), meaning that $X(s)$ is a soliton solution of ICF. □

The following result reformulates Theorem 2.1 as a system of differential equations when the curves are spacelike or timelike, respectively.

Proposition 3.1. *Let $X\colon I \to dS_2$ be a regular curve parametrized by the arc length s. Let e be one of the vectors $\{(-1,0,0), (-1,1,0), (0,0,1)\}$ such that, up to isometries of dS_2, any nonzero vector in $\mathbb{R}^3_1$ can be written in the form*

$$\mathbf{v} = ae \text{ for some } a > 0. \tag{3.1}$$

By decomposing e with respect to the moving frame $\{X(s), T(s), N(s)\}$, we define the following functions:

$$\alpha(s) = \langle X(s), e\rangle, \quad \tau(s) = \langle T(s), e\rangle, \quad \eta(s) = \langle N(s), e\rangle, \tag{3.2}$$

where T and N denote the unit tangent and normal vector fields of X, respectively. Set $\psi(s) := (\alpha(s), \tau(s), \eta(s))$. Then the following statements hold.

(1) X is a spacelike soliton solution to the ICF if and only if $\psi(s)$ satisfies

$$\begin{cases}\alpha'(s) = \tau(s), \\ \tau'(s) = -\dfrac{\eta(s)}{a\,\tau(s)} - \alpha(s), \\ \eta'(s) = -\dfrac{1}{a},\end{cases} \tag{3.3}$$

for some constant $a > 0$ and $\psi(s) \in H \cup C \cup S$ for all $s \in I$, where

$$\begin{aligned} H &= \{(\alpha,\tau,\eta) \in \mathbb{R}^3 : \alpha^2 + \tau^2 - \eta^2 = -1,\ |\eta| \geq 1\}, \\ C &= \{(\alpha,\tau,\eta) \in \mathbb{R}^3 \setminus \{0\} : \alpha^2 + \tau^2 - \eta^2 = 0,\ \eta \neq 0\}, \\ S &= \{(\alpha,\tau,\eta) \in \mathbb{R}^3 : \alpha^2 + \tau^2 - \eta^2 = 1\}. \end{aligned} \tag{3.4}$$

(2) X is a timelike soliton solution to the ICF if and only if $\psi(s)$ satisfies

$$\begin{cases}\alpha'(s) = \tau(s), \\ \tau'(s) = \dfrac{\eta(s)}{a\,\tau(s)} + \alpha(s), \\ \eta'(s) = \dfrac{1}{a},\end{cases} \tag{3.5}$$

for some constant $a > 0$ and $\psi(s) \in H \cup C \cup S$ for all $s \in I$, where

$$\begin{aligned} H &= \{(\alpha, \tau, \eta) \in \mathbb{R}^3 : \alpha^2 - \tau^2 + \eta^2 = -1, \ |\tau| \geq 1\}, \\ C &= \{(\alpha, \tau, \eta) \in \mathbb{R}^3 \setminus \{0\} : \alpha^2 - \tau^2 + \eta^2 = 0, \ \tau \neq 0\}, \\ S &= \{(\alpha, \tau, \eta) \in \mathbb{R}^3 : \alpha^2 - \tau^2 + \eta^2 = 1\}. \end{aligned} \tag{3.6}$$

Proof. Assume first that $X \colon I \to dS_2$ is a spacelike (resp. timelike) soliton solution to the ICF. By Theorem 2.1, a constant vector $\mathbf{v}$ exists such that

$$\kappa^{-1}(s) = \langle T(s), \mathbf{v} \rangle.$$

Up to an isometry of dS_2, the vector $\mathbf{v}$ may be chosen as in (3.1). With the notation introduced in (3.2), this relation can be rewritten as

$$\kappa(s) = \frac{1}{a\,\tau(s)}, \quad a > 0, \tag{3.7}$$

where $\tau(s) \neq 0$ for all $s \in I$ (see Remark 2.2).

Substituting (3.7) into the Frenet equations of a spacelike (resp. timelike) curve in dS_2 yields precisely system (3.3) (resp. (3.5)). Moreover, expressing the constant vector e in the Frenet frame of X gives

$$e = \alpha X + \tau T - \eta N \text{ (resp. } e = \alpha X - \tau T + \eta N),$$

according to the causal character of X. Taking the Minkowski inner product with itself, we obtain

$$\langle e, e \rangle = \alpha^2 + \tau^2 - \eta^2 \text{ (resp. } \langle e, e \rangle = \alpha^2 - \tau^2 + \eta^2),$$

which implies that $\psi(s) \in H \cup C \cup S$.

Conversely, suppose that $\psi(s)$ satisfies (3.3) (resp. (3.5)) with $\psi(s) \in H \cup C \cup S$. Comparing the system with the Frenet equations yields

$$\left(\kappa(s) - \frac{1}{a\tau(s)}\right)\tau(s) = 0, \quad \left(\kappa(s) - \frac{1}{a\tau(s)}\right)\eta(s) = 0.$$

Since τ and η cannot vanish simultaneously (otherwise, $\alpha(s) = \pm 1$ for all $s \in I$ leads a contradiction), we conclude that

$$\kappa^{-1}(s) = a\tau(s) = \langle T(s), \mathbf{v} \rangle$$

for some constant vector $\mathbf{v}$ as in (3.1). By Theorem 2.1, the curve X is therefore a spacelike (resp. timelike) soliton solution to the ICF. $\square$

Proof of Theorem 2.3. The proof consists of four steps.

Step 1: Fix a constant $a > 0$, and choose the initial data $\psi(w_0) \in H \cup C \cup S$ such that $\tau(w_0) \neq 0$ for any fixed $w_0 \in I$.

Step 2: Since the right-hand side of system (3.3) (resp. (3.5)) is smooth near $\psi(w_0)$, by the Picard–Lindelöf theorem, there is a unique solution $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ defined on a maximal interval.

Step 3: Define $\kappa(s) = \frac{1}{a\tau(s)}$. Choose the initial data $(X(w_0), T(w_0), N(w_0))$ satisfying

$$\begin{aligned} \alpha(w_0)X(w_0) + \tau(w_0)T(w_0) - \eta(w_0)N(w_0) = e, \\ (\text{resp. } \alpha(w_0)X(w_0) - \tau(w_0)T(w_0) + \eta(w_0)N(w_0) = e). \end{aligned} \tag{3.8}$$

Integrating the corresponding Frenet equations, we obtain a unique spacelike (resp. timelike) curve, i.e., $X\colon I \to dS_2$ whose geodesic curvature is $\kappa(s)$.

Step 4: For a nonzero vector $\mathbf{v} \in \mathbb{R}^3_1$, it can be written in the form given in (3.1) up to an isometry of dS_2. Using the notation of Proposition 3.1, we have

$$\kappa(s) = \frac{1}{a\tau(s)} = \frac{1}{\langle T(s), \mathbf{v}\rangle}.$$

Hence, by Theorem 2.1, the curve X is a soliton solution to the ICF.

Finally, since $\psi(s)$ lies on the two-dimensional surfaces H, C, and S, the set of all such soliton solutions forms a two-parameter family. □

Corollary 3.2. *Let $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ be a solution of system (3.3) (resp. (3.5)) satisfying $\psi(s) \in H \cup C \cup S$, a spacelike (resp. timelike) curve $X\colon I \to dS_2$ parametrized by an arc length s exists, such that its tangent and normal unit vector fields T and N satisfy* (3.2).

Proof. Define $\kappa(s) = \frac{1}{a\tau(s)}$, $a > 0$. Following the same reasoning as in the proof of Theorem 2.3, such a curve exists. Furthermore, Eq (3.2) is satisfied due to (3.8) and

$$\begin{aligned} \frac{d}{ds}(\alpha(s)X(s) + \tau(s)T(s) - \eta(s)N(s)) = 0, \\ (resp.\ \frac{d}{ds}(\alpha(s)X(s) - \tau(s)T(s) + \eta(s)N(s)) = 0). \end{aligned}$$

This completes the proof. □

Remark 3.3. *The systems* (3.3) *and* (3.5) *are singular at points where τ vanishes. At such points, the associated spacelike (resp. timelike) soliton solutions to the ICF cease to be regular, since the geodesic curvature blows up. In order to restrict to the regular curve, we distinguish two cases in which τ is strictly positive and strictly negative.*

4. Spacelike soliton solutions to the ICF in dS_2

In this section, we first study trivial constant-curvature solutions (Lemma 4.1). We then analyze nontrivial solutions based on the constant $\gamma = \langle e, e\rangle$. For the cases $\gamma = -1$ (timelike $\mathbf{v}$) and $\gamma = 0$ (lightlike $\mathbf{v}$), we determine the singular endpoints (Lemma 4.2) and describe the asymptotic behavior (Lemma 4.3). We next consider the case $\gamma = 1$ (spacelike $\mathbf{v}$), where we analyze the possible numbers of critical points of τ together with the corresponding asymptotic behavior (Lemma 4.7, Corollary 4.13).

4.1. Trivial solutions

Lemma 4.1. *Let $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ be a solution of system* (3.3) *satisfying $\psi(s) \in H \cup C \cup S$. If $\tau(s)$ is a nonzero constant, then ψ necessarily lies in S and is given by*

$$\psi(s) = (\pm(s - w_0) \mp \eta(w_0),\ \pm 1,\ -(s - w_0) + \eta(w_0))$$

for any fixed $w_0 \in I$. In this case, the associated curves are spacelike planar curves with geodesic curvature $\kappa(s) = \pm 1$.

Proof. Assume that $\tau(s) \equiv b$ for some nonzero constant b. Then system (3.3) reduces to a linear system for α and η, yielding

$$\eta(s) = -\frac{1}{a}(s - w_0) + \eta(w_0), \quad \alpha(s) = -\frac{1}{ab}\,\eta(s),$$

for any fixed $w_0 \in I$. Substituting these expressions into

$$\alpha^2(s) + \tau^2(s) - \eta^2(s) = \gamma, \quad \gamma \in \{-1, 0, 1\},$$

we obtain

$$\left(\frac{1}{a^2 b^2} - 1\right)\eta^2(s) = -b^2 + \gamma.$$

The left-hand side is constant only if

$$\frac{1}{a^2 b^2} = 1 \quad \text{and} \quad \gamma = b^2.$$

It follows that $b = \pm 1$, $a = 1$, and hence $\psi(s) \in S$.

Consequently

$$\eta(s) = -(s - w_0) + \eta(w_0), \quad \alpha(s) = \pm(s - w_0) \mp \eta(w_0),$$

and the solution takes the form

$$\psi(s) = (\pm(s - w_0) \mp \eta(w_0),\ \pm 1,\ -(s - w_0) + \eta(w_0)) \in S.$$

This solution represents a trivial trajectory of system (3.3).

Finally, by Corollary 3.2, the corresponding curve X satisfies

$$\kappa(s) = \frac{1}{a\tau(s)} = \pm 1,$$

which shows that X is a spacelike planar curve with a constant geodesic curvature. □

4.2. Solutions for a timelike or lightlike v

Lemma 4.2. *Let $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ be a solution of system* (3.3) *satisfying $\psi(s) \in H \cup C$, defined on its maximal interval $I = (w_-, w_+)$. Then when $\eta(s) \geq 1$ (resp. $\eta(s) > 0$), system (3.3) develops a singularity at a finite point w_+, while when $\eta(s) \leq -1$ (resp. $\eta(s) < 0$), system (3.3) develops a singularity at a finite point w_-. Moreover, the corresponding spacelike soliton curve $X\colon I \to dS_2$ to the ICF, whose geodesic curvature is given by*

$$\kappa(s) = \frac{1}{a\tau(s)}, \quad a > 0,$$

terminates at this singular point. The location of the singular endpoint is determined by the constant a and the initial value $\eta(w_0)$ for any fixed $w_0 \in I$.

Proof. By Corollary 3.2, $\psi(s)$ corresponds to a spacelike curve $X\colon I \to dS_2$ whose geodesic curvature is $\kappa(s) = \frac{1}{a\tau(s)}$, $a > 0$. Furthermore, by Proposition 3.1, this curve is a soliton solution to the ICF.

By the definition $\eta(s) = \langle N(s),\ e\rangle$ in (3.2), together with the Lorentzian inner product between the timelike vector field $N(s)$ and the timelike vector e (resp. the lightlike vector e), we obtain $|\eta(s)| \geq 1$ (resp. $\eta(s) \neq 0$) when $\psi(s) \in H$ (resp. $\psi(s) \in C$).

We first consider the case where $\eta(s) \geq 1$ (resp. $\eta(s) > 0$). Since $\eta'(s) = -\frac{1}{a}$, the function $\eta(s)$ is strictly decreasing. Hence, there is a finite limit

$$\lim_{s\to w_+} \eta(s) = \eta_+, \quad \text{with } \eta_+ \geq 1 \text{ (resp. 0)}.$$

From $\eta^2(s) = \alpha^2(s) + \tau^2(s) - \gamma$, it follows that both α and τ remain bounded as $s \to w_+$. Since α is monotonic (by $\alpha'(s) = \tau(s)$ and Remark 2.2), the limit α_+ exists and is finite. Consequently, the limit

$$\lim_{s\to w_+} \tau(s) = \tau_+$$

also exists.

Assume that $\tau_+ \neq 0$. Then $\psi(s)$ remains in a compact subset of the open region

$$\Omega = \{(\alpha, \tau, \eta) \in \mathbb{R}^3 : \tau \neq 0\}$$

near w_+. Since the vector field $\psi'(s)$ is smooth in Ω, the classical ODE extendibility theorem implies that $\psi(s)$ can be extended smoothly beyond w_+ if $w_+ < +\infty$, contradicting the maximality of I.

On the other hand, if $w_+ = +\infty$, then $\eta'(s) = -\frac{1}{a}$ yields

$$\lim_{s\to w_+} \eta(s) = -\infty,$$

which contradicts $\eta(s) \geq 1$ (resp. $\eta(s) > 0$). Therefore, in either case, we have

$$\tau_+ = 0. \tag{4.1}$$

Thus, system (3.3) develops a singularity at w_+, and $\alpha_+^2 - \eta_+^2 = \gamma$ holds. Moreover, the geodesic curvature of the corresponding curve X satisfies

$$\lim_{s\to w_+} \kappa(s) = +\infty \quad \left(\text{resp. } \lim_{s\to w_+} \kappa(s) = -\infty\right)$$

as $\tau(s) > 0$ (resp. $\tau(s) < 0$) for all $s \in I$. Since $\eta'(s) = -\frac{1}{a}$, the explicit formula

$$\eta(s) = -\frac{1}{a}(s - w_0) + \eta(w_0),$$

for any fixed $w_0 \in I$, implies that $w_+ \leq a(-1 + \eta(w_0)) + w_0$ (resp. $w_+ < a\eta(w_0) + w_0$).

An analogous argument applies, as the case $\eta(s) \leq -1$ (resp. $\eta(s) < 0$) shows that

$$\lim_{s\to w_-} \tau(s) = 0,$$

where $w_- \geq a(1+\eta(w_0)) + w_0$ (resp. $w_- > a\eta(w_0) + w_0$) for any fixed $w_0 \in I$. Moreover, the geodesic curvature of the corresponding curve X satisfies

$$\lim_{s\to w_-} \kappa(s) = +\infty \quad (\text{resp.} \quad \lim_{s\to w_-} \kappa(s) = -\infty)$$

as $\tau(s) > 0$ (resp. $\tau(s) < 0$) for all $s \in I$.

Thus, one end of the soliton curve X is a singular endpoint where the geodesic curvature blows up and the solution cannot be extended further. □

The following result describes the behavior at the other end of the solution, namely the end that is not singular, as characterized in Lemma 4.2.

Lemma 4.3. *Given a solution $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ of system* (3.3) *satisfying $\psi(s) \in H \cup C$ defined on the maximal interval $I = (w_-, w_+)$. Then, when $\eta(s) \geq 1$ (resp. $\eta(s) > 0$), we have $w_- = -\infty$, while when $\eta(s) \leq -1$ (resp. $\eta(s) < 0$), we have $w_+ = +\infty$. Moreover, the following hold.*

(i) If $\tau(s) > 0$ for all $s \in I$, then

$$\lim_{s\to-\infty} \alpha(s) = \lim_{s\to-\infty} -\eta(s) = -\infty, \ \lim_{s\to-\infty} \tau(s) = \frac{1}{a},$$

$$(\textit{resp.} \lim_{s\to+\infty} \alpha(s) = \lim_{s\to+\infty} -\eta(s) = +\infty, \ \lim_{s\to+\infty} \tau(s) = \frac{1}{a}).$$

(ii) If $\tau(s) < 0$ for all $s \in I$, then

$$\lim_{s\to-\infty} \alpha(s) = \lim_{s\to-\infty} -\eta(s) = +\infty, \ \lim_{s\to-\infty} \tau(s) = -\frac{1}{a},$$

$$(\textit{resp.} \lim_{s\to+\infty} \alpha(s) = \lim_{s\to+\infty} -\eta(s) = -\infty, \ \lim_{s\to+\infty} \tau(s) = -\frac{1}{a}).$$

Proof. Suppose that τ has a critical point $s_0 \in I$, so that $\tau'(s_0) = 0$. Differentiating τ' and using system (3.3), we obtain

$$\tau''(s_0) = \tau(s_0)\left(\frac{1}{a^2\tau^2(s_0)} - 1\right).$$

It follows that

(i) If $\tau(s) > 0$ for all $s \in I$, then s_0 is a local maximum (resp. minimum) point of τ if and only if $\tau(s_0) > \frac{1}{a}$ (resp. $\tau(s_0) < \frac{1}{a}$);

(ii) If $\tau(s) < 0$ for all $s \in I$, then s_0 is a local maximum (resp. minimum) point of τ if and only if $\tau(s_0) > -\frac{1}{a}$ (resp. $\tau(s_0) < -\frac{1}{a}$).

When $\tau(s) > 0$ on I, by the characterization of the critical points established above, any local maximum satisfies $\tau > \frac{1}{a}$ and any local minimum satisfies $\tau < \frac{1}{a}$.

Suppose that τ reaches two local maxima $s_1 < s_2$. Then, by the characterization of the critical points and the intermediate value property, there is a local minimum point s_0 so that $s_1 < s_0 < s_2$. Consequently, the points $b, d \in I$ exist such that $s_1 < b < s_0 < d < s_2$ and

$$\begin{aligned} \tau(b) &= \frac{1}{a},\ \tau'(b) = -(\eta+\alpha)(b) < 0, \\ \tau(d) &= \frac{1}{a},\ \tau'(d) = -(\eta+\alpha)(d) > 0, \end{aligned} \tag{4.2}$$

when substituting $\tau(b) = \tau(d) = \frac{1}{a}$ into $\tau'(s) = -\frac{\eta(s)}{a\tau(s)} - \alpha(s)$.

From $\eta^2(s) - \alpha^2(s) = \tau^2(s) - \gamma$, we obtain

$$(\eta^2 - \alpha^2)(b) = (\eta^2 - \alpha^2)(d) = \frac{1}{a^2} - \gamma > 0.$$

Hence, $|\eta(s)| > |\alpha(s)|$ at both points. Combining this with the fact that $\eta + \alpha$ has the opposite signs at b and d by (4.2), we conclude that

$$\eta(b) > -\alpha(b) > 0 \text{ and } \eta(d) < -\alpha(d) < 0. \tag{4.3}$$

First, consider the case where $\eta(s) \geq 1$ (resp. $\eta(s) > 0$). Condition (4.3) implies that a point b satisfying (4.2) may exist at most once, whereas a point d satisfying (4.2) cannot exist, since η is strictly decrease and does not change sign on I.

By Lemma 4.2, a singularity must occur at w_+.

At the opposite of I, there are two possibilities (determined by whether b exists or not).

Case 1: Assume that $\tau(s) \neq \frac{1}{a}$ for all $s \in I$. By the classical ODE extendibility theorem and the maximality of I, the solution of system (3.3) can be extended backward to $w_- = -\infty$. By the characterization of the critical points, we get

$$\tau(s) < \frac{1}{a} \text{ and } \tau'(s) < 0,\ s \in (-\infty, w_+).$$

It follows that

$$\lim_{s\to-\infty} \tau(s) = \frac{1}{a} \text{ and } \lim_{s\to-\infty} \tau'(s) = \lim_{s\to-\infty} -(\eta+\alpha)(s) = 0.$$

Since $\alpha'(s) = \tau(s) > 0$ and $\eta'(s) = -\frac{1}{a}$ for all $s \in I$, we get

$$\lim_{s\to-\infty} \alpha(s) = \lim_{s\to-\infty} -\eta(s) = -\infty.$$

Case 2: Assume that the unique point b exists. Then there is at most one maximum point of τ that may lie to the left of b. There are two sub–cases further (determined by whether the global maximum point exists or not).

Case 2.1: The unique maximum point s_1 exists. By the characterization of the critical points, we get

$$\frac{1}{a} < \tau(s) < \tau(s_1) \text{ and } \tau'(s) > 0,\ s \in (w_-, s_1), \tag{4.4}$$

then $\tau_- \in [\frac{1}{a}, \tau(s_1))$ exists such that

$$\lim_{s\to w_-} \tau(s) = \tau_- \text{ and } \lim_{s\to w_-} \tau'(s) = 0.$$

By the classical ODE extendibility and the maximality of I, we get $w_- = -\infty$. Since $\alpha'(s) = \tau(s) > 0$ and $\eta'(s) = -\frac{1}{a}$ for all $s \in I$, we get

$$\lim_{s\to -\infty} \alpha(s) = -\infty \text{ and } \lim_{s\to -\infty} \eta(s) = +\infty. \tag{4.5}$$

Using $\alpha^2(s) + \tau^2(s) - \eta^2(s) = \gamma$ and Eq (4.4), we compute

$$\lim_{s\to -\infty} \frac{\eta^2(s)}{\alpha^2(s)} = \lim_{s\to -\infty} \frac{\alpha^2(s) - \tau^2(s) + \gamma}{\alpha^2(s)} = 1.$$

Since both $\eta(s)$ and $-\alpha(s)$ diverge to $+\infty$, L'Hospital's rule applies, and then we obtain

$$1 = \lim_{s\to -\infty} -\frac{\eta(s)}{\alpha(s)} = \lim_{s\to -\infty} -\frac{-\frac{1}{a}}{\tau(s)},$$

which implies

$$\tau_- = \frac{1}{a}.$$

Substituting it into $\tau'(s)$, we also have

$$\lim_{s\to -\infty} \alpha(s) = \lim_{s\to -\infty} -\eta(s) = -\infty.$$

Case 2.2: τ is decreasing on I.

Assume that τ admits a finite limit $\tau_- > \frac{1}{a}$ as $s \to w_-$. By the classical ODE extendibility and the maximality of I, we get $w_- = -\infty$. It follows that

$$\frac{1}{a} = \tau(b) < \tau(s) < \tau_- \text{ and } \tau'(s) < 0, \ s \in (-\infty, b). \tag{4.6}$$

Since $\alpha'(s) = \tau(s) > 0$ and $\eta'(s) = -\frac{1}{a} < 0$, it follows that

$$\lim_{s\to -\infty} \alpha(s) = -\infty, \quad \lim_{s\to -\infty} \eta(s) = +\infty.$$

From (4.6) and because both $\eta(s)$ and $-\alpha(s)$ diverge to $+\infty$, L'Hospital's rule applies, yielding

$$\tau_- = \frac{1}{a},$$

which contradicts (4.6). Therefore, we must have

$$\lim_{s\to -\infty} \tau(s) = +\infty.$$

Moreover, from

$$\tau'(s) = -\frac{\eta(s)}{a\tau(s)} - \alpha(s) < 0,$$

we deduce that

$$-\frac{\eta(s)}{a\tau(s)} < \alpha(s) < 0$$

for a sufficiently negative s. If we substitute $\eta^2(s) = \alpha^2(s) + \tau^2(s) - \gamma$, this implies

$$\alpha^2(s) < \frac{\eta^2(s)}{a^2\tau^2(s)} = \frac{\alpha^2(s) + \tau^2(s) - \gamma}{a^2\tau^2(s)}.$$

Rearranging yields

$$a^2\tau^2(s)\alpha^2(s) < \alpha^2(s) + \tau^2(s) - \gamma,$$

which is equivalent to

$$1 < \frac{1}{a^2\tau^2(s)} + \frac{1}{a^2\alpha^2(s)} + \frac{-\gamma}{a^2\tau^2(s)\alpha^2(s)}.$$

Letting $s \to -\infty$, the left-hand side equals 1, while the right-hand side tends to 0, which yields a contradiction.

The only remaining case is that τ diverges to infinity with w_- being finite. From $\eta^2(s) = \alpha^2(s) + \tau^2(s) - \gamma$ and $\eta'(s) = -\frac{1}{a}$, we obtain

$$\lim_{s \to w_-} \eta(s) = +\infty.$$

Since $\eta(s) = -\frac{1}{a}(s - w_0) + \eta(w_0)$ for any fixed $w_0 \in I$, this forces $w_- = -\infty$, a contradiction.

In conclusion, Case 2.2 cannot occur.

For the other case where $\eta(s) \leq -1$ (resp. $\eta(s) < 0$) on I, we proceed analogously by considering two cases, depending on the existence of the point d, as in the case $\eta(s) \geq 1$ (resp. $\eta(s) > 0$) on I.

An analogous argument applies to the case $\tau(s) < 0$, which completes the proof of the stated properties. □

Corollary 4.4. *As a direct consequence of Lemmas 4.2 and 4.3, the function τ must vanish exactly once on I. Hence, system (3.3) admits a unique singular point.*

Example 4.5. Choose a lightlike vector $\mathbf{v} = 2(-1, 1, 0)$ and consider the case where τ is strictly decreasing on I, corresponding to Case 1 in the proof of Lemma 4.3 (see Figure 1).

- Left: The functions $\alpha(s)$, $\tau(s)$, and $\eta(s)$ associated with the ODE system (3.3). Here, since $\eta(s) > 0$, $\tau(s)$ develops a singularity at w_+ and $\tau(s) \to \frac{1}{a} = \frac{1}{2}$ along the other end, as described in Lemmas 4.2 and 4.3.
- Right: The corresponding ICF soliton X on the de Sitter surface and its evolution under the ICF over a short time interval. The curve develops a singular endpoint at w_+ and tends to infinity along the other side.

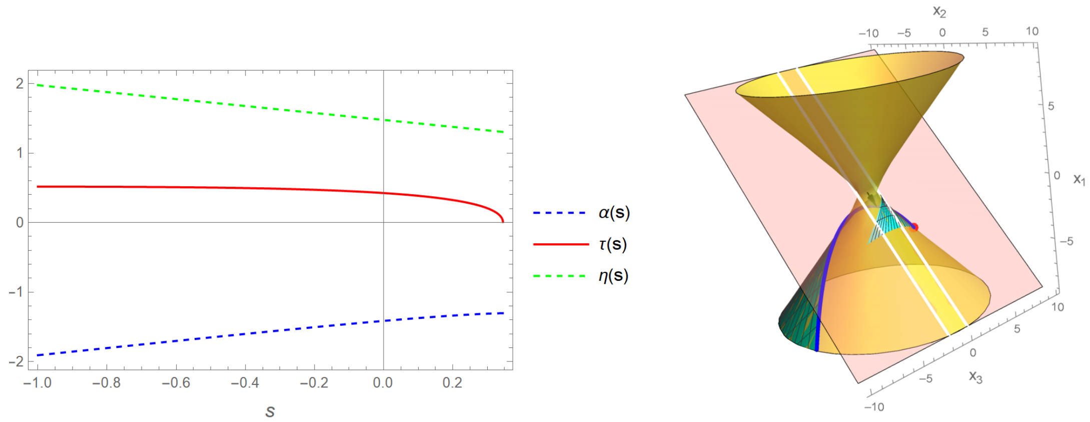


Figure 1. X (blue), the singular boundary point (red), and the ICF (cyan).

Example 4.6. Choose a timelike vector $\mathbf{v} = 8(-1, 0, 0)$ and consider the case where τ attains a global maximum, corresponding to Case 2.1 in the proof of Lemma 4.3 (see Figure 2).

- Left: The functions $\alpha(s)$, $\tau(s)$, and $\eta(s)$ associated with ODE system (3.3). Here, since $\eta(s) < -1$, and $\tau(s)$ develops a singularity at w_- and $\tau(s) \to \frac{1}{a} = \frac{1}{8}$ along the other end, as described in Lemmas 4.2 and 4.3.
- Right: The corresponding ICF soliton X on the de Sitter surface and its evolution under the ICF over a short time interval. The curve develops a singular endpoint at w_- and tends to infinity along the other side.

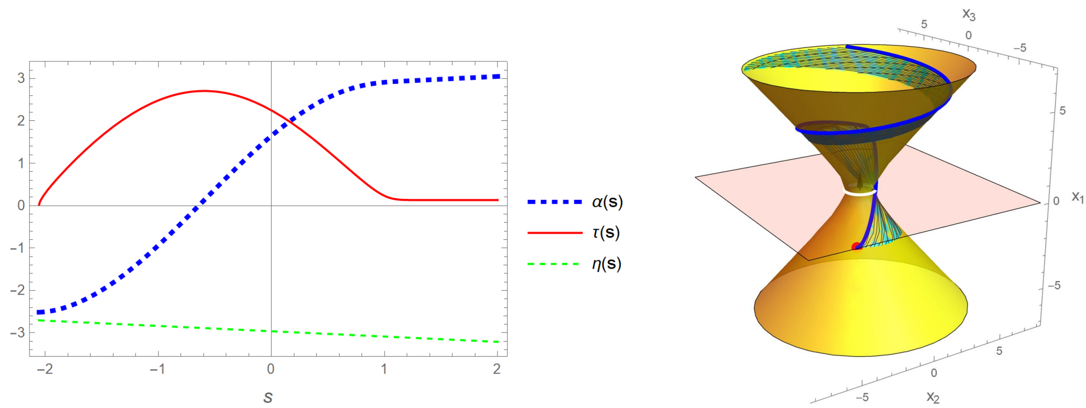


Figure 2. X (blue), the singular boundary point (red), and the ICF (cyan).

4.3. Solutions for a spacelike $\mathbf{v}$

Lemma 4.7. *Let $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ be a nontrivial solution of system (3.3) satisfying $\psi(s) \in S$ defined on the maximal interval $I = (w_-, w_+)$. Then either τ develops a singularity at exactly one finite endpoint of I, or τ converges to $\frac{1}{a}$ along every nonsingular end of I.*

Proof. We assume that $\tau(s) > 0$ on I. The case where $\tau(s) < 0$ is treated analogously.

Step 1: Classification of critical points of τ.

Let s_0 be a critical point of τ. We then have

$$\tau''(s_0) = \tau(s_0)\left(\frac{1}{a^2\tau^2(s_0)} - 1\right)$$

by considering system (3.3). Hence, we have the following

- s_0 is a local maximum point of τ if and only if $\tau(s_0) > \frac{1}{a}$;
- s_0 is a local minimum point of τ if and only if $\tau(s_0) < \frac{1}{a}$.

Step 2: τ admits at most three local critical points.

Suppose, by contradiction, that τ has four critical points $s_1 < s_2 < s_3 < s_4$, where s_1, s_3 are local maxima and s_2, s_4 are local minima. By continuity, the points $b, d, e \in I$ exist such that $s_1 < b < s_2 < d < s_3 < e < s_4$, and

$$\tau(b) = \tau(d) = \tau(e) = \frac{1}{a},\ \tau'(b) < 0,\ \tau'(d) > 0,\ \tau'(e) < 0. \tag{4.7}$$

At these points, substituting $\tau = \frac{1}{a}$ into $\tau' = -\frac{\eta}{a\tau} - \alpha$ yields $\tau' = -(\eta + \alpha)$.

From $\eta^2(s) - \alpha^2(s) = \tau^2(s) - 1$, it follows that

$$(\eta^2 - \alpha^2)(b) = (\eta^2 - \alpha^2)(d) = (\eta^2 - \alpha^2)(e) = \frac{1}{a^2} - 1. \tag{4.8}$$

At Points b, d and e, we distinguish three cases based on the sign of (4.8).

Case 1: If $a < 1$, then $|\eta| > |\alpha|$. Combined with (4.7), this implies

$$\eta(b) > 0,\ \eta(d) < 0,\ \eta(e) > 0. \tag{4.9}$$

Since $\eta'(s) = -\frac{1}{a} < 0$, η cannot change sign twice. Therefore, b, d, e cannot coexist, which contradicts the assumption that four critical points exist. Consequently, τ admits at most three critical points in this case.

Case 2: If $a > 1$, then $|\eta| < |\alpha|$. By combining this with (4.7), we obtain

$$\alpha(b) > 0,\ \alpha(d) < 0,\ \alpha(e) > 0.$$

Since $\alpha' = \tau > 0$, α cannot change sign twice. This again yields a contradiction, showing that b, d, e cannot coexist. Therefore, τ admits at most three critical points.

Case 3: If $a = 1$, then $|\eta| = |\alpha|$. Combined with (4.7), this yields $\eta = \alpha$ at b, d, e. Since α is strictly increasing and η is strictly decreasing on I, at most one point at which $\eta = \alpha$ can exist. Hence, τ admits at most two critical points.

Step 3: The case where τ admits three critical points.

Assume that $a < 1$ (resp. $a > 1$) and that τ admits exactly two local maximum (resp. minimum) points $s_1 < s_3$ (resp. $s_2 < s_4$) on I. By the analysis in Step 2, τ is strictly increasing (resp. decreasing) on (w_-, s_1) (resp. on (w_-, s_2)) and strictly decreasing (resp. increasing) on (s_3, w_+) (resp. on (s_4, w_+)). Moreover

$$\begin{aligned} &\frac{1}{a} < \tau(s) < \tau(s_1),\ s \in (w_-, s_1), \quad \frac{1}{a} < \tau(s) < \tau(s_3),\ s \in (s_3, w_+); \\ &(\text{resp. } \tau(s_2) < \tau(s) < \frac{1}{a},\ s \in (w_-, s_2), \quad \tau(s_4) < \tau(s) < \frac{1}{a},\ s \in (s_4, w_+)). \end{aligned} \tag{4.10}$$

It follows that the limits

$$\tau_- := \lim_{s\to w_-} \tau(s) \in [\frac{1}{a}, \tau(s_1)) \ (\text{resp.} \tau_- \in (\tau(s_2), \frac{1}{a}])$$

and

$$\tau_+ := \lim_{s\to w_+} \tau(s) \in [\frac{1}{a}, \tau(s_3)) \ (\text{resp.} \tau_+ \in (\tau(s_4), \frac{1}{a}]),$$

exist. Since τ is monotone and convergent near the endpoints, then $\lim_{s\to w_\pm} \tau'(s) = 0$.

Because the right-hand side of system (3.3) is smooth and all components of $\psi(s)$ remain bounded on any finite sub–interval of I, the ODE extendibility theorem implies that $I = \mathbb{R}$. Moreover, since $\alpha' = \tau > 0$ and $\eta' = -\frac{1}{a} < 0$, we obtain

$$\lim_{s\to\pm\infty} \alpha(s) = \pm\infty, \ \lim_{s\to\pm\infty} \eta(s) = \mp\infty.$$

Using $\alpha^2(s) + \tau^2(s) - \eta^2(s) = 1$ together with (4.10), we compute

$$\lim_{s\to\pm\infty} \frac{\eta^2(s)}{\alpha^2(s)} = \lim_{s\to\pm\infty} \frac{\alpha^2(s) + \tau^2(s) - 1}{\alpha^2(s)} = 1.$$

Since both $\eta(s)$ and $-\alpha(s)$ diverge to $-\infty$ (resp. $+\infty$) as $s \to +\infty$ (resp. $s \to -\infty$). L'Hospital's rule applies, yielding

$$1 = \lim_{s\to\pm\infty} -\frac{\eta(s)}{\alpha(s)} = \lim_{s\to\pm\infty} -\frac{-\frac{1}{a}}{\tau(s)}.$$

Consequently

$$\tau_- = \tau_+ = \frac{1}{a}.$$

Substituting this into the equation for $\tau'(s)$, we also obtain

$$\lim_{s\to\pm\infty} \alpha(s) = \lim_{s\to\pm\infty} -\eta(s) = \pm\infty.$$

Example 4.8. Choose a spacelike vector $\mathbf{v} = 0.5(0, 0, 1)$, so that $a = 0.5 < 1$ (see Figure 3).

- Left: The function $\tau(s)$ associated with the ODE system (3.3). Here, $\tau \to \frac{1}{a} = 2$ along both ends, corresponding to Step 3 in the proof of Lemma 4.7.
- Right: The corresponding complete ICF soliton X on the de Sitter surface and its evolution under the ICF over a short time interval.

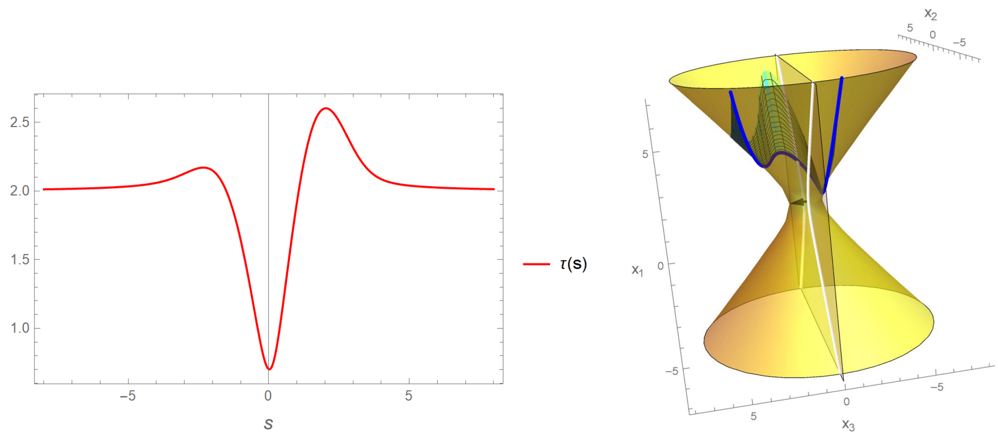


Figure 3. Complete soliton curve X (blue) and the ICF (cyan).

Step 4: The case where τ admits exactly two critical points.

Let s_1 be a local maximum point and s_2 be a local minimum point with $s_1 < s_2$. Then, by continuity, a point $b \in (s_1, s_2)$ exists such that

$$\tau(b) = \frac{1}{a} \text{ and } \tau'(b) < 0.$$

The case where $s_2 < s_3$ with s_2 being a local minimum point and s_3 being a local maximum point is treated analogously.

According to Step 2, three cases may occur.

Case 1: If $a < 1$.

We show that the situation where τ admits exactly two critical points cannot occur by excluding all possible end behaviors of τ.

Step 4.1: Exclusion of the finite–endpoint singularity at w_-.

Suppose, for contradiction, that τ develops a singularity at w_-. From

$$\tau'(s) = -\frac{\eta(s)}{a\tau(s)} - \alpha(s) > 0 \text{ as } s \to w_-,$$

it follows that $\eta(s) < 0$ as $s \to w_-$. Since $\eta' = -\frac{1}{a} < 0$ on I, this implies

$$\eta(s) < 0 \quad \text{for all } s \in I.$$

This contradicts the fact that $\eta(b) > 0$ by (4.9). Hence, τ cannot develop a singularity at w_-.

Step 4.2: Exclusion of divergence at w_+.

Suppose, for contradiction, that

$$\lim_{s \to w_+} \tau(s) = +\infty.$$

Using $\eta^2(s) = \alpha^2(s) + \tau^2(s) - 1$ and $\eta'(s) = -\frac{1}{a} < 0$, we obtain

$$\lim_{s \to w_+} \eta(s) = -\infty,$$

which forces $w_+ = +\infty$ due to $\eta(s) = -\frac{1}{a}(s - w_0) + \eta(w_0)$ for any fixed $w_0 \in I$. Moreover, since $\alpha'(s) = \tau(s) > 0$, it follows that

$$\lim_{s \to +\infty} \alpha(s) = +\infty.$$

For a sufficiently large s, we get $\tau'(s) = -\frac{\eta(s)}{a\tau(s)} - \alpha(s) > 0$. Combining this with $\alpha^2(s)+\tau^2(s)-\eta^2(s) = 1$ yields an inequality of the form

$$1 < \frac{1}{a^2\tau^2(s)} + \frac{1}{a^2\alpha^2(s)} - \frac{1}{a^2\tau^2(s)\alpha^2(s)} \to 0 \text{ as } s \to +\infty,$$

which is impossible. Therefore, τ cannot diverge at w_+.

Step 4.3: Exclusion of convergence at two ends.

The only remaining possibility is that

$$\lim_{s \to \pm\infty} \alpha(s) = \lim_{s \to \pm\infty} -\eta(s) = \pm\infty, \quad \lim_{s \to \pm\infty} \tau(s) = \frac{1}{a}.$$

Since $(\eta - \alpha)'(s) = -\frac{1}{a} - \tau(s) < 0$ on I, we obtain

$$\lim_{s \to \mp\infty} (\eta - \alpha)(s) = \pm\infty, \tag{4.11}$$

while

$$(\eta^2 - \alpha^2)(s) = \tau^2(s) - 1 \to \frac{1}{a^2} - 1 > 0 \text{ as } s \to \pm\infty \tag{4.12}$$

implies

$$\lim_{s \to \mp\infty} (\eta + \alpha)(s) = 0.$$

On the other hand, due to $\tau(s) > \frac{1}{a}$ on $(-\infty, s_1)$ and $\tau(s) < \frac{1}{a}$ on $(s_2, +\infty)$ by the characterization of the critical points in Step 1, we derive

$$(\eta + \alpha)'(s) > 0 \text{ on } (-\infty, s_1) \quad \text{and} \quad (\eta + \alpha)'(s) < 0 \text{ on } (s_2, +\infty),$$

which implies $\eta + \alpha > 0$ on $(-\infty, s_1) \cup (s_2, +\infty)$. This contradicts the sign information implied by (4.11) and (4.12).

Consequently, when $a < 1$, the case where τ admits exactly two critical points cannot occur.

Case 2: If $a > 1$.

In contrast to Case 1, there is no restriction on the sign of $\eta(b)$. As a result, the argument used in Step 4.1 of Case 1 to exclude boundary blow-up no longer applies. However, the reasoning used in Steps 4.2 and 4.3 of Case 1 remains valid in the present setting. Hence, τ must develop a singular endpoint at w_-, while the asymptotic behavior occurs at the opposite endpoint.

Example 4.9. Choose a spacelike vector $\mathbf{v} = 2(0, 0, 1)$, so that $a = 2 > 1$ (see Figure 4).

- Left: The function $\tau(s)$ associated with the ODE system (3.3). Here, $\tau(s)$ develops a singularity at w_- and $\tau(s) \to \frac{1}{a} = \frac{1}{2}$ along the other end, corresponding to Case 2 of Step 4 in the proof of Lemma 4.7.

- Right: The corresponding ICF soliton X on the de Sitter surface and its evolution under the ICF over a short time interval. The curve develops a singular endpoint at w_- and tends to infinity on the other side.

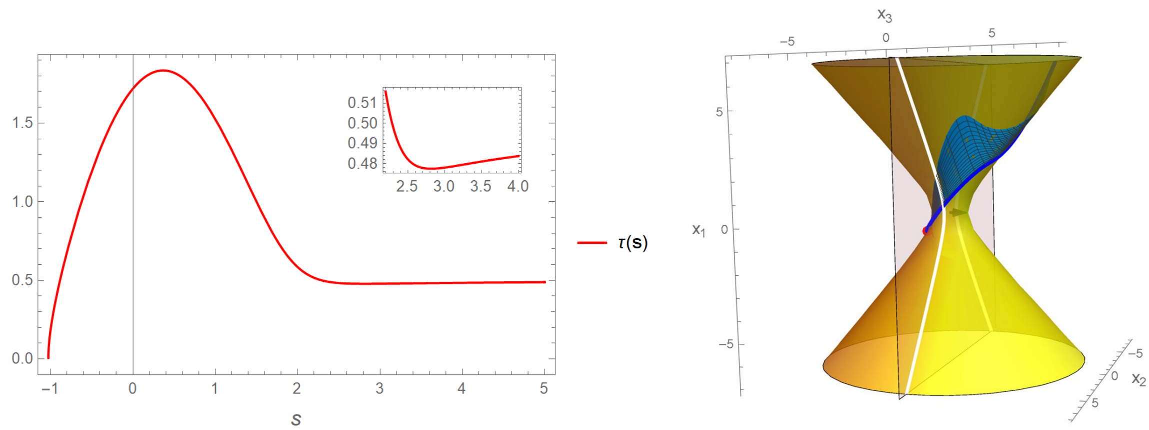


Figure 4. X (blue), the singular boundary point (red), and the ICF (cyan).

Case 3: If $a = 1$.

From (4.7) and (4.8), we obtain $\eta(b) = \alpha(b)$. Moreover,

$$\tau'(b) = -2\eta(b) < 0,$$

which implies $\eta(b) > 0$. Therefore, the arguments used in Step 4.1 of Case 1 to rule out the formation of a singularity at w_- and in Step 4.2 of Case 1 to exclude global divergence at w_+ apply verbatim.

In contrast to Cases 1 and 2, the relation

$$(\eta^2 - \alpha^2)(s) = \tau^2(s) - 1 \to \frac{1}{a^2} - 1 = 0 \text{ as } s \to \pm\infty$$

without leading to a contradiction.

Consequently, asymptotic behavior occurs at both ends.

Example 4.10. Choose a spacelike vector $\mathbf{v} = (0, 0, 1)$, so that $a = 1$ (see Figure 5).

- Left: The function $\tau(s)$ associated with the ODE system (3.3). Here, $\tau \to \frac{1}{a} = 1$ along both ends, corresponding to Case 3 of Step 4 in the proof of Lemma 4.7.
- Right: The corresponding complete ICF soliton X on the de Sitter surface and its evolution under the ICF over a short time interval.

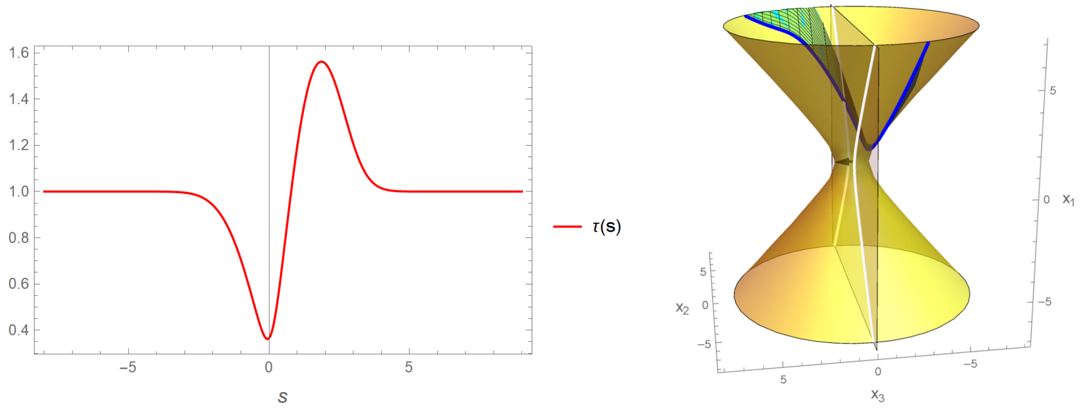


Figure 5. Complete soliton curve X (blue) and the ICF (cyan).

Step 5: The case where τ admits exactly one critical point.

Suppose first that τ admits a unique critical point s_0, which is a maximum.

Step 5.1: Exclusion of singularities at both endpoints.

Suppose that the values of τ are singular at both endpoints w_- and w_+. Using

$$\tau'(s) = -\frac{\eta(s)}{a\tau(s)} - \alpha(s) > 0 \text{ near } w_- \quad\text{and}\quad \tau'(s) < 0 \text{ near } w_+,$$

together with the fact that η is strictly decreasing, we can deduce that

$$\eta(s) < 0 \text{ and } \eta(s) > 0 \text{ for all } s \in I$$

occurs at the same time, which is impossible.

Step 5.2: Exclusion of the case with no singularity.

Suppose that τ does not develop any singularity at either endpoint. Since the vector field in system (3.3) is smooth, the solution extends to the whole real line, $I = \mathbb{R}$. Moreover, as $\alpha' = \tau > 0$ and $\eta' = -\frac{1}{a} < 0$, we have

$$\lim_{s\to\pm\infty} \alpha(s) = \lim_{s\to\pm\infty} -\eta(s) = \pm\infty, \ \lim_{s\to\pm\infty} \tau(s) = \frac{1}{a},$$

and the function $\eta + \alpha$ must vanish twice.

On the other hand, since $\tau(s) > \frac{1}{a}$ for all $s \in I$, we obtain

$$(\eta + \alpha)'(s) = \tau - \frac{1}{a} > 0,$$

which implies that $\eta + \alpha$ can vanish at most once.

As a conclusion of Steps 5.1 and 5.2, τ must develop a singular endpoint and the asymptotic behavior occurs at the other end.

Example 4.11. Choose a spacelike vector $\mathbf{v} = 0.2(0, 0, 1)$, so that $a = 0.2$ (see Figure 6).

- Left: The function $\tau(s)$ associated with the ODE system (3.3). Here, $\tau(s)$ develops a singularity at one endpoint and $\tau(s) \to \frac{1}{a} = 5$ along the other end, corresponding to Step 5.2 in the proof of Lemma 4.7.
- Right: The corresponding ICF soliton X on the de Sitter surface and its evolution under the ICF over a short time interval. The curve develops a singular endpoint on one side and tends to infinity along the other side.

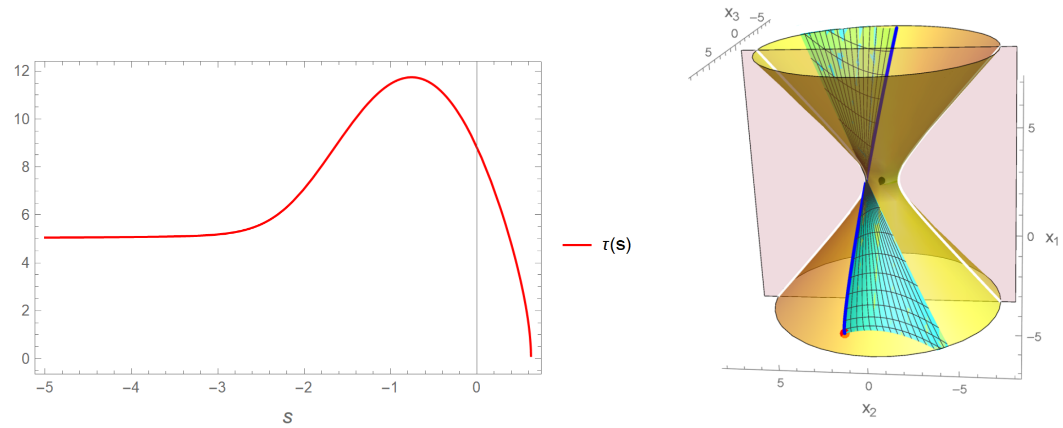


Figure 6. X (blue), the singular boundary point (red), and the ICF (cyan).

Step 5.3: Exclusion of the case where the unique critical point is minimum.

Suppose that the unique critical point of τ is minimum.

If τ diverges at either endpoint, an argument analogous to Step 4.2 yields a contradiction. If τ converges to $\frac{1}{a}$ at both ends, a similar argument to that in Step 5.2 again yields a contradiction.

Hence, τ cannot admit a global minimum as its unique critical point.

Step 6: The case where τ has no critical point.

In this case, τ is monotone on I.

If τ diverges at the end away from 0, an argument analogous to Step 4.2 yields a contradiction. Therefore, at this end, τ must exhibit asymptotic behavior.

On the other hand, since τ is monotone and cannot converge to $\frac{1}{a}$ at both ends without admitting a critical point, it follows that τ must develop a singularity.

Example 4.12. Choose a spacelike vector $\mathbf{v} = 3(0, 0, 1)$, so that $a = 3$ (see Figure 7).

- Left: The function $\tau(s)$ associated with the ODE system (3.3). Here, $\tau(s)$ develops a singularity at one endpoint and $\tau(s) \to \frac{1}{a} = \frac{1}{3}$ along the other end, corresponding to Step 6 in the proof of Lemma 4.7.
- Right: The corresponding ICF soliton X on the de Sitter surface and its evolution under the ICF over a short time interval. The curve develops a singular endpoint on one side and tends to infinity along the other side.

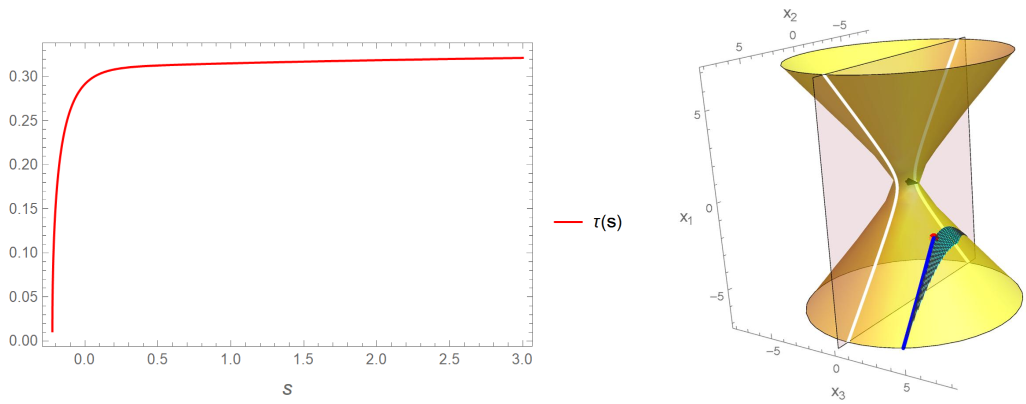


Figure 7. X (blue), the singular boundary point (red), and the ICF (cyan).

□

As an immediate consequence of Lemma 4.7, we obtain the following corollary.

Corollary 4.13. *Let $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ be a nontrivial solution of system* (3.3) *satisfying $\psi(s) \in S$ on its maximal interval I.*

(i) If $\tau(s) > 0$ for all $s \in I$, then along every nonsingular end of I, we have

$$\lim_{s\to\pm\infty} \alpha(s) = \lim_{s\to\pm\infty} -\eta(s) = \pm\infty, \ \lim_{s\to\pm\infty} \tau(s) = \frac{1}{a}.$$

(ii) If $\tau(s) < 0$ for all $s \in I$, along every nonsingular end of I, we have

$$\lim_{s\to\pm\infty} \alpha(s) = \lim_{s\to\pm\infty} -\eta(s) = \mp\infty, \ \lim_{s\to\pm\infty} \tau(s) = -\frac{1}{a}.$$

The solution may terminate at one finite endpoint if τ develops a singularity there.

The qualitative behavior of nontrivial spacelike soliton solutions in dS_2 are described as follows.

Theorem 4.14. *Let $\mathbf{v} \in \mathbb{R}^3_1$ be an arbitrary nonzero vector, and let $X\colon I \to dS_2$ be a nontrivial spacelike soliton curve of the ICF on the de Sitter surface, satisfying the soliton condition*

$$\kappa^{-1}(s) = \langle T(s), \mathbf{v}\rangle,$$

where $T(s)$ is the unit tangent vector field of X and $\kappa(s)$ denotes its geodesic curvature. Then the qualitative behavior of X depends on the causal character of $\mathbf{v}$.

(1) If $\mathbf{v}$ is timelike or lightlike, then X encounters a singularity at one end of I, where $\kappa(s)$ blows up. At the opposite end, X escapes to infinity with respect to the direction $\mathbf{v}$, and $\kappa(s)$ tends to ± 1.

(2) If $\mathbf{v}$ is spacelike, then one of the following occurs.

*(a) **Complete case.** The curve X is complete, namely $I = \mathbb{R}$. Both ends of X diverge to infinity in opposite directions with respect to $\mathbf{v}$, and $\kappa(s)$ tends to ± 1.*

(b) ***Incomplete case.*** *X has a finite singular endpoint at one end of I, where $\kappa(s)$ blows up, while the other end of X escapes to infinity with respect to* $\mathbf{v}$*, and $\kappa(s)$ tends to ± 1.*

Moreover, in the timelike or lightlike case, the maximal interval I is determined by $\mathbf{v}$ *and the initial normal vector of X. In the spacelike case, however, no such determination of I is possible.*

Proof of Theorem 4.14. Let $\mathbf{v} \in \mathbb{R}^3_1 \setminus \{0\}$ be given. Up to isometries of the de Sitter surface dS_2, $\mathbf{v}$ is expressed in the form of (3.1). By Theorem 2.3, a spacelike soliton solution $X\colon I \to dS_2$ of the ICF exists, whose curvature is given by

$$\kappa(s) = \frac{1}{a\,\tau(s)},\ a > 0,$$

and which corresponds to a nontrivial solution $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ of system (3.3) in the sense of Proposition 3.1.

(1) If $\mathbf{v}$ is timelike or lightlike, ($\psi(s) \in H \cup C$).

Lemma 4.2 implies that τ necessarily develops a singularity at one finite endpoint of I; that is, $\tau(s) \to 0$ as s approaches this endpoint. Consequently, the geodesic curvature blows up there, and the soliton curve X terminates.

At the opposite end of I, Lemma 4.3 shows that

$$|\alpha(s)| \to +\infty,\ |\tau(s)| \to \frac{1}{a}$$

as $s \to +\infty$ or $s \to -\infty$. Geometrically, this means that X escapes to infinity with respect to the direction $\mathbf{v}$, while the geodesic curvature satisfies $\kappa(s) \to \pm 1$.

Moreover, the boundedness of w_+ (resp. w_-) in Lemma 4.2, combined with the unbounded result of w_- (resp. w_+) in Lemma 4.3, demonstrates that the maximal interval I is determined by the values of a and the initial normal vector of X.

(2) If $\mathbf{v}$ is spacelike, ($\psi(s) \in S$).

By Lemma 4.7, the function τ admits exactly two possible behaviors along each endpoint of the maximal interval I.

- *Singular case.* τ develops a singularity at an endpoint, namely

$$\lim_{s \to w_-} \tau(s) = 0 \quad (\text{resp. } \lim_{s \to w_+} \tau(s) = 0).$$

  Since $\kappa(s) = \frac{1}{a\tau(s)}$, it follows immediately that

$$\lim_{s \to w_-} |\kappa(s)| = +\infty \quad (\text{resp. } \lim_{s \to w_+} |\kappa(s)| = +\infty).$$

  Moreover, using $\alpha^2(s) + \tau^2(s) - \eta^2(s) = 1$, we obtain

$$\lim_{s \to w_-} (\alpha^2 - \eta^2)(s) = 1 \quad (\text{resp. } \lim_{s \to w_+} (\alpha^2 - \eta^2)(s) = 1).$$

  Since this identity does not impose any restriction on $\eta(s)$. Consequently, the location of the singular endpoint of I cannot be determined.

- *Regular case.* If $\tau(s) \to \pm\frac{1}{a}$ along a nonsingular end of I, then Corollary 4.13 implies that X escapes to infinity with respect to the direction $\mathbf{v}$, while the geodesic curvature satisfies $\kappa(s) \to \pm 1$.

Depending on whether the regular behavior occurs at one or both ends, this corresponds precisely to the complete and incomplete cases described in the theorem. □

5. Timelike soliton solutions to the ICF in dS_2

In this section, we consider the qualitative behavior of timelike soliton solutions to the ICF in dS_2 by analyzing the solutions of system (3.5).

Lemma 5.1. *Let $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ be a solution of system* (3.5) *satisfying $\psi(s) \in H \cup C \cup S$. Then no trivial solution of system* (3.5) *exists for which $\tau(s)$ (and hence the geodesic curvature $\kappa(s)$) is a nonzero constant.*

Proof. Suppose, by contradiction, that $\tau(s) \equiv b \neq 0$ is a constant. Then system (3.5) yields

$$\tau'(s) = \frac{1}{ab}\eta(s) + \alpha(s) = 0,\ \alpha'(s) = b,\ \eta'(s) = \frac{1}{a},$$

and hence

$$\eta(s) = \frac{1}{a}(s - w_0) + \eta(w_0),\ \alpha(s) = -\frac{1}{ab}\eta(s),$$

for any fixed $w_0 \in I$. Using $\alpha^2(s) - \tau^2(s) + \eta^2(s) = \gamma \in \{-1, 0, 1\}$, we obtain

$$\left(\frac{1}{a^2b^2} + 1\right)\eta^2(s) = b^2 + \gamma,$$

which implies that $\eta(s)$ is a constant function. This contradicts $\eta'(s) = \frac{1}{a} \neq 0$. Therefore, no such trivial solution exists. □

Lemma 5.2. *Let $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ be a solution of system* (3.5) *satisfying $\psi(s) \in H \cup C \cup S$, defined on its maximal interval $I = (w_-, w_+)$. If $s_0 \in I$ is a critical point of τ, then the sign of $\tau(s)$ determines its nature: s_0 is a global minimum of τ if $\tau(s) > 0$ and a global maximum if $\tau(s) < 0$.*

Proof. Assume that $s_0 \in I$ satisfies $\tau'(s_0) = 0$. Differentiating the second equation in system (3.5) and evaluating at $s = s_0$, we obtain

$$\tau''(s_0) = \tau(s_0)\left(\frac{1}{a^2\tau^2(s_0)} + 1\right).$$

It follows that the sign of $\tau''(s_0)$ is completely determined by the sign of $\tau(s_0)$. Consequently, s_0 is a local minimum of τ when $\tau(s_0) > 0$, and a local maximum when $\tau(s_0) < 0$.

Moreover, by Remark 3.3, τ does not change sign on I. Therefore, all critical points of τ, if any, must be of the same type. By the continuity of τ, any critical point of τ is necessarily global, which completes the proof. □

The qualitative behavior of timelike soliton solutions to the ICF in dS_2 is characterized as follows.

Theorem 5.3. *Let* $\mathbf{v} \in \mathbb{R}_1^3$ *be an arbitrary nonzero vector, and let* $X\colon I \to dS_2$ *be a timelike soliton curve of the ICF on the de Sitter surface, satisfying the soliton condition*

$$\kappa^{-1}(s) = \langle T(s), \mathbf{v} \rangle,$$

where $T(s)$ *denotes the tangent vector field of* X *and* $\kappa(s)$ *denotes its geodesic curvature. Then one of the following situations occurs.*

(a) ***Complete case.*** *The curve* X *is complete, namely* $I = \mathbb{R}$*. Both ends of* X *escape to infinity in opposite directions with respect to* $\mathbf{v}$*, and the geodesic curvature tend to zero.*

(b) ***Incomplete case.*** *The curve* X *has a finite endpoint at one end of* I*, while the other end escapes to infinity with respect to* $\mathbf{v}$ *and has a vanishing geodesic curvature. The geometry of the finite endpoint depends on the causal character of* $\mathbf{v}$*.*

   (1) If $\mathbf{v}$ *is timelike,* X *extends to a regular endpoint lying on the spacelike geodesic of* dS_2*, and* $\kappa(s)$ *converges to the finite value* $\pm\frac{1}{|\mathbf{v}|}$*.*

   (2) If $\mathbf{v}$ *is lightlike,* X *terminates at a singular endpoint lying on the lightlike geodesic of* dS_2*, where* $\kappa(s)$ *blows up.*

   (3) If $\mathbf{v}$ *is spacelike,* X *ends at a singular point whose height with respect to* $\mathbf{v}$ *takes values in* $[-1, 1]$*, and* $\kappa(s)$ *blows up.*

Moreover, in the incomplete case, the maximal interval of existence I *is determined by a constant depending on* $\mathbf{v}$ *and the initial normal vector of* X*.*

Proof of Theorem 5.3. Let $\mathbf{v} \in \mathbb{R}_1^3 \setminus \{0\}$ be given. Up to isometries of the de Sitter surface dS_2, $\mathbf{v}$ is expressed in the form of (3.1). By Theorem 2.3, there is a timelike soliton solution $X\colon I \to dS_2$ of the ICF, whose geodesic curvature satisfies

$$\kappa(s) = \frac{1}{a\,\tau(s)}, \; a > 0,$$

and which corresponds, via Proposition 3.1, to a solution $\psi(s) = (\alpha(s), \tau(s), \eta(s))$ of system (3.5).

Define $\gamma := \langle \psi(s), \psi(s) \rangle$. By (3.6), we have $\gamma = -1$ when $\psi(s) \in H$; $\gamma = 0$ when $\psi(s) \in C$, and $\gamma = 1$ when $\psi(s) \in S$. From Remark 3.3, τ does not change sign on I. Without loss of generality, we assume that $\tau(s) > 0$ for all $s \in I$. The case $\tau(s) < 0$ is being analogous.

According to Lemma 5.2, exactly one of the following three situations occurs.

Case 1: τ admits a global minimum at some $s_0 \in I$.

Since τ stays uniformly away from zero, the solution remains in a compact subset of the open region

$$\Omega = \{(\alpha, \tau, \eta) : \tau \neq 0\},$$

on which the vector field of system (3.5) is smooth. Hence, by the ODE extendibility theorem, the maximal interval of existence is $I = \mathbb{R}$.

Moreover, from $\alpha' = \tau > 0$ and $\eta' = \frac{1}{a} > 0$, we obtain

$$\lim_{s\to\pm\infty} \alpha(s) = \pm\infty, \; \lim_{s\to\pm\infty} \eta(s) = \pm\infty.$$

Using $\tau^2(s) = \alpha^2(s) + \eta^2(s) - \gamma$, it follows that

$$\lim_{s\to\pm\infty} \tau(s) = +\infty, \quad \lim_{s\to\pm\infty} \kappa(s) = 0.$$

Example 5.4. Choose a spacelike vector $\mathbf{v} = 0.3(0, 0, 1)$ and consider the case where τ reaches a global minimum point (see Figure 8).

- Left: The functions $\alpha(s)$, $\tau(s)$, and $\eta(s)$ associated with the ODE system (3.5). Here, $\tau(s) \to +\infty$ along both ends, corresponding to Case 1 in the proof of Theorem 5.3.
- Right: The corresponding complete ICF soliton X on the de Sitter surface and its evolution under the ICF over a short time interval.

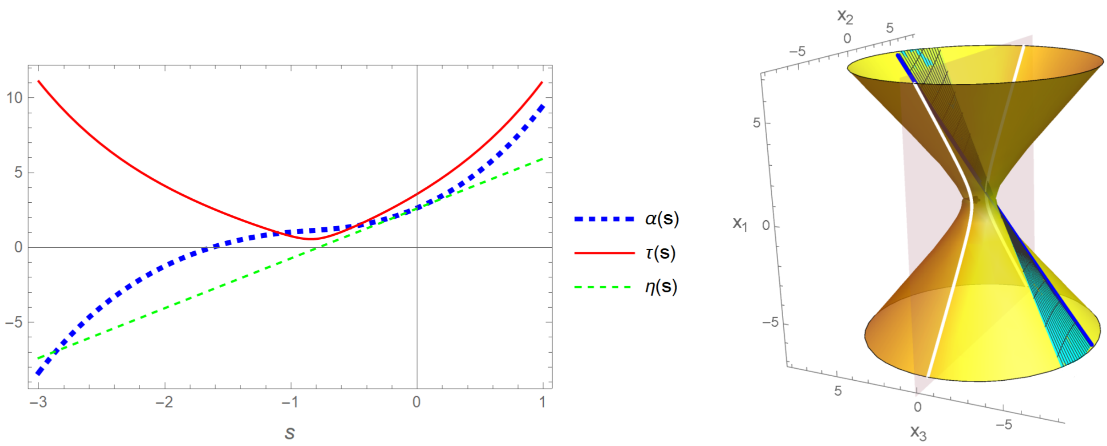


Figure 8. Complete soliton curve X (blue) and the ICF (cyan).

Case 2: τ is strictly decreasing on $I = (w_-, w_+)$.

We first analyze the behavior of the solution as $s \to w_-$. Since $\tau(s) > 0$ on I, the same argument as in Case 1 applies to (w_-, s_0), which yields

$$w_- = -\infty, \quad \lim_{s\to-\infty} \alpha(s) = -\infty, \quad \lim_{s\to-\infty} \eta(s) = -\infty.$$

Moreover

$$\lim_{s\to-\infty} \tau(s) = +\infty, \quad \lim_{s\to-\infty} \kappa(s) = 0.$$

We next investigate the behavior at the opposite end $s \to w_+$. Since τ is positive and strictly decreasing, the limit

$$\tau_+ := \lim_{s\to w_+} \tau(s)$$

exists with $\tau_+ \geq 0$. The precise value of τ_+ depends on the causal character of the vector $\mathbf{v}$ or, equivalently, the set of $\psi(s)$.

Case 2.1: If $\mathbf{v}$ is timelike, $(\psi(s) \in H)$. In this case,

$$\tau(s) \geq 1 \text{ for all } s \in I,$$

and hence $\tau_+ \geq 1$. Assume, for contradiction, that $\tau_+ > 1$. Since τ remains bounded away from zero, the solution stays in a compact subset of the open region

$$\Omega = \{(\alpha, \tau, \eta) : \tau > 1\},$$

on which the vector field of system (3.5) is smooth. Hence, by the ODE extendibility theorem, we must have $w_+ = +\infty$.

Consequently, using $\alpha' = \tau > 0$ and $\eta' = \frac{1}{a} > 0$, we obtain

$$\lim_{s \to +\infty} \alpha(s) = +\infty, \ \lim_{s \to +\infty} \eta(s) = +\infty.$$

Substituting into $\tau^2(s) = \alpha^2(s) + \eta^2(s) + 1$ yields

$$\lim_{s \to +\infty} \tau(s) = +\infty,$$

which contradicts the assumption that $\tau(s) \to \tau_+$ (finite). Therefore, $\tau_+ = 1$.

As a consequence

$$\lim_{s \to w_+} \kappa(s) = \frac{1}{a}.$$

Moreover, from $\tau^2(s) = \alpha^2(s) + \eta^2(s) + 1$, we deduce

$$\lim_{s \to w_+} \alpha(s) = 0, \ \lim_{s \to w_+} \eta(s) = 0, \ \lim_{s \to w_+} \tau'(s) = 0,$$

where $w_+ = -a\,\eta(w_0) + w_0$ follows from $\eta(s) = \frac{1}{a}(s - w_0) + \eta(w_0)$ for any fixed $w_0 \in I$.

Case 2.2: If $\mathbf{v}$ is lightlike, ($\psi(s) \in C$). In this case, we have

$$\tau(s) > 0 \text{ for all } s \in I,$$

and hence $\tau_+ \geq 0$. If $\tau_+ > 0$, then τ remains bounded away from zero near w_+, and the solution stays in a compact subset of the open region

$$\Omega = \{(\alpha, \tau, \eta) : \tau \neq 0\},$$

where the vector field of system (3.5) is smooth. By the ODE extendibility theorem, we have $w_+ = +\infty$, which leads to the same contradiction as in Case 2.1. Hence we have

$$\tau_+ = 0, \ \lim_{s \to w_+} \kappa(s) = +\infty.$$

Using $\tau^2(s) = \alpha^2(s) + \eta^2(s)$, we obtain

$$\lim_{s \to w_+} \alpha(s) = 0, \ \lim_{s \to w_+} \eta(s) = 0,$$

where $w_+ = -a\,\eta(w_0) + w_0$ follows from $\eta(s) = \frac{1}{a}(s - w_0) + \eta(w_0)$ for any fixed $w_0 \in I$.

Example 5.5. Choose a lightlike vector $\mathbf{v} = 2(-1, 1, 0)$ and consider the case where τ is strictly decreasing on I (see Figure 9).

- Left: The functions $\alpha(s)$, $\tau(s)$, and $\eta(s)$ associated with the ODE system (3.5). Here, $\tau(s)$ develops a singularity at w_+ and $\tau(s) \to +\infty$ along the other end, corresponding to Case 2.2 in the proof of Theorem 5.3.
- Right: The corresponding ICF soliton X on the de Sitter surface and its evolution under the ICF over a short time interval. The curve develops a singularity at w_+ and tends to infinity along the other end.

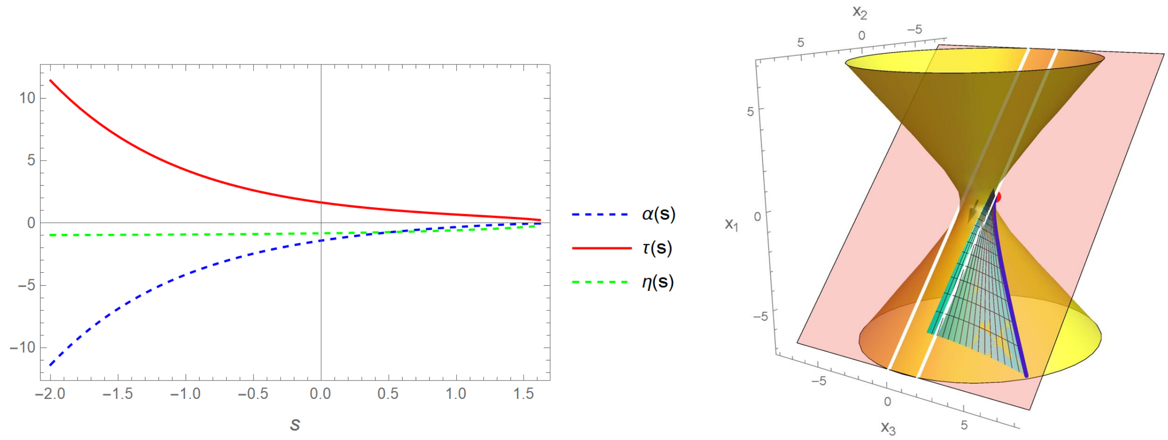


Figure 9. X (blue), the singular boundary point (red), and the ICF (cyan).

Case 2.3: If $\mathbf{v}$ is spacelike, ($\psi(s) \in S$).

Since $\tau(s) > 0$ for all $s \in I$, as in Case 2.2, we obtain

$$\tau_+ = 0, \lim_{s \to w_+} \kappa(s) = +\infty.$$

From $\tau^2(s) = \alpha^2(s) + \eta^2(s) - 1$, we conclude that

$$\lim_{s \to w_+} (\alpha^2 + \eta^2)(s) = 1,$$

which implies

$$-1 \leq \lim_{s \to w_+} \alpha(s) \leq 1, \ -1 \leq \lim_{s \to w_+} \eta(s) \leq 1.$$

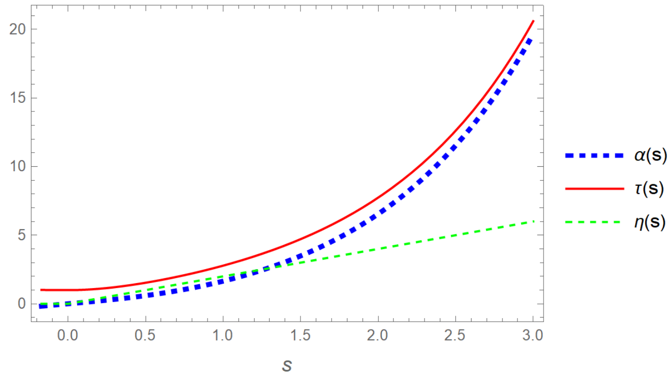
The endpoint satisfies $a(-1 - \eta(w_0)) + w_0 \leq w_+ \leq a(1 - \eta(w_0)) + w_0$ since $\eta(s) = \frac{1}{a}(s - w_0) + \eta(w_0)$ for any fixed $w_0 \in I$.

Case 3: τ is strictly increasing on $I = (w_-, w_+)$. An analogous argument as Case 2 yields the corresponding conclusions.

Example 5.6. Choose a timelike vector $\mathbf{v} = 0.5(-1, 0, 0)$ and consider the case where τ is strictly increasing on I (see Figure 10).

- Left: The functions $\alpha(s)$, $\tau(s)$, and $\eta(s)$ associated with the ODE system (3.5). Here, $\tau(s) \to 1$ as $s \to w_-$ and $\tau(s) \to +\infty$ along the other end, corresponding to Case 3 in the proof of Theorem 5.3.

- Right: The corresponding ICF soliton X on the de Sitter surface and its evolution under the ICF over a short time interval. The curve has a regular endpoint at w_- and tends to infinity along the other side.



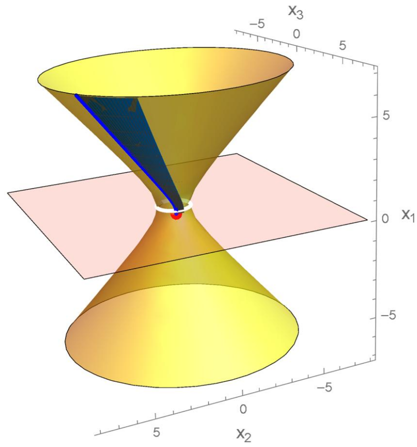


Figure 10. X (blue), the regular boundary point (red), and the ICF (cyan).

Case 1 corresponds to the complete situation described in the theorem: The solution exists for all $s \in \mathbb{R}$, both ends of X escape to infinity in opposite directions with respect to $\mathbf{v}$, and the geodesic curvature tends to zero.

Cases 2 and 3 correspond to the incomplete situation in the theorem. In both, one end of X escapes to infinity with vanishing geodesic curvature, while the other reaches a finite endpoint whose geometry depends on the causal character of $\mathbf{v}$.

- If $\mathbf{v}$ is timelike, the endpoint is regular, lying on the spacelike geodesic of dS_2, where $\kappa(s)$ converges to $\pm\frac{1}{|\mathbf{v}|}$.
- If $\mathbf{v}$ is lightlike, the endpoint is singular, lying on the lightlike geodesic of dS_2, where $\kappa(s)$ blows up.
- If $\mathbf{v}$ is spacelike, the endpoint is a singularity whose height function remains bounded in $[-1, 1]$, and $\kappa(s)$ blows up.

Finally, the bounds for w_- and w_+ obtained in Cases 2 and 3 show that, in the incomplete case, the maximal interval of existence I is determined by the vector $\mathbf{v}$ and the initial normal vector of X. □

Remark 5.7. *A notable difference from the spacelike soliton solutions is that, in the timelike soliton solutions to the ICF, the causal character of the vector* $\mathbf{v}$ *plays a much less significant role. Regardless of whether* $\mathbf{v}$ *is timelike, lightlike, or spacelike,* τ *is either monotone or admits a unique critical point.*

6. Explanations of the graphical illustrations

Throughout the paper, we illustrate representative spacelike and timelike soliton solutions to the ICF in dS_2 corresponding to the various cases of the classification. These solutions satisfy the following geometric conditions:

- $\langle X(s), X(s)\rangle = 1$; that is, the curves in dS_2;
- $\langle T(s), T(s)\rangle = 1$ (resp. $\langle T(s), T(s)\rangle = -1$); that is, the unit tangent vector fields of the curves are spacelike (resp. timelike);
- The geodesic curvature $\kappa(s) = \langle T'(s), N(s)\rangle$ and $\kappa^{-1}(s) = \langle T(s), \mathbf{v}\rangle$ for some vector $\mathbf{v} \in \mathbb{R}^3_1\backslash\{0\}$ by Theorem 2.1.

In addition, we demonstrate how these solutions evolve through the isometries of dS_2 within a short time period, as depicted in the figures. The white curves are the spacelike geodesic Γ_1, the lightlike geodesic Γ_2, or the timelike geodesic Γ_3 of dS_2 when $\mathbf{v}$ is timelike, lightlike, or spacelike, respectively. On the pink planes, the height of $X(s)$ vanish.

7. Conclusions

We first establish an equivalent characterization of nontrivial spacelike and timelike ICF soliton curves on the de Sitter surface (Theorem 2.1). We then prove the existence of such solitons for any vector $\mathbf{v}$ (Theorem 2.3). Finally, we analyze their qualitative behavior and obtain a classification according to the causal character of $\mathbf{v}$ (Theorems 4.14 and 5.3).

Use of Generative-AI tools declaration

ChatGPT and DeepSeek were used exclusively for English language translation and grammatical refinement of the manuscript. The tools were not used in the research process. The author has reviewed and take full responsibility for the final manuscript.

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Conflict of interest

The author declares that he has no conflict of interest.

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