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Research article

Reflected backward stochastic evolution equations with jumps in infinite dimensional domains

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Abstract: In this paper, we investigated a class of reflected backward stochastic evolution equations with jumps (RBSEEJs). Specifically, the solutions to these equations are constrained to the unit ball of a separable Hilbert space H. We established the existence and uniqueness of such solutions via the penalization method. Finally, an illustrative example demonstrates the applicability of the established theory.

Keywords: backward stochastic evolution equation; Poisson random mesasure; reflection; penalization; projection
Mathematics Subject Classification: 35R60, 60H15

1. Introduction

The theory of reflected backward stochastic differential equations (RBSDEs) emerged from the seminal work of El-Karoui et al. [1]. In this foundational study, the authors not only established the existence and uniqueness of solutions but also uncovered a pivotal insight: The first component of the solution coincides precisely with the value function of a related optimal stopping problem. This contribution catalyzed subsequent advances, notably the extension by Cvitanic and Karatzas [2] to RBSDEs with dual barriers, where solutions are confined between two stochastic boundaries. Their work established a fundamental connection between RBSDE solutions and equilibrium strategies in zero-sum Dynkin games. Since these early breakthroughs, research on RBSDEs has flourished, witnessing substantial theoretical refinements. Scholars have dedicated significant efforts to broadening the scope of the original results, exploring more intricate reflection mechanisms (see, e.g., [3–5]) and accommodating a wider spectrum of nonlinear generators (see, e.g., [6, 7]). Peng and Xu [8] pioneered the introduction of nonlinear expectations through RBSDEs and showcased a key application of constrained RBSDEs: the pricing of American options in the incomplete markets.

On the other hand, recent decades have seen a growing body of literature on reflected stochastic partial differential equations (RSPDEs) and reflected backward stochastic partial differential equations (RBSPDEs), with substantial progress in both areas. For RSPDEs, Nualart and Pardoux [9] pioneered the study of reflected solutions to the heat equation under Dirichlet boundary conditions, driven by additive space-time white noise. Donati-Martin and Pardoux [10] extended this framework by incorporating nonlinearity into drift and diffusion coefficients. Subsequent studies [11] and [12] generalized these results to multidimensional settings.

In RBSPDEs, Yang et al. [13] established foundational results for Brownian-driven systems, proving well-posedness under super-parabolicity or parabolicity conditions. Taking this a step further, Yang [14] generalized the theory to discontinuous frameworks by replacing Brownian diffusion with a compensated Poisson random measure. Beyond these advances, significant work has examined solution properties of RSPDEs [15–17]. Of particular note is a compelling study by Brzeźniak and Zhang [18], which delved into the well-posedness of reflected stochastic evolution equations within infinite-dimensional domains, marking a significant contribution to the field.

To the best of our knowledge, reflected backward stochastic evolution equations remain relatively unexplored. This gap motivates our investigation of reflected backward stochastic evolution equations (RBSEEs) driven by both an infinite-dimensional Brownian motion and an independent compensated Poisson random measure, where the solution u is constrained to the unit ball of a separable Hilbert space H. The primary objective of this paper is to establish existence and uniqueness for such solutions. Our proof employs the penalization method. Building on results from [19], we first prove existence and uniqueness for solutions of the penalized equations. We then derive a series of a priori estimates to demonstrate convergence of these penalized solutions to the solution of the target RBSEEJ. We mention that our approach follows the idea of Brzeźniak and Zhang [18]. The main novelties of this paper are twofold. First, we extend the framework of Brzeźniak and Zhang [18] to the backward case, and this backward structure poses new challenges. Second, we incorporate jumps via a compensated Poisson random measure. The presence of jumps creates technical difficulties in the penalization argument, particularly in controlling the nonnegative terms arising when applying Itô's formula to $\|u - \pi(u)\|_H^2$; see Lemma 3.2.

This paper is structured as follows: Section 2 precisely formulates the problem and introduces preliminary concepts relevant to the RBSEEJ. Section 3 presents the penalized system and derives a series of a priori estimates for its solutions. Finally, Section 4 establishes the existence and uniqueness of solutions to the RBSEEJ.

2. Preliminaries

Let V and H be two separable Hilbert spaces, with V continuously and densely embedded in H. Identifying H with its dual, we obtain the Gelfand triple:

$$V \subset H \cong H^* \subset V^*, \tag{2.1}$$

where V^* stands for the topological dual of V. Let $\mathcal{A} : V \to V^*$ be a bounded linear operator satisfying the coercivity hypothesis: There exist constants $\alpha > 0$ and $\lambda \geq 0$ such that

$$2\langle \mathcal{A}u, u\rangle + \lambda\|u\|_H^2 \geq \alpha\|u\|_V^2, \quad \forall\ u \in V, \tag{2.2}$$

where $\langle \mathcal{A}u, u\rangle = \mathcal{A}u(u)$ denotes the action of $\mathcal{A}u \in V^*$ on $u \in V$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let K be a separable Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle_K$ and norm $\|\cdot\|_K^2$. Let W be a cylindrical Brownian motion with covariance space K. Specifically, for any $k \in K$, $\langle W_t, k\rangle_K$ is a real-valued Brownian motion satisfying $E[\langle W_t, k\rangle_K^2] = t\|k\|_K^2$. This admits the decomposition:

$$W_t = \sum_{i=1}^{\infty} B_t^i e_i, \tag{2.3}$$

where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis of K and $\{B^i\}_{i=1}^{\infty}$ is a family of independent real-valued Brownian motions. Let $\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$ be the compensated Poisson random measure on $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$, where N is the Poisson random measure with Lévy measure ν associated with a Lévy process η. Precisely, $N(t, A) = \sum_{0<s\leq t} I_A(\Delta\eta_s)$ and $E[N(t, A)] = \nu(A)t$, where $\Delta\eta_s = \eta_s - \eta_{s-}$. Define

$$\mathcal{F}_t = \sigma(W_s, \eta_s; s \leq t) \vee \mathcal{N},$$

where $\mathcal{N}$ is the collection of all P-null sets in $\mathcal{F}$. Let us introduce the following notations:

- Let $\mathcal{L}_2(K; H)$ be the space of Hilbert-Schmidt operators from K to H, equipped with inner product and norm

$$\langle S_1, S_2\rangle_{\mathcal{L}_2} := \sum_{i=1}^{\infty}\langle S_1 e_i, S_2 e_i\rangle_H \text{ and } \|S_1\|_{\mathcal{L}_2}^2 := \sum_{i=1}^{\infty}\|S_1 e_i\|_H^2,$$

  where $\{e_i, i \geq 1\}$ is an orthonormal basis of K. For simplicity, we write $\mathcal{L}_2(K, H)$ as $\mathcal{L}_2$.
- $D([0, T]; H)$ stands for the space of all càdlàg paths from $[0, T]$ into H.
- $L^2(\nu)$ represents the space of H-valued functions that are square integrable with respect to measure ν, i.e.,

$$\|q\|_\nu^2 := \int_{\mathbb{R}_0} \|q(z)\|_H^2 \nu(dz) < \infty, \quad q \in L^2(\nu).$$

- Let $S^2([0, T]; H)$ be the space of H-valued adapted processes u satisfying

$$E\Big[\sup_{0\leq t\leq T}\|u_t\|_H^2\Big] < \infty.$$

- For a Banach space $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$, $L^2([0, T]; \mathcal{W})$ denotes the space of $\mathcal{W}$-valued adapted processes u such that

$$E\Big[\int_0^T \|u_s\|_{\mathcal{W}}^2 ds\Big] < \infty.$$

- Let $\mathcal{H}_T$ denote the space of V-valued adapted processes u satisfying

$$\|u\|_{\mathcal{H}_T}^2 := E[\sup_{0\leq t\leq T}\|u_t\|_H^2] + E\Big[\int_0^T \|u_s\|_V^2 ds\Big] < \infty.$$

In this paper, we consider the following RBSEEJ:

$$\begin{cases} du_t = \mathcal{A}u_t dt - f(t,u_t,p_t,q_t)dt + p_t dW_t + \int_{\mathbb{R}_0} q_t(z)\tilde{N}(dt,dz) - dL_t,\ t \in [0,T], \\ u_T = \xi \in \bar{B}_1, \end{cases} \tag{2.4}$$

where $\bar{B}_1$ denotes the closure of the open unit ball B_1 in the separable Hilbert space H.

We now rigorously define the solution to the RBSEEJ (2.4).

Definition 2.1. *A quadruple (u,p,q,L) is said to be a solution of the RBSEEJ (2.4) if the following conditions are satisfied:*

i. $(u,p,q) \in \mathcal{H}_T \times L^2([0,T];\mathcal{L}_2) \times L^2([0,T];L^2(\nu))$ and $u_t \in \bar{B}_1$ for all $t \in [0,T]$;
ii. L is an H-valued adapted process with $L(0) = 0$ and bounded variation, i.e.,
$$E\Big[\big(\mathrm{Var}(L)_H([0,T])\big)^2\Big] < \infty,$$
where $\mathrm{Var}(L)_H([0,T]) := \sup \sum_{i=1}^{n} \|L(t_i) - L(t_{i-1})\|_H$ and the supremum is taken over all partitions $0 = t_0 < t_1 < \cdots < t_n = T$ of the interval $[0,T]$;
iii. for all $t \in [0,T]$,
$$u_t = \xi - \int_t^T \mathcal{A}u_s ds + \int_t^T f(s,u_s,p_s,q_s)ds - \int_t^T p_s dW_s - \int_t^T \int_{\mathbb{R}_0} q_s(z)\tilde{N}(ds,dz) + L_T - L_t,\ a.s.;$$
iv. (minimality condition) for any $\phi \in D([0,T];H)$ with values in $\bar{B}_1$,
$$\int_0^T \langle \phi_t - u_t, dL_t \rangle_H \geq 0,\ a.s.,$$
where the integral on the left-hand side is the Riemiann-Stieltjes integral of the H-valued function $\phi - u$ with respect to an H-valued bounded-variation function L.

We introduce the following assumptions to be used in this paper.

($\mathbf{H}_\xi$) Let $\xi : \Omega \to H$ be a square-integrable $\mathcal{F}_T$-measurable random variable satisfying $\xi \in \bar{B}_1$.

($\mathbf{H}_f$) (i) There exists a constant $C_f > 0$ such that for all $(\omega,t) \in \Omega \times [0,T]$ and any $(u_1,p_1,q_1),(u_2,p_2,q_2) \in H \times \mathcal{L}_2 \times L^2(\nu)$,
$$\|f(\omega,t,u_1,p_1,q_1) - f(\omega,t,u_2,p_2,q_2)\|_H \leq C_f(\|u_1-u_2\|_H + \|p_1-p_2\|_{\mathcal{L}_2} + \|q_1-q_2\|_\nu).$$
(ii) Set $f_0(\omega,t) = f(\omega,t,0,0,0)$. Then $f_0 \in L^2([0,T];H)$, i.e., $E\Big[\int_0^T \|f_0(t)\|_H^2 dt\Big] < \infty$.

Let $\pi(y)$ be the projection of $y \in H$ onto $\bar{B}_1$, defined as

$$\pi(y) = \begin{cases} y, & if\ \|y\|_H \leq 1, \\ \frac{y}{\|y\|_H}, & if\ \|y\|_H > 1. \end{cases}$$

Note that $\pi(y)$ and $y - \pi(y)$ always share the same direction as y. In particular,

$$y - \pi(y) = I_{\{\|y\|_H > 1\}}\Big(1 - \frac{1}{\|y\|_H}\Big)y. \tag{2.5}$$

Furthermore, the function π satisfies the following properties (see [19]), which will be used later:

(P.1) $\|\pi(x) - \pi(y)\|_H \leq 2\|x - y\|_H$ for all $x, y \in H$;
(P.2) $(x, x - \pi(x)) = \|x\|_H\|x - \pi(x)\|_H$ and $(\pi(x), x - \pi(x)) = \|x - \pi(x)\|_H$ for all $x \in H$;
(P.3) $(x - y, x - \pi(x)) \geq 0$ for all $x \in H$ and $y \in \bar{B}_1$.

3. The penalized systems and some estimates

In order to establish the existence and uniqueness of solutions for the RBSEEJ (2.4), we consider the following penalized backward stochastic evolution equations with jumps (BSEEJ) for each $n \in \mathbb{N}^+$:

$$\begin{cases} du_t^n = \mathcal{A}u_t^n dt - f(t, u_t^n, p_t^n, q_t^n)dt + p_t^n dW_t + \int_{\mathbb{R}_0} q_t(z)\tilde{N}(dt, dz) + n(u_t^n - \pi(u_t^n))dt, \\ u_T^n = \xi. \end{cases} \tag{3.1}$$

Set $\hat{f}(t, u, p, q) = f(t, u, p, q) + n(u - \pi(u))$. By property (P.1) and assumptions $(\mathbf{H}_\xi)$ and $(\mathbf{H}_f)$, both ξ and $\hat{f}$ satisfy conditions of Theorem 4.1 in [19]. Consequently, the penalized BSEEJ (3.1) admits a unique solution (u^n, p^n, q^n) by virtue of Theorem 4.1 in [19].

Let $L_t^n = -\int_0^t n(u_s^n - \pi(u_s^n))ds$. Our aim is to prove that $\{(u^n, p^n, q^n, L^n)\}_{n\geq 1}$ converges to the solution of the RBSEEJ (2.4). To this end, we need to develop several essential technical lemmas. In what follows, C denotes a generic constant that may vary from line to line. The following lemma provides a uniform bound of the sequence (u^n, p^n, q^n).

Lemma 3.1. *There exists a positive constant K_1, depending only on T, α, λ, and C_f such that*

$$\sup_{n\geq 1} E\Big[\sup_{0\leq t\leq T}\|u_t^n\|_H^2 + \int_0^T (\|u_t^n\|_V^2 + \|p_t^n\|_{\mathcal{L}_2}^2 + \|q_t^n\|_\nu^2)dt\Big] \leq K_1 E\Big[\|\xi\|_H^2 + \int_0^T \|f_0(t)\|_H^2 dt\Big].$$

Proof. Applying Itô's formula to $\|u_t^n\|_H^2$, we have

$$\begin{aligned} &\|u_t^n\|_H^2 + \int_t^T \|p_s^n\|_{\mathcal{L}_2}^2 ds + \int_t^T \int_{\mathbb{R}_0} \|q_s^n(z)\|_H^2 \nu(dz)ds + 2\int_t^T \langle u_s^n, n(u_s^n - \pi(u_s^n))\rangle_H ds \\ =&\|\xi\|_H^2 - 2\int_t^T \langle \mathcal{A}u_s^n, u_s^n\rangle ds + 2\int_t^T \langle u_s^n, f(s, u_s^n, p_s^n, q_s^n)\rangle_H ds \\ &- 2\int_t^T \langle u_s^n, p_s^n dW_s\rangle_H - \int_t^T \int_{\mathbb{R}_0} (\|u_{s-}^n + q_s^n(z)\|_H^2 - \|u_{s-}^n\|_H^2)\tilde{N}(ds, dz). \end{aligned}$$

From the coercivity condition (4.6), we obtain

$$-2\int_t^T \langle \mathcal{A}u_s^n, u_s^n\rangle ds \leq -\alpha \int_t^T \|u_s^n\|_V^2 ds + \lambda \int_t^T \|u_s^n\|_H^2 ds.$$

By property (P.2), the penalization term satisfies

$$2\int_t^T \langle u_s^n, n(u_s^n - \pi(u_s^n))\rangle_H ds = 2n\int_t^T \|u_s^n\|_H \|u_s^n - \pi(u_s^n)\|_H ds \geq 0.$$

Using the Cauchy-Schwarz inequality and the Lipschitz condition $(\mathbf{H}_f)$, we have

$$2\int_t^T \langle u_s^n, f(s, u_s^n, p_s^n, q_s^n)\rangle_H ds$$

$$\leq 2\int_t^T \|u_s^n\|_H\|f(s,u_s^n,p_s^n,q_s^n)-f_0(s)\|_H ds+2\int_t^T \|u_s^n\|_H\|f_0(s)\|_H ds$$
$$\leq (1+2C_f+4C_f^2)\int_t^T \|u_s^n\|_H^2 ds+\int_t^T \|f_0(s)\|_H^2 ds+\frac{1}{2}\int_t^T \|p_s^n\|_{\mathcal{L}_2}^2 ds+\frac{1}{2}\int_t^T \|q_s^n\|_\nu^2 ds.$$

Combining these estimates, we get

$$\begin{aligned}
&\|u_t^n\|_H^2+\alpha\int_t^T \|u_s^n\|_V^2 ds+\frac{1}{2}\int_t^T \|p_s^n\|_{\mathcal{L}_2}^2 ds+\frac{1}{2}\int_t^T \|q_s^n(z)\|_\nu^2 ds\\
\leq &\|\xi\|_H^2+(1+\lambda+2C_f+4C_f^2)\int_t^T \|u_s^n\|_H^2 ds+\int_t^T \|f_0(s)\|_H^2 ds\\
&-2\int_t^T \langle u_s^n,p_s^n dW_s\rangle_H-\int_t^T\int_{\mathbb{R}_0}(\|u_{s-}^n+q_s^n(z)\|_H^2-\|u_{s-}^n\|_H^2)\tilde{N}(ds,dz).
\end{aligned}$$

Taking expectations on both sides and applying Gronwall's inequality, we obtain

$$\begin{aligned}
\sup_{0\leq t\leq T} E[\|u_t^n\|_H^2]+E\Big[\int_0^T(\|u_s^n\|_V^2+\int_0^T \|p_s^n\|_{\mathcal{L}_2}^2 ds+\int_0^T \|q_s^n\|_\nu^2 ds\Big]&\\
\leq CE\Big[\|\xi\|_H^2+\int_0^T \|f_0(s)\|_H^2 ds\Big].& \qquad (3.2)
\end{aligned}$$

Next, we estimate $E[\sup_{0\leq t\leq T}\|u_t^n\|_H^2]$ using the Burkholder-Davis-Gundy inequality. For the Brownian motion term, we have

$$E\Big[\sup_{0\leq t\leq T}\Big|\int_t^T \langle u_s^n,p_s^n dW_s\rangle_H\Big|\Big]\leq \frac{1}{4}E\Big[\sup_{0\leq t\leq T}\|u_t^n\|_H^2\Big]+CE\Big[\int_0^T \|p_s^n\|_{\mathcal{L}_2}^2 ds\Big].$$

For the jump term, define $M_t=\int_0^t\int_{\mathbb{R}_0}(\|u_{s-}^n+q_s^n(z)\|_H^2-\|u_{s-}^n\|_H^2)\tilde{N}(ds,dz)$. Its quadratic variation satisfies:

$$\begin{aligned}
[M,M]_t^{\frac{1}{2}}=&\Big(\int_0^t\int_{\mathbb{R}_0}(\|u_{s-}^n+q_s^n(z)\|_H^2-\|u_{s-}^n\|_H^2)^2 N(ds,dz)\Big)^{\frac{1}{2}}\\
=&\Big(\sum_{0\leq s\leq t}(2\langle u_{s-}^n,q_s^n(\Delta\eta_s)\rangle_H+\|q_s^n(\Delta\eta_s)\|_H^2)^2\Big)^{\frac{1}{2}}\\
\leq &C\Big(\sum_{0\leq s\leq t}(\langle u_{s-}^n,q_s^n(\Delta\eta_s)\rangle_H^2+\|q_s^n(\Delta\eta_s)\|_H^4)\Big)^{\frac{1}{2}}\\
\leq &C\Big(\sum_{0\leq s\leq t}\langle u_{s-}^n,q_s^n(\Delta\eta_s)\rangle_H^2\Big)^{\frac{1}{2}}+C\Big(\sum_{0\leq s\leq t}\|q_s^n(\Delta\eta_s)\|_H^4\Big)^{\frac{1}{2}}\\
\leq &C\sup_{0\leq s\leq t}\|u_{s-}^n\|_H\Big(\sum_{0\leq s\leq t}\|q_s^n(\Delta\eta_s)\|_H^2\Big)^{\frac{1}{2}}+C\sum_{0\leq s\leq t}\|q_s^n(\Delta\eta_s)\|_H^2\\
\leq &\varepsilon E[\sup_{0\leq s\leq t}\|u_{s-}^n\|_H^2]+C_\varepsilon\sum_{0\leq s\leq t}\|q_s^n(\Delta\eta_s)\|_H^2,
\end{aligned}$$

where $\varepsilon>0$. Using Burkholder-Davis-Gundy's inequality and choosing a suitable ε, we have

$$E\Big[\sup_{0\leq t\leq T}\Big|\int_t^T\int_{\mathbb{R}_0}(\|u_{s-}^n+q_s^n(z)\|_H^2-\|u_{s-}^n\|_H^2)\tilde{N}(ds,dz)\Big|\Big]$$

$$
\begin{aligned}
&= E\Big[\sup_{0\le t\le T}|M_T - M_t|\Big]\\
&\le 2E\Big[\sup_{0\le t\le T}|M_t|\Big]\\
&\le CE\Big[[M,M]_T^{\frac{1}{2}}\Big]\\
&\le \frac{1}{4}E[\sup_{0\le t\le T}\|u_t^n\|_H^2] + CE\Big[\int_0^T \|q_s^n\|_\nu^2 ds\Big].
\end{aligned}
$$

Combining these with estimate (3.2) yields

$$E[\sup_{0\le t\le T}\|u_t^n\|_H^2] \le CE\Big[\|\xi\|_H^2 + \int_0^T \|f_0(s)\|_H^2 ds\Big].$$

Thus, the desired result is proven. □

To estimate $\|u^n - \pi(u^n)\|_H^2$, we need the following result, which provides a key inequality to control the nonnegative jump term arising from Itô's formula.

Lemma 3.2. *For any $u, q \in H$, the following inequality holds:*

$$\|u + q - \pi(u+q)\|_H^2 - \|u - \pi(u)\|_H^2 - 2\langle u - \pi(u), q\rangle_H \ge 0.$$

Proof. Set $z = u + q$. It suffices to show that

$$\|z - \pi(z)\|_H^2 - \|u - \pi(u)\|_H^2 - 2\langle u - \pi(u), z - u\rangle_H \ge 0.$$

We proceed by considering four cases:

(1) If $z, u \in \bar{B}_1$, then $z = \pi(z)$ and $u = \pi(u)$. It is clear that

$$\|z - \pi(z)\|_H^2 - \|u - \pi(u)\|_H^2 - 2\langle u - \pi(u), z - u\rangle_H = 0,$$

which satisfies the inequality.

(2) If $z \notin \bar{B}_1$ and $u \in \bar{B}_1$, we have

$$\|z - \pi(z)\|_H^2 - \|u - \pi(u)\|_H^2 - 2\langle u - \pi(u), z - u\rangle_H = \|z - \pi(z)\|_H^2 \ge 0.$$

(3) If $z \in \bar{B}_1$ and $u \notin \bar{B}_1$, then $z = \pi(z)$ and $\|z\|_H \le 1$. Using property (P.2), we obtain

$$
\begin{aligned}
&\|z - \pi(z)\|_H^2 - \|u - \pi(u)\|_H^2 - 2\langle u - \pi(u), z - u\rangle_H\\
=& -\|u - \pi(u)\|_H^2 - 2\langle u - \pi(u), z - \pi(u) + \pi(u) - u\rangle_H\\
=& \|u - \pi(u)\|_H^2 - 2\langle u - \pi(u), z\rangle_H + 2\langle u - \pi(u), \pi(u)\rangle_H\\
\ge& \|u - \pi(u)\|_H^2 + 2\|u - \pi(u)\|_H(1 - \|z\|_H)\\
\ge& 0.
\end{aligned}
$$

(4) If $z, u \notin \bar{B}_1$, we use the identity $z - u = z - \pi(z) - (u - \pi(u)) + \pi(z) - \pi(u)$ and property (P.2) to obtain

$$\|z - \pi(z)\|_H^2 - \|u - \pi(u)\|_H^2 - 2\langle u - \pi(u), z - u\rangle_H$$

$$
\begin{aligned}
&= \|z-\pi(z)\|_H^2 - \|u-\pi(u)\|_H^2 - 2\langle u-\pi(u), z-\pi(z)-(u-\pi(u))+\pi(z)-\pi(u)\rangle_H \\
&= \|z-\pi(z)\|_H^2 + \|u-\pi(u)\|_H^2 - 2\langle u-\pi(u), z-\pi(z)\rangle_H + 2\langle u-\pi(u), \pi(u)-\pi(z)\rangle_H \\
&= \|z-\pi(z)-(u-\pi(u))\|_H^2 + 2\langle u-\pi(u), \pi(u)\rangle_H - 2\langle u-\pi(u), \pi(z)\rangle_H \\
&\geq \|z-\pi(z)-(u-\pi(u))\|_H^2 + 2\|u-\pi(u)\|_H - 2\|u-\pi(u)\|_H\|\pi(z)\|_H \\
&\geq 0.
\end{aligned}
$$

Thus, the desired result is proven. □

We now begin to obtain an estimate for the distance between u^n and its projection $\pi(u^n)$ onto $\bar{B}_1$. This is crucial for the proof of both the convergence of u^n and the reflecting constraint $u_t \in \bar{B}_1$; see Theorem 4.1 below.

Lemma 3.3. *There exists a positive constant K_2, depending only on T, α, λ, and C_f, such that*

$$
E[\|u_t^n - \pi(u_t^n)\|_H^2] + nE\Big[\int_0^T \|u_s^n - \pi(u_s^n)\|_H^2 ds\Big] \leq \frac{K_2}{n} E\Big[\|\xi\|_H^2 + \int_0^T \|f_0(s)\|_H^2 ds\Big]. \tag{3.3}
$$

Proof. Let $g(u) = \|u-\pi(u)\|_H^2$ for $u \in H$. Simple calculations show that its first and second Fréchet derivatives are:

$$
g'(u)(v) = 2\langle u-\pi(u), v\rangle_H, \text{ for } u, v \in H,
$$

and

$$
g''(u)(v,w) = 2I_{\{\|u\|_H>1\}}\Big[\langle w, v\rangle_H\Big(1-\frac{1}{\|u\|_H}\Big) + \langle u, v\rangle_H\langle u, w\rangle_H \frac{1}{\|u\|_H^3}\Big], \text{ for } u, v, w \in H.
$$

Applying Itô's formula to $g(u_s^n)$ and noting (2.3) and $g(u_T^n) = \|\xi - \pi(\xi)\|_H^2 = 0$, we obtain

$$
\begin{aligned}
g(u_t^n) = &-\int_t^T g'(u_s^n)(\mathcal{A}u_s^n)ds + \int_t^T g'(u_s^n)(f(s,u_s^n,p_s^n,q_s^n))ds - n\int_t^T g'(u_s^n)(u_s^n - \pi(u_s^n))ds \\
&- \frac{1}{2}\sum_{i=1}^{\infty}\int_t^T g''(u_s^n)(p_s^n e_i, p_s^n e_i)ds - \int_t^T\int_{\mathbb{R}_0}[g(u_{s-}^n + q_s^n(z)) - g(u_{s-}^n)]\tilde{N}(ds,dz) \\
&- \sum_{i=1}^{\infty}\int_t^T g'(u_s^n)(p_s^n e_i)dB_s^i - \int_t^T\int_{\mathbb{R}_0}[g(u_{s-}^n + q_s^n(z)) - g(u_{s-}^n) - g'(u_{s-}^n)(q_s^n(z))]\nu(dz)ds.
\end{aligned}
$$

By property (P.2), (2.5), and the coercivity condition (4.6), we have

$$
\begin{aligned}
-\int_t^T g'(u_s^n)(\mathcal{A}u_s^n)ds &= -2\int_t^T \langle \mathcal{A}u_s^n, u_s^n - \pi(u_s^n)\rangle ds \\
&= -2\int_t^T I_{\{\|u_s^n\|_H>1\}}\langle \mathcal{A}u_s^n, u_s^n - \frac{u_s^n}{\|u_s^n\|_H}\rangle ds \\
&= -2\int_t^T I_{\{\|u_s^n\|_H>1\}}\Big(1-\frac{1}{\|u_s^n\|_H}\Big)\langle \mathcal{A}u_s^n, u_s^n\rangle ds \\
&\leq \int_t^T I_{\{\|u_s^n\|_H>1\}}\Big(1-\frac{1}{\|u_s^n\|_H}\Big)(-\alpha\|u_s^n\|_V^2 + \lambda\|u_s^n\|_H^2)ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \lambda \int_t^T I_{\{\|u_s^n\|_H>1\}}\Big(1-\frac{1}{\|u_s^n\|_H}\Big)\|u_s^n\|_H^2 ds\\
&=\lambda \int_t^T \|u_s^n-\pi(u_s^n)\|_H\|u_s^n\|_H ds\\
&\leq \frac{n}{2}\int_t^T \|u_s^n-\pi(u_s^n)\|_H^2 ds+\frac{\lambda^2}{2n}\int_t^T \|u_s^n\|_H^2 ds.
\end{aligned}
$$

Using the Lipschitz condition $(\mathbf{H}_f)$ and Young's inequality:

$$
\begin{aligned}
&2\int_t^T \langle u_s^n-\pi(u_s^n), f(s,u_s^n,p_s^n,q_s^n)\rangle_H ds\\
\leq&\frac{n}{2}\int_t^T \|u_s^n-\pi(u_s^n)\|_H^2 ds+\frac{2}{n}\int_t^T \|f(s,u_s^n,p_s^n,q_s^n)\|_H^2 ds\\
\leq&\frac{n}{2}\int_t^T \|u_s^n-\pi(u_s^n)\|_H^2 ds+\frac{C}{n}\int_t^T \Big(\|u_s^n\|_H^2+\|p_s^n\|_{\mathcal{L}_2}^2+\|q_s^n\|_\nu^2+\|f_0(s)\|_H^2\Big)ds.
\end{aligned}
$$

The penalization term simplifies to

$$
-n\int_t^T g'(u_s^n)(u_s^n-\pi(u_s^n))ds=-2n\int_t^T \|u_s^n-\pi(u_s^n)\|_H^2 ds.
$$

Recall the second Fréchet derivative of g, and we get

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^\infty\int_t^T g''(u_s^n)(p_s^ne_i,p_s^ne_i)ds\\
=&\int_t^T I_{\{\|u_s^n\|_H>1\}}\|p_s^n\|_{\mathcal{L}_2}^2\Big(1-\frac{1}{\|u_s^n\|_H}\Big)ds+\int_t^T I_{\{\|u_s^n\|_H>1\}}\frac{1}{\|u_s^n\|_H^3}\sum_{i=1}^\infty\langle u_s^n,p_s^ne_i\rangle_H^2 ds\\
\geq&0.
\end{aligned}
$$

By Lemma 3.2, we have

$$
\begin{aligned}
&\int_t^T\int_{\mathbb{R}_0}[g(u_{s-}^n+q_s^n(z))-g(u_{s-}^n)-g'(u_{s-}^n)(q_s^n(z))]\nu(dz)ds\\
=&\int_t^T\int_{\mathbb{R}_0}[\|u_{s-}^n+q_s^n(z)-\pi(u_{s-}^n+q_s^n(z))\|_H^2-\|u_{s-}^n-\pi(u_{s-}^n)\|_H^2-2\langle u_{s-}^n-\pi(u_{s-}^n),q_s^n(z)\rangle_H]\nu(dz)ds\\
\geq&0.
\end{aligned}
$$

Taking expectations and combining all terms, it follows that

$$
\begin{aligned}
&E[\|u_t^n-\pi(u_t^n)\|_H^2]+nE\Big[\int_0^T \|u_s^n-\pi(u_s^n)\|_H^2 ds\Big]\\
\leq&\frac{C}{n}\Big(E[\sup_{0\leq t\leq T}\|u_t^n\|_H^2]+E\Big[\int_0^T(\|p_s^n\|_{\mathcal{L}_2}^2+\|q_s^n\|_\nu^2+\|f_0(s)\|_H^2)ds\Big]\Big).
\end{aligned}
$$

This implies (3.3) by invoking Lemma 3.1. □

4. The uniqueness and existence

Theorem 4.1. *Assume that* $(\mathbf{H}_\xi)$ *and* $(\mathbf{H}_f)$ *hold. Then RBSEEJ* (2.4) *admits a unique solution.*

Proof. **Existence.** Let $L^n_t = -\int_0^t n(u^n_s - \pi(u^n_s))ds$. Then the penalized Eq (3.1) can be rewritten as

$$\begin{cases} du^n_t = \mathcal{A}u^n_t dt - f(t, u^n_t, p^n_t, q^n_t)dt + p^n_t dW_t + \int_{\mathbb{R}_0} q_t(z)\tilde{N}(dt, dz) - dL^n_t, \\ u^n_T = \xi. \end{cases} \tag{4.1}$$

We will show that $\{(u^n, p^n, q^n, L^n)\}_{n\geq 1}$ converges to some quadruple (u, p, q, L), which is a solution to the RBSEEJ (2.4). The proof is organized into two steps.
Step 1. We claim that

$$\begin{aligned} \lim_{n,m\to\infty} \Big\{ & E[\sup_{0\leq t\leq T} \|u^n_t - u^m_t\|^2_H] + E[\sup_{0\leq t\leq T} \|L^n_t - L^m_t\|^2_{V^*}] \\ & + E\Big[\int_0^T (\|u^n_s - u^m_s\|^2_V + \|p^n_s - p^m_s\|^2_{\mathcal{L}_2} + \|q^n_s - q^m_s\|^2_\nu)ds\Big]\Big\} = 0. \end{aligned} \tag{4.2}$$

Indeed, by Itô's formula, we have

$$\begin{aligned} & \|u^n_t - u^m_t\|^2_H + \int_t^T \|p^n_s - p^m_s\|^2_{\mathcal{L}_2} ds + \int_t^T \|q^n_s - q^m_s\|^2_\nu ds \\ = & -2\int_t^T \langle \mathcal{A}(u^n_s - u^m_s), u^n_s - u^m_s\rangle ds + 2\int_t^T \langle u^n_s - u^m_s, f(s, u^n_s, p^n_s, q^n_s) - f(s, u^m_s, p^m_s, q^m_s)\rangle_H ds \\ & - 2n\int_t^T \langle u^n_s - u^m_s, u^n_s - \pi(u^n_s)\rangle_H ds + 2m\int_t^T \langle u^n_s - u^m_s, u^m_s - \pi(u^m_s)\rangle_H ds \\ & - 2\int_t^T \langle u^n_s - u^m_s, (p^n_s - p^m_s)dW_s\rangle_H - \int_t^T\int_{\mathbb{R}_0} (\|q^n_s - q^m_s\|^2_H + \langle u^n_{s-} - u^m_{s-}, q^n_s - q^m_s\rangle_H)\tilde{N}(ds, dz). \end{aligned}$$

By the coercivity condition (4.6), we get

$$-2\int_t^T \langle \mathcal{A}(u^n_s - u^m_s), u^n_s - u^m_s\rangle ds \leq -\alpha\int_t^T \|u^n_s - u^m_s\|^2_V ds + \lambda\int_t^T \|u^n_s - u^m_s\|^2_H ds.$$

By Cauchy-Schwartz's inequality and the Lipschitz condition $(\mathbf{H}_f)$, we have

$$\begin{aligned} & 2\int_t^T \langle u^n_s - u^m_s, f(s, u^n_s, p^n_s, q^n_s) - f(s, u^m_s, p^m_s, q^m_s)\rangle_H ds \\ \leq & (2C_f + 4C_f^2)\int_t^T \|u^n_s - u^m_s\|^2_H ds + \frac{1}{2}\int_t^T \|p^n_s - p^m_s\|^2_{\mathcal{L}_2} ds + \frac{1}{2}\int_t^T \|q^n_s - q^m_s\|^2_\nu ds. \end{aligned}$$

It follows from (P.3) that

$$\begin{aligned} & - 2n\int_t^T \langle u^n_s - u^m_s, u^n_s - \pi(u^n_s)\rangle_H ds \\ = & -2n\int_t^T \langle u^n_s - \pi(u^m_s), u^n_s - \pi(u^n_s)\rangle_H ds - 2n\int_t^T \langle \pi(u^m_s) - u^m_s, u^n_s - \pi(u^n_s)\rangle_H ds \end{aligned}$$

$$\leq 2n\int_t^T \langle u_s^n-\pi(u_s^n), u_s^m-\pi(u_s^m)\rangle_H ds.$$

Similarly,

$$2m\int_t^T \langle u_s^n-u_s^m, u_s^m-\pi(u_s^m)\rangle_H ds \leq 2m\int_t^T \langle u_s^n-\pi(u_s^n), u_s^m-\pi(u_s^m)\rangle_H ds.$$

Combining these estimates, we get

$$\begin{aligned}
&\|u_t^n-u_t^m\|_H^2+\alpha\int_t^T \|u_s^n-u_s^m\|_V^2 ds+\frac{1}{2}\int_t^T \|p_s^n-p_s^m\|_{\mathcal{L}_2}^2 ds+\frac{1}{2}\int_t^T \|q_s^n-q_s^m\|_\nu^2 ds\\
\leq&(\lambda+2C_f+4C_f^2)\int_t^T \|u_s^n-u_s^m\|_H^2 ds+2(n+m)\int_t^T \langle u_s^n-\pi(u_s^n), u_s^m-\pi(u_s^m)\rangle_H ds\\
&-2\int_t^T \langle u_s^n-u_s^m,(p_s^n-p_s^m)dW_s\rangle_H-\int_t^T\int_{\mathbb{R}_0}(\|q_s^n-q_s^m\|_H^2+\langle u_{s-}^n-u_{s-}^m, q_s^n-q_s^m\rangle_H)\tilde{N}(ds,dz).
\end{aligned}$$

Taking expectation and using Gronwall's inequality and Lemma 3.3, we have

$$\sup_{0\leq t\leq T} E[\|u_t^n-u_t^m\|_H^2]+E\Big[\int_0^T (\|u_s^n-u_s^m\|_V^2+\|p_s^n-p_s^m\|_{\mathcal{L}_2}^2+\|q_s^n-q_s^m\|_\nu^2)ds\Big]\leq \frac{C}{n}+\frac{C}{m}.$$

Using the Burkholder-Davis-Gundy inequality to estimate the martingale terms, we find

$$E[\sup_{0\leq t\leq T}\|u_t^n-u_t^m\|_H^2]\leq \frac{C}{n}+\frac{C}{m}.$$

Letting $n,m\to\infty$, we can obtain

$$\lim_{n,m\to\infty}\Big\{E[\sup_{0\leq t\leq T}\|u_t^n-u_t^m\|_H^2]+E\Big[\int_0^T (\|u_s^n-u_s^m\|_V^2+\|p_s^n-p_s^m\|_{\mathcal{L}_2}^2+\|q_s^n-q_s^m\|_\nu^2)ds\Big]\Big\}=0. \tag{4.3}$$

This implies (4.2), provided we can show that

$$\lim_{n,m\to\infty} E[\sup_{0\leq t\leq T}\|L_t^n-L_t^m\|_{V^*}^2]=0. \tag{4.4}$$

Let us observe that

$$u_t^n=\xi-\int_t^T \mathcal{A}u_s^n ds+\int_t^T f(s,u_s^n,p_s^n,q_s^n)ds-\int_t^T p_s^n dW_s-\int_t^T\int_{\mathbb{R}_0} q_s^n(z)\tilde{N}(ds,dz)+L_T^n-L_t^n, \tag{4.5}$$

which implies that

$$\begin{aligned}
L_t^n-L_t^m=&u_0^n-u_0^m-(u_t^n-u_t^m)+\int_0^t \mathcal{A}(u_s^n-u_s^m)ds-\int_0^t (f(s,u_s^n,p_s^n,q_s^n)-f(s,u_s^m,p_s^m,q_s^m))ds\\
&+\int_0^t (p_s^n-p_s^m)dW_s+\int_0^t\int_{\mathbb{R}_0}(q_s^n(z)-q_s^m(z))\tilde{N}(ds,dz).
\end{aligned}$$

By (4.3), $(\mathbf{H}_f)$, and the fact that $\mathcal{A}$ is a bounded linear operator from V to V^*, we can obtain the desired result (4.4).

Step 2. From Step 1, it is clear that $\{(u^n, p^n, q^n, L^n)\}_{n\geq 1}$ converges to (u, p, q, L) in $\mathcal{H}_T \times L^2([0,T];\mathcal{L}_2)\times L^2([0,T];L^2(\nu))\times C([0,T];V^*)$. In this step, we will show that (u, p, q, L) is a solution of the RBSEEJ (2.4).

First, it follows from Fatou's lemma and Lemma 3.3 that for any $t \in [0,T]$,

$$E[\|u_t - \pi(u_t)\|_H^2] \leq \lim_{n\to\infty} E[\|u_t^n - \pi(u_t^n)\|_H^2] = 0.$$

This implies that for any $t \in [0,T]$, $u_t \in \bar{B}_1$. Thus, we obtain Definition 2.1-i.

We now show that L satisfies Definition 2.1-ii. It is easy to see that the mapping $\|\cdot\|_H : V^* \to \mathbb{R}^+$ (with the convention $\|h\|_H = \infty$ for $h \in V^*\backslash H$) is lower semi-continuous. Since

$$\|L_t\|_H \leq \liminf_{n\to\infty} \|L_t^n\|_H \leq \liminf_{n\to\infty} n\int_0^t \|u_s^n - \pi(u_s^n)\|_H ds,$$

we can deduce from Lemma 3.3 that L is an H-valued process. According to the arguments in the proof of Lemma 4.5 in [19], we can prove that the function

$$\mathrm{Var}(\cdot)_H([0,T]) : v \in C([0,T],V^*) \to \mathrm{Var}(v)_H([0,T]) \in [0,+\infty]$$

is lower semi-continuous. In other words, for any sequence $\{v^n, n\geq 1\}$ such that $v^n \to v$ in $C([0,T],V^*)$, we have

$$\mathrm{Var}(v)_H([0,T]) \leq \liminf_{n\to\infty} \mathrm{Var}(v^n)_H([0,T]).$$

This implies

$$\mathrm{Var}(L)_H([0,T]) \leq \liminf_{n\to\infty} \mathrm{Var}(L^n)_H([0,T]) \leq \liminf_{n\to\infty} \int_0^T n\|u_s^n - \pi(u_s^n)\|_H ds.$$

Hence, by Fatou's lemma and Lemma 3.3, we have

$$E[(\mathrm{Var}(L)_H([0,T]))^2] \leq \sup_n E\Big[\int_0^T n^2\|u_s^n - \pi(u_s^n)\|_H^2 ds\Big] < \infty.$$

This implies Definition 2.1-ii.

Passing to the limit in the BSEEJ (4.5), we have

$$u_t = \xi - \int_t^T \mathcal{A}u_s ds + \int_t^T f(s, u_s, p_s, q_s)ds - \int_t^T p_s dW_s - \int_t^T \int_{\mathbb{R}_0} q_s(z)\tilde{N}(ds,dz) + L_T - L_t.$$

Thus, we obtain Definition 2.1-iii.

We finally prove Definition 2.1-iv. For any $\phi \in D([0,T];H)$ with values in $\bar{B}_1$, we obtain from (P.3) that

$$\int_0^T \langle \phi_t - u_t^n, dL_t^n\rangle_H = n\int_0^T \langle u_t^n - \phi_t, u_t^n - \pi(u_t^n)\rangle_H dt \geq 0.$$

Following the reasoning in the proof of Lemma 4.5 in [19], we can establish that

$$\int_0^T \langle \phi_t - u_t, dL_t\rangle_H = \lim_{n\to\infty} \int_0^T \langle \phi_t - u_t^n, dL_t^n\rangle_H,$$

even though here, ϕ, u, u^n are valued in $D([0,T];H)$ rather than $C([0,T];H)$. Indeed, let $F = \phi - u$ and $F^n = \phi - u^n$. Then, we have

$$E\Big[\Big|\int_0^T \langle F_t, dL_t\rangle_H - \int_0^T \langle F_t^n, dL_t^n\rangle_H\Big|\Big] \leq E\Big[\Big|\int_0^T \langle F_t - F_t^n, dL_t\rangle_H\Big|\Big] + E\Big[\Big|\int_0^T \langle F_t^n, d(L_t - L_t^n)\rangle_H\Big|\Big].$$

By the strong convergence $F^n \to F$ in $S^2([0,T];H)$ and the fact that L has bounded variation, we obtain

$$E\Big[\Big|\int_0^T \langle F_t - F_t^n, dL_t\rangle_H\Big|\Big] \to 0.$$

Moreover, since F^n is bounded in $S^2([0,T];H)$ and $L^n \to L$ in $C([0,T],V^*)$, we have

$$E\Big[\Big|\int_0^T \langle F_t^n, d(L_t - L_t^n)\rangle_H\Big|\Big] \to 0.$$

This implies Definition 2.1-iv. Thus, (u,p,q,L) is a solution to the RBSEEJ (2.4).

Uniqueness. Let (u,p,q,L) and (u',p',q',L') be two solutions of the RBSEEJ (2.4). Applying Itô's formula, we have

$$\begin{aligned}
&\|u_t - u_t'\|_H^2 + \int_t^T (\|p_s - p_s'\|_{\mathcal{L}^2}^2 + \|q_s - q_s'\|_\nu^2)ds \\
=&\|\xi - \xi'\|_H^2 - 2\int_t^T \langle \mathcal{A}(u_s - u_s'), u_s - u_s'\rangle ds \\
&+ 2\int_t^T \langle u_s - u_s', f(s,u_s,p_s,q_s) - f'(s,u_s',p_s',q_s')\rangle_H ds \\
&+ 2\int_t^T \langle u_s - u_s', dL_s\rangle_H - 2\int_t^T \langle u_s - u_s', dL_s'\rangle_H \\
&- 2\int_t^T \langle u_s - u_s', (p_s - p_s')dW_s\rangle_H \\
&- \int_t^T \int_{\mathbb{R}_0} (\|u_{s-} - u_{s-}' + q_s(z) - q_s'(z)\|_H^2 - \|u_{s-} - u_{s-}'\|_H^2)\tilde{N}(ds,dz).
\end{aligned}$$

Note that

$$-2\int_t^T \langle \mathcal{A}(u_s - u_s'), u_s - u_s'\rangle ds \leq -\alpha \int_t^T \|u_s - u_s'\|_V^2 ds + \lambda \int_t^T \|u_s - u_s'\|_H^2 ds.$$

Definition 2.1-iv and the fact $u, u' \in \bar{B}_1$ imply

$$2\int_t^T \langle u_s - u_s', dL_s\rangle_H - 2\int_t^T \langle u_s - u_s', dL_s'\rangle_H \leq 0.$$

Moreover,

$$
\begin{aligned}
&2\int_t^T \langle u_s - u_s', f'(s,u_s,p_s,q_s) - f'(s,u_s',p_s',q_s')\rangle_H ds \\
\leq &(1+2C_f+4C_f^2)\int_t^T \|u_s-u_s'\|_H^2 ds + \frac{1}{2}\int_t^T \|p_s-p_s'\|_{\mathcal{L}_2}^2 ds + \frac{1}{2}\int_t^T \|q_s-q_s'\|_\nu^2 ds.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&E[\|u_t-u_t'\|_H^2] + E\Big[\int_0^T (\alpha\|u_s-u_s'\|_V^2 + \frac{1}{2}\|p_s-p_s'\|_{\mathcal{L}_2}^2 + \frac{1}{2}\|q_s-q_s'\|_\nu^2)ds\Big] \\
\leq &(1+\lambda+2C_f+4C_f^2)E\Big[\int_0^T \|u_s-u_s'\|_H^2 ds\Big].
\end{aligned}
$$

Using Gronwall's inequality, it follows that

$$
\sup_{0\leq t\leq T} E[\|u_t-u_t'\|_H^2] + E\Big[\int_0^T (\|u_s-u_s'\|_V^2 + \|p_s-p_s'\|_{\mathcal{L}_2}^2 + \|q_s-q_s'\|_\nu^2)ds\Big] = 0,
$$

proving the uniqueness. □

Example 4.4 Let $H = L^2(\mathbb{R}^d)$, and set

$$
V = H_2^1(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) : \nabla u \in L^2(\mathbb{R}^d;\mathbb{R}^d)\}.
$$

It is clear that

$$
V \subset H \cong H^* \subset V^*.
$$

Denote by $a(x) = (a_{ij}(x))$ a matrix-valued function on $\mathbb{R}^d$ satisfying the uniform ellipticity condition:

$$
\frac{1}{c}I_d \leq a(x) \leq cI_d, \ \text{ for some constant } \ c > 0,
$$

where I_d denotes the $d\times d$ identity matrix. Let $f(x)$ be a vector field on $\mathbb{R}^d$ with $f \in L^p(\mathbb{R}^d)$ for some $p > d$. Define

$$
\mathcal{A}u = -\mathrm{div}(a(x)\nabla u(x)) + f(x)\cdot\nabla u(x).
$$

Then condition (4.6) is fulfilled for $(H, V, \mathcal{A})$, i.e., there exist constants $\alpha > 0$ and $\lambda \geq 0$ such that

$$
2\langle \mathcal{A}u, u\rangle + \lambda\|u\|_H^2 \geq \alpha\|u\|_V^2, \ \ \forall\, u \in V. \tag{4.6}
$$

In this setting, the RBSEEJ (2.4) has the following form:

$$
\begin{cases}
du_t = -(\mathrm{div}(a\nabla u_t) + f\cdot\nabla u_t)dt - g(t,u_t,p_t,q_t)dt + p_t dW_t + \int_{\mathbb{R}_0} q_t(z)\tilde{N}(dt,dz) - dL_t, \\
u_T = \xi \in \bar{B}_1.
\end{cases} \tag{4.7}
$$

Thus, for any drift coefficient $g(t,u,p,q)$ satisfying $(\mathbf{H}_f)$ and any terminal random variable ξ satisfying $(\mathbf{H}_\xi)$, it follows from the results in Section 4 that (4.7) has a unique solution.

5. Conclusions

This paper investigated the well-posedness of solutions to RBSEEJs by using a penalization method. The framework was driven by an infinite dimensional Brownian motion and an independent compensated Poisson random measure, providing a theoretical foundation for constrained stochastic systems. The results have implications for mathematical finance and optimal control. Future work will focus on applications to Lévy markets, optimal control, and numerical schemes for such RBSEEs.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no competing interests.

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