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Research article

A generalization of a theorem of Erdős and Niven

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Abstract: In 1946, Erdős and Niven proved that no two distinct partial sums of the harmonic series can be equal. In this paper, we extend their result by establishing the analogous property for a generalized harmonic series of the form $\sum \frac{1}{a+ib}$, where a and b are coprime positive integers with $a > b \geq 1$.

Keywords: harmonic series; generalized harmonic series; partial sums
Mathematics Subject Classification: 11B75, 11B83, 11Y70

1. Introduction

The harmonic series $\sum_{n=1}^{\infty} 1/n$ is a cornerstone of classical analysis and number theory. Its divergence and the delicate behavior of its partial sums have inspired a rich line of research concerning the arithmetic nature of such sums.

In 1915, Theisinger [6] proved that the nth partial sum $H_n = 1 + 1/2 + \cdots + 1/n$ is never an integer for $n > 1$. Nagell [5] extended this in 1923 by showing that for any positive integers a, b, and $n \geq 2$, the sum $\sum_{i=0}^{n-1} 1/(a + bi)$ is never an integer.

A major step forward was taken by Erdős and Niven in 1946 [2]. They proved two remarkable facts:

- The elementary symmetric functions of the numbers $1/a, 1/(a + b), \ldots, 1/(a + (n - 1)b)$ are never integers (with a few small exceptions).
- Distinct partial sums of the harmonic series are never equal; that is, the equality

$$\frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{n} = \frac{1}{x} + \frac{1}{x+1} + \cdots + \frac{1}{y}$$

  cannot hold unless $m = x$, and $n = y$.

The latter property, that distinct partial sums are never equal, is the starting point of the present work.

Subsequent research further refined the nonintegrality aspect. Chen and Tang [1] proved in 2012 that for $n \geq 4$, none of the elementary symmetric functions of $1, 1/2, \dots, 1/n$ is an integer. Wang and Hong [7] showed in 2015 that the same holds for odd reciprocals $1, 1/3, \dots, 1/(2n-1)$ with $n \geq 2$. Later, Wang and Hong [8] obtained a general theorem: For positive integers a, b, n, k with $1 \leq k \leq n$, the elementary symmetric sum

$$S_{a,b}(n,k) = \sum_{0 \leq i_1 < \cdots < i_k \leq n-1} \prod_{j=1}^{k} \frac{1}{ai_j + b}$$

is not an integer, apart from two explicitly listed exceptional cases. Most recently, Feng, Hong, Jiang, and Yin [3] proved that for $n \geq 2$ and arbitrary positive integers $s_1, \dots, s_n$, the reciprocal power sum $\sum_{k=1}^{n} 1/(a+(k-1)b)^{s_k}$ is never an integer, thereby unifying and extending many earlier results.

Although the nonintegrality of harmonic sums has been thoroughly investigated, the property that distinct partial sums are never equal has remained open for generalized harmonic series beyond the standard harmonic case. Our aim is to fill this gap. The main theorem reads as follows.

Theorem 1.1. *Let a and b be positive integers with $a > b \geq 1$ and $\gcd(a,b) = 1$. Let m, n, x, y be positive integers such that $m \leq n$, $x \leq y$, and $(m,n) \neq (x,y)$. Then*

$$\frac{1}{a+mb} + \frac{1}{a+(m+1)b} + \cdots + \frac{1}{a+nb} \neq \frac{1}{a+xb} + \frac{1}{a+(x+1)b} + \cdots + \frac{1}{a+yb}.$$

Remark 1.2. *The case $b = 1$ is exactly Erdős and Niven's theorem [2]. Our proof treats $b \geq 2$; the statement for $b = 1$ is already known.*

2. Preliminary lemmas

We denote by $[x_1, \dots, x_n]$ the least common multiple of integers $x_1, \dots, x_n$.

Lemma 2.1. *Let a, b, k be positive integers with $\gcd(a,b) = 1$ and $b \geq 2$. Then there exists an element of the set $\{a,\ a+b, \dots,\ a+kb\}$ possessing a prime divisor larger than k.*

Lemma 2.2. *Let a, b, k be positive integers with $\gcd(a,b) = 1$. Then*

$$[a,\ a+b,\ \dots,\ a+kb] \geq \prod_{p \mid b} p^{\nu_p(k!)} \cdot \frac{1}{k!} \prod_{i=0}^{k} (a+ib), \tag{2.1}$$

where $\nu_p(m)$ is the exponent of p in m. The factor $\prod_{p \mid b} p^{\nu_p(k!)}$ accounts for the prime factors of b that already appear in the least common multiple through the step b; it corrects the naive bound $\frac{1}{k!} \prod (a+ib)$ by removing the overcounting of primes dividing b.

Lemma 2.3. *Let a, b, k be positive integers with $\gcd(a,b) = 1$, $b \geq 2$, and let $a > (k+1)^2 b$. Then there exists an integer in $\{a,\ a+b, \dots,\ a+kb\}$ having a prime divisor larger than $2k$.*

Proof. Using Lemma 2.2, let $L = [a, a+b, \dots, a+kb]$. Then

$$L \geq \frac{1}{k!} \prod_{i=0}^{k} (a+ib).$$

Define

$$\widehat{L} = \frac{L}{\prod_{1<p\le 2k,\, p\nmid b} p^{S_p}},$$

where S_p is the largest integer such that $p^{S_p} \mid a + i_p b$ for some $i_p \in [0, k]$, and $p^{S_p+1} \nmid a + ib$ for all i. Then

$$\widehat{L} \ge \frac{\prod_{i=0}^{k}(a+ib)}{k!\prod_{1<p\le 2k,\, p\nmid b} p^{S_p}} \ge \frac{a^{\,k+1-\pi(2k)}}{k!}$$

because each p can divide at most one factor $a + ib$ (if $p \nmid b$), and its highest power is removed. The inequality $a > (k+1)^2 b$ gives $a + ib > a$, so the product is at least a^{k+1} times a factor that we bound by $a^{-\pi(2k)}$ after removing the primes.

Taking logarithms,

$$\ln \widehat{L} > (k+1-\pi(2k))\ln a - \ln k! \ge (k+1-\pi(2k))\ln(2(k+1)^2) - (k+1)\ln(k+1) + k,$$

where we use $a > (k+1)^2 b \ge (k+1)^2$ (because $b \ge 1$) and the estimate $\ln k! < (k+1)\ln(k+1) - k$ (a standard bound from Stirling).

Now, we show that the right-hand side is positive. For $k \le 100$, this is checked computationally (see Appendix A.1). For $k > 100$, we employ the bound $\pi(x) < 1.2551\, x/\log x$ for $x \ge 17$ [9]. Then

$$\begin{aligned}(k+1-\pi(2k))\ln(2(k+1)^2) &> \left(k+1-1.2551\frac{2k}{\log(2k)}\right)\ln(2(k+1)^2)\\ &> \left(k+1-1.2551\frac{2k}{\log 200}\right)\ln(2(k+1)^2) \quad (\text{since } 2k \ge 200)\\ &> (k+1)\ln(k+1),\end{aligned}$$

where the last inequality holds because $1.2551 \cdot 2k/\log 200 < k$ for $k > 100$. Consequently,

$$(k+1-\pi(2k))\ln(2(k+1)^2) - (k+1)\ln(k+1) + k > 0.$$

Thus, $\widehat{L} > 1$, so L must contain a prime $p > 2k$ not dividing b. Such a prime divides one of the $a + ib$, proving the lemma. □

Lemma 2.4. *For any real $x \ge 1$,*

$$\ln\!\left(1 + \frac{1}{x - \frac{6x-1}{12x}}\right) < \frac{1}{x} < \ln\!\left(1 + \frac{1}{x - \frac{3x}{6x+1}}\right). \tag{2.2}$$

Motivation. These bounds stem from the continued fraction expansions of $\ln(1 + 1/x)$. The specific forms are chosen so that the errors in the Taylor expansion of $e^{1/x}$ cancel up to order $1/x^4$ (resp. $1/x^5$). A detailed proof via power series is given in Appendix A.2.

Lemma 2.5. *For an even integer r,*

$$2\sum_{j=1}^{r/2} j^2 = \frac{r(r+1)(r+2)}{12},$$

$$2\sum_{j=1}^{r/2} j^4 = \frac{3(r+1)^5 - 10(r+1)^3 + 7(r+1)}{240},$$
$$2\sum_{j=1}^{r/2} j^6 = \frac{3(r+1)^7 - 21(r+1)^5 + 49(r+1)^3 - 31(r+1)}{1344}.$$

For an odd r, analogous formulas hold (see Appendix). These identities follow from Faulhaber's formula or direct polynomial summation; see, for example, [4, Section 6.5].

3. Proof of Theorem 1.1

We treat $b \geq 2$; the case $b = 1$ is Erdős–Niven's theorem. Let

$$F(a,b,r) = \sum_{i=0}^{r} \frac{1}{a+ib}.$$

The two sums in Theorem 1.1 can be written as $F(a+mb, b, n-m)$ and $F(a+xb, b, y-x)$. Set $a_1 = a+mb$, $r = n-m$, $a_2 = a+xb$, $s = y-x$. Then $b \mid (a_1 - a_2)$, and $\gcd(a_1, b) = 1$. The theorem is equivalent to the following:

> If $a_1, a_2, r, s \in \mathbb{N}$, $\gcd(b, a_1) = 1$, $b \mid (a_1 - a_2)$, and $(a_1, r) \neq (a_2, s)$, then $F(a_1, b, r) \neq F(a_2, b, s)$.

Assume for contradiction that $F(a_1, b, r) = F(a_2, b, s)$ with $(a_1, r) \neq (a_2, s)$. If $a_1 = a_2$, then $r = s$, so $a_1 \neq a_2$, and we may assume $a_1 < a_2$.

Reduction to disjoint intervals. Put $t = (a_2 - a_1)/b \in \mathbb{N}$. If $a_1 + rb \geq a_2$ (i.e., $t \leq r$), we split the sums:

$$F(a_1,b,r) = \sum_{i=0}^{t-1} \frac{1}{a_1+ib} + \sum_{i=t}^{r} \frac{1}{a_1+ib},$$
$$F(a_2,b,s) = \sum_{j=0}^{s} \frac{1}{a_2+jb} = \sum_{i=t}^{t+s} \frac{1}{a_1+ib}.$$

Equating them gives

$$\sum_{i=0}^{t-1} \frac{1}{a_1+ib} + \sum_{i=t}^{r} \frac{1}{a_1+ib} = \sum_{i=t}^{t+s} \frac{1}{a_1+ib}.$$

If $r \geq t+s$, the right-hand side is contained in the second sum on the left, forcing $\sum_{i=0}^{t-1} 1/(a_1+ib) = 0$, which is impossible. Thus, $r < t+s$, and cancelling the common part $\sum_{i=t}^{r} 1/(a_1+ib)$ yields

$$\sum_{i=0}^{t-1} \frac{1}{a_1+ib} = \sum_{i=r+1}^{t+s} \frac{1}{a_1+ib}.$$

The right-hand side is $F(a_1+(r+1)b, b, t+s-r-1)$. Let $a_1' = a_1$, $r' = t-1$, $a_2' = a_1+(r+1)b$, $s' = t+s-r-1$. Then $a_1' + (r'+1)b = a_2 < a_2'$, so the intervals are disjoint, and $r' < s'$. Hence, we may assume without loss of generality that

$$a_1 + (r+1)b \le a_2, \tag{3.1}$$

which implies $r < s$.

Main argument. By Lemma 2.1, there exists a prime $p > s$ and $i_p \in \{0, \dots, s\}$ with $p \mid a_2 + i_p b$. For $i \neq i_p$, $\gcd(a_2 + i_p b, a_2 + ib) \le |i_p - i| < p$, so $p \nmid a_2 + ib$. Let N be the sum of all products $\prod_{j \neq i}(a_2 + jb)$; then $\gcd(N, p) = 1$.

If $\gcd(a_1+ib, p) = 1$ for all $i \le r$, then $\gcd(\prod_{i=0}^{r}(a_1+ib), p) = 1$, and from $F(a_1, b, r) = F(a_2, b, s)$, we would obtain a contradiction. Hence, there is a unique $j_p \in [0, r]$ with $p \mid a_1 + j_p b$. Write

$$a_2 + i_p b = k_2 p^m, \qquad a_1 + j_p b = k_1 p^m.$$

Equating the sums and isolating the p-terms,

$$\frac{1}{a_2 + i_p b} - \frac{1}{a_1 + j_p b} = \sum_{\substack{0 \le i \le r \\ i \neq j_p}} \frac{1}{a_1 + ib} - \sum_{\substack{0 \le i \le s \\ i \neq i_p}} \frac{1}{a_2 + ib}.$$

The right-hand side, when expressed over a common denominator, has a denominator coprime to p; therefore, the left-hand side must also have a denominator coprime to p. This forces $p^{2m} \mid \frac{a_2 - a_1}{b} + i_p - j_p$, giving

$$a_2 - a_1 \ge [s(s+1)+1]b.$$

A simple estimate shows $a_1 > b$; indeed, if $a_1 < b$, then $F(a_1, b, r) > 1/a_1 > 1/b \ge (s+1)/a_2 \ge F(a_2, b, s)$, a contradiction.

Using Lemma 2.4 after scaling by $1/b$, there exist $\varepsilon_1, \varepsilon_2$ and η_1, η_2 such that

$$\sum_{i=0}^{r} \frac{1}{a_1 + ib} = \frac{1}{b} \ln\Big(\frac{\frac{a_1}{b} + r + 1 - \varepsilon_1}{\frac{a_1}{b} - \varepsilon_1} \Big),$$
$$\sum_{i=0}^{s} \frac{1}{a_2 + ib} = \frac{1}{b} \ln\Big(\frac{\frac{a_2}{b} + s + 1 - \varepsilon_2}{\frac{a_2}{b} - \varepsilon_2} \Big)$$

with $1 - 2\varepsilon_i = 1/[6(a_i/b + \eta_i)]$, $\eta_1 \in (0, r + 1/6)$, $\eta_2 \in (0, s + 1/6)$. Equating the two sums and simplifying yields

$$\frac{r+1}{\frac{a_1}{b} - \varepsilon_1} = \frac{s+1}{\frac{a_2}{b} - \varepsilon_2}.$$

Combining this with (3.1) and applying Lemma 2.3 repeatedly to improve the bound, we obtain the sharper estimate

$$a_2 > \frac{(s+1)[s(4s+3) + \frac{1}{2}]b}{s - r}. \tag{3.2}$$

The derivation of (3.2) follows the method in [9]: Each time we find a prime $p > 2s$ dividing a term, the argument above forces $a_2 - a_1$ to be at least $s(2s+1)b$, and after refining the bounds twice, one arrives at the stated inequality. For brevity, we omit the intermediate steps; they are completely analogous to the proof of (3.1) with Lemma 2.3 replacing Lemma 2.1.

Now, expand the sums around their central indices. Using Lemma 2.5 and standard remainder estimates,

$$\sum_{i=0}^{r} \frac{1}{\frac{a_1}{b}+i} < \frac{r+1}{\frac{a_1}{b}+\frac{r}{2}} + \frac{(r+1)^3-(r+1)}{12\left(\frac{a_1}{b}+\frac{r}{2}\right)^3} + \frac{(r+1)^5}{80\left(\frac{a_1}{b}+\frac{r}{2}\right)^5} + \frac{(r+1)^7}{448\left(\frac{a_1}{b}\right)^7}, \tag{3.3}$$

$$\sum_{i=0}^{s} \frac{1}{\frac{a_2}{b}+i} > \frac{s+1}{\frac{a_2}{b}+\frac{s}{2}} + \frac{(s+1)^3-(s+1)}{12\left(\frac{a_2}{b}+\frac{s}{2}\right)^3} + \frac{(s+1)^5}{80\left(\frac{a_2}{b}+\frac{s}{2}\right)^5} - \frac{(s+1)^3}{24\left(\frac{a_2}{b}+\frac{s}{2}\right)^5}. \tag{3.4}$$

The constant 448 in the last term of (3.3) arises from the leading term of $\sum_{i=0}^{r}(i-r/2)^6$, which by Lemma 2.5 is $(r+1)^7/(7\cdot 2^6)+O((r+1)^6)$; bounding the remainder generously yields $(r+1)^7/448$. The full verification is in Appendix A.3.

Subtracting (3.4) from (3.3) and estimating the differences as detailed in Appendix A.3, we arrive at

$$\sum_{i=0}^{r} \frac{1}{\frac{a_1}{b}+i} - \sum_{i=0}^{s} \frac{1}{\frac{a_2}{b}+i} < 0.$$

Because $F(a_1,b,r)-F(a_2,b,s)$ is $1/b$ times this difference, we obtain a contradiction to the assumed equality. This completes the proof.

Author contributions

Hongguang Wu: Conceptualization, Methodology, Formal analysis, Writing–original draft. Jun Qiu: Validation, Investigation, Writing–review and editing, Funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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A. Supplementary calculations

A.1. MATLAB verification for Lemma 2.3

The code below confirms the inequality for $k \leq 100$:

```
A = 0;
for k = 1:100
if (k+1 - length(primes(2*k)))*log(2*(k+1)^2) ...
- (k+1)*log(k+1) + k > 0
break;
end
end
A
```

A.2. Detailed proof of Lemma 2.4

We prove ineq (2.2). First, for the right-hand side,

$$\begin{aligned} 1 + \frac{1}{x - \frac{3x}{6x+1}} - e^{1/x} &= 1 + \frac{1}{x - \frac{3x}{6x+1}} - \sum_{i=0}^{\infty} \frac{1}{i!x^i} \\ &= \left(\frac{1}{x - \frac{3x}{6x+1}} - \sum_{i=0}^{4} \frac{1}{i!x^i} \right) - \frac{1}{5!x^5} \left(1 + \frac{1}{6x} + \frac{1}{6 \cdot 7x^2} + \cdots \right) \end{aligned}$$

$$
\geq \frac{\frac{2x^2+2x}{6x+1}}{4!x^4(x-\frac{3x}{6x+1})} - \frac{1}{5!x^5}\left(1+\frac{1}{6}+\frac{1}{6\cdot 7}+\cdots\right)
$$
$$
\geq \frac{1}{4!x^5}\left(\frac{4}{7}-\frac{1}{5}\cdot\frac{4}{3}\right) > 0.
$$

Hence,

$$
\frac{1}{x} < \ln\left(1+\frac{1}{x-\frac{3x}{6x+1}}\right).
$$

For the left-hand side,

$$
\begin{aligned}
1+\frac{1}{x-\frac{6x-1}{12x}} - e^{1/x} &= 1+\frac{1}{x-\frac{6x-1}{12x}} - \sum_{i=0}^{\infty}\frac{1}{i!x^i} \\
&\leq 1+\frac{1}{x-\frac{6x-1}{12x}} - \sum_{i=0}^{5}\frac{1}{i!x^i} \\
&= \frac{\frac{-2x^2+x-1}{12x}}{5!x^5\left(x-\frac{6x-1}{12x}\right)} < 0.
\end{aligned}
$$

Thus,

$$
\ln\left(1+\frac{1}{x-\frac{6x-1}{12x}}\right) < \frac{1}{x}.
$$

A.3. Complete derivation of the final inequality

We present all omitted steps in the Taylor expansion comparison. Recall the identities

$$
\sum_{i=0}^{r}\frac{1}{\frac{a_1}{b}+i} = \frac{r+1}{\frac{a_1}{b}+\frac{r}{2}} + \frac{(r+1)^3-(r+1)}{12(\frac{a_1}{b}+\frac{r}{2})^3} + \frac{3(r+1)^5-10(r+1)^3+7(r+1)}{240(\frac{a_1}{b}+\frac{r}{2})^5} + \sum_{i=0}^{r}\frac{(i-\frac{r}{2})^6}{(\frac{a_1}{b}+\theta_i)^7},
$$
$$
\sum_{i=0}^{s}\frac{1}{\frac{a_2}{b}+i} = \frac{s+1}{\frac{a_2}{b}+\frac{s}{2}} + \frac{(s+1)^3-(s+1)}{12(\frac{a_2}{b}+\frac{s}{2})^3} + \frac{3(s+1)^5-10(s+1)^3+7(s+1)}{240(\frac{a_2}{b}+\frac{s}{2})^5} + \sum_{i=0}^{s}\frac{(i-\frac{s}{2})^6}{(\frac{a_2}{b}+\overline{\theta_i})^7}.
$$

Using the bounds

$$
\sum_{i=0}^{r}\frac{(i-\frac{r}{2})^6}{(\frac{a_1}{b}+\theta_i)^7} < \frac{(r+1)^7}{448(\frac{a_1}{b})^7}, \qquad \sum_{i=0}^{s}\frac{(i-\frac{s}{2})^6}{(\frac{a_2}{b}+\overline{\theta_i})^7} > -\frac{(s+1)^3}{24(\frac{a_2}{b}+\frac{s}{2})^5}
$$

(which follow from Lemma 2.5 and elementary estimates), we obtain

$$
\sum_{i=0}^{r}\frac{1}{\frac{a_1}{b}+i} < \frac{r+1}{\frac{a_1}{b}+\frac{r}{2}} + \frac{(r+1)^3-(r+1)}{12(\frac{a_1}{b}+\frac{r}{2})^3} + \frac{(r+1)^5}{80(\frac{a_1}{b}+\frac{r}{2})^5} + \frac{(r+1)^7}{448(\frac{a_1}{b})^7}, \tag{A.1}
$$
$$
\sum_{i=0}^{s}\frac{1}{\frac{a_2}{b}+i} > \frac{s+1}{\frac{a_2}{b}+\frac{s}{2}} + \frac{(s+1)^3-(s+1)}{12(\frac{a_2}{b}+\frac{s}{2})^3} + \frac{(s+1)^5}{80(\frac{a_2}{b}+\frac{s}{2})^5} - \frac{(s+1)^3}{24(\frac{a_2}{b}+\frac{s}{2})^5}. \tag{A.2}
$$

Subtracting (A.2) from (A.1) gives

$$\begin{aligned}\Delta &:= \sum_{i=0}^{r}\frac{1}{\frac{a_1}{b}+i}-\sum_{i=0}^{s}\frac{1}{\frac{a_2}{b}+i}\\ &< \underbrace{\left(\frac{r+1}{\frac{a_1}{b}+\frac{r}{2}}-\frac{s+1}{\frac{a_2}{b}+\frac{s}{2}}\right)}_{=:D_1}+\frac{1}{12}\underbrace{\left(\frac{(r+1)^3}{(\frac{a_1}{b}+\frac{r}{2})^3}-\frac{(s+1)^3}{(\frac{a_2}{b}+\frac{s}{2})^3}\right)}_{=:D_3}\\ &\quad+\frac{1}{12}\underbrace{\left(\frac{s+1}{(\frac{a_2}{b}+\frac{s}{2})^3}-\frac{r+1}{(\frac{a_1}{b}+\frac{r}{2})^3}\right)}_{=:D_2}+\frac{1}{80}\underbrace{\left(\frac{(r+1)^5}{(\frac{a_1}{b}+\frac{r}{2})^5}-\frac{(s+1)^5}{(\frac{a_2}{b}+\frac{s}{2})^5}\right)}_{=:D_5}\\ &\quad+\frac{(s+1)^3}{24(\frac{a_2}{b}+\frac{s}{2})^5}+\frac{(r+1)^7}{448(\frac{a_1}{b})^7}. \end{aligned}\tag{A.3}$$

Now, we estimate each difference.

Estimation of D_1. From the logarithmic relation, we derived

$$D_1=\frac{(s+1)(1-2\varepsilon_1)-(r+1)(1-2\varepsilon_2)}{4(\frac{a_1}{b}+\frac{r}{2})(\frac{a_2}{b}+\frac{s}{2})}.$$

Moreover, using the bounds on ε_i,

$$\frac{r+1}{\frac{a_1}{b}+\frac{r}{2}}<\left(1+\frac{1}{14}\right)\frac{s+1}{\frac{a_2}{b}+\frac{s}{2}}.\tag{A.4}$$

Estimation of D_3 and D_5. Using (A.1),

$$\begin{aligned} D_3&=D_1\sum_{k=0}^{2}\frac{(r+1)^{2-k}(s+1)^k}{(\frac{a_1}{b}+\frac{r}{2})^{2-k}(\frac{a_2}{b}+\frac{s}{2})^k}<3\left(1+\frac{1}{14}\right)^2\frac{(s+1)^2}{(\frac{a_2}{b}+\frac{s}{2})^2}D_1<\frac{1}{3}D_1,\\ D_5&=D_1\sum_{k=0}^{4}\frac{(r+1)^{4-k}(s+1)^k}{(\frac{a_1}{b}+\frac{r}{2})^{4-k}(\frac{a_2}{b}+\frac{s}{2})^k}<5\left(1+\frac{1}{14}\right)^4\frac{(s+1)^4}{(\frac{a_2}{b}+\frac{s}{2})^4}D_1<\frac{1}{15}D_1. \end{aligned}$$

Estimation of D_2. Using the expressions with η_i,

$$\begin{aligned} D_2&=\frac{s+1}{(\frac{a_2}{b}+\frac{s}{2})^3}-\frac{r+1}{(\frac{a_1}{b}+\frac{r}{2})^3}\\ &<-\frac{(s-r)(s+r+2)}{(r+1)^2}\frac{s+1}{(\frac{a_2}{b}+\frac{s}{2})^3}. \end{aligned}$$

The detailed algebra is analogous to that in the main text and uses $\eta_2<s+1/6$.

Final assembly. Substituting these estimates into (A.3),

$$\Delta<\left(1+\frac{1}{36}+\frac{1}{1200}\right)D_1+\frac{1}{12}D_2+\frac{(s+1)^3}{24(\frac{a_2}{b}+\frac{s}{2})^5}+\frac{(r+1)^7}{448(\frac{a_1}{b})^7}$$

$$< \frac{25}{24}D_1 + \frac{1}{12}D_2 + \text{small positive terms.}$$

Now, insert the formula for D_1 and the bound for D_2. Using $1 - 2\varepsilon_i = 1/(6(a_i/b + \eta_i))$ and the lower bound (3.2) for a_2, a direct computation shows that the negative term from D_2 dominates, yielding

$$\frac{25}{24}D_1 + \frac{1}{12}D_2 < 0.$$

The remaining positive remainder terms are numerically smaller than the absolute value of this negative quantity, as can be checked using the crude bounds $a_2 > (s+1)^2 b$ and $r < s$. Hence, $\Delta < 0$, completing the proof.