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Research article

Complex deformable Rolle, mean value, Flett, and Sahoo-Riedel theorems in the complex plane

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Abstract: In this paper, we proved analogues of Rolle's theorem, the mean value theorem, Flett's theorem, and the Sahoo-Riedel theorem for λ-deformable analytic functions in the complex plane. The main tool was the identity

$$D_\lambda f(z) - \delta f(z) = \lambda f'(z),$$

which connects the λ-complex deformable derivative to the classical complex derivative. A key observation was that the natural vanishing condition in the deformable setting is not $D_\lambda f(z) = 0$, but $D_\lambda f(z) - \delta f(z) = 0$. We also characterized constant functions in terms of the complex deformable derivative and proved that the complex deformable Rolle theorem and the complex deformable mean value theorem are equivalent. Unlike Flett's theorem, the Sahoo-Riedel theorem requires no boundary condition on the derivative and thus applies to all λ-deformable analytic functions. All results reduce to their classical complex counterparts when $\lambda = 1$.

Keywords: complex deformable calculus; Rolle's theorem; mean value theorem; Flett's theorem; Sahoo-Riedel theorem; λ-deformable analytic function
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1. Introduction

Rolle's theorem and the mean value theorem are fundamental results in real analysis. They have many applications in the study of differential equations, optimization, and numerical analysis [1–3]. In the real case, these theorems are well-understood. However, their extensions to the complex plane are not straightforward. It is well-known that the classical Rolle theorem fails for holomorphic functions. A simple example is the function

$$f(z) = e^z - 1,$$

which vanishes at $z = 2k\pi i$ for every $k \in \mathbb{Z}$. However, its derivative $f'(z) = e^z$ has no zero in the complex plane. Therefore, one cannot expect a point c such that $f'(c) = 0$ in the complex setting.

The first complex analogue of Rolle's theorem was given by Evard and Jafari [4]. They proved that if f is holomorphic on an open convex subset Ω of $\mathbb{C}$ and $f(a) = f(b) = 0$ for two distinct points $a, b \in \Omega$, then there exist points z_1, z_2 on the open line segment (a, b) such that

$$\mathrm{Re}(f'(z_1)) = 0 \quad \text{and} \quad \mathrm{Im}(f'(z_2)) = 0.$$

As a consequence, they also obtained a complex mean value theorem. Later, Çakmak and Tiryaki [5] proved that these two theorems are equivalent in the complex plane.

Beyond the Rolle and mean value theorems, there are other important refinements in real analysis. In 1958, Flett [6] proved that if $f : [a, b] \to \mathbb{R}$ is differentiable and $f'(a) = f'(b)$, then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(c) - f(a)}{c - a}.$$

This result is a refinement of the mean value theorem that uses a boundary condition on the derivative instead of the function values. Sahoo and Riedel [3] later removed this boundary condition entirely, proving that for any differentiable $f : [a, b] \to \mathbb{R}$, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(c) - f(a)}{c - a} + \frac{f'(b) - f'(a)}{2(b - a)}(c - a).$$

This makes the Sahoo-Riedel theorem more general than Flett's theorem. A complex analogue of Flett's theorem for holomorphic functions was established by Davitt et al. [7], who proved a version that reduces to the complex Flett theorem when $f'(a) = f'(b)$. However, no complex analogue of these results exists in the deformable setting.

Fractional and generalized derivatives have attracted considerable attention in recent decades [8–10]. Among these, Khalil et al. [11] introduced the conformable derivative, which is defined by a limit and satisfies

$$T_\alpha(f)(x) = \lim_{h \to 0} \frac{f(x + hx^{1-\alpha}) - f(x)}{h}.$$

Zulfeqarr et al. [12] introduced the deformable derivative for real-valued functions. For a real function f and a parameter $\lambda \in (0, 1]$ with $\lambda + \delta = 1$, the deformable derivative is defined by

$$D_\lambda f(t) = \lim_{\varepsilon \to 0} \frac{(1 + \varepsilon\delta) f(t + \varepsilon\lambda) - f(t)}{\varepsilon}.$$

A key property of this derivative is the identity

$$D_\lambda f(t) = \delta f(t) + \lambda f'(t),$$

which shows that the deformable derivative is a linear combination of the function and its classical derivative. In particular, the real deformable Rolle theorem of Zulfeqarr et al. [12] states that if $f : [a, b] \to \mathbb{R}$ is continuous, differentiable, and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $D_\lambda f(c) = \delta f(c)$, which is equivalent to $f'(c) = 0$.

The conformable derivative was extended to the complex plane by Uçar and Özgür [13]. They studied the complex conformable Rolle theorem and mean value theorem for α-holomorphic functions.

The conformable versions of Flett's theorem and the Sahoo-Riedel theorem for real functions were obtained by Uçar [14]. The complex deformable derivative was introduced by Çakmak [15]. For a complex-valued function f and $\lambda \in (0, 1]$ with $\lambda + \delta = 1$, it is defined by

$$D_\lambda f(z) = \lim_{\varepsilon \to 0} \frac{(1 + \varepsilon\delta) f(z + \lambda\varepsilon) - f(z)}{\varepsilon}.$$

When $\lambda = 1$, this reduces to the classical complex derivative. It was shown in [15] that f is λ-complex deformable differentiable at a point if and only if it is complex differentiable at that point, and

$$D_\lambda f(z) = \delta f(z) + \lambda f'(z).$$

This identity is the main tool of the present paper.

The present paper extends these results to the complex deformable setting. We prove complex deformable analogues of Rolle's theorem, the mean value theorem, Flett's theorem, and the Sahoo-Riedel theorem for λ-deformable analytic functions, extending the work of [12] on real deformable derivatives and [13, 14] on conformable derivatives to the complex deformable case. Specifically, we identify $D_\lambda f(z) = \delta f(z)$ as the correct vanishing condition in the deformable setting, and use the identity $D_\lambda f - \delta f = \lambda f'$ to reduce each theorem to its classical counterpart. The proofs apply real deformable theorems from [12, 14] to auxiliary real-valued functions. All results recover the classical complex results when $\lambda = 1$, including the results of [4, 5, 7].

A key observation is that the natural vanishing condition in the deformable setting is not $D_\lambda f(z) = 0$, but

$$D_\lambda f(z) - \delta f(z) = 0,$$

which is equivalent to $f'(z) = 0$. This is in contrast to the conformable case, where the vanishing condition $T_\alpha(f)(z) = 0$ is directly equivalent to $f'(z) = 0$. The presence of the term $\delta f(z)$ is a fundamental feature of the deformable derivative that distinguishes it from both the classical derivative and the conformable derivative. In particular, a naive attempt to copy the classical or conformable proofs without recognizing this feature would lead to incorrect statements, since $D_\lambda f(z) = 0$ does not characterize zeros of f' and does not imply that f is constant.

The paper is organized as follows. In Section 2, we recall the definition and basic properties of the complex deformable derivative. In Section 3, we prove the complex deformable Rolle theorem. In Section 4, we prove the complex deformable mean value theorem, characterize constant functions, and establish the equivalence of the Rolle and mean value theorems. In Section 5, we prove the complex deformable Flett theorem. In Section 6, we prove the complex deformable Sahoo-Riedel theorem.

2. Preliminaries

Throughout this section, we recall the basic definitions and results from [15].

Let $z = x + iy$, where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$. For two distinct points $a, b \in \mathbb{C}$, we define

$$[a, b] = \{a + t(b - a) : 0 \le t \le 1\} \quad \text{and} \quad (a, b) = \{a + t(b - a) : 0 < t < 1\}.$$

Definition 2.1. *Let f be a complex-valued function and let $\lambda \in (0, 1]$. Set $\delta = 1 - \lambda$. The* λ-complex deformable derivative *of f at z is defined by*

$$D_\lambda f(z) = \lim_{\varepsilon \to 0} \frac{(1 + \varepsilon\delta) f(z + \lambda\varepsilon) - f(z)}{\varepsilon},$$

provided the limit exists.

Definition 2.2. *A function f is called λ-deformable analytic in a domain Ω if $D_\lambda f(z)$ exists for every $z \in \Omega$.*

Theorem 2.3. *Let f be complex differentiable at $z_0 \in \mathbb{C}$. Then f is λ-complex deformable differentiable at z_0, and*

$$D_\lambda f(z_0) = \delta f(z_0) + \lambda f'(z_0).$$

Theorem 2.4. *A function f is λ-complex deformable differentiable at z_0 if and only if it is complex differentiable at z_0.*

Corollary 2.5. *A function f is λ-deformable analytic in Ω if and only if it is analytic in Ω.*

Remark 2.6. *For every λ-deformable analytic function f,*

$$D_\lambda f(z) - \delta f(z) = \lambda f'(z)$$

for every $z \in \Omega$. In particular, $D_\lambda f(z) - \delta f(z) = 0$ if and only if $f'(z) = 0$.

Remark 2.7. *The complex conformable derivative of Uçar and Özgür [13] satisfies $T_\alpha(f)(z) = \lambda f'(z)$, so the condition $T_\alpha(f)(z) = 0$ is equivalent to $f'(z) = 0$. For the complex deformable derivative, the condition $D_\lambda f(z) = 0$ gives $\lambda f'(z) + \delta f(z) = 0$, which is not equivalent to $f'(z) = 0$. This is why all results in this paper are stated in terms of $D_\lambda f(z) - \delta f(z)$ rather than $D_\lambda f(z)$ itself. The deformable derivative has structural advantages over the conformable derivative [11, 16]. The conformable derivative $T_\alpha(f)(z) = z^{1-\alpha} f'(z)$ is undefined at $z = 0$, whereas the deformable derivative $D_\lambda f(z) = \delta f(z) + \lambda f'(z)$ is defined on all of $\mathbb{C}$. Moreover, the deformable derivative interpolates between the function ($\lambda = 0$) and its classical derivative ($\lambda = 1$). Conformable derivatives in the complex setting have been studied in [13, 17], and fractional derivatives in the complex plane appear in [18].*

3. Complex deformable Rolle theorem

In classical complex analysis, Rolle's theorem does not hold in the usual sense. Evard and Jafari [4] overcame this by considering the real and imaginary parts of f' separately. In the conformable setting, Uçar and Özgür [13] followed the same approach. Here we apply it to the complex deformable derivative.

In the real deformable setting, Zulfeqarr et al. [12] proved that if $f : [a, b] \to \mathbb{R}$ is continuous, differentiable, and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that

$$D_\lambda f(c) = \delta f(c),$$

which is equivalent to $f'(c) = 0$. The proof reduces to the classical real Rolle theorem via the identity $D_\lambda f = \delta f + \lambda f'$. We follow the same strategy in the complex setting. By Corollary 2.5, every λ-deformable analytic function is analytic, so the Cauchy-Riemann equations are available. By Remark 2.6, the natural quantity to consider is $D_\lambda f - \delta f = \lambda f'$. This reduces the problem to finding zeros of $\operatorname{Re} f'$ and $\operatorname{Im} f'$, which is achieved by applying the real deformable Rolle theorem of Zulfeqarr et al. [12] to two auxiliary real-valued functions constructed from u and v.

Theorem 3.1 (Complex deformable Rolle theorem)**.** *Let $0 < \lambda \leq 1$, $\delta = 1 - \lambda$, and let Ω be an open convex subset of $\mathbb{C}$. Let f be λ-deformable analytic in Ω. Suppose $a, b \in \Omega$, $a \neq b$, and $f(a) = f(b)$. Then there exist $z_1, z_2 \in (a, b)$ such that*

$$\operatorname{Re}(D_\lambda f(z_1) - \delta f(z_1)) = 0 \quad \textit{and} \quad \operatorname{Im}(D_\lambda f(z_2) - \delta f(z_2)) = 0.$$

Equivalently,

$$\operatorname{Re} D_\lambda f(z_1) = \delta \operatorname{Re} f(z_1) \quad \textit{and} \quad \operatorname{Im} D_\lambda f(z_2) = \delta \operatorname{Im} f(z_2).$$

Proof. By Corollary 2.5, f is analytic in Ω. Write $f = u + iv$, $a = a_1 + ia_2$, and $b = b_1 + ib_2$, and set $p = b_1 - a_1$, $q = b_2 - a_2$. Since $a \neq b$, we have $p^2 + q^2 \neq 0$, and since Ω is convex, $[a, b] \subset \Omega$.

Define the real-valued function

$$\phi(t) = p\, u(a + t(b - a)) + q\, v(a + t(b - a)), \qquad t \in [0, 1].$$

Since $f(a) = f(b)$, we have $\phi(0) = \phi(1)$. Since f is analytic, u and v are differentiable, hence ϕ is continuous and differentiable on $[0, 1]$. Thus ϕ satisfies all the hypotheses of the real deformable Rolle theorem of Zulfeqarr et al. [12], so there exists $t_1 \in (0, 1)$ such that

$$D_\lambda \phi(t_1) = \delta \phi(t_1).$$

Since $D_\lambda \phi(t_1) = \delta \phi(t_1) + \lambda \phi'(t_1)$, this gives $\phi'(t_1) = 0$. Setting $z_1 = a + t_1(b - a)$ and applying the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$, a direct computation gives

$$\phi'(t_1) = (p^2 + q^2)\, u_x(z_1).$$

Since $p^2 + q^2 \neq 0$, we get $u_x(z_1) = \operatorname{Re} f'(z_1) = 0$. By Remark 2.6,

$$\operatorname{Re}(D_\lambda f(z_1) - \delta f(z_1)) = \lambda \operatorname{Re} f'(z_1) = 0.$$

Now define the real-valued function

$$\psi(t) = p\, v(a + t(b - a)) - q\, u(a + t(b - a)), \qquad t \in [0, 1].$$

Since $f(a) = f(b)$, we have $\psi(0) = \psi(1)$. By the same argument, ψ is continuous and differentiable on $[0, 1]$. By the real deformable Rolle theorem [12], there exists $t_2 \in (0, 1)$ such that $D_\lambda \psi(t_2) = \delta \psi(t_2)$, which gives $\psi'(t_2) = 0$. Setting $z_2 = a + t_2(b - a)$ and applying the Cauchy-Riemann equations gives

$$\psi'(t_2) = (p^2 + q^2)\, v_x(z_2).$$

Hence $v_x(z_2) = \operatorname{Im} f'(z_2) = 0$, and by Remark 2.6,

$$\operatorname{Im}(D_\lambda f(z_2) - \delta f(z_2)) = \lambda \operatorname{Im} f'(z_2) = 0. \qquad \square$$

Remark 3.2. *The conclusion of Theorem 3.1 can equivalently be written as*

$$\operatorname{Re} D_\lambda f(z_1) = \delta \operatorname{Re} f(z_1) \quad \textit{and} \quad \operatorname{Im} D_\lambda f(z_2) = \delta \operatorname{Im} f(z_2).$$

When f is real-valued, taking $z_1 = c$ recovers the real deformable Rolle theorem of Zulfeqarr et al. [12], namely, $D_\lambda f(c) = \delta f(c)$. When $\lambda = 1$, we have $\delta = 0$ and the conclusion reduces to $\operatorname{Re} f'(z_1) = 0$ and $\operatorname{Im} f'(z_2) = 0$, which is the classical complex Rolle theorem of Evard and Jafari [4].

Remark 3.3. *In Theorem 3.1, $z_1 = z_2$ if and only if $f'(z_0) = 0$ at that common point z_0, since this requires $\operatorname{Re} f'(z_0) = 0$ and $\operatorname{Im} f'(z_0) = 0$ simultaneously. For example, taking $f(z) = \sin(\pi z)$, $a = 0$, $b = 2$, we have $f(0) = f(2) = 0$ and $f'(1/2) = 0$, so one may take $z_1 = z_2 = 1/2$. In general, z_1 and z_2 need not coincide: for $f(z) = e^{2\pi iz} - 1$, $a = 0$, $b = 1$ (see Example 3.4), $\operatorname{Re} f'$ vanishes at $1/2$ and $\operatorname{Im} f'$ vanishes at $1/4$, so $z_1 \neq z_2$.*

The following example illustrates Theorem 3.1. The real and imaginary parts of $D_\lambda f(x) - \delta f(x)$ are shown in Figures 1 and 2, where the zeros at $z_1 = 1/2$ and $z_2 = 1/4$ confirm the conclusion of the theorem.

Example 3.4. *Let $f(z) = e^{2\pi iz} - 1$, $\lambda = 1/2$, $\delta = 1/2$, $a = 0$, $b = 1$. Then $f(0) = f(1) = 0$. Since $f'(z) = 2\pi i\, e^{2\pi iz}$, restricting to the real axis gives $\operatorname{Re} f'(x) = -2\pi \sin 2\pi x$, which vanishes at $x = 1/2$. Also $\operatorname{Im} f'(x) = 2\pi \cos 2\pi x$, which vanishes at $x = 1/4$. Setting $z_1 = 1/2$ and $z_2 = 1/4$, Remark 2.6 gives*

$$\operatorname{Re}(D_\lambda f(z_1) - \tfrac{1}{2} f(z_1)) = \tfrac{1}{2} \operatorname{Re} f'(z_1) = 0$$

and

$$\operatorname{Im}(D_\lambda f(z_2) - \tfrac{1}{2} f(z_2)) = \tfrac{1}{2} \operatorname{Im} f'(z_2) = 0,$$

confirming Theorem 3.1.

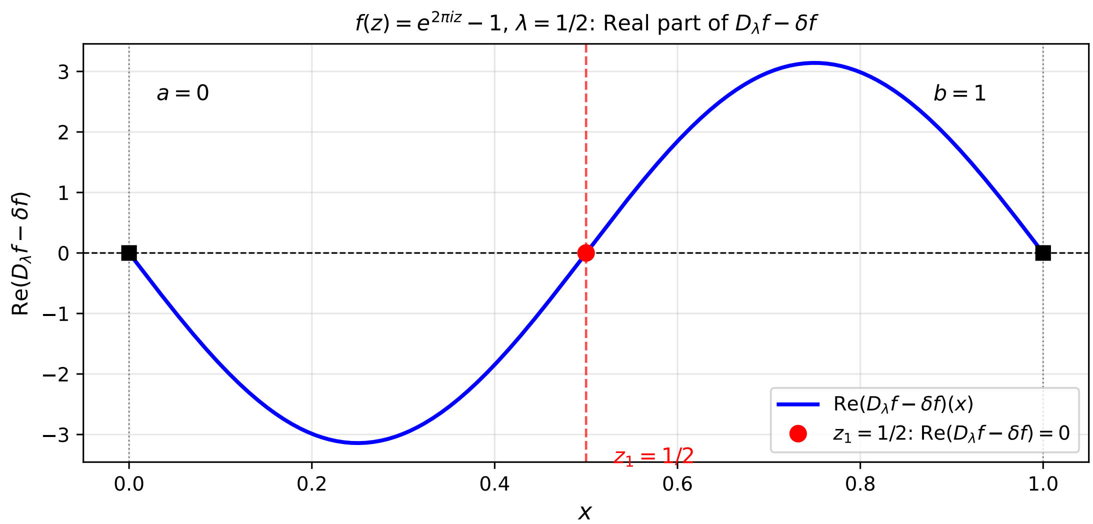

Figure 1. Real part of $D_\lambda f(x) - \delta f(x)$ for $f(z) = e^{2\pi iz} - 1$, $\lambda = 1/2$, $a = 0$, $b = 1$. The red dot at $z_1 = 1/2$ marks the point where $\operatorname{Re}(D_\lambda f(z_1) - \delta f(z_1)) = 0$, confirming Theorem 3.1.

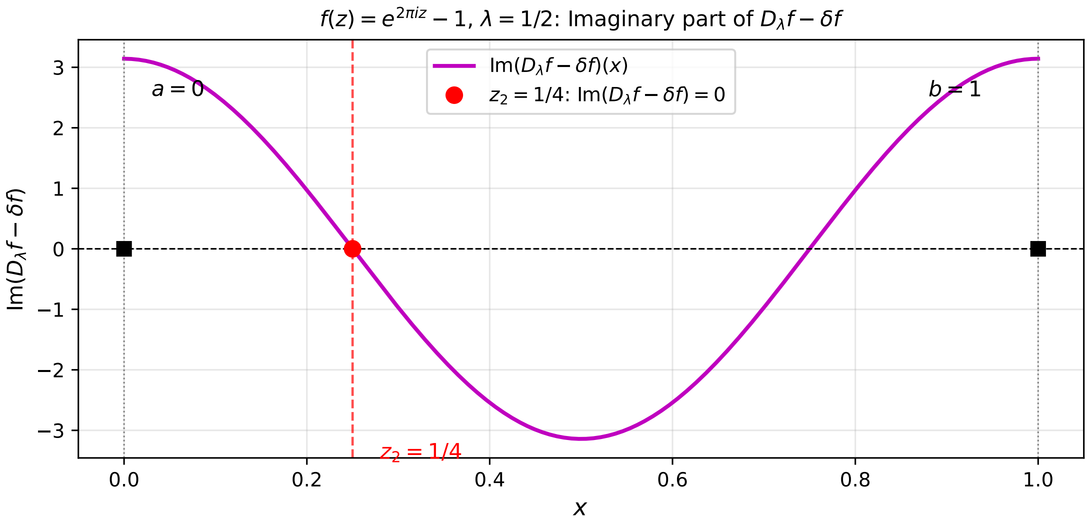

Figure 2. Imaginary part of $D_\lambda f(x) - \delta f(x)$ for $f(z) = e^{2\pi i z} - 1$, $\lambda = 1/2$, $a = 0$, $b = 1$. The red dot at $z_2 = 1/4$ marks the point where $\mathrm{Im}(D_\lambda f(z_2) - \delta f(z_2)) = 0$, confirming Theorem 3.1.

4. Complex deformable mean value theorem and related results

The mean value theorem follows from Rolle's theorem by subtracting a linear function that forces the boundary values to coincide. We apply the same idea here. In the real deformable setting, Zulfeqarr et al. [12] obtained the mean value theorem as a consequence of their Rolle theorem. We follow the same approach in the complex setting.

Theorem 4.1 (Complex deformable mean value theorem). *Let $0 < \lambda \leq 1$, $\delta = 1 - \lambda$, and let Ω be an open convex subset of $\mathbb{C}$. Let f be λ-deformable analytic in Ω. If $a, b \in \Omega$ and $a \neq b$, then there exist $z_1, z_2 \in (a, b)$ such that*

$$\mathrm{Re}(D_\lambda f(z_1) - \delta f(z_1)) = \lambda \,\mathrm{Re}\left(\frac{f(b) - f(a)}{b - a}\right)$$

and

$$\mathrm{Im}(D_\lambda f(z_2) - \delta f(z_2)) = \lambda \,\mathrm{Im}\left(\frac{f(b) - f(a)}{b - a}\right).$$

Proof. Set $A = (f(b) - f(a))/(b - a)$ and define $g(z) = f(z) - f(a) - A(z - a)$. Then g is λ-deformable analytic in Ω with $g(a) = g(b) = 0$. By Theorem 3.1, there exist $z_1, z_2 \in (a, b)$ such that

$$\mathrm{Re}(D_\lambda g(z_1) - \delta g(z_1)) = 0 \quad \text{and} \quad \mathrm{Im}(D_\lambda g(z_2) - \delta g(z_2)) = 0.$$

By Remark 2.6,

$$D_\lambda g(z) - \delta g(z) = \lambda g'(z) = \lambda(f'(z) - A) = (D_\lambda f(z) - \delta f(z)) - \lambda A.$$

Taking real parts at z_1 and imaginary parts at z_2 gives the result. □

Remark 4.2. *When $\lambda = 1$, Theorems 3.1 and 4.1 reduce to the classical results of Evard and Jafari [4].*

Remark 4.3. *The factor λ on the right-hand side of Theorem 4.1 has a natural interpretation. Since* $\mathrm{Re}(D_\lambda f(z_1) - \delta f(z_1)) = \lambda \,\mathrm{Re}\, f'(z_1)$ *by Remark 2.6, dividing both sides by λ gives* $\mathrm{Re}\, f'(z_1) = \mathrm{Re}(\frac{f(b)-f(a)}{b-a})$*, which is the classical complex mean value condition of Evard and Jafari [4]. Thus the factor λ reflects the deformable scaling on both sides and does not change the geometric content of the theorem.*

The following result characterizes constant functions in the deformable setting.

Corollary 4.4. *Let $0 < \lambda \leq 1$, $\delta = 1 - \lambda$, and Ω be a connected domain in $\mathbb{C}$. Let f be λ-deformable analytic in Ω. If $D_\lambda f(z) = \delta f(z)$ for every $z \in \Omega$, then f is constant.*

Proof. By Remark 2.6, the hypothesis gives $\lambda f'(z) = 0$ on Ω. Since $\lambda > 0$, we have $f'(z) = 0$ on Ω, and since Ω is connected, f is constant. □

Remark 4.5. *The condition $D_\lambda f(z) = 0$ does not imply that f is constant. It gives $\lambda f'(z) + \delta f(z) = 0$, which has nonzero solutions $f(z) = Ce^{-(\delta/\lambda)z}$. The correct condition is $D_\lambda f(z) = \delta f(z)$. This is consistent with the real deformable case: Zulfeqarr et al. [12] showed that $D_\lambda f(c) = \delta f(c)$ is equivalent to $f'(c) = 0$, which is the correct vanishing condition.*

We now show that Theorems 3.1 and 4.1 are equivalent.

Theorem 4.6. *Let $0 < \lambda \leq 1$, $\delta = 1 - \lambda$, and Ω be an open convex subset of $\mathbb{C}$. For λ-deformable analytic functions in Ω, Theorems 3.1 and 4.1 are equivalent.*

Proof. Assume Theorem 3.1 holds. For f with $a, b \in \Omega$ and $a \neq b$, set $A = (f(b) - f(a))/(b - a)$ and $g(z) = f(z) - f(a) - A(z - a)$. Then $g(a) = g(b) = 0$, and Theorem 3.1 gives $z_1, z_2 \in (a, b)$ with

$$\mathrm{Re}(D_\lambda g(z_1) - \delta g(z_1)) = 0 \quad \text{and} \quad \mathrm{Im}(D_\lambda g(z_2) - \delta g(z_2)) = 0.$$

Since $D_\lambda g - \delta g = (D_\lambda f - \delta f) - \lambda A$ by Remark 2.6, the conclusion of Theorem 4.1 follows.

Conversely, assume Theorem 4.1 holds. If $f(a) = f(b)$, then $(f(b) - f(a))/(b - a) = 0$, and Theorem 4.1 directly gives the conclusion of Theorem 3.1. □

5. Complex deformable Flett theorem

Flett's theorem [6] is a refinement of the mean value theorem. Unlike the mean value theorem, it does not require $f(a) = f(b)$. Instead, it imposes a boundary condition on the derivative. The conformable version of Flett's theorem for real functions was proved by Uçar [14]. In this section, we prove a complex deformable version.

As in the previous sections, the natural boundary condition involves $D_\lambda f - \delta f$ rather than $D_\lambda f$ itself. The conclusion is stated in the same spirit as Theorems 3.1 and 4.1, using two points on the segment. In the proof, we apply the real deformable Flett theorem to two auxiliary real-valued functions, following the same strategy as in the proof of Theorem 3.1.

Theorem 5.1 (Complex deformable Flett theorem)**.** *Let $0 < \lambda \leq 1$, $\delta = 1 - \lambda$, and let Ω be an open convex subset of $\mathbb{C}$. Let f be λ-deformable analytic in Ω. Suppose $a, b \in \Omega$, $a \neq b$, and*

$$D_\lambda f(a) - \delta f(a) = D_\lambda f(b) - \delta f(b).$$

Then there exist $z_1, z_2 \in (a,b)$ *such that*

$$\mathrm{Re}(D_\lambda f(z_1) - \delta f(z_1)) = \lambda\, \mathrm{Re}\left(\frac{f(z_1) - f(a)}{z_1 - a}\right)$$

and

$$\mathrm{Im}(D_\lambda f(z_2) - \delta f(z_2)) = \lambda\, \mathrm{Im}\left(\frac{f(z_2) - f(a)}{z_2 - a}\right).$$

Proof. By Remark 2.6, the boundary condition is equivalent to $f'(a) = f'(b)$. By Corollary 2.5, f is analytic in Ω. Write $f = u + iv$, $a = a_1 + ia_2$, and $b = b_1 + ib_2$, and set $p = b_1 - a_1$, $q = b_2 - a_2$. Since $a \neq b$, we have $p^2 + q^2 \neq 0$, and since Ω is convex, $[a,b] \subset \Omega$.

Consider the real-valued function

$$F(t) = p\, u(a + t(b-a)) + q\, v(a + t(b-a)), \qquad t \in [0,1].$$

Since f is analytic, u and v are infinitely differentiable, so F is infinitely differentiable on $[0,1]$. In particular, F'' exists. A direct computation using the chain rule and the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ gives

$$F'(t) = (p^2 + q^2)\, u_x(a + t(b-a)).$$

Since $f'(a) = f'(b)$, we have $u_x(a) = u_x(b)$, so $F'(0) = F'(1)$. Thus F satisfies all the hypotheses of the real deformable Flett theorem of Uçar [14], so there exists $t_1 \in (0,1)$ such that

$$F'(t_1) = \frac{F(t_1) - F(0)}{t_1}.$$

Set $z_1 = a + t_1(b-a)$, so that $z_1 - a = t_1(b-a)$. We have $F'(t_1) = (p^2+q^2)\, u_x(z_1)$ and since $z_1 - a = t_1(p + iq)$,

$$\begin{aligned}
\mathrm{Re}\left(\frac{f(z_1) - f(a)}{z_1 - a}\right) &= \mathrm{Re}\left(\frac{f(z_1) - f(a)}{t_1(p + iq)}\right) \\
&= \frac{1}{t_1}\, \mathrm{Re}\left(\frac{(p - iq)(f(z_1) - f(a))}{p^2 + q^2}\right) \\
&= \frac{p(u(z_1) - u(a)) + q(v(z_1) - v(a))}{t_1(p^2 + q^2)}.
\end{aligned}$$

Therefore $F(t_1) - F(0) = (p^2 + q^2) \cdot t_1 \cdot \mathrm{Re}\left(\frac{f(z_1) - f(a)}{z_1 - a}\right)$. Combining with $F'(t_1) = (p^2 + q^2) u_x(z_1)$ and dividing by $p^2 + q^2$,

$$u_x(z_1) = \mathrm{Re}\, f'(z_1) = \mathrm{Re}\left(\frac{f(z_1) - f(a)}{z_1 - a}\right).$$

By Remark 2.6,

$$\mathrm{Re}(D_\lambda f(z_1) - \delta f(z_1)) = \lambda\, \mathrm{Re}\, f'(z_1) = \lambda\, \mathrm{Re}\left(\frac{f(z_1) - f(a)}{z_1 - a}\right).$$

For the imaginary part, consider the real-valued function

$$G(t) = p\, v(a + t(b-a)) - q\, u(a + t(b-a)), \qquad t \in [0,1].$$

By the Cauchy-Riemann equations, $G'(t) = (p^2 + q^2)\, v_x(a + t(b - a))$. Since $f'(a) = f'(b)$ implies $v_x(a) = v_x(b)$, we have $G'(0) = G'(1)$. By the real deformable Flett theorem [14], there exists $t_2 \in (0, 1)$ such that $G'(t_2) = (G(t_2) - G(0))/t_2$. Set $z_2 = a + t_2(b - a)$. Since

$$\operatorname{Im}\left(\frac{f(z_2) - f(a)}{z_2 - a}\right) = \frac{p(v(z_2) - v(a)) - q(u(z_2) - u(a))}{t_2(p^2 + q^2)} = \frac{G(t_2) - G(0)}{t_2(p^2 + q^2)},$$

the same argument gives

$$v_x(z_2) = \operatorname{Im} f'(z_2) = \operatorname{Im}\left(\frac{f(z_2) - f(a)}{z_2 - a}\right).$$

By Remark 2.6,

$$\operatorname{Im}(D_\lambda f(z_2) - \delta f(z_2)) = \lambda \operatorname{Im} f'(z_2) = \lambda \operatorname{Im}\left(\frac{f(z_2) - f(a)}{z_2 - a}\right).$$

The proof is complete. □

Remark 5.2. *When $\lambda = 1$, the boundary condition reduces to $f'(a) = f'(b)$, and the conclusion takes the form*

$$\operatorname{Re} f'(z_1) = \operatorname{Re}\left(\frac{f(z_1) - f(a)}{z_1 - a}\right) \quad \textit{and} \quad \operatorname{Im} f'(z_2) = \operatorname{Im}\left(\frac{f(z_2) - f(a)}{z_2 - a}\right).$$

This coincides with the Corollary of Davitt et al. [7], which is the classical complex Flett theorem for holomorphic functions.

Remark 5.3. *Theorem 5.1 generalizes the conformable Flett theorem of Uçar [14] to the complex deformable setting. The boundary condition $D_\lambda f(a) - \delta f(a) = D_\lambda f(b) - \delta f(b)$ is equivalent to $f'(a) = f'(b)$ by Remark 2.6, which is the natural deformable analogue of the conformable boundary condition $g^{(\alpha)}(a) = g^{(\alpha)}(b)$.*

Example 5.4. *Let $f(z) = z^3/3$, $\lambda = 1/2$, $\delta = 1/2$, $a = -1$, $b = 1$. Then $f'(z) = z^2$, so $f'(-1) = 1 = f'(1)$. The boundary condition in deformable form is*

$$D_\lambda f(-1) - \delta f(-1) = \lambda f'(-1) = \tfrac{1}{2} \quad \textit{and} \quad D_\lambda f(1) - \delta f(1) = \lambda f'(1) = \tfrac{1}{2},$$

so the hypothesis of Theorem 5.1 is satisfied. Restricting to the real axis, the condition $\operatorname{Re} f'(x) = \operatorname{Re}((f(x) - f(-1))/(x + 1))$ becomes $x^2 = (x^2 - x + 1)/3$, which gives $2x^2 + x - 1 = 0$, so $x = 1/2$. Taking $z_1 = 1/2 \in (-1, 1)$ confirms the real part condition of Theorem 5.1.

6. Complex deformable Sahoo-Riedel theorem

The Sahoo-Riedel theorem [3] is a generalization of Flett's theorem that requires no boundary condition on the derivative. The conformable version for real functions was proved by Uçar [14]. In this section, we prove a complex deformable version by applying the real deformable Sahoo-Riedel theorem to two auxiliary real-valued functions, following the same strategy as in the proofs of Theorems 3.1 and 5.1.

Theorem 6.1 (Complex deformable Sahoo-Riedel theorem)**.** *Let $0 < \lambda \le 1$, $\delta = 1 - \lambda$, and Ω be an open convex subset of $\mathbb{C}$. Let f be λ-deformable analytic in Ω. If $a, b \in \Omega$ and $a \neq b$, then there exist $z_1, z_2 \in (a, b)$ such that*

$$\mathrm{Re}(D_\lambda f(z_1) - \delta f(z_1)) = \lambda\,\mathrm{Re}\left(\frac{f(z_1) - f(a)}{z_1 - a}\right) + \frac{\mathrm{Re}(D_\lambda f(b) - \delta f(b)) - \mathrm{Re}(D_\lambda f(a) - \delta f(a))}{2(b-a)} \cdot (z_1 - a)$$

and

$$\mathrm{Im}(D_\lambda f(z_2) - \delta f(z_2)) = \lambda\,\mathrm{Im}\left(\frac{f(z_2) - f(a)}{z_2 - a}\right) + \frac{\mathrm{Im}(D_\lambda f(b) - \delta f(b)) - \mathrm{Im}(D_\lambda f(a) - \delta f(a))}{2(b-a)} \cdot (z_2 - a).$$

Remark 6.2. *Since $z_1 = a + t_1(b-a)$ for some $t_1 \in (0, 1)$, the ratio $(z_1 - a)/(b - a) = t_1$ is a real number. Hence all terms on the right-hand side of Theorem 6.1 are real quantities.*

Proof. By Corollary 2.5, f is analytic in Ω. Write $f = u + iv$, $a = a_1 + ia_2$, and $b = b_1 + ib_2$, and set $p = b_1 - a_1$, $q = b_2 - a_2$. Since $a \neq b$, we have $p^2 + q^2 \neq 0$, and since Ω is convex, $[a, b] \subset \Omega$.

Consider the real-valued function

$$F(t) = p\,u(a + t(b-a)) + q\,v(a + t(b-a)), \qquad t \in [0, 1].$$

Since f is analytic, F is infinitely differentiable on $[0, 1]$. In particular, F'' exists. By the Cauchy-Riemann equations, $F'(t) = (p^2 + q^2)\,u_x(a + t(b-a))$. By the real Sahoo-Riedel theorem [3], there exists $t_1 \in (0, 1)$ such that

$$F(t_1) - F(0) = t_1 F'(t_1) - \frac{F'(1) - F'(0)}{2}\,t_1^2.$$

Set $z_1 = a + t_1(b-a)$, so $z_1 - a = t_1(b-a)$ and $t_1 = (z_1 - a)/(b-a)$. As shown in the proof of Theorem 5.1,

$$F(t_1) - F(0) = (p^2 + q^2) \cdot t_1 \cdot \mathrm{Re}\left(\frac{f(z_1) - f(a)}{z_1 - a}\right).$$

Also,

$$F'(t_1) = (p^2 + q^2)\,\mathrm{Re}\,f'(z_1), \quad F'(1) - F'(0) = (p^2 + q^2)(\mathrm{Re}\,f'(b) - \mathrm{Re}\,f'(a)).$$

Substituting and dividing by $(p^2 + q^2) \cdot t_1$,

$$\mathrm{Re}\left(\frac{f(z_1) - f(a)}{z_1 - a}\right) = \mathrm{Re}\,f'(z_1) - \frac{\mathrm{Re}\,f'(b) - \mathrm{Re}\,f'(a)}{2} \cdot t_1.$$

Rearranging and substituting $t_1 = (z_1 - a)/(b - a)$,

$$\mathrm{Re}\,f'(z_1) = \mathrm{Re}\left(\frac{f(z_1) - f(a)}{z_1 - a}\right) + \frac{\mathrm{Re}\,f'(b) - \mathrm{Re}\,f'(a)}{2(b-a)} \cdot (z_1 - a).$$

Multiplying by λ and using Remark 2.6,

$$\mathrm{Re}(D_\lambda f(z_1) - \delta f(z_1)) = \lambda\,\mathrm{Re}\left(\frac{f(z_1) - f(a)}{z_1 - a}\right) + \frac{\mathrm{Re}(D_\lambda f(b) - \delta f(b)) - \mathrm{Re}(D_\lambda f(a) - \delta f(a))}{2(b-a)} \cdot (z_1 - a).$$

For the imaginary part, consider the real-valued function

$$G(t) = p\,v(a + t(b - a)) - q\,u(a + t(b - a)), \qquad t \in [0, 1].$$

By the Cauchy-Riemann equations, $G'(t) = (p^2 + q^2)\,v_x(a + t(b - a))$. By the real Sahoo-Riedel theorem, there exists $t_2 \in (0, 1)$ such that $G(t_2) - G(0) = t_2 G'(t_2) - (G'(1) - G'(0))t_2^2/2$. Set $z_2 = a + t_2(b - a)$. Since

$$G(t_2) - G(0) = (p^2 + q^2) \cdot t_2 \cdot \mathrm{Im}\left(\frac{f(z_2) - f(a)}{z_2 - a}\right),$$

the same computation gives

$$\mathrm{Im}\, f'(z_2) = \mathrm{Im}\left(\frac{f(z_2) - f(a)}{z_2 - a}\right) + \frac{\mathrm{Im}\, f'(b) - \mathrm{Im}\, f'(a)}{2(b - a)} \cdot (z_2 - a).$$

Multiplying by λ and using Remark 2.6 gives the result. The proof is complete. □

Remark 6.3. *When $\lambda = 1$, the theorem reduces to*

$$\mathrm{Re}\, f'(z_1) = \mathrm{Re}\left(\frac{f(z_1) - f(a)}{z_1 - a}\right) + \frac{\mathrm{Re}\, f'(b) - \mathrm{Re}\, f'(a)}{2(b - a)} \cdot (z_1 - a),$$
$$\mathrm{Im}\, f'(z_2) = \mathrm{Im}\left(\frac{f(z_2) - f(a)}{z_2 - a}\right) + \frac{\mathrm{Im}\, f'(b) - \mathrm{Im}\, f'(a)}{2(b - a)} \cdot (z_2 - a).$$

This coincides with Theorem 2 of Davitt et al. [7], which is the classical complex Sahoo-Riedel theorem for holomorphic functions.

Remark 6.4. *Unlike Theorem 5.1, Theorem 6.1 requires no boundary condition on f. This makes it a stronger result in the sense that it applies to all λ-deformable analytic functions without restriction.*

Example 6.5. *Let $f(z) = z^2$, $\lambda = 1/2$, $\delta = 1/2$, $a = 0$, $b = 1$. Then $f'(z) = 2z$, so $f'(0) = 0$ and $f'(1) = 2$. Restricting to the real axis and substituting into the Sahoo-Riedel identity,*

$$2x = x + \frac{2 - 0}{2} \cdot x = 2x,$$

which holds for all $x \in (0, 1)$. In deformable form,

$$\mathrm{Re}(D_\lambda f(x) - \delta f(x)) = \tfrac{1}{2}\,\mathrm{Re}\, f'(x) = x$$

and

$$\lambda\,\mathrm{Re}\left(\frac{f(x) - f(0)}{x}\right) + \frac{\mathrm{Re}(D_\lambda f(1) - \delta f(1)) - \mathrm{Re}(D_\lambda f(0) - \delta f(0))}{2 \cdot 1} \cdot x = \tfrac{1}{2}x + \tfrac{1}{2} \cdot 1 \cdot x = x,$$

confirming Theorem 6.1 for any $z_1 = x \in (0, 1)$.

7. Conclusions

In this paper, we proved complex deformable analogues of four classical theorems: Rolle's theorem, the mean value theorem, Flett's theorem, and the Sahoo-Riedel theorem. All results were obtained for λ-deformable analytic functions in the complex plane, and all reduce to their classical complex counterparts when $\lambda = 1$.

The central identity throughout the paper is

$$D_\lambda f(z) - \delta f(z) = \lambda f'(z),$$

which connects the complex deformable derivative to the classical complex derivative. This identity reveals that the natural vanishing condition in the deformable setting is $D_\lambda f(z) = \delta f(z)$, not $D_\lambda f(z) = 0$. This is a fundamental difference between the deformable derivative and both the classical derivative and the conformable derivative. In all proofs, the real deformable theorems of Zulfeqarr et al. [12] and Uçar [14] were applied to auxiliary real-valued functions, extending these strategies to the complex plane.

The results of this paper extend and unify the work of Evard and Jafari [4], Çakmak and Tiryaki [5], and Davitt et al. [7] on classical complex mean value theorems, and the work of Uçar [14] on conformable Flett and Sahoo-Riedel theorems, to the complex deformable setting. In particular, when $\lambda = 1$, Theorems 5.1 and 6.1 recover the results of Davitt et al. [7] for holomorphic functions.

We also showed that the complex deformable Rolle theorem and the complex deformable mean value theorem are equivalent, extending the equivalence result of Çakmak and Tiryaki [5] to the deformable setting. Furthermore, we characterized constant functions by the condition $D_\lambda f(z) = \delta f(z)$, and showed that the weaker condition $D_\lambda f(z) = 0$ does not imply constancy.

The results of this paper suggest several directions for further study. First, it would be natural to investigate whether a complex deformable analogue of other refinements of the mean value theorem can be established. Second, the mean value theorem proved here can be used to estimate the growth of λ-deformable analytic functions, and the constant function characterization may be useful in the study of complex deformable differential equations [15]. Third, the theorems proved here may serve as tools in optimization problems involving deformable calculus, in the same spirit as classical mean value theorems in optimization [1]. Fourth, the extension of these results to the case $\lambda > 1$, using the higher-order deformable derivative defined in [15], remains an open problem.

Use of Generative-AI tools declaration

AI tools were used to assist in the language editing and structural organization of the manuscript. All mathematical content, proofs, and scientific conclusions are entirely the work of the author and were verified independently.

Conflict of interest

The author declares that there are no competing interests.

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