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Research article

Non-parabolic spatial hybrid framed curves and their applications in the spatial hybrid number space

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Abstract: In this paper, we define non-parabolic spatial hybrid framed curves in the spatial hybrid number space, which may have singularities, and prove the existence and uniqueness theorem for non-parabolic spatial hybrid framed curves. We construct the Frenet type frame by linear combination of vectors in the original moving frame. As applications, we define evolutes, involutes, pedal and contrapedal curves of non-parabolic spatial hybrid framed curves and discuss their relations.

Keywords: non-parabolic spatial hybrid framed curves; evolutes; involutes; pedal curves; contrapedal curves
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1. Introduction

The set of hybrid numbers, a noncommutative number system denoted by $\mathbb{H}$ or $\mathbb{K}$, unifies and generalizes complex numbers, dual numbers, and hyperbolic numbers. The hybrid number system was defined by Özdemir in [1]. A hybrid number is a number created with any combination of complex, dual, and hyperbolic numbers. There have been a large quantity of results about algebra and geometry on the hybrid number system. In algebra, Özdemir researched representations of hybrid numbers such as the matrix representation, the polar representation, and the exponential representation. He also gave formulas to find roots of the n-th degree of a hybrid number. Analogous to the similarity of matrices, Öztürk and Özdemir defined the similarity of hybrid numbers and investigated properties of similar hybrid numbers [2]. The hybridian product can induce the scalar product g of two hybrid numbers. The hybrid number system $\mathbb{H}$ endowed with the scalar product g is called the hybrid number space. It is a four-dimensional linear space and isomorphic to the $(\mathbb{R}^4, g)$. All hybrid numbers whose scalar parts vanish form a subspace of $\mathbb{H}$, called the spatial hybrid number space. In the spatial hybrid number space, numbers can be divided into elliptic, hyperbolic, and parabolic according to the scalar product. Akbıyık gave Frenet-Serret formulas for non-parabolic spatial hybrid curves and non-lightlike hybrid curves [3]. These are basic and useful tools to deal with properties of regular curves. Alo derived Cartan formulas for lightlike hybrid curves in the spatial hybrid number space and the hybrid number space [4]. Using the Cartan frame, Alo introduced Bertrand curves of lightlike hybrid curves.

In the history of differential geometry, people usually researched regular curves, but singular curves are common in our life, such as movement trajectories of objects and edges of buildings. We cannot establish the Frenet frame of a singular curve at its singularity by the traditional method since the derivative of the curve is a zero vector here. Bishop proposed that there is more than one way to frame a curve [5]. Although Bishop only considered regular curves, people found that his idea can be applied to singular curves. Fukunaga and Takahashi proved the existence and uniqueness theorems for Legendre curves, whose base curves may have singularities [6]. Increasing the dimension of the space, Honda and Takahashi defined framed curves in the Euclidean n-space [7]. In brief, a framed curve is a smooth curve with a moving frame along it. One can find more appropriate frames for some framed curves. Honda and Takahashi raised the Frenet type frame for framed curves in the Euclidean 3-space [8, 9]. It can be regarded as the generalization of the Frenet frame for regular curves. So far, questions related to framed curves can be handled and abundant results are produced [10–12]. Most of existing studies on framed curves are in the Euclidean space, the Minkowski space, and their subspaces [13–15]. Hybrid curves which people studied are all regular. So, we study curves which are allowed to have singularities in the hybrid number space.

In the present paper, we consider non-parabolic spatial hybrid framed curves and curves generated by them. In Section 2, we recall some basic knowledge of hybrid numbers. In Section 3, we define non-parabolic spatial hybrid framed curves in the spatial hybrid number space and give the fundamental theorem of non-parabolic spatial hybrid framed curves. As applications, we define curves generated by non-parabolic spatial hybrid framed curves and research their relations in Section 4. Finally, we give an example to show our results.

All maps and manifolds considered here are differentiable of class C^∞.

2. Preliminaries

Let

$$\mathbb{H} = \{a + b\boldsymbol{i} + c\boldsymbol{\varepsilon} + d\boldsymbol{h} \mid a, b, c, d \in \mathbb{R}\}$$

be the set of hybrid numbers and

$$\mathbb{H}^p = \{b\boldsymbol{i} + c\boldsymbol{\varepsilon} + d\boldsymbol{h} \in \mathbb{H}\}$$

be a subset of $\mathbb{H}$. $\mathbb{H}^p$ is a three-dimensional vector space on $\mathbb{R}$ with the basis $\{\boldsymbol{i}, \boldsymbol{\varepsilon}, \boldsymbol{h}\}$. For any $\boldsymbol{H}_1 = a_1 + b_1\boldsymbol{i} + c_1\boldsymbol{\varepsilon} + d_1\boldsymbol{h},\ \boldsymbol{H}_2 = a_2 + b_2\boldsymbol{i} + c_2\boldsymbol{\varepsilon} + d_2\boldsymbol{h} \in \mathbb{H}$, their hybridian product is

$$\begin{aligned}\boldsymbol{H}_1\boldsymbol{H}_2 = &\, a_1a_2 - b_1b_2 + b_1c_2 + b_2c_1 + d_1d_2 + a_1(b_2\boldsymbol{i} + c_2\boldsymbol{\varepsilon} + d_2\boldsymbol{h}) + a_2(b_1\boldsymbol{i} + c_1\boldsymbol{\varepsilon} + d_1\boldsymbol{h}) \\ &+ (b_1d_2 - b_2d_1)\boldsymbol{i} + (b_1d_2 - b_2d_1 - c_1d_2 + c_2d_1)\boldsymbol{\varepsilon} + (b_2c_1 - b_1c_2)\boldsymbol{h}.\end{aligned}$$

The hybridian product is associative but noncommutative. If $\boldsymbol{H}_1, \boldsymbol{H}_2 \in \mathbb{H}^p$, then their hybridian product can be simplified to

$$\begin{aligned}\boldsymbol{H}_1\boldsymbol{H}_2 = &- b_1b_2 + b_1c_2 + b_2c_1 + d_1d_2 + (b_1d_2 - b_2d_1)\boldsymbol{i} \\ &+ (b_1d_2 - b_2d_1 - c_1d_2 + c_2d_1)\boldsymbol{\varepsilon} + (b_2c_1 - b_1c_2)\boldsymbol{h}.\end{aligned}$$

The conjugate of $\boldsymbol{H}_1$ is $\bar{\boldsymbol{H}}_1 = a_1 - b_1\boldsymbol{i} - c_1\boldsymbol{\varepsilon} - d_1\boldsymbol{h}$. The scalar product of $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ is defined by

$$\begin{aligned} g(\boldsymbol{H}_1, \boldsymbol{H}_2) &= \frac{1}{2}(\boldsymbol{H}_1\bar{\boldsymbol{H}}_2 + \boldsymbol{H}_2\bar{\boldsymbol{H}}_1) \\ &= a_1a_2 + b_1b_2 - b_1c_2 - b_2c_1 - d_1d_2. \end{aligned}$$

When $\boldsymbol{H}_1, \boldsymbol{H}_2 \in \mathbb{H}^p$,

$$\begin{aligned} g(\boldsymbol{H}_1, \boldsymbol{H}_2) &= b_1b_2 - b_1c_2 - b_2c_1 - d_1d_2 \\ &= (b_1, c_1, d_1)\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} b_2 \\ c_2 \\ d_2 \end{pmatrix}. \end{aligned}$$

We still denote $g|_{\mathbb{H}^p\times\mathbb{H}^p}$ by g and call $(\mathbb{H}^p, g)$ the spatial hybrid number space. We can identify $(\mathbb{H}^p, g)$ with $(\mathbb{R}^3, g)$. $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ are g-orthogonal if $g(\boldsymbol{H}_1, \boldsymbol{H}_2) = 0$. For any nonzero $\boldsymbol{H} \in \mathbb{H}^p$, it is called elliptic, hyperbolic, or parabolic if $g(\boldsymbol{H}, \boldsymbol{H})$ is positive, negative, or zero, respectively. The norm of $\boldsymbol{H}$ is defined by $\|\boldsymbol{H}\| = \sqrt{|g(\boldsymbol{H}, \boldsymbol{H})|}$. The vector product of $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ is defined by

$$\begin{aligned} \boldsymbol{H}_1 \times \boldsymbol{H}_2 &= \frac{1}{2}(\boldsymbol{H}_1\bar{\boldsymbol{H}}_2 - \boldsymbol{H}_2\bar{\boldsymbol{H}}_1) \\ &= (b_2d_1 - b_1d_2)\boldsymbol{i} + (b_2d_1 - b_1d_2 - c_2d_1 + c_1d_2)\boldsymbol{\varepsilon} + (b_1c_2 - b_2c_1)\boldsymbol{h} \\ &= (b_1, c_1, d_1)\begin{pmatrix} -d_2 & -d_2 & c_2 \\ 0 & d_2 & -b_2 \\ b_2 & b_2 - c_2 & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{\varepsilon} \\ \boldsymbol{h} \end{pmatrix}. \end{aligned}$$

Proposition 2.1. *Take* $\boldsymbol{H}_k = b_k\boldsymbol{i} + c_k\boldsymbol{\varepsilon} + d_k\boldsymbol{h} \in \mathbb{H}^p,\ k = 1, 2, 3, 4$, *then*

(1) $g(\boldsymbol{H}_1 \times \boldsymbol{H}_2, \boldsymbol{H}_3) = -\det\begin{pmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{pmatrix}$.

(2) $(\boldsymbol{H}_1 \times \boldsymbol{H}_2) \times \boldsymbol{H}_3 = g(\boldsymbol{H}_1, \boldsymbol{H}_3)\boldsymbol{H}_2 - g(\boldsymbol{H}_2, \boldsymbol{H}_3)\boldsymbol{H}_1$.

(3) $g(\boldsymbol{H}_1 \times \boldsymbol{H}_2, \boldsymbol{H}_3 \times \boldsymbol{H}_4) = \det\begin{pmatrix} g(\boldsymbol{H}_1, \boldsymbol{H}_3) & g(\boldsymbol{H}_1, \boldsymbol{H}_4) \\ g(\boldsymbol{H}_2, \boldsymbol{H}_3) & g(\boldsymbol{H}_2, \boldsymbol{H}_4) \end{pmatrix}$.

Proof.

(1)

$$\begin{aligned} & g(\boldsymbol{H}_1 \times \boldsymbol{H}_2, \boldsymbol{H}_3) \\ &= (b_2d_1 - b_1d_2, b_2d_1 - b_1d_2 - c_2d_1 + c_1d_2, b_1c_2 - b_2c_1)\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} b_3 \\ c_3 \\ d_3 \end{pmatrix} \\ &= (c_2d_1 - c_1d_2, b_1d_2 - b_2d_1, b_2c_1 - b_1c_2)\begin{pmatrix} b_3 \\ c_3 \\ d_3 \end{pmatrix} \\ &= -\det\begin{pmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{pmatrix}. \end{aligned}$$

The scalar triple product can be expressed by the determinant.

(2)

$$(\boldsymbol{H}_1 \times \boldsymbol{H}_2) \times \boldsymbol{H}_3 = (b_1, c_1, d_1) \begin{pmatrix} -d_2 & -d_2 & c_2 \\ 0 & d_2 & -b_2 \\ b_2 & b_2 - c_2 & 0 \end{pmatrix} \begin{pmatrix} -d_3 & -d_3 & c_3 \\ 0 & d_3 & -b_3 \\ b_3 & b_3 - c_3 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{\varepsilon} \\ \boldsymbol{h} \end{pmatrix}$$

$$= \begin{pmatrix} d_3(b_1 d_2 - b_2 d_1) + b_3(b_1 c_2 - b_2 c_1) \\ d_3(c_1 d_2 - c_2 d_1) + (b_3 - c_3)(b_1 c_2 - b_2 c_1) \\ (b_3 - c_3)(b_1 d_2 - b_2 d_1) - b_3(c_1 d_2 - c_2 d_1) \end{pmatrix}^T \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{\varepsilon} \\ \boldsymbol{h} \end{pmatrix},$$

$$g(\boldsymbol{H}_1, \boldsymbol{H}_3)\boldsymbol{H}_2 = (b_1 b_3 - b_1 c_3 - b_3 c_1 - d_1 d_3)(b_2 \boldsymbol{i} + c_2 \boldsymbol{\varepsilon} + d_2 \boldsymbol{h}),$$

$$g(\boldsymbol{H}_2, \boldsymbol{H}_3)\boldsymbol{H}_1 = (b_2 b_3 - b_2 c_3 - b_3 c_2 - d_2 d_3)(b_1 \boldsymbol{i} + c_1 \boldsymbol{\varepsilon} + d_1 \boldsymbol{h}).$$

So

$$(\boldsymbol{H}_1 \times \boldsymbol{H}_2) \times \boldsymbol{H}_3 = g(\boldsymbol{H}_1, \boldsymbol{H}_3)\boldsymbol{H}_2 - g(\boldsymbol{H}_2, \boldsymbol{H}_3)\boldsymbol{H}_1.$$

The vector triple product can be expressed by the real linear combination of the first two numbers.

(3) According to (1) and (2), we have

$$\begin{aligned} & g(\boldsymbol{H}_1 \times \boldsymbol{H}_2, \boldsymbol{H}_3 \times \boldsymbol{H}_4) \\ =& g((\boldsymbol{H}_3 \times \boldsymbol{H}_4) \times \boldsymbol{H}_1, \boldsymbol{H}_2) \\ =& g(g(\boldsymbol{H}_3, \boldsymbol{H}_1)\boldsymbol{H}_4 - g(\boldsymbol{H}_4, \boldsymbol{H}_1)\boldsymbol{H}_3, \boldsymbol{H}_2) \\ =& g(\boldsymbol{H}_1, \boldsymbol{H}_3) g(\boldsymbol{H}_2, \boldsymbol{H}_4) - g(\boldsymbol{H}_1, \boldsymbol{H}_4) g(\boldsymbol{H}_2, \boldsymbol{H}_3). \end{aligned}$$

3. Non-parabolic spatial hybrid framed curves

Denote $\Delta = \{(\boldsymbol{\nu}_1, \boldsymbol{\nu}_2) \in \mathbb{H}^p \times \mathbb{H}^p \mid \|\boldsymbol{\nu}_1\| = \|\boldsymbol{\nu}_2\| = 1, g(\boldsymbol{\nu}_1, \boldsymbol{\nu}_2) = 0\}$. To study smooth curves which may have singularities in the spatial hybrid number space, we define non-parabolic spatial hybrid framed curves in $\mathbb{H}^p$.

Definition 3.1. Let I be an interval in $\mathbb{R}$. $(\gamma, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : I \to \mathbb{H}^p \times \Delta$ is called a non-parabolic spatial hybrid framed curve if $g(\gamma'(t), \boldsymbol{\nu}_1(t)) = g(\gamma'(t), \boldsymbol{\nu}_2(t)) = 0$ for all $t \in I$. We call $\gamma : I \to \mathbb{H}^p$ a non-parabolic spatial hybrid framed base curve if there exists $(\boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : I \to \Delta$ such that $(\gamma, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$ is a non-parabolic spatial hybrid framed curve.

Let $\boldsymbol{\mu}(t) = \boldsymbol{\nu}_1(t) \times \boldsymbol{\nu}_2(t)$. Then $\{\boldsymbol{\nu}_1(t), \boldsymbol{\nu}_2(t), \boldsymbol{\mu}(t)\}$ is a g-orthogonal frame along γ. The Frenet type formulas are

$$\begin{pmatrix} \boldsymbol{\nu}_1'(t) \\ \boldsymbol{\nu}_2'(t) \\ \boldsymbol{\mu}'(t) \end{pmatrix} = \begin{pmatrix} 0 & l(t) & m(t) \\ -\delta_1 \delta_2 l(t) & 0 & n(t) \\ -\delta_2 m(t) & -\delta_1 n(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\nu}_1(t) \\ \boldsymbol{\nu}_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \ \gamma'(t) = \alpha(t)\boldsymbol{\mu}(t),$$

where

$$\delta_1 = g(\boldsymbol{\nu}_1(t), \boldsymbol{\nu}_1(t)), \ \delta_2 = g(\boldsymbol{\nu}_2(t), \boldsymbol{\nu}_2(t)), \ l(t) = \delta_2 g(\boldsymbol{\nu}_1'(t), \boldsymbol{\nu}_2(t)),$$

$$m(t) = \delta_1 \delta_2 g(\boldsymbol{\nu}_1'(t), \boldsymbol{\mu}(t)), \ n(t) = \delta_1 \delta_2 g(\boldsymbol{\nu}_2'(t), \boldsymbol{\mu}(t)), \ \alpha(t) = \delta_1 \delta_2 g(\gamma'(t), \boldsymbol{\mu}(t)).$$

The map $(l, m, n, \alpha) : I \to \mathbb{R}^4$ is called the curvature of $(\gamma, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$. We say $(\gamma, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$ is an elliptic (or a hyperbolic) spatial hybrid framed curve if $\delta_1 \delta_2 = 1$ (or $\delta_1 \delta_2 = -1$).

It is obvious that $t_0 \in I$ is a singularity of γ if and only if $\alpha(t_0) = 0$.

Now we give the fundamental theory of non-parabolic spatial hybrid framed curves. We first recall hybridian orthogonal matrices.

Definition 3.2. *[16]* A matrix $A \in \mathbb{R}^{3\times 3}$ is called hybridian orthogonal if $A^T G A = G$, where

$$G = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In an n-dimensional vector space, an $n \times n$ symmetric matrix is a bilinear form. The matrix G is the matrix of the scalar product g in the spatial hybrid number space. A hybridian orthogonal matrix is special if its determinant equals 1. Special hybridian orthogonal matrices form a group and keep the scalar product and the vector product.

Proposition 3.1. *[16] All spatial hybridian orthogonal matrices form a group.*

Proof. Define

$$\mathcal{G} = \{A \in \mathbb{R}^{3\times 3} \mid A^T G A = G, \det(A) = 1\}.$$

For any $A, B \in \mathcal{G}$, we have

$$(AB)^T G(AB) = B^T A^T G A B = B^T G B = G$$

and

$$\det(AB) = \det(A)\det(B) = 1.$$

So, $AB \in \mathcal{G}$.

The associative law holds naturally. The unit matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{G}$ is the unit element. The inverse element of A is A^{-1} and

$$(A^{-1})^T G A^{-1} = (A^T)^{-1} A^T G A A^{-1} = G.$$

So, $\mathcal{G}$ is a group.

Proposition 3.2. *[16] For any $A \in \mathcal{G}$ and any $\boldsymbol{H}_1, \boldsymbol{H}_2 \in \mathbb{H}^p$, we have $g(A\boldsymbol{H}_1, A\boldsymbol{H}_2) = g(\boldsymbol{H}_1, \boldsymbol{H}_2)$ and $(A\boldsymbol{H}_1) \times (A\boldsymbol{H}_2) = A(\boldsymbol{H}_1 \times \boldsymbol{H}_2)$.*

Proof.

$$g(A\boldsymbol{H}_1, A\boldsymbol{H}_2) = (A\boldsymbol{H}_1)^T G(A\boldsymbol{H}_2) = \boldsymbol{H}_1^T A^T G A \boldsymbol{H}_2 = \boldsymbol{H}_1^T G \boldsymbol{H}_2 = g(\boldsymbol{H}_1, \boldsymbol{H}_2).$$

For any $\boldsymbol{H}_3 \in \mathbb{H}^p$, we have

$$\begin{aligned}
& g((A\boldsymbol{H}_1) \times (A\boldsymbol{H}_2), \boldsymbol{H}_3) \\
&= -\det(A\boldsymbol{H}_1, A\boldsymbol{H}_2, \boldsymbol{H}_3) \\
&= -\det(G^{-1}A^T G)\det(A\boldsymbol{H}_1, A\boldsymbol{H}_2, \boldsymbol{H}_3) \\
&= -\det(\boldsymbol{H}_1, \boldsymbol{H}_2, G^{-1}A^T G\boldsymbol{H}_3) \\
&= g(\boldsymbol{H}_1 \times \boldsymbol{H}_2, G^{-1}A^T G\boldsymbol{H}_3) \\
&= \boldsymbol{H}_3^T G^T A (G^{-1})^T G(\boldsymbol{H}_1 \times \boldsymbol{H}_2) \\
&= \boldsymbol{H}_3^T G A(\boldsymbol{H}_1 \times \boldsymbol{H}_2) \\
&= g(\boldsymbol{H}_3, A(\boldsymbol{H}_1 \times \boldsymbol{H}_2)).
\end{aligned}$$

This implies $(A\boldsymbol{H}_1) \times (A\boldsymbol{H}_2) = A(\boldsymbol{H}_1 \times \boldsymbol{H}_2)$.

Definition 3.3. Let $(\gamma, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : I \to \mathbb{H}^p \times \Delta$ and $(\tilde{\gamma}, \tilde{\boldsymbol{\nu}}_1, \tilde{\boldsymbol{\nu}}_2) : I \to \mathbb{H}^p \times \Delta$ be two non-parabolic spatial hybrid framed curves. We call them congruent through a special hybridian orthogonal matrix if there exist $A \in \mathcal{G}$ and $\boldsymbol{H}_0 \in \mathbb{H}^p$ such that

$$\tilde{\gamma}(t) = A(\gamma(t)) + \boldsymbol{H}_0,\ \tilde{\boldsymbol{\nu}}_1(t) = A(\boldsymbol{\nu}_1(t)),\ \tilde{\boldsymbol{\nu}}_2(t) = A(\boldsymbol{\nu}_2(t))$$

for any $t \in I$.

By Proposition 3.2, we know when two non-parabolic spatial hybrid framed curves are congruent, their curvatures coincide. Next, we claim the existence and uniqueness theorem of non-parabolic spatial hybrid framed curves. The theorem shows that four smooth functions and two signs can uniquely determine a non-parabolic spatial hybrid framed curve.

Theorem 3.1. *(The existence and uniqueness theorem) Given smooth functions $l, m, n, \alpha : I \to \mathbb{R}$ and constants $(\delta_1, \delta_2) \in \{(1, -1), (-1, 1), (-1, -1)\}$, there exists a non-parabolic spatial hybrid framed curve $(\gamma, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : I \to \mathbb{H}^p \times \Delta$, whose curvature is (l, m, n, α) and $g(\boldsymbol{\nu}_1(t), \boldsymbol{\nu}_1(t)) = \delta_1,\ g(\boldsymbol{\nu}_2(t), \boldsymbol{\nu}_2(t)) = \delta_2$. The curve is unique under special hybridian orthogonal matrices.*

Proof. Fix some $t_0 \in I$. Consider a system

$$\begin{pmatrix} \boldsymbol{\nu}_1'(t) \\ \boldsymbol{\nu}_2'(t) \\ \boldsymbol{\mu}'(t) \end{pmatrix} = \begin{pmatrix} 0 & l(t) & m(t) \\ -\delta_1\delta_2 l(t) & 0 & n(t) \\ -\delta_2 m(t) & -\delta_1 n(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\nu}_1(t) \\ \boldsymbol{\nu}_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix} \tag{3.1}$$

with the initial value

$$\begin{aligned} &g(\boldsymbol{\nu}_1(t_0), \boldsymbol{\nu}_1(t_0)) = \delta_1, \quad g(\boldsymbol{\nu}_2(t_0), \boldsymbol{\nu}_2(t_0)) = \delta_2, \quad g(\boldsymbol{\mu}(t_0), \boldsymbol{\mu}(t_0)) = \delta_1\delta_2, \\ &g(\boldsymbol{\nu}_1(t_0), \boldsymbol{\nu}_2(t_0)) = 0, \quad g(\boldsymbol{\nu}_1(t_0), \boldsymbol{\mu}(t_0)) = 0, \quad g(\boldsymbol{\nu}_2(t_0), \boldsymbol{\mu}(t_0)) = 0. \end{aligned}$$

l, m, n are smooth, so the solution of Eq (3.1) exists.

Let $(\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \boldsymbol{\mu}) : I \to \mathbb{H}^p \times \mathbb{H}^p \times \mathbb{H}^p$ be a solution of Eq (3.1). Define functions $a_1, a_2, ..., a_6 : I \to \mathbb{R}$ by

$$\begin{aligned} &a_1(t) = g(\boldsymbol{\nu}_1(t), \boldsymbol{\nu}_1(t)), \quad a_2(t) = g(\boldsymbol{\nu}_2(t), \boldsymbol{\nu}_2(t)), \quad a_3(t) = g(\boldsymbol{\mu}(t), \boldsymbol{\mu}(t)), \\ &a_4(t) = g(\boldsymbol{\nu}_1(t), \boldsymbol{\nu}_2(t)), \quad a_5(t) = g(\boldsymbol{\nu}_1(t), \boldsymbol{\mu}(t)), \quad a_6(t) = g(\boldsymbol{\nu}_2(t), \boldsymbol{\mu}(t)). \end{aligned}$$

Consider the following system:

$$\begin{pmatrix} a_1'(t) \\ a_2'(t) \\ a_3'(t) \\ a_4'(t) \\ a_5'(t) \\ a_6'(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 2l(t) & 2m(t) & 0 \\ 0 & 0 & 0 & -2\delta_1\delta_2 l(t) & 0 & 2n(t) \\ 0 & 0 & 0 & 0 & -2\delta_2 m(t) & -2\delta_1 n(t) \\ -\delta_1\delta_2 l(t) & l(t) & 0 & 0 & n(t) & m(t) \\ -\delta_2 m(t) & 0 & m(t) & -\delta_1 n(t) & 0 & l(t) \\ 0 & -\delta_1 n(t) & n(t) & -\delta_2 m(t) & -\delta_1\delta_2 l(t) & 0 \end{pmatrix} \begin{pmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \\ a_4(t) \\ a_5(t) \\ a_6(t) \end{pmatrix}$$

with the initial value

$$a_1(t_0) = \delta_1,\ a_2(t_0) = \delta_2,\ a_3(t_0) = \delta_1\delta_2,\ a_4(t_0) = 0,\ a_5(t_0) = 0,\ a_6(t_0) = 0.$$

It has the unique solution

$$a_1(t) = \delta_1,\ a_2(t) = \delta_2,\ a_3(t) = \delta_1\delta_2,\ a_4(t) = 0,\ a_5(t) = 0,\ a_6(t) = 0.$$

That is to say, $(\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \boldsymbol{\mu}) : I \to \Delta \times \mathbb{H}^p$ and $g(\boldsymbol{\nu}_1(t), \boldsymbol{\mu}(t)) = g(\boldsymbol{\nu}_2(t), \boldsymbol{\mu}(t)) = 0$.

Take $\gamma(t) = \int \alpha(t)\boldsymbol{\mu}(t)\mathrm{d}t$, then $(\gamma, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : I \to \mathbb{H}^p \times \Delta$ is a non-parabolic spatial hybrid framed curve with the curvature (l, m, n, α) and $g(\boldsymbol{\nu}_1(t), \boldsymbol{\nu}_1(t)) = \delta_1,\ g(\boldsymbol{\nu}_2(t), \boldsymbol{\nu}_2(t)) = \delta_2$.

Let $(\gamma, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : I \to \mathbb{H}^p \times \Delta$ and $(\tilde{\gamma}, \tilde{\boldsymbol{\nu}}_1, \tilde{\boldsymbol{\nu}}_2) : I \to \mathbb{H}^p \times \Delta$ be two non-parabolic spatial hybrid framed curves with the same curvature (l, m, n, α) and

$$g(\boldsymbol{\nu}_1(t), \boldsymbol{\nu}_1(t)) = \delta_1,\ g(\boldsymbol{\nu}_2(t), \boldsymbol{\nu}_2(t)) = \delta_2,\ g(\tilde{\boldsymbol{\nu}}_1(t), \tilde{\boldsymbol{\nu}}_1(t)) = \delta_1,\ g(\tilde{\boldsymbol{\nu}}_2(t), \tilde{\boldsymbol{\nu}}_2(t)) = \delta_2.$$

Then there exist $A \in \mathcal{G}$ and $\boldsymbol{H}_0 \in \mathbb{H}^p$ such that

$$\tilde{\gamma}(t_0) = A(\gamma(t_0)) + \boldsymbol{H}_0,\ \tilde{\boldsymbol{\nu}}_1(t_0) = A(\boldsymbol{\nu}_1(t_0)),\ \tilde{\boldsymbol{\nu}}_2(t_0) = A(\boldsymbol{\nu}_2(t_0))$$

for some $t_0 \in I$. Then

$$\tilde{\boldsymbol{\mu}}(t_0) = \tilde{\boldsymbol{\nu}}_1(t_0) \times \tilde{\boldsymbol{\nu}}_2(t_0) = A(\boldsymbol{\nu}_1(t_0) \times \boldsymbol{\nu}_2(t_0)) = A(\boldsymbol{\mu}(t_0)).$$

$(A\gamma + \boldsymbol{H}_0, A\boldsymbol{\nu}_1, A\boldsymbol{\nu}_2, A\boldsymbol{\mu}) : I \to \mathbb{H}^p \times \Delta \times \mathbb{H}^p$ and $(\tilde{\gamma}, \tilde{\boldsymbol{\nu}}_1, \tilde{\boldsymbol{\nu}}_2, \tilde{\boldsymbol{\mu}}) : I \to \mathbb{H}^p \times \Delta \times \mathbb{H}^p$ are both solutions of the system

$$\begin{cases} \gamma'(t) = \alpha(t)\boldsymbol{\mu}(t), \\ \boldsymbol{\nu}_1'(t) = l(t)\boldsymbol{\nu}_2(t) + m(t)\boldsymbol{\mu}(t), \\ \boldsymbol{\nu}_2'(t) = -\delta_1\delta_2 l(t)\boldsymbol{\nu}_1(t) + n(t)\boldsymbol{\mu}(t), \\ \boldsymbol{\mu}'(t) = -\delta_2 m(t)\boldsymbol{\nu}_1(t) - \delta_1 n(t)\boldsymbol{\nu}_2(t). \end{cases}$$

So, $(A\gamma + \boldsymbol{H}_0, A\boldsymbol{\nu}_1, A\boldsymbol{\nu}_2, A\boldsymbol{\mu}) = (\tilde{\gamma}, \tilde{\boldsymbol{\nu}}_1, \tilde{\boldsymbol{\nu}}_2, \tilde{\boldsymbol{\mu}})$. That is, $(\gamma, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$ and $(\tilde{\gamma}, \tilde{\boldsymbol{\nu}}_1, \tilde{\boldsymbol{\nu}}_2)$ are congruent under special hybridian orthogonal matrices.

For a non-parabolic spatial hybrid framed curve $(\gamma, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : I \to \mathbb{H}^p \times \Delta$ with the curvature (l, m, n, α), any linear combination of $\boldsymbol{\nu}_1(t)$ and $\boldsymbol{\nu}_2(t)$ is g-orthogonal to $\gamma'(t)$. When $\delta_1 m^2(t) + \delta_2 n^2(t) \neq 0$ for all $t \in I$, we take

$$\begin{pmatrix} \boldsymbol{n}_1(t) \\ \boldsymbol{n}_2(t) \end{pmatrix} = \frac{1}{\sqrt{\sigma(\delta_1 m^2(t) + \delta_2 n^2(t))}} \begin{pmatrix} \sigma\delta_1 m(t) & \sigma\delta_2 n(t) \\ -n(t) & m(t) \end{pmatrix} \begin{pmatrix} \boldsymbol{\nu}_1(t) \\ \boldsymbol{\nu}_2(t) \end{pmatrix},$$

where $\sigma = \mathrm{sgn}(\delta_1 m^2(t) + \delta_2 n^2(t))$. Then

$$
\begin{aligned}
g(\boldsymbol{n}_1(t),\boldsymbol{n}_1(t)) &= \frac{1}{\sigma(\delta_1 m^2(t)+\delta_2 n^2(t))} g(\sigma\delta_1 m(t)\boldsymbol{\nu}_1(t)+\sigma\delta_2 n(t)\boldsymbol{\nu}_2(t), \sigma\delta_1 m(t)\boldsymbol{\nu}_1(t)+\sigma\delta_2 n(t)\boldsymbol{\nu}_2(t))\\
&= \frac{1}{\sigma(\delta_1 m^2(t)+\delta_2 n^2(t))}(\delta_1 m^2(t)+\delta_2 n^2(t))\\
&= \sigma,\\
g(\boldsymbol{n}_2(t),\boldsymbol{n}_2(t)) &= \frac{1}{\sigma(\delta_1 m^2(t)+\delta_2 n^2(t))} g(-n(t)\boldsymbol{\nu}_1(t)+m(t)\boldsymbol{\nu}_2(t), -n(t)\boldsymbol{\nu}_1(t)+m(t)\boldsymbol{\nu}_2(t))\\
&= \frac{1}{\sigma(\delta_1 m^2(t)+\delta_2 n^2(t))}(\delta_1 n^2(t)+\delta_2 m^2(t))\\
&= \sigma\delta_1\delta_2,\\
g(\boldsymbol{n}_1(t),\boldsymbol{n}_2(t)) &= \frac{1}{\sigma(\delta_1 m^2(t)+\delta_2 n^2(t))} g(\sigma\delta_1 m(t)\boldsymbol{\nu}_1(t)+\sigma\delta_2 n(t)\boldsymbol{\nu}_2(t), -n(t)\boldsymbol{\nu}_1(t)+m(t)\boldsymbol{\nu}_2(t))\\
&= 0,\\
\boldsymbol{n}_1(t)\times\boldsymbol{n}_2(t) &= \frac{1}{\sigma(\delta_1 m^2(t)+\delta_2 n^2(t))}(\sigma\delta_1 m(t)\boldsymbol{\nu}_1(t)+\sigma\delta_2 n(t)\boldsymbol{\nu}_2(t))\times(-n(t)\boldsymbol{\nu}_1(t)+m(t)\boldsymbol{\nu}_2(t))\\
&= \boldsymbol{\mu}(t).
\end{aligned}
$$

So $\{\boldsymbol{n}_1(t),\boldsymbol{n}_2(t),\boldsymbol{\mu}(t)\}$ is also a g-orthogonal frame along γ, called the Frenet type frame. The Frenet type formulas are

$$
\begin{pmatrix}\boldsymbol{n}_1'(t)\\ \boldsymbol{n}_2'(t)\\ \boldsymbol{\mu}'(t)\end{pmatrix} = \begin{pmatrix} 0 & L(t) & M(t)\\ -\delta_1\delta_2 L(t) & 0 & 0\\ -\sigma\delta_1\delta_2 M(t) & 0 & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{n}_1(t)\\ \boldsymbol{n}_2(t)\\ \boldsymbol{\mu}(t)\end{pmatrix},\ \gamma'(t)=\alpha(t)\boldsymbol{\mu}(t),
$$

where

$$
L(t) = \frac{\delta_1\delta_2 m(t)n'(t)-\delta_1\delta_2 m'(t)n(t)}{\sigma(\delta_1 m^2(t)+\delta_2 n^2(t))}+\sigma\delta_1 l(t),\ M(t)=\sqrt{\sigma(\delta_1 m^2(t)+\delta_2 n^2(t))}.
$$

The $\boldsymbol{n}_1$ direction and $\boldsymbol{n}_2$ direction are called the principal normal direction and the binormal direction of γ, respectively. Let γ be a non-parabolic regular curve in $\mathbb{H}^p$ and $\{\boldsymbol{T}(t),\boldsymbol{N}(t),\boldsymbol{B}(t)\}$ be its Frenet-Serret frame. Then the vectors in $\{\boldsymbol{T}(t),\boldsymbol{N}(t),\boldsymbol{B}(t)\}$ and $\{\boldsymbol{\mu}(t),\boldsymbol{n}_1(t),\boldsymbol{n}_2(t)\}$ differ by at most a sign.

4. Curves generated by non-parabolic spatial hybrid framed curves

In this section, we define evolutes of non-parabolic spatial hybrid framed curves using distance squared functions. As its inverse process, we define involutes and discuss their relationship.

Definition 4.1. Let $(\gamma,\boldsymbol{\nu}_1,\boldsymbol{\nu}_2): I\to\mathbb{H}^p\times\Delta$ be a non-parabolic spatial hybrid framed curve. The distance squared function of $(\gamma,\boldsymbol{\nu}_1,\boldsymbol{\nu}_2)$ is defined by $F: I\times\mathbb{H}^p\to\mathbb{R}$,

$$
F(t,\boldsymbol{x}) = g(\gamma(t)-\boldsymbol{x},\gamma(t)-\boldsymbol{x}).
$$

For some fixed $\boldsymbol{x}\in\mathbb{H}^p$, $F(t,\boldsymbol{x})$ is denoted by $f_x(t)$.

Proposition 4.1. *Let $(\gamma,\boldsymbol{\nu}_1,\boldsymbol{\nu}_2): I\to\mathbb{H}^p\times\Delta$ be a non-parabolic spatial hybrid framed curve with the curvature (l,m,n,α). Assume $\alpha(t)\Big(m(t)n'(t)-m'(t)n(t)+\delta_1\delta_2 l(t)m^2(t)+l(t)n^2(t)\Big)\neq 0$ for any $t\in I$. Then the followings hold:*

(1) $f_x'(t_0) = 0$ *if and only if there exist* $\lambda_1, \lambda_2 \in \mathbb{R}$ *such that* $\gamma(t_0) - \boldsymbol{x} = \lambda_1 \boldsymbol{\nu}_1(t_0) + \lambda_2 \boldsymbol{\nu}_2(t_0)$.
(2) $f_x'(t_0) = f_x''(t_0) = 0$ *if and only if* $\gamma(t_0) - \boldsymbol{x} = \lambda_1 \boldsymbol{\nu}_1(t_0) + \lambda_2 \boldsymbol{\nu}_2(t_0)$ *and* $\lambda_1 m(t_0) + \lambda_2 n(t_0) = \alpha(t_0)$.
(3) $f_x'(t_0) = f_x''(t_0) = f_x'''(t_0) = 0$ *if and only if*

$$\begin{aligned}\gamma(t_0) - \boldsymbol{x} = &\frac{n'(t_0)\alpha(t_0) - n(t_0)\alpha'(t_0) + \delta_1\delta_2 l(t_0)m(t_0)\alpha(t_0)}{m(t_0)n'(t_0) - m'(t_0)n(t_0) + \delta_1\delta_2 l(t_0)m^2(t_0) + l(t_0)n^2(t_0)}\boldsymbol{\nu}_1(t_0)\\ &+ \frac{m(t_0)\alpha'(t_0) - m'(t_0)\alpha(t_0) + l(t_0)n(t_0)\alpha(t_0)}{m(t_0)n'(t_0) - m'(t_0)n(t_0) + \delta_1\delta_2 l(t_0)m^2(t_0) + l(t_0)n^2(t_0)}\boldsymbol{\nu}_2(t_0).\end{aligned}$$

Proof. By Definition 4.1, we can get

$$\begin{aligned}\frac{1}{2}f_x'(t) &= \alpha(t)g(\gamma(t) - \boldsymbol{x}, \boldsymbol{\mu}(t)),\\ \frac{1}{2}f_x''(t) &= \alpha'(t)g(\gamma(t) - \boldsymbol{x}, \boldsymbol{\mu}(t)) + \delta_1\delta_2\alpha^2(t) - \alpha(t)g(\gamma(t) - \boldsymbol{x}, \delta_2 m(t)\boldsymbol{\nu}_1(t) + \delta_1 n(t)\boldsymbol{\nu}_2(t)),\\ \frac{1}{2}f_x'''(t) &= \left(-\delta_2 m'(t)\alpha(t) + \delta_2 l(t)n(t)\alpha(t) - 2\delta_2 m(t)\alpha'(t)\right) g(\gamma(t) - \boldsymbol{x}, \boldsymbol{\nu}_1(t))\\ &\quad - \left(\delta_1 n'(t)\alpha(t) + \delta_2 l(t)m(t)\alpha(t) + 2\delta_1 n(t)\alpha'(t)\right) g(\gamma(t) - \boldsymbol{x}, \boldsymbol{\nu}_2(t))\\ &\quad - \left(\delta_2 m^2(t)\alpha(t) + \delta_1 n^2(t)\alpha(t) + \alpha''(t)\right) g(\gamma(t) - \boldsymbol{x}, \boldsymbol{\mu}(t)) + 3\delta_1\delta_2\alpha(t)\alpha'(t).\end{aligned}$$

So,

(1) $f_x'(t_0) = 0$ if and only if there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\gamma(t_0) - \boldsymbol{x} = \lambda_1 \boldsymbol{\nu}_1(t_0) + \lambda_2 \boldsymbol{\nu}_2(t_0)$.
(2) $f_x'(t_0) = f_x''(t_0) = 0$ if and only if $\gamma(t_0) - \boldsymbol{x} = \lambda_1 \boldsymbol{\nu}_1(t_0) + \lambda_2 \boldsymbol{\nu}_2(t_0)$ and $\lambda_1 m(t_0) + \lambda_2 n(t_0) = \alpha(t_0)$.
(3) $f_x'(t_0) = f_x''(t_0) = f_x'''(t_0) = 0$ if and only if $\gamma(t_0) - \boldsymbol{x} = \lambda_1 \boldsymbol{\nu}_1(t_0) + \lambda_2 \boldsymbol{\nu}_2(t_0)$ and

$$\begin{cases} \lambda_1 m(t_0) + \lambda_2 n(t_0) = \alpha(t_0),\\ \lambda_1 \left(m'(t_0) - l(t_0)n(t_0)\right) + \lambda_2 \left(n'(t_0) + \delta_1\delta_2 l(t_0)m(t_0)\right) = \alpha'(t_0).\end{cases}$$

This means

$$\begin{aligned}\gamma(t_0) - \boldsymbol{x} = &\frac{n'(t_0)\alpha(t_0) - n(t_0)\alpha'(t_0) + \delta_1\delta_2 l(t_0)m(t_0)\alpha(t_0)}{m(t_0)n'(t_0) - m'(t_0)n(t_0) + \delta_1\delta_2 l(t_0)m^2(t_0) + l(t_0)n^2(t_0)}\boldsymbol{\nu}_1(t_0)\\ &+ \frac{m(t_0)\alpha'(t_0) - m'(t_0)\alpha(t_0) + l(t_0)n(t_0)\alpha(t_0)}{m(t_0)n'(t_0) - m'(t_0)n(t_0) + \delta_1\delta_2 l(t_0)m^2(t_0) + l(t_0)n^2(t_0)}\boldsymbol{\nu}_2(t_0).\end{aligned}$$

Definition 4.2. Let $(\gamma, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : I \to \mathbb{H}^p \times \Delta$ be a non-parabolic spatial hybrid framed curve with the curvature (l, m, n, α). Assume $m(t)n'(t) - m'(t)n(t) + \delta_1\delta_2 l(t)m^2(t) + l(t)n^2(t) \neq 0$ for any $t \in I$. The evolute of $(\gamma, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$ is defined by $\mathcal{E}v_\gamma : I \to \mathbb{H}^p$,

$$\begin{aligned}\mathcal{E}v_\gamma(t) = \gamma(t) &- \frac{n'(t)\alpha(t) - n(t)\alpha'(t) + \delta_1\delta_2 l(t)m(t)\alpha(t)}{m(t)n'(t) - m'(t)n(t) + \delta_1\delta_2 l(t)m^2(t) + l(t)n^2(t)}\boldsymbol{\nu}_1(t)\\ &- \frac{m(t)\alpha'(t) - m'(t)\alpha(t) + l(t)n(t)\alpha(t)}{m(t)n'(t) - m'(t)n(t) + \delta_1\delta_2 l(t)m^2(t) + l(t)n^2(t)}\boldsymbol{\nu}_2(t).\end{aligned}$$

Let γ be a non-parabolic regular curve with the Frenet-Serret frame $\{\boldsymbol{T}(t), \boldsymbol{N}(t), \boldsymbol{B}(t)\}$ and the curvature (κ_1, κ_2). Then γ is a non-parabolic spatial hybrid framed base curve. The expression $m(t_0)n'(t_0) - m'(t_0)n(t_0) + \delta_1\delta_2 l(t_0)m^2(t_0) + l(t_0)n^2(t_0)$ equals zero if and only if $\kappa_2(t_0) = 0$.

Remark 4.1. Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{H}^p \times \Delta$ be a non-parabolic spatial hybrid framed curve with the curvature $(L, M, 0, \alpha)$. Assume $L(t) \neq 0$ for any $t \in I$. Then its evolute is

$$\mathcal{E}v_\gamma(t) = \gamma(t) - \frac{\alpha(t)}{M(t)}\boldsymbol{n}_1(t) - \delta_1\delta_2\frac{M(t)\alpha'(t) - M'(t)\alpha(t)}{L(t)M^2(t)}\boldsymbol{n}_2(t).$$

The geometric meaning of the evolute is the trajectory of centers of osculating spheres of γ. Differentiating the above equation, we can get

$$\begin{aligned}\mathcal{E}v'_\gamma(t) &= \alpha(t)\boldsymbol{\mu}(t) - \frac{\alpha'(t)M(t) - \alpha(t)M'(t)}{M^2(t)}\boldsymbol{n}_1(t) - \frac{\alpha(t)}{M(t)}(L(t)\boldsymbol{n}_2(t) + M(t)\boldsymbol{\mu}(t))\\ &\quad - \delta_1\delta_2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{M(t)\alpha'(t) - M'(t)\alpha(t)}{L(t)M^2(t)}\right)\boldsymbol{n}_2(t) + \frac{\alpha'(t)M(t) - \alpha(t)M'(t)}{M^2(t)}\boldsymbol{n}_1(t)\\ &= \left(-\frac{\alpha(t)L(t)}{M(t)} - \delta_1\delta_2\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{M(t)\alpha'(t) - M'(t)\alpha(t)}{L(t)M^2(t)}\right)\right)\boldsymbol{n}_2(t).\end{aligned}$$

So $(\mathcal{E}v_\gamma, \boldsymbol{n}_1, \boldsymbol{\mu})$ is a non-parabolic spatial hybrid framed curve.

Now we consider the inverse process of evolutes.

Definition 4.3. Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{H}^p \times \Delta$ be a non-parabolic spatial hybrid framed curve with the curvature $(L, M, 0, \alpha)$. The involute of $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ is defined by $\mathcal{I}nv_\gamma : I \to \mathbb{H}^p$,

$$\mathcal{I}nv_\gamma(t) = \gamma(t) + f_1(t)\boldsymbol{n}_1(t) + f_2(t)\boldsymbol{\mu}(t),$$

where (f_1, f_2) is the solution of the system

$$\begin{cases} f_1'(t) = \sigma\delta_1\delta_2 M(t) f_2(t), \\ f_2'(t) = -M(t) f_1(t) - \alpha(t). \end{cases}$$

It is obvious that

$$\begin{aligned}\mathcal{I}nv'_\gamma(t) =& \alpha(t)\boldsymbol{\mu}(t) + \sigma\delta_1\delta_2 M(t) f_2(t)\boldsymbol{n}_1(t) + f_1(t)L(t)\boldsymbol{n}_2(t) + f_1(t)M(t)\boldsymbol{\mu}(t)\\ &- (M(t)f_1(t) + \alpha(t))\,\boldsymbol{\mu}(t) - \sigma\delta_1\delta_2 M(t) f_2(t)\boldsymbol{n}_1(t)\\ =& f_1(t)L(t)\boldsymbol{n}_2(t).\end{aligned}$$

So $(\mathcal{I}nv_\gamma, \boldsymbol{n}_1, \boldsymbol{\mu})$ is a non-parabolic spatial hybrid framed curve. Note that $\boldsymbol{n}_1(t) \times \boldsymbol{\mu}(t) = -\sigma\boldsymbol{n}_2(t)$. Then $\{\boldsymbol{n}_1(t), \boldsymbol{\mu}(t), -\sigma\boldsymbol{n}_2(t)\}$ is a g-orthogonal frame along $\mathcal{I}nv_\gamma$. The Frenet type formulas are

$$\begin{pmatrix} \boldsymbol{n}_1'(t) \\ \boldsymbol{\mu}'(t) \\ -\sigma\boldsymbol{n}_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & M(t) & -\sigma L(t) \\ -\sigma\delta_1\delta_2 M(t) & 0 & 0 \\ \sigma\delta_1\delta_2 L(t) & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n}_1(t) \\ \boldsymbol{\mu}(t) \\ -\sigma\boldsymbol{n}_2(t) \end{pmatrix},$$

$$\mathcal{I}nv'_\gamma(t) = (-\sigma f_1(t)L(t))(-\sigma\boldsymbol{n}_2(t)).$$

Theorem 4.1. *Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{H}^p \times \Delta$ be a non-parabolic spatial hybrid framed curve with the curvature $(L, M, 0, \alpha)$, where $L(t) \neq 0$ for any $t \in I$. $\mathcal{I}nv_\gamma$ is its involute. Then the evolute of $(\mathcal{I}nv_\gamma, \boldsymbol{n}_1, \boldsymbol{\mu})$ is γ.*

Proof.

$$\begin{aligned}\mathcal{E}v_{\mathcal{I}nv_\gamma}(t) =&\mathcal{I}nv_\gamma(t) - \frac{-\sigma f_1(t)L(t)}{-\sigma L(t)}\boldsymbol{n}_1(t) - \sigma\delta_1\delta_2\frac{\sigma L(t)(\sigma f_1(t)L(t))' - \sigma L'(t)\sigma f_1(t)L(t)}{L^2(t)M(t)}\boldsymbol{\mu}(t)\\ =&\mathcal{I}nv_\gamma(t) - f_1(t)\boldsymbol{n}_1(t) - \sigma\delta_1\delta_2\frac{f_1'(t)}{M(t)}\boldsymbol{\mu}(t)\\ =&\gamma(t).\end{aligned}$$

From the Theorem 4.1, we know the composite map of $\mathcal{I}nv_\gamma$ and $\mathcal{E}v_\gamma$ is γ. So, we call involutes the inverse process of evolutes.

For a non-parabolic spatial hybrid framed curve, we can define its pedal and contrapedal curve by g-orthogonal projection. Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ be a non-parabolic spatial hybrid framed curve and $\boldsymbol{p} \in \mathbb{H}^p$ be a fixed point. As for each plane spanned by $\boldsymbol{n}_1(t)$ and $\boldsymbol{\mu}(t)$ at $\gamma(t)$, we take a unique g-orthogonal projection of $\boldsymbol{p}$ on it. We can get a curve, called the pedal curve. Similarly, changing $\boldsymbol{\mu}(t)$ to $\boldsymbol{n}_2(t)$, we can get the contrapedal curve.

Definition 4.4. Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{H}^p \times \Delta$ be a non-parabolic spatial hybrid framed curve with the curvature $(L, M, 0, \alpha)$. $\boldsymbol{p} \in \mathbb{H}^p$ is a fixed point.

(1) The pedal curve of $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ relative to $\boldsymbol{p}$ is defined by $\mathcal{P}e_{\gamma,p} : I \to \mathbb{H}^p$,

$$\mathcal{P}e_{\gamma,p}(t) = \boldsymbol{p} - \sigma\delta_1\delta_2 g(\boldsymbol{p} - \gamma(t), \boldsymbol{n}_2(t))\boldsymbol{n}_2(t).$$

(2) The contrapedal curve of $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ relative to $\boldsymbol{p}$ is defined by $\mathcal{CP}e_{\gamma,p} : I \to \mathbb{H}^p$,

$$\mathcal{CP}e_{\gamma,p}(t) = \boldsymbol{p} - \delta_1\delta_2 g(\boldsymbol{p} - \gamma(t), \boldsymbol{\mu}(t))\boldsymbol{\mu}(t).$$

There are relationships among evolutes, involutes, pedal and contrapedal curves.

Theorem 4.2. *Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : I \to \mathbb{H}^p \times \Delta$ be a non-parabolic spatial hybrid framed curve with the curvature $(L, M, 0, \alpha)$, where $L(t) \neq 0$ for any $t \in I$. $\boldsymbol{p} \in \mathbb{H}^p$ is a fixed point. Then the pedal curve of the evolute $(\mathcal{E}v_\gamma, \boldsymbol{n}_1, \boldsymbol{\mu})$ coincides with the contrapedal curve of $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$. The contrapedal curve of the involute $(\mathcal{I}nv_\gamma, \boldsymbol{n}_1, \boldsymbol{\mu})$ coincides with the pedal curve of $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$.*

Proof. The binormal direction of $\mathcal{E}v_\gamma$ is the $\boldsymbol{\mu}$ direction. So the pedal curve of the evolute $(\mathcal{E}v_\gamma, \boldsymbol{n}_1, \boldsymbol{\mu})$ relative to $\boldsymbol{p}$ is

$$\begin{aligned}\mathcal{P}e_{\mathcal{E}v_\gamma,p}(t) &= \boldsymbol{p} - \delta_1\delta_2 g(\boldsymbol{p} - \mathcal{E}v_\gamma(t), \boldsymbol{\mu}(t))\boldsymbol{\mu}(t)\\ &= \boldsymbol{p} - \delta_1\delta_2 g(\boldsymbol{p} - \gamma(t), \boldsymbol{\mu}(t))\boldsymbol{\mu}(t)\\ &= \mathcal{CP}e_{\gamma,p}(t).\end{aligned}$$

The tangent direction of $\mathcal{I}nv_\gamma$ is the $\boldsymbol{n}_2$ direction. So the contrapedal curve of the involute $(\mathcal{I}nv_\gamma, \boldsymbol{n}_1, \boldsymbol{\mu})$ relative to $\boldsymbol{p}$ is

$$\begin{aligned}\mathcal{CP}e_{\mathcal{I}nv_\gamma,p}(t) &= \boldsymbol{p} - \sigma\delta_1\delta_2 g(\boldsymbol{p} - \mathcal{I}nv_\gamma(t), \boldsymbol{n}_2(t))\boldsymbol{n}_2(t)\\ &= \boldsymbol{p} - \sigma\delta_1\delta_2 g(\boldsymbol{p} - \gamma(t), \boldsymbol{n}_2(t))\boldsymbol{n}_2(t)\\ &= \mathcal{P}e_{\gamma,p}(t).\end{aligned}$$

Theorem 4.2 is to say that the evolute-involute process can connect the pedal and contrapedal curve. Let $\gamma_i : I \to \mathbb{H}^p$ $(i = 1, 2, 3)$ be smooth curves, where γ_1 and γ_2 are non-parabolic spatial hybrid framed base curves. $\boldsymbol{p} \in \mathbb{H}^p$ is a fixed point. Then Figure 1 commutes.

$$\begin{tikzcd} \gamma_1 \arrow[rr, shift left, "\mathcal{E}v\to"] \arrow[dr, "\mathcal{CP}e"'] & & \gamma_2 \arrow[ll, shift left, "\leftarrow\mathcal{I}nv"] \arrow[dl, "\mathcal{P}e"] \\ & \gamma_3 & \end{tikzcd}$$

Figure 1. $\mathcal{E}v$ is taking the evolute, $\mathcal{I}nv$ is taking the involute, $\mathcal{P}e$ is taking the pedal curve relative to $\boldsymbol{p}$, and $\mathcal{CP}e$ is taking the contrapedal curve relative to $\boldsymbol{p}$.

5. Example

Example 5.1. Let $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2) : [0, 2\pi] \to \mathbb{H}^p \times \Delta$ be

$$\begin{aligned}\gamma(t) &= \left(\sin^3 t - \frac{\sqrt{10}}{4}\cos 2t\right)\boldsymbol{i} + (\sin^3 t)\boldsymbol{\varepsilon} + (\cos^3 t)\boldsymbol{h},\\ \boldsymbol{n}_1(t) &= (\cos t)\boldsymbol{i} + (\cos t)\boldsymbol{\varepsilon} + (\sin t)\boldsymbol{h},\\ \boldsymbol{n}_2(t) &= (3 + \sqrt{10}\sin t)\boldsymbol{i} + (\sqrt{10}\sin t)\boldsymbol{\varepsilon} - (\sqrt{10}\cos t)\boldsymbol{h}.\end{aligned}$$

Then

$$\boldsymbol{\mu}(t) = (\sqrt{10} + 3\sin t)\boldsymbol{i} + (3\sin t)\boldsymbol{\varepsilon} - (3\cos t)\boldsymbol{h}.$$

The Frenet type formulas are

$$\begin{pmatrix}\boldsymbol{n}_1'(t)\\ \boldsymbol{n}_2'(t)\\ \boldsymbol{\mu}'(t)\end{pmatrix} = \begin{pmatrix}0 & -\sqrt{10} & 3\\ \sqrt{10} & 0 & 0\\ 3 & 0 & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{n}_1(t)\\ \boldsymbol{n}_2(t)\\ \boldsymbol{\mu}(t)\end{pmatrix},\ \gamma'(t) = \sin t\cos t\boldsymbol{\mu}(t).$$

$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ are singularities of γ.

Then the evolute of $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ is

$$\mathcal{E}v_\gamma(t) = \left(\frac{2}{3}\sin^3 t - \frac{3\sqrt{10}}{20}\cos 2t\right)\boldsymbol{i} + \left(\frac{2}{3}\sin^3 t\right)\boldsymbol{\varepsilon} + \left(\frac{2}{3}\cos^3 t\right)\boldsymbol{h}.$$

The involute of $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ is

$$\mathcal{I}nv_\gamma(t) = \gamma(t) + f_1(t)\boldsymbol{n}_1(t) + f_2(t)\boldsymbol{\mu}(t),$$

where (f_1, f_2) satisfies

$$\begin{cases} f_1'(t) = -3f_2(t),\\ f_2'(t) = -3f_1(t) - \sin t\cos t.\end{cases}$$

That is,

$$\begin{cases} f_1(t) = -\dfrac{3}{26}\sin 2t + c_1 e^{3t} + c_2 e^{-3t},\\ f_2(t) = \dfrac{1}{13}\cos 2t - c_1 e^{3t} + c_2 e^{-3t},\end{cases}$$

where $c_1, c_2 \in \mathbb{R}$. If we take $c_1 = c_2 = 0$, the involute of $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ is

$$\mathcal{I}nv_\gamma(t) = \left(\frac{10}{13}\sin^3 t - \frac{9\sqrt{10}}{52}\cos 2t\right)\boldsymbol{i} + \left(\frac{10}{13}\sin^3 t\right)\boldsymbol{\varepsilon} + \left(\frac{10}{13}\cos^3 t\right)\boldsymbol{h}.$$

We draw them in Figure 2.

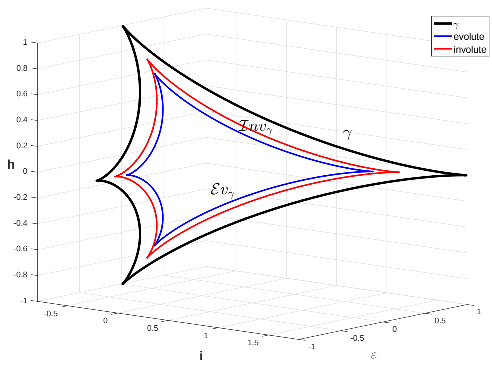

Figure 2. A non-parabolic spatial hybrid framed base curve (black) with its evolute (blue) and involute (red). The evolute and the involute have singularities.

Take $\boldsymbol{p} = \boldsymbol{0} \in \mathbb{H}^p$, and the pedal curve of $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ relative to $\boldsymbol{p}$ is

$$\mathcal{P}e_{\gamma,p}(t) = \cos 2t\left(\left(-\frac{3\sqrt{10}}{4} - \frac{5}{2}\sin t\right)\boldsymbol{i} - \left(\frac{5}{2}\sin t\right)\boldsymbol{\varepsilon} + \left(\frac{5}{2}\cos t\right)\boldsymbol{h}\right).$$

The contrapedal curve of $(\gamma, \boldsymbol{n}_1, \boldsymbol{n}_2)$ relative to $\boldsymbol{p}$ is

$$\mathcal{CP}e_{\gamma,p}(t) = \frac{1}{2}\cos 2t\left((\sqrt{10} + 3\sin t)\boldsymbol{i} + (3\sin t)\boldsymbol{\varepsilon} - (3\cos t)\boldsymbol{h}\right).$$

We draw them in Figure 3.

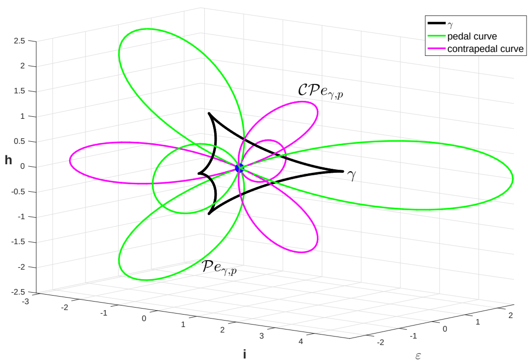

Figure 3. A non-parabolic spatial hybrid framed base curve (black) with its pedal curve (green) and contrapedal curve (magenta). The pedal curve and the contrapedal curve are regular.

6. Conclusions

In the present paper, we define non-parabolic spatial hybrid framed curves and establish the fundamental theory for them. Four smooth functions and two signs can uniquely determine a non-parabolic spatial hybrid framed curve. As applications, we define curves generated by non-parabolic spatial hybrid framed curves and discuss their properties. In the future, we will study properties and applications of framed curves in the hybrid number space.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest in this paper.

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