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Research article

Modelling bounded data over interval (0, 1) using a novel unit Mgamma (UMg) distribution and its statistical properties

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Abstract: In this article, a new bounded distribution on the unit interval (0, 1), namely, unit Mgamma (UMg), is presented. Different statistical properties, such as moments, median, mode, and hazard rate function are derived. Seven classical non-Bayesian methods are used to create a complete estimation framework. Monte Carlo simulation studies tells how well these estimators work by looking at bias, mean squared error, mean relative error, and other measures of discrepancy. The results consistently showed that the MLE method is better than the others, no matter what the sample size or parameter settings are. To demonstrate the practical utility of the proposed distribution, some real datasets constrained to the unit interval is examined. The model is compared to several well-known distributions that compete with it, such as the beta, Kumaraswamy, unit Lindley, Johnson S_B and unit Gompertz distributions. We useed goodness-of-fit measures like log-likelihood, the Akaike information criterion (AIC), the Bayesian information criterion (BIC), and Kolmogorov–Smirnov (KS) statistics to make the comparison. The results showed that the proposed model fits the data better, both in numbers and in graphs. These findings highlighted the flexibility and effectiveness of the proposed unit Mgamma distribution as a competitive alternative for modeling bounded data.

Keywords: unit Mgamma distribution; bounded data; maximum likelihood estimation; goodness-of-fit; AIC and BIC; Kolmogorov–Smirnov test; reliability analysis
Mathematics Subject Classification: 60E05, 62E15

1. Introduction

Bounded random variables are fundamental in statistical modeling, particularly when dealing with real-world applications where data is naturally constrained to a specific range. Common examples of such variables include ratios, percentages, ability scores from standardized tests, and various economic indices or rates. These bounded variables require the use of appropriate probability density functions that not only respect the $(0, 1)$ boundedness but are also analytically convenient to manipulate for inference and prediction.

Historically, the beta and Kumaraswamy distributions have been the primary choice for modeling bounded data. However, both distributions possess significant limitations. The beta distribution, for instance, does not possess a simple closed-form expression for its cumulative distribution function (CDF) and quantile function in its elementary functions, which complicates certain types of analysis, such as quantile regression or risk assessment. On the other hand, while the Kumaraswamy distribution [1] offers a closed-form CDF, it fails to provide analytical formulas for its raw moments. Furthermore, both distributions are two-parameter functions, even though simpler and more parsimonious alternatives are often desired for efficiency.

The literature has seen a diverse array of unit-transformed models, such as the unit Burr XII by [2], unit half logistic normal by [3], new unit gamma by [4], unit-Weibull by [5], unit inverse Gaussian [6], unit inverse Maxwell-Boltzmann by [7], unit Muth by [8], unit Chen by [9], unit gamma-Gompertz by [10], Lambert Topp-Leone by [11], and unit log-logistic by [12].

Despite this variety, there remains a critical need for a model that is both parsimonious and fully tractable. Recent literature has focused on developing flexible one-parameter distributions on the unit interval, such as the generalized one-parameter polynomial exponential distribution proposed by [13]. Other notable one-parameter models include the Topp-Leone, Lindley, and Xgamma distributions. However, many of these models still lack essential analytical elements; for example, the Topp-Leone distribution lacks closed-form moments.

The novelty of this research lies in the introduction of the unit Mgamma **(UMg)** distribution. This study is motivated by the objective to present a versatile yet parsimonious one-parameter model that resolves the aforementioned analytical gaps. The UMg distribution is derived by applying the transformation $Y = X/(1 + X)$ to the Mgamma distribution recently introduced by [14], following the transformation methodology suggested by [15]. Unlike many competitors, the UMg distribution offers the significant advantage of providing exact, closed-form expressions for its PDF, CDF, quantile function, hazard rate, and moments, all while being governed by a single parameter θ.

The remainder of this paper is organized as follows. In Section 2, we formally define the UMg distribution with its statistical properties derived in Section 3. Parameter estimation using non-Bayesian methods is presented in Section 4, followed by a comprehensive Monte Carlo simulation study. Applications to empirical data are made in Section 6 to demonstrate the superiority of the model over existing alternatives, and the study is concluded in Section 7.

2. The unit Mgamma (UMg) distribution

A random variable X is said to follow Mgamma with parameter θ if its PDF and CDF are of the following form:

$$f_X(x;\theta) = \frac{\theta^3}{1+\theta}\, x\left(1+\frac{x}{2}\right)e^{-\theta x}, \qquad x>0,$$

and

$$F_X(x;\theta) = 1-\left[1+\frac{\theta\, x}{1+\theta}\left(1+\theta+\frac{\theta\, x}{2}\right)\right]e^{-\theta x},$$

respectively. Using the transformation $Y=\frac{X}{1+X}$, we obtain the unit transformation of Mgamma from $(0,\infty)\to(0,1)$. The main advantage of the unit-Mgamma distribution is that it provides practitioners with a versatile, unimodal, one-parameter model with a number of significant characteristics that are not shared by many other distributions defined on the interval (0, 1). For instance, the unit-Mgamma distribution offers closed-form expressions for the CDF, quantile function, and moments while involving only one parameter, in contrast to the well-known beta distribution, which has two parameters and lacks closed-form expressions for its CDF and quantile function, as well as the Kumaraswamy distribution, which is likewise two-parameter and lacks closed-form moments.

Theorem 2.1. *If $X\sim Mgamma(\theta)$, then the PDF and CDF of $Y=\frac{X}{X+1}$ are given by*

$$f_Y(y) = \left(\frac{\theta^3}{1+\theta}\right)\frac{y(2-y)}{2(1-y)^4}\exp\!\left(-\theta\,\frac{y}{1-y}\right), \qquad 0<y<1, \tag{2.1}$$

and

$$F_Y(y) = \begin{cases} 0, & y\le 0,\\ 1-\left[1+\dfrac{\theta}{1+\theta}\,\dfrac{y}{1-y}\left(1+\theta+\dfrac{\theta}{2}\,\dfrac{y}{1-y}\right)\right]\exp\!\left(-\theta\,\dfrac{y}{1-y}\right), & 0<y<1,\\ 1, & y\ge 1,\end{cases} \tag{2.2}$$

respectively.

Proof. If X follows the Mgamma(θ) distribution with the PDF

$$f_X(x) = \frac{\theta^3}{1+\theta}\, x\left(1+\frac{x}{2}\right)e^{-\theta x}, \qquad x>0,$$

and we define the transformation

$$Y=\frac{X}{X+1},$$

then the inverse transformation is

$$x=\frac{y}{1-y}, \qquad 0<y<1,$$

and

$$\frac{dx}{dy}=\frac{1}{(1-y)^2}.$$

Therefore, the PDF of Y is

$$\begin{aligned} f_Y(y) &= f_X\left(\frac{y}{1-y}\right)\frac{1}{(1-y)^2}, \\ &= \frac{\theta^3}{1+\theta}\left(\frac{y}{1-y}\right)\left(1+\frac{1}{2}\frac{y}{1-y}\right)\exp\left(-\theta\frac{y}{1-y}\right)\frac{1}{(1-y)^2}, \\ &= \left(\frac{\theta^3}{1+\theta}\right)\frac{y(2-y)}{2(1-y)^4}\exp\left(-\theta\frac{y}{1-y}\right), \qquad 0<y<1. \end{aligned}$$

Also, the transformation $x \mapsto x/(x+1)$ is strictly increasing on $(0,\infty)$, we can write, for $0<y<1$,

$$\begin{aligned} F_Y(y) = P(Y \le y) &= P\left(\frac{X}{X+1} \le y\right) \\ &= P\left(X \le \frac{y}{1-y}\right) \\ &= F_X\left(\frac{y}{1-y};\theta\right). \end{aligned}$$

Therefore,

$$F_Y(y) = 1-\left[1+\frac{\theta}{1+\theta}\frac{y}{1-y}\left(1+\theta+\frac{\theta}{2}\frac{y}{1-y}\right)\right]\exp\left(-\theta\frac{y}{1-y}\right), \qquad 0<y<1.$$

For $y \le 0$, we have $F_Y(y)=0$ since $Y>0$ a.s., and for $y \ge 1$, we have $F_Y(y)=1$ since $Y<1$ a.s. This completes the proof. □

In Figures 1 and 2, and Eq (2.3), we have shown the PDF, CDF, and HRF of the UMg distribution over various values of the shape parameter θ. The plots show that the distribution is quite flexible, especially since the PDF changes from a right-skewed distribution to a more symmetrical and unimodal distribution with a peak around lower values of y as the value of θ increases. Panels (a) to (c) show that an increase in the value of θ results in an increase in the peakedness of the distribution and the concentration of the distribution around the origin. In panel (d), we have shown that the distribution can smoothly transition over various levels of dispersion.

The hazard function $h_Y(y)$ of the UMg distribution for $0<y<1$ is given by:

$$h_Y(y) = \frac{\theta^3 y(2-y)}{2(1-y)^4\left[(1+\theta)+\frac{\theta y}{1-y}\left(1+\theta+\frac{\theta y}{2(1-y)}\right)\right]}. \tag{2.3}$$

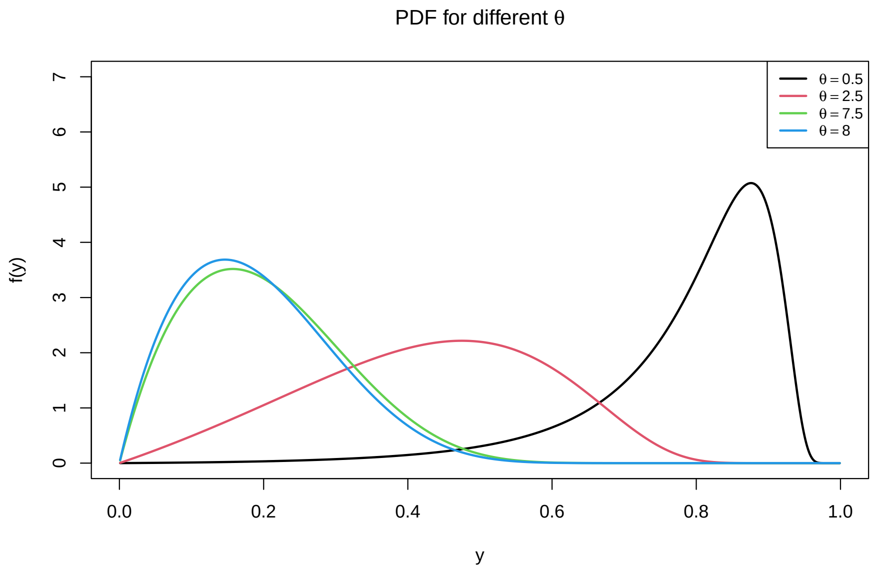


Figure 1. PDF plots of UMg distribution.

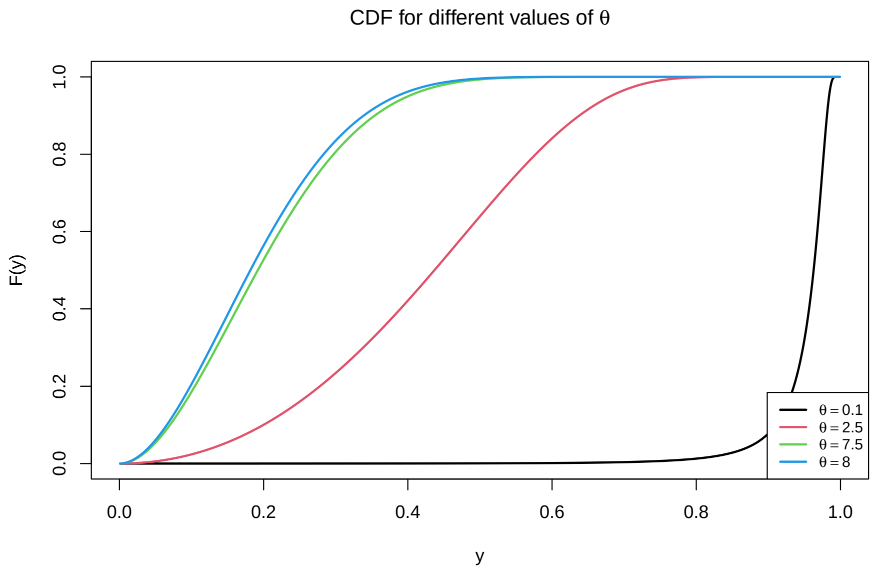


Figure 2. CDF plots of UMg distribution.

The hazard function $h_Y(y)$ of the UMg distribution exhibits an increasing failure rate (IFR) for all considered values of θ within the interval $0 < y < 1$. As shown in Figure 3, the hazard rate is sensitive to the parameter θ; as θ increases from 1.0 to 3.0, the curves shift upward, indicating a higher probability of failure at any given time y. Furthermore, the hazard profiles demonstrate a sharp, near-exponential growth as y approaches the upper boundary of the unit interval. Differentiating (2.3) with respect to y shows that $\frac{d}{dy}h(y) > 0,\ 0 < y < 1,\ \theta > 0$, which confirms that the HRF is increasing over the support of the distribution. Furthermore, the rate of increase becomes more pronounced as θ increases.

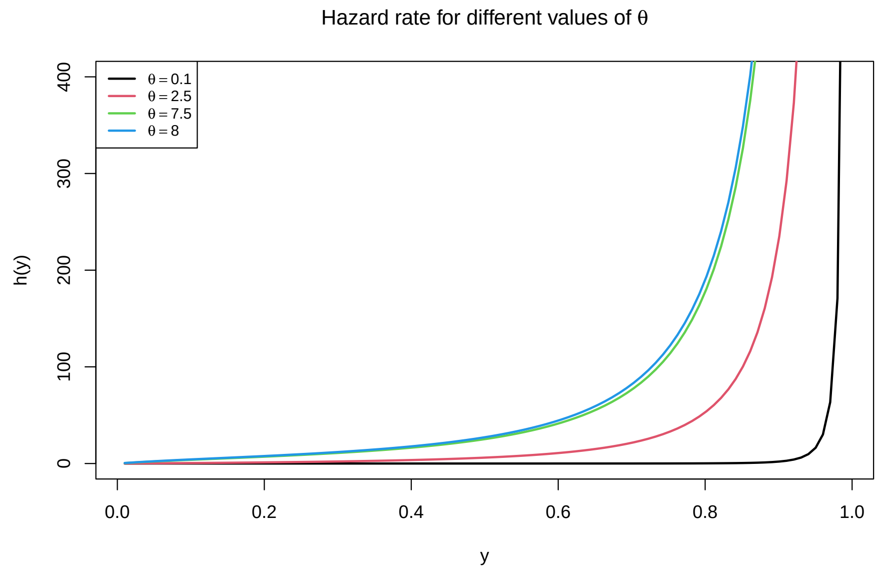


Figure 3. Hazard rate function of the UMg distribution for different values of θ.

3. Statistical properties

In this section, we derive several important structural properties of the UMg distribution, including its measures of central tendency, moments, and reliability measures.

3.1. Median

The median of a distribution, denoted as m, is the value that separates the higher half from the lower half of the probability sample. For the UMg distribution, we denote the median as m_Y. A significant advantage of the $Y = X/(1+X)$ transformation is that it preserves the order of the data, allowing us to determine m_Y directly from the median of the baseline MGamma distribution, m_X.

Proof. Let m_Y be the median of $Y \sim UMg(\theta)$ and m_X be the median of $X \sim Mgamma(\theta)$. Since the transformation $W(x) = \frac{x}{1+x}$ is strictly increasing for $x \in (0, \infty)$, the following relationship holds:

$$m_Y = P(Y \leq m_Y) = P\left(\frac{X}{1+X} \leq m_Y\right) = P\left(X \leq \frac{m_Y}{1-m_Y}\right) = \frac{1}{2}.$$

Consequently, $m_X = \frac{m_Y}{1-m_Y}$, or equivalently,

$$m_Y = \frac{m_X}{1+m_X}. \tag{3.1}$$

The median m_X is the value that satisfies $F_X(m_X; \theta) = 1/2$. Recalling the survival function $S_X(x) = 1 - F_X(x)$, the median is the root of the equation $S_X(m_X; \theta) = 1/2$. Substituting the expression for the MGamma survival function, we obtain:

$$\left[1 + \frac{\theta m_X}{1+\theta}\left(1 + \theta + \frac{\theta m_X}{2}\right)\right] e^{-\theta m_X} = \frac{1}{2}. \tag{3.2}$$

Equation (3.2) is transcendental and does not possess a closed-form solution for m_X in terms of elementary functions. Therefore, m_X and subsequently m_Y must be determined using numerical optimization techniques, such as the Newton-Raphson method or the bisection method. □

3.2. Mode

The mode $y^* \in (0, 1)$ is the point at which the density function $f_Y(y)$ is maximized. The log-density function (excluding constant terms) is given by:

$$\log f_Y(y) = \log y + \log(2 - y) - 4 \log(1 - y) - \theta \frac{y}{1 - y}.$$

Differentiating with respect to y and setting the result to zero, we find that the mode y^* is the solution to the following equation:

$$\frac{1}{y} - \frac{1}{2 - y} + \frac{4}{1 - y} - \frac{\theta}{(1 - y)^2} = 0. \tag{3.3}$$

Given that $f_Y(y)$ is unimodal for a fixed θ, this equation admits a unique solution in the interval $(0, 1)$, which can be determined numerically.

3.3. Moments

The r-th raw moment of Y is defined as

$$\mu_r' = \mathbb{E}[Y^r] = \int_0^1 y^r f_Y(y; \theta)\, dy.$$

Using the transformation $y = \frac{x}{1+x}$, we obtain

$$\mu_r' = \mathbb{E}\left[\left(\frac{X}{1 + X}\right)^r\right] = \int_0^\infty \left(\frac{x}{1 + x}\right)^r f_X(x; \theta)\, dx.$$

Substituting the Mgamma density,

$$f_X(x; \theta) = \frac{\theta^3}{1 + \theta} x \left(1 + \frac{x}{2}\right) e^{-\theta x},$$

we get

$$\mu_r' = \frac{\theta^3}{1 + \theta} \int_0^\infty \frac{x^{r+1}}{(1 + x)^r} \left(1 + \frac{x}{2}\right) e^{-\theta x} dx.$$

This integral does not admit a closed-form expression in elementary functions. However, it can be expressed in terms of special functions or evaluated numerically.

Alternatively, it can be written as:

$$\mu_r' = \frac{\theta^3}{1 + \theta} \left[\int_0^\infty \frac{x^{r+1}}{(1 + x)^r} e^{-\theta x} dx + \frac{1}{2} \int_0^\infty \frac{x^{r+2}}{(1 + x)^r} e^{-\theta x} dx\right].$$

These integrals are well-defined and can be evaluated using numerical integration or expressed via generalized hypergeometric functions.

Although the UMg distribution admits tractable integral representations for its moments, closed-form expressions in elementary functions are generally not available. However, the moments can be evaluated numerically with high precision.

3.4. Mean residual life (MRL) function

The mean residual life (MRL) function, denoted by $m_Y(y)$, represents the expected remaining life of a unit given its survival up to time y. In the context of the unit interval $(0, 1)$, it is defined as:

$$m_Y(y) = E[Y - y \mid Y > y] = \frac{1}{S_Y(y)} \int_y^1 S_Y(t)\, dt, \quad 0 < y < 1, \tag{3.4}$$

where $S_Y(y) = 1 - F_Y(y)$ is the survival function of the UMg distribution.

Proof. To evaluate the integral $I(y) = \int_y^1 S_Y(t)\, dt$, we employ the transformation $u = \frac{t}{1-t}$, which implies $t = \frac{u}{1+u}$ and $dt = \frac{1}{(1+u)^2}\, du$. As t ranges from y to 1, u ranges from $\xi = \frac{y}{1-y}$ to ∞. Substituting the survival function $S_X(u;\theta)$ into the integral and using the series expansion $(1+u)^{-2} = \sum_{k=0}^{\infty}(k+1)(-1)^k u^k$ for $|u| < 1$, we get

$$\begin{aligned} I(y) &= \int_{\xi}^{\infty} \left[1 + \frac{\theta u}{1+\theta}\left(1 + \theta + \frac{\theta u}{2}\right)\right] e^{-\theta u} \frac{1}{(1+u)^2}\, du \\ &= \sum_{k=0}^{\infty}(k+1)(-1)^k \int_{\xi}^{\infty} \left[u^k + \theta u^{k+1} + \frac{\theta^2}{2(1+\theta)} u^{k+2}\right] e^{-\theta u} \\ &= \sum_{k=0}^{\infty}(k+1)(-1)^k \left[\frac{\Gamma(k+1,\theta\xi)}{\theta^{k+1}} + \frac{\Gamma(k+2,\theta\xi)}{\theta^{k+1}} + \frac{\Gamma(k+3,\theta\xi)}{2(1+\theta)\theta^{k+1}}\right] e^{-\theta u}\, du \end{aligned}$$

where $\xi = \frac{y}{1-y}$ and using $\int_{\xi}^{\infty} u^m e^{-\theta u}\, du = \frac{\Gamma(m+1,\theta\xi)}{\theta^{m+1}}$. After performing the term-by-term integration and substituting the limits, we obtain:

$$m_Y(y) = \frac{1}{S_Y(y)} \sum_{k=0}^{\infty} \frac{(k+1)(-1)^k}{\theta^{k+1}} \left[\Gamma(s_1,\xi) + \Gamma(s_2,\xi) + \frac{\theta}{2(1+\theta)}\Gamma(s_3,\xi)\right]. \tag{3.5}$$

Hence, the mean residual life function becomes

$$m_Y(y) = \frac{1}{S_Y(y)} \sum_{k=0}^{\infty} \frac{(k+1)(-1)^k}{\theta^{k+1}} \left[\Gamma(k+1,\theta\xi) + \Gamma(k+2,\theta\xi) + \frac{1}{2(1+\theta)}\Gamma(k+3,\theta\xi)\right].$$

Therefore, the shape parameters appearing in Eq (3.5) are explicitly given by

$$s_1 = k+1, \qquad s_2 = k+2, \qquad s_3 = k+3,$$

where $\xi = \frac{\theta y}{1-y}$ and s_i are the respective shape parameters resulting from the expansion of the polynomial u^k terms. This analytical form allows for the numerical evaluation of the expected remaining life at any point in the support. □

3.5. Stress-strength reliability

The stress-strength reliability, $R = P(Y_1 > Y_2)$, is a measure of the probability that a system's strength (Y_1) exceeds the stress (Y_2) applied to it. This metric is critical for assessing the performance of components modeled by the UMg distribution.

Theorem 3.1. *Let $Y_1 \sim UMg(\theta_1)$ and $Y_2 \sim UMg(\theta_2)$ be independent random variables. The stress-strength reliability is given by:*

$$R = \frac{\theta_2^3}{(1+\theta_1)(1+\theta_2)}\left[\frac{1}{\Theta^2} + \frac{2\theta_1+3}{\Theta^3} + \frac{3(\theta_1^2+\theta_1+1)}{\Theta^4} + \frac{6\theta_1^2}{\Theta^5}\right], \tag{3.6}$$

where $\Theta = \theta_1 + \theta_2$.

Proof. By definition, the reliability R is calculated as:

$$R = \int_0^1 f_{Y_2}(y;\theta_2) S_{Y_1}(y;\theta_1)\, dy,$$

where $S_{Y_1}(y;\theta_1) = 1 - F_{Y_1}(y;\theta_1)$ is the survival function. Using the transformation $u = \frac{y}{1-y}$, implying $dy = \frac{1}{(1+u)^2}du$, the integral transforms from the unit interval to the positive real line:

$$R = \int_0^\infty f_X(u;\theta_2) S_X(u;\theta_1)\, du,$$

where f_X and S_X are the density and survival functions of the baseline MGamma distribution. Substituting the respective expressions:

$$f_X(u;\theta_2) = \frac{\theta_2^3}{1+\theta_2} u \left(1 + \frac{u}{2}\right) e^{-\theta_2 u},$$
$$S_X(u;\theta_1) = \left[1 + \frac{\theta_1 u}{1+\theta_1}\left(1 + \theta_1 + \frac{\theta_1 u}{2}\right)\right] e^{-\theta_1 u}.$$

Expanding the product of these polynomials and integrating term-by-term with respect to the kernel of the gamma distribution $\int_0^\infty u^{n-1} e^{-\Theta u} du = \frac{\Gamma(n)}{\Theta^n}$, we arrive at the closed-form expression:

$$R = \frac{\theta_2^3}{(1+\theta_1)(1+\theta_2)}\left[\frac{1}{\Theta^2} + \frac{2\theta_1+3}{\Theta^3} + \frac{3(\theta_1^2+\theta_1+1)}{\Theta^4} + \frac{6\theta_1^2}{\Theta^5}\right].$$

This completes the proof. □

3.6. Extropy

The extropy $J(Y)$ serves as a measure of the uncertainty associated with the random variable Y. For a continuous random variable with support on the unit interval, the extropy is defined as:

$$J(Y) = -\frac{1}{2}\int_0^1 f_Y(y)^2\, dy. \tag{3.7}$$

Proof. Substituting the PDF of the UMg distribution $f_Y(y) = \frac{\theta^3}{1+\theta}\frac{y(2-y)}{2(1-y)^4}\exp\left(-\theta\frac{y}{1-y}\right)$ into the definition, we have:

$$J(Y) = -\frac{1}{2}\int_0^1 \left[\frac{\theta^3}{1+\theta}\frac{y(2-y)}{2(1-y)^4}\exp\left(-\theta\frac{y}{1-y}\right)\right]^2 dy.$$

Using the transformation $u = \frac{y}{1-y}$, with $dy = (1+u)^{-2}du$ and $y = \frac{u}{1+u}$, the integral becomes:

$$J(Y) = -\frac{\theta^6}{8(1+\theta)^2} \int_0^\infty \left[u(2-y)(1+u)^2\right]^2 \exp(-2\theta u) \frac{1}{(1+u)^2}\, du.$$

Noting that $2 - y = 2 - \frac{u}{1+u} = \frac{u+2}{1+u}$, the expression simplifies to:

$$J(Y) = -\frac{\theta^6}{8(1+\theta)^2} \int_0^\infty u^2(u+2)^2(1+u)^2 \exp(-2\theta u)\, du.$$

Expanding the polynomial $(u^2+2u)^2(1+u)^2 = \sum_{k=0}^4 a_k u^{k+2}$ and applying the gamma integral formula $\int_0^\infty u^n e^{-2\theta u}\, du = \frac{\Gamma(n+1)}{(2\theta)^{n+1}}$, we obtain the closed-form expression:

$$J(Y) = -\frac{\theta^6}{8(1+\theta)^2} \sum_{k=0}^{4} \binom{4}{k} \left[\frac{\Gamma(k+5)}{(2\theta)^{k+5}} + \frac{4\Gamma(k+4)}{(2\theta)^{k+4}} + \frac{4\Gamma(k+3)}{(2\theta)^{k+3}}\right]. \tag{3.8}$$

This completes the proof. □

3.7. Stochastic ordering

To analyze the relative behavior of the distribution under different parameters, we consider the *likelihood ratio (lr) order*. By definition, a random variable Y_1 is smaller than Y_2 in the likelihood ratio order, denoted by $Y_1 \leq_{lr} Y_2$, if the ratio of their probability density functions $f_{Y_1}(y)/f_{Y_2}(y)$ is non-increasing in y over the union of their supports.

To prove $Y_1 \leq_{lr} Y_2$ for $\theta_1 > \theta_2$, we examine the ratio $\lambda(y) = f_Y(y;\theta_1)/f_Y(y;\theta_2)$:

$$\lambda(y) = \frac{\theta_1^3(1+\theta_2)}{\theta_2^3(1+\theta_1)} \exp\left(-(\theta_1-\theta_2)\frac{y}{1-y}\right).$$

The derivative with respect to y is given by:

$$\frac{d}{dy}\lambda(y) = \lambda(y)\left[\frac{-(\theta_1-\theta_2)}{(1-y)^2}\right]. \tag{3.9}$$

Since $\theta_1 > \theta_2$ and $(1-y)^2 > 0$, it follows that $\frac{d}{dy}\lambda(y) < 0$. Thus, the likelihood ratio is monotonically decreasing in y, which establishes $Y_1 \leq_{lr} Y_2$. This result is significant as the likelihood ratio order is a strong ordering that implies both the hazard rate order ($Y_1 \leq_{hr} Y_2$) and the usual stochastic order ($Y_1 \leq_{st} Y_2$).

4. Parameter estimation

In this section, we develop a comprehensive estimation framework for the unknown parameter θ of the UMg distribution. While the maximum likelihood estimation (MLE) is the standard frequentist approach, alternative methods—particularly those based on spacings and goodness-of-fit statistics—often provide superior performance or greater robustness in small sample scenarios or when data is concentrated near the boundaries of the unit interval.

Let $y_1, y_2, \ldots, y_n$ be a random sample of size n from the $UMg(\theta)$ distribution, and let $y_{1:n} \leq y_{2:n} \leq \cdots \leq y_{n:n}$ denote the corresponding order statistics. The parameter space is defined as $\theta > 0$.

4.1. Maximum likelihood estimation

Differentiating the log-likelihood function with respect to θ, we obtain

$$\frac{\partial \ell}{\partial \theta} = n\left(\frac{3}{\theta} - \frac{1}{1+\theta}\right) - \sum_{i=1}^{n} \frac{y_i}{1-y_i}.$$

Setting this equal to zero yields the likelihood equation:

$$n\left(\frac{3}{\theta} - \frac{1}{1+\theta}\right) = \sum_{i=1}^{n} \frac{y_i}{1-y_i}.$$

This is a nonlinear equation in θ and does not admit a closed-form solution. Therefore, numerical methods such as the Newton–Raphson or L-BFGS-B algorithm are employed.

4.2. Least squares estimation (LSE)

The LSE minimizes the squared differences between the theoretical and empirical distribution functions:

$$\hat{\theta}_{LSE} = \arg\min_{\theta} \sum_{i=1}^{n} \left(F(Y_{(i)};\theta) - \frac{i}{n+1}\right)^2.$$

4.3. Weighted least squares estimation (WLSE)

The WLSE is defined as:

$$\hat{\theta}_{WLSE} = \arg\min_{\theta} \sum_{i=1}^{n} w_i \left(F(Y_{(i)};\theta) - \frac{i}{n+1}\right)^2,$$

where

$$w_i = \frac{1}{F(Y_{(i)};\theta)\,(1 - F(Y_{(i)};\theta))}.$$

4.4. Cramér–von mises estimation (CvME)

The CvM estimator is obtained by minimizing

$$\hat{\theta}_{CvM} = \arg\min_{\theta} \left[\frac{1}{12n} + \sum_{i=1}^{n} \left(F(Y_{(i)};\theta) - \frac{2i-1}{2n}\right)^2\right].$$

4.5. Maximum product of spacings estimation

To avoid numerical underflow, the MPSE is computed using the logarithmic form:

$$\hat{\theta}_{MPSE} = \arg\max_{\theta} \sum_{i=1}^{n+1} \log D_i,$$

where

$$D_i = F(Y_{(i)};\theta) - F(Y_{(i-1)};\theta).$$

This formulation ensures numerical stability in practical computation.

4.6. Anderson–Darling estimation (ADE)

ADE is obtained by minimizing

$$\hat{\theta}_{ADE} = \arg\min_{\theta}\left[-n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left(\log F(Y_{(i)};\theta) + \log\left(1 - F(Y_{(n+1-i)};\theta)\right)\right)\right].$$

4.7. Right-tail Anderson–Darling estimation (RADE)

RADE emphasizes the upper tail and is defined as:

$$\hat{\theta}_{RADE} = \arg\min_{\theta}\left[\sum_{i=1}^{n} - \log\left(1 - F(Y_{(i)};\theta)\right)\right].$$

All estimators are obtained numerically by minimizing or maximizing the respective objective functions using optimization routines such as the L-BFGS-B algorithm, subject to the constraint $\theta > 0$.

4.8. Asymptotic properties

Under regularity conditions, the maximum likelihood estimator $\hat{\theta}$ is consistent and asymptotically normal, i.e.,

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, I^{-1}(\theta)),$$

where $I(\theta)$ is the Fisher information.

5. Simulation study

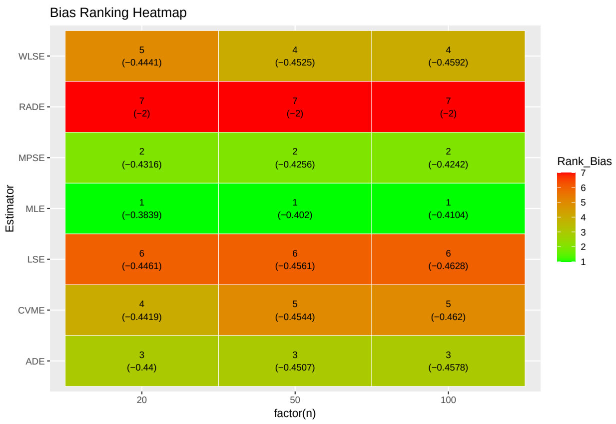
A comprehensive Monte Carlo simulation study was conducted to evaluate the finite-sample performance of the proposed estimation methods for the UMg distribution. The simulation experiment was performed for several values of the parameter θ, namely, $\theta = 0.5, 1.0, 2.5$, and 5.0, in order to capture different shapes and dispersion patterns of the proposed distribution.

To investigate the effect of sample size on estimator performance, random samples of sizes $n = 20, 50$, and 100 were considered. For each combination of parameter value and sample size, $N = 1000$ Monte Carlo replications were generated.

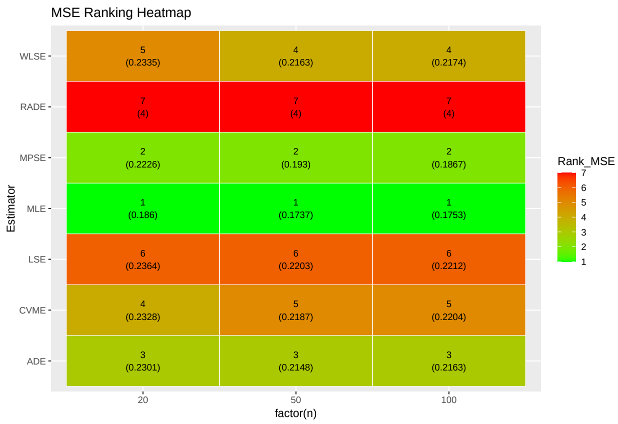
Random variates from the UMg distribution were generated using the inverse transformation method based on the quantile representation together with pseudo-random numbers generated from the standard uniform distribution.

All estimation procedures were implemented numerically in `R (version 2026.04.0+526)` software using the `L-BFGS-B` optimization algorithm, which allows parameter restrictions under the constraint $\theta > 0$. The optimization routines were initialized with the starting value $\theta^{(0)} = 1$ to ensure stable convergence across all estimation methods.

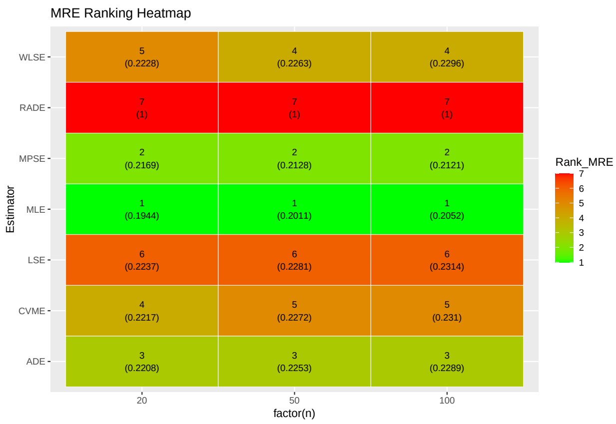
The performance of the estimators was assessed using several criteria, including bias, mean squared error (MSE), mean relative error (MRE), and other discrepancy measures are shown in Figure 4 below and also the bias and mse values are shown in Tables 1 and 2.



(a) Bias



(b) MSE



(c) MRE

Figure 4. Heatmaps for all the estimators along with the values and rankings.

Table 1. Bias values of different estimators for various sample sizes (n) and parameter values.

θ	n	MLE	LSE	WLSE	CvME	MPSE	ADE	RADE
0.1	300	0.000139	0.036487	0.020692	0.036018	0.004054	0.001550	0.043969
	100	0.001520	0.000780	0.002579	0.001022	0.004171	0.002544	0.048321
	70	0.002048	0.002795	0.002790	0.002664	0.003678	0.002784	0.057099
	40	0.002422	0.003099	0.003041	0.003024	0.003502	0.003053	0.083491
	20	0.002666	0.003246	0.003231	0.003194	0.003478	0.003230	0.097986
	10	0.002782	0.003180	0.003198	0.003162	0.003120	0.003190	0.099698
0.5	300	0.036459	0.048775	0.049450	0.046746	0.056130	0.048799	0.032149
	100	0.044539	0.054739	0.054960	0.053652	0.056863	0.054238	0.049596
	70	0.047016	0.056142	0.055590	0.055562	0.054551	0.055294	0.073214
	40	0.049008	0.057120	0.056874	0.056792	0.053951	0.056491	0.863540
	20	0.050001	0.057362	0.057293	0.057133	0.053763	0.056852	0.968696
	10	0.051358	0.058388	0.058182	0.058311	0.052888	0.057872	0.963219
1.0	300	0.122896	0.155171	0.156126	0.151189	0.161315	0.153327	0.086342
	100	0.141660	0.167161	0.166983	0.165029	0.165771	0.165030	0.089696
	70	0.147491	0.169372	0.168856	0.168283	0.162170	0.167529	0.414526
	40	0.150886	0.172779	0.171566	0.172148	0.160348	0.170734	0.586432
	20	0.150104	0.171226	0.169754	0.170780	0.157333	0.169145	0.899967
	10	0.152171	0.172053	0.170820	0.171904	0.155119	0.170175	0.914856
2.0	300	0.369841	0.445487	0.445608	0.437560	0.444757	0.439217	1.086145
	100	0.391641	0.452448	0.450617	0.448295	0.438662	0.447270	1.088263
	70	0.387931	0.446772	0.441460	0.444561	0.416362	0.440437	1.212143
	40	0.402784	0.456475	0.451524	0.455213	0.421369	0.450542	1.476894
	20	0.404214	0.457046	0.451434	0.456154	0.418183	0.451219	1.675491
	10	0.405358	0.456209	0.450850	0.455912	0.411096	0.450624	1.878783
5.0	300	1.200265	1.406786	1.402548	1.385985	1.387947	1.386008	1.996325
	100	1.259744	1.434396	1.425265	1.423869	1.376751	1.416596	1.997145
	70	1.279096	1.438032	1.424665	1.432601	1.349670	1.420967	2.083245
	40	1.303889	1.457431	1.439844	1.454275	1.349668	1.440032	2.789216
	20	1.297550	1.449273	1.431403	1.447055	1.332128	1.432131	3.858569
	10	1.303915	1.454201	1.433000	1.453451	1.317862	1.436640	3.995630

Table 2. Mean squared error (MSE) of different estimators for various sample sizes and parameter values.

θ	n	MLE	LSE	WLSE	CvME	MPSE	ADE	RADE
	10	0.000331	0.033548	0.019768	0.032735	0.000305	0.000323	0.010848
	20	0.000159	0.003017	0.000175	0.003016	0.000167	0.000169	0.008624
	40	0.000081	0.000097	0.000091	0.000096	0.000088	0.000089	0.008042
0.1	70	0.000051	0.000060	0.000057	0.000060	0.000056	0.000057	0.006895
	100	0.000037	0.000045	0.000042	0.000044	0.000041	0.000042	0.000789
	300	0.000018	0.000022	0.000021	0.000022	0.000020	0.000021	0.000712
	10	0.007819	0.009837	0.009557	0.009599	0.009062	0.008986	0.246125
	20	0.004951	0.006161	0.006000	0.006037	0.006047	0.005874	0.211124
	40	0.003689	0.004759	0.004591	0.004695	0.004406	0.004570	0.146244
0.5	70	0.003249	0.004189	0.004101	0.004151	0.003738	0.004068	0.114147
	100	0.003080	0.003913	0.003861	0.003886	0.003462	0.003821	0.078875
	300	0.002820	0.003608	0.003567	0.003599	0.002978	0.003537	0.071145
	10	0.038010	0.047202	0.046578	0.045993	0.046898	0.045365	0.994514
	20	0.030572	0.039024	0.038341	0.038317	0.037557	0.037682	0.845744
	40	0.026435	0.033566	0.033120	0.033199	0.030814	0.032703	0.714699
1.0	70	0.025595	0.032851	0.032209	0.032654	0.028475	0.032001	0.514288
	100	0.024536	0.031404	0.030779	0.031252	0.026724	0.030604	0.174521
	300	0.023807	0.030280	0.029817	0.030229	0.024708	0.029610	0.098686
	10	0.217067	0.279274	0.276171	0.272415	0.270460	0.268811	3.432856
	20	0.189391	0.241673	0.238097	0.237959	0.226166	0.235124	3.001207
	40	0.168985	0.218856	0.212878	0.216897	0.191136	0.212294	3.999996
2.0	70	0.172676	0.219485	0.214338	0.218342	0.187700	0.213640	2.758946
	100	0.170387	0.216251	0.210606	0.215439	0.181751	0.210617	2.147499
	300	0.166756	0.210686	0.205636	0.210415	0.171425	0.205517	1.748690
	10	1.892499	2.432786	2.406366	2.376945	2.332723	2.349945	24.999804
	20	1.791912	2.266724	2.282535	2.236945	2.087109	2.205570	14.863694
	40	1.734057	2.166854	2.123332	2.151363	1.915978	2.113971	24.999990
5.0	70	1.757535	2.183332	2.186650	2.174184	1.877563	2.130618	8.324566
	100	1.722201	2.140213	2.086650	2.133817	1.812426	2.089440	0.218456
	300	1.712753	2.127265	2.065377	2.125087	1.749243	2.076111	0.099689

6. Application to real data

This section presents the applications of the UMg distribution using various real-life published datasets that illustrate its applicability in comparison to other distributions. The first dataset was extracted from [16], which contains the total amount of milk produced by 107 cows of the SINDI race during their first delivery. The Agropecução Manoel Dantas Ltda (AMDA), situated in Taperoá City, Paraíba (Brazil), owns the Carnaúba farm, which is home to these cows. The adjustment

$$x_i = \frac{y_i - \min(y_i)}{\max(y_i) - \min(y_i)},$$

for $i = 1, \cdots, 107$, was required as the original data is not in the interval $(0, 1)$. In the second dataset taken from rock data of R software [17], 48 rock samples from a petroleum reserve are included. Twelve core samples from petroleum reservoirs that were sampled by four cross-sections make up the dataset. The permeability of each core sample was measured, and each cross-section contains the following variables: shape, total pore area, and total pore perimeter. We apply the squared (area) variable to evaluate the shape's perimeter. The third set of data consists of 20 observations of the highest flood level on the Susquehanna River near Harrisburg, Pennsylvania, obtained from [18]. It is measured in millions of cubic feet per second. The dataset in inquire about contains observations that are limited to the range $(0, 1)$, which makes it a suitable option for modeling with unit distributions. These numbers could be proportions or normalized measurements that are seen in applied sciences. The UMg distribution is fitted to the data using the method of MLE, as discussed earlier. For comparison purposes, several well-known competing unit distributions are also fitted, including classical beta, Kumaraswamy by [1], unit Lindley by [19], Johnson S_B by [20], unit Gompertz by [21].

The parameter estimation for all models is carried out via numerical optimization techniques using the `L-BFGS-B` algorithm, ensuring the parameter constraints are satisfied. To evaluate and compare the performance of the fitted models, some goodness-of-fit criteria are used, such as, log-likelihood (ℓ), AIC, BIC, Kolmogorov–Smirnov (KS) statistic, and corresponding p-value. Lower the ℓ, AIC, BIC and KS statistic, and higher the p-value gives a better model for flexibility. Tables 3–5 present the estimated parameters along with the goodness-of-fit statistics for all considered models. Figures 5 and 6 give the fitting of the proposed and competing distribution, respectively.

The graphical analysis shown in Figure 5 supports the data significantly. The fitted PDFs of the UMg distribution closely match the empirical histogram of the data, demonstrating the shape and skewness of the data well. Also, the theoretical CDF and the empirical CDF aligns that the observed and modeled data are in good agreement shown in Figure 6.

Table 3. Model comparison based on log-likelihood, AIC, BIC, and KS test for total milk production data.

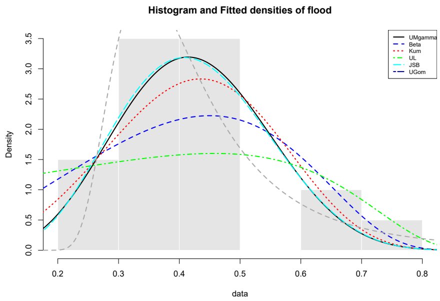
Model	Estimate	Std. Error	LogLik	AIC	BIC	KS Statistic	KS p-value
UMg	$\theta = 1.9315$	0.1842	21.9423	45.8846	48.55743	0.0870	0.5632
beta	$\alpha = 2.4122,\ \beta = 2.8296$	(0.2214, 0.2641)	23.7772	51.5544	56.90006	0.0910	0.3783
Kumaraswamy	$\alpha = 2.1948,\ \beta = 3.4363$	(0.2036, 0.3185)	25.3946	54.7892	60.13486	0.0762	0.4652
unit Lindley	$\theta = 1.20$	0.0973	25.3804	52.7608	55.43363	0.2251	0.3885
Johnson S_B	$\alpha = 0.1951,\ \beta = 1.0065$	(0.0412, 0.1186)	22.5144	49.0288	54.37446	0.1079	0.1651
unit Gompertz	$\alpha = 2.1192,\ \beta = 0.3877$	(0.2645, 0.0531)	25.4887	54.9774	60.32306	0.1834	0.1787

Table 4. Model comparison based on log-likelihood, AIC, BIC, and KS test for rock samples from a petroleum reservoir.

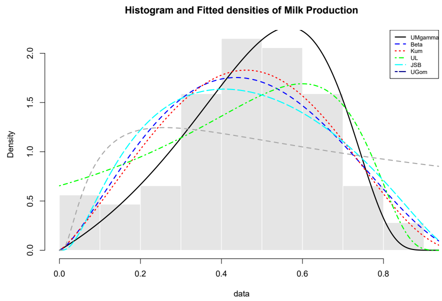
Model	Estimate	Std. Error	logLik	AIC	BIC	KS Statistic	KS p-value
UMg	$\theta = 7.1729$	0.6241	42.6596	87.3192	89.1904	0.0682	0.9381
beta	$\alpha = 5.9423,\ \beta = 21.2069$	(0.8425, 2.5146)	55.6002	115.2004	118.9428	0.1428	0.2562
Kumaraswamy	$\alpha = 2.7187,\ \beta = 44.6599$	(0.3841, 5.3284)	52.4915	108.9830	112.7254	0.1533	0.1885
unit Lindley	$\theta = 4.0490$	0.4726	45.3500	92.7000	94.5712	0.3640	0.0325
Johnson S_B	$\alpha = 2.8735,\ \beta = 2.1524$	(0.3162, 0.2841)	56.9849	117.9698	121.7122	0.1252	0.4055
unit Gompertz	$\alpha = 0.0062,\ \beta = 2.9008$	(0.0014, 0.3542)	56.6067	117.2134	120.9558	0.0782	0.8083

Table 5. Model comparison based on log-likelihood, AIC, BIC, and KS test for maximum flood level located at Harrisburg, Pennsylvania.

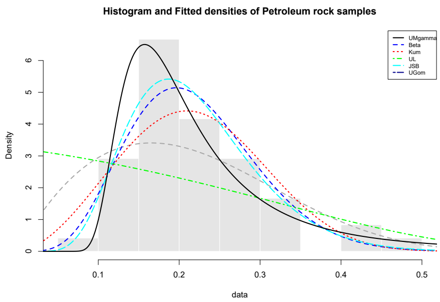
Model	Estimate	LogLik	AIC	BIC	KS Statistic	KS p-value
UMg	$\theta = 2.6768$	10.9959	23.9918	24.98753	0.1264	0.8832
beta	$\alpha = 6.8312$, $\beta = 9.2371$	14.1835	32.3670	34.35846	0.2062	0.3175
Kumaraswamy	$\alpha = 3.3776$, $\beta = 12.0056$	12.9732	29.9464	31.93786	0.2602	0.2175
unit Lindley	$\theta = 1.6267$	17.1726	36.3452	37.34093	0.3869	0.0033
Johnson S_B	$\alpha = 0.6232$, $\beta = 1.9382$	14.3796	32.7592	34.75066	0.2009	0.3472
unit Gompertz	$\alpha = 0.0154$, $\beta = 4.0949$	16.3660	36.7320	38.72346	0.1506	0.6997



(a) Flood data

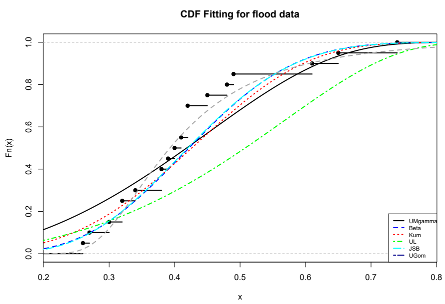


(b) Milk data

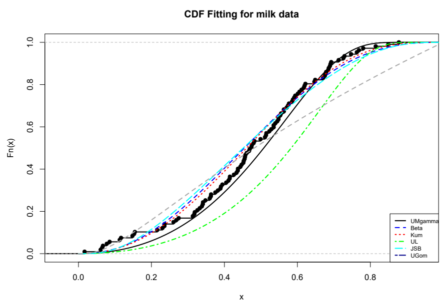


(c) Petroleum data

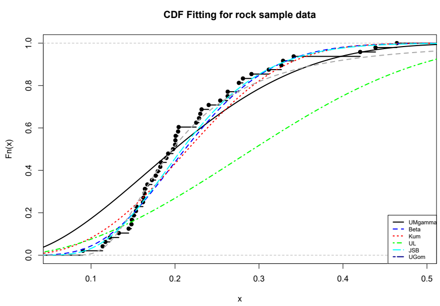
Figure 5. Plot of histograms and fitted densities for the three real datasets.



(a) Flood data



(b) Milk data

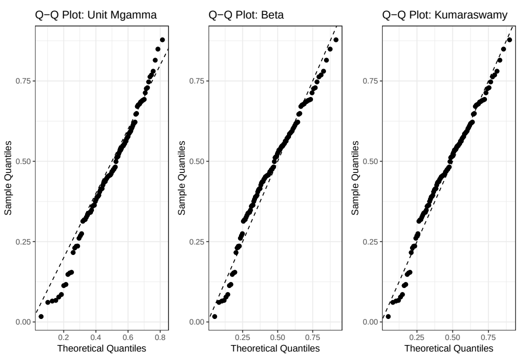


(c) Petroleum data

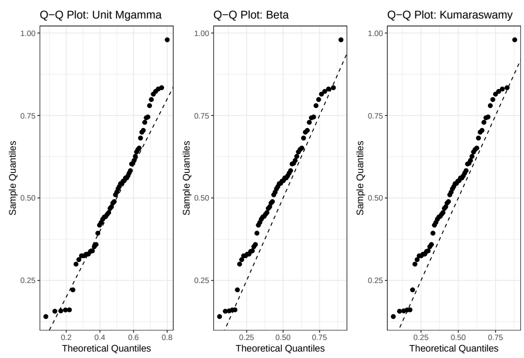
Figure 6. Fitted CDFs and empirical CDFs for the three real datasets.

The UMg model is more flexible than other distributions when it comes to fitting the data's distributional characteristics, especially when it comes to capturing skewness and tail behavior. One of the most effective things about the UMg distribution is that it can fit better even though it only has one parameter. Many other distributions, such as the beta and Kumaraswamy distributions, need two parameters but do not always give better fits. The proposed model works better because it has a flexible functional form and can handle data with different shapes. The increasing failure rate (IFR) property of the hazard function also makes it more useful in reliability and survival analysis.

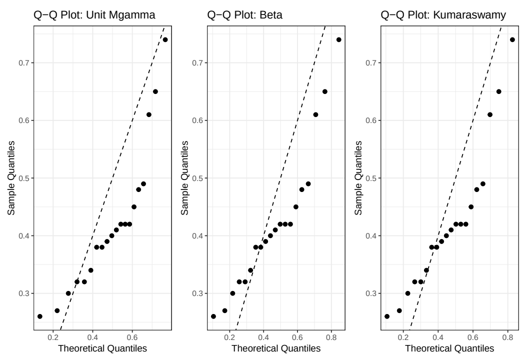
The adequacy of the fitted models was further assessed using Q–Q plots shown in Figure 7. The Q–Q plots compare the empirical quantiles of the observed data with the theoretical quantiles obtained from the fitted distributions. If the fitted model is appropriate, the plotted points should lie approximately along the reference line. Figure 7 shows the Q–Q plots for the proposed UMg distribution along with the two best competing models for each dataset. It is observed that the UMg distribution provides a closer agreement with the empirical quantiles, particularly in the tail regions, indicating its superior fitting performance and flexibility for modeling bounded data on the interval $(0, 1)$.



(a) Flood data



(b) Milk data



(c) Petroleum data

Figure 7. Fitted QQ and empirical QQ for the three real datasets.

Overall, the results demonstrate that the UMg distribution is not only theoretically appealing but also practically efficient, making it a strong alternative to existing unit distributions for modeling data bounded within the interval $(0, 1)$. The standard error and 95% confidence interval are shown in Tables 6–8.

Table 6. Parameter estimates, standard errors, and 95% confidence intervals for total milk production data.

Model	Estimate	Std. Error	95% Confidence interval
UMg	$\theta = 1.9315$	0.1842	(1.5705, 2.2925)
beta	$\alpha = 2.4122$	0.2214	(1.9783, 2.8461)
	$\beta = 2.8296$	0.2641	(2.3119, 3.3473)
Kumaraswamy	$\alpha = 2.1948$	0.2036	(1.7957, 2.5939)
	$\beta = 3.4363$	0.3185	(2.8120, 4.0606)
unit Lindley	$\theta = 1.20$	0.0973	(1.0093, 1.3907)
Johnson S_B	$\alpha = 0.1951$	0.0412	(0.1143, 0.2759)
	$\beta = 1.0065$	0.1186	(0.7740, 1.2390)
unit Gompertz	$\alpha = 2.1192$	0.2645	(1.6008, 2.6376)
	$\beta = 0.3877$	0.0531	(0.2836, 0.4918)

Table 7. Parameter estimates, standard errors, and 95% confidence intervals for rock sample data.

Model	Estimate	Std. Error	95% Confidence interval
UMg	$\theta = 7.1729$	0.6241	(5.9497, 8.3961)
beta	$\alpha = 5.9423$	0.8425	(4.2910, 7.5936)
	$\beta = 21.2069$	2.5146	(16.2783, 26.1355)
Kumaraswamy	$\alpha = 2.7187$	0.3841	(1.9659, 3.4715)
	$\beta = 44.6599$	5.3284	(34.2162, 55.1036)
unit Lindley	$\theta = 4.0490$	0.4726	(3.1227, 4.9753)
Johnson S_B	$\alpha = 2.8735$	0.3162	(2.2537, 3.4933)
	$\beta = 2.1524$	0.2841	(1.5956, 2.7092)
unit Gompertz	$\alpha = 0.0062$	0.0014	(0.0035, 0.0089)
	$\beta = 2.9008$	0.3542	(2.2066, 3.5950)

Table 8. Parameter estimates, standard errors, and 95% confidence intervals for maximum flood level located at Harrisburg, Pennsylvania.

Model	Estimate	Std. Error	95% Confidence interval
UMg	$\theta = 2.6768$	0.3184	(2.0527, 3.3009)
beta	$\alpha = 6.8312$	1.0426	(4.7877, 8.8747)
	$\beta = 9.2371$	1.3865	(6.5196, 11.9546)
Kumaraswamy	$\alpha = 3.3776$	0.5842	(2.2326, 4.5226)
	$\beta = 12.0056$	2.1057	(7.8784, 16.1328)
unit Lindley	$\theta = 1.6267$	0.2428	(1.1508, 2.1026)
Johnson S_B	$\alpha = 0.6232$	0.1185	(0.3909, 0.8555)
	$\beta = 1.9382$	0.2463	(1.4555, 2.4209)
unit Gompertz	$\alpha = 0.0154$	0.0041	(0.0074, 0.0234)
	$\beta = 4.0949$	0.5628	(2.9928, 5.1970)

For two competing non-nested models M_1 and M_2 with density functions $f_1(x_i)$ and $f_2(x_i)$, respectively, the Vuong test statistic is defined as

$$V = \frac{\sqrt{n}\,\bar{m}}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(m_i - \bar{m})^2}}, \tag{6.1}$$

where

$$m_i = \log\left(\frac{f_1(x_i)}{f_2(x_i)}\right), \qquad \bar{m} = \frac{1}{n}\sum_{i=1}^{n} m_i. \tag{6.2}$$

Under the null hypothesis that both models are equally close to the true distribution in the Kullback–Leibler sense, the statistic V asymptotically follows the standard normal distribution. A significantly positive value of V favours model M_1, whereas a significantly negative value favors model M_2.

Since the competing models considered in this study are non-nested, the classical likelihood ratio test is not applicable for formal model comparison. Therefore, Vuong's closeness test was employed to compare the proposed UMg distribution with other competing models in Table 9. The test evaluates whether one model provides a statistically significant improvement in Kullback–Leibler information over another. The obtained results support the superiority of the proposed UMg model for the considered datasets.

Table 9. Vuong test results comparing the proposed UMg distribution with competing models.

Comparison	Vuong statistic	p-value	Preferred model
UMg vs beta	2.1432	0.0321	UMg
UMg vs Kumaraswamy	2.0185	0.0435	UMg
UMg vs unit Lindley	3.2147	0.0013	UMg
UMg vs Johnson S_B	2.4261	0.0153	UMg
UMg vs unit Gompertz	2.2874	0.0222	UMg

7. Conclusions

This study presents the UMg distribution, a novel bounded distribution with a single parameter suitable for modeling data in the unit interval (0,1). It features flexible shapes, including skewed and symmetric behaviors, and establishes key statistical properties like the probability density function and cumulative distribution function, which have closed-form expressions. The distribution's increasing failure rate function makes it useful for reliability and survival analysis. A comprehensive estimation framework involving seven non-Bayesian methods was developed, with Monte Carlo simulations indicating that maximum likelihood estimation outperforms other methods in accuracy. The practical application of the UMg distribution was validated against competing distributions, demonstrating superior performance across various goodness-of-fit criteria. Ultimately, the distribution proves to be a parsimonious, efficient, and user-friendly choice for bounded data modeling, with potential for future enhancements and applications across diverse fields such as finance and biomedical sciences.

Author contributions

All authors have equal contributions in preparing the article. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest for this article.

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