AIMS Mathematics

AIMS Mathematics, 11(6): 18665–18691.
DOI: 10.3934/math.2026759
Received: 01 April 2026
Revised: 14 May 2026
Accepted: 20 May 2026
Published: 25 June 2026

https://www.aimspress.com/journal/Math

*Research article*

# A geometric perspective on the inextensible flows and energy of curves in 4-dimensional pseudo-Galilean space

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**Abstract:** In this study, inextensible flows of curves in pseudo-Galilean 4-space were comprehensively characterized, and the necessary and sufficient conditions governing these curve flows were established as a coupled system of partial differential equations. By defining the directional derivatives with respect to the parameter $t$ and the arc-length parameter $s$ in accordance with the Serret-Frenet frame, we derived the extended Serret-Frenet relations specifically adapted to the metric in $G_1^4$. This formulation ensured the preservation of the curve's arc length during deformation, effectively modeling the motion of inelastic filaments. Beyond the kinematic description, we utilized the Sasaki metric framework on the tangent bundle to formulate the bending elastic energy functionals and pseudo-angle variations for the moving frame fields. These energy profiles were evaluated along both spatial $s-$lines and dynamic $t-$lines, explicitly demonstrating how geometric constraints and metric indices differentiated the system's energy distribution and energy transfer mechanisms. Furthermore, the structural integrity of the flow was evaluated through Lyapunov stability criteria, proving that the inextensibility constraint stabilized the higher-dimensional variations and rendered the system orbitally stable. The framework was validated via helicoidal trajectory simulations, aligning derived equations with physical thresholds. This work provided a rigorous mathematical basis for inelastic rod and fiber dynamics in 4D pseudo-Galilean space.

**Keywords:** pseudo-Galilean 4-space $G_1^4$; inextensible flow; pseudo-angle; energy
**Mathematics Subject Classification:** 53A10, 53A35, 74B05, 53C80

## 1. Introduction

Inextensible curves in pseudo-Galilean 4-space $G_1^4$ are a particularly interesting topic in differential geometry. In classical differential geometry, inextensible curves often arise in the modelling of elastic rods or strings and can be thought of as curves whose arc length does not change with time. Given the unique metric structure of $G_1^4$, the application of this concept presents some differences in $G_1^4$.

The study of inextensible curves in $G_1^4$ contributes to a better understanding of the geometric structure of space. Furthermore, such curves may have potential applications in modelling certain systems in classical mechanics or certain physical models. The differential equations of inextensible curves and their solutions are important for understanding how motions in $G_1^4$ behave under constraints; the inextensibility of a curve means that its arc length is insensitive to changes in parameters during a curve deformation. Also, the inextensibility of a curve is related to the norm of the curve's tangent vector being constant.

The relationship between inextensible curves and energy in pseudo-Galilean 4-space $G_1^4$ is an important topic for modeling physical systems and studying differential geometric structures. As we have seen, the concept of inextensibility relates to the curve being "inextensible", that is, the norm of the tangent vector is constant. Energy, on the other hand, is often associated with the state of a system and can be used to derive the equations of motion. Also, in classical mechanics, the equations of motion are derived using a system's Lagrangian or Hamiltonian. Energy is often closely related to the Hamiltonian. Geometric constraints, such as inextensibility, limit the possible motions of the system, and these constraints are reflected in the energy expressions.

The Lagrangian or Hamiltonian of a physical system modelling a curve in $G_1^4$ typically contains terms involving the square of the norm of the tangent vector. For example, the kinetic energy of a simple particle can be related to the norm of its velocity vector according to the metric in $G_1^4$. When considering a curve $\Omega$ in $G_1^4$, we may want to define concepts such as the kinetic energy of the curve. In $G_1^4$, the tangent vector $\Omega' = T$ represents the velocity, and the square of the norm of the tangent vector $\|T\|^2$ is analogous to the square of the velocity. However, due to the indefinite nature of the metric in $G_1^4$, $\|T\|^2$ can be positive (or negative, zero). If the curve is inextensible, $\|T\|^2 = A$ =constant, which creates a constraint on the energy expression. This constant can be positive ($A > 0$, space-like), negative ($A < 0$, time-like), or zero ($A = 0$, null). That is,

1) If the curve is spacelike inextensible, the kinetic energy term is positive and constant. This may evoke a similarity to constant-speed motion in classical mechanics, but the metric structure of $G_1^4$ complicates this analogy.

2) If the curve is timelike inextensible, the kinetic energy term is negative and constant. This situation may have no direct analogue in classical physics but is related to the timelike motion in the theory of relativity.

3) If the curve is null inextensible, the kinetic energy term is zero. This is analogous to the situation of massless particles moving at the speed of light and plays a special role in the expression for energy.

The kinematics of magnetic flows associated with Killing magnetic fields on lightlike cones were rigorously examined in [1], leading to the identification of distinct magnetic curves determined by their corresponding Killing vectors. Following this trajectory, physical profiles, including energy distributions and electromagnetic field vectors, were fundamentally analyzed within structured geometric domains and specific cone spaces in [2]. In the context of higher-dimensional geometries, the structural behavior of moving frames was investigated by establishing the total energy and pseudo-angle variations of Frenet vector fields within pseudo-Euclidean spaces $R_v^n$ [3]. Concurrently, the theoretical framework governing harmonic and minimal vector fields, alongside geometric maps on tangent bundles, has been extensively documented in the literature [4]. Such trajectory descriptions frequently rely on constrained variational methods; for instance, Capovilla et al. [5] developed foundational Hamiltonian structures for space curves subject to local energy penalties. In a similar vein, particle dynamics under external environmental influences extend to classical relativistic problems, such as the exact motion of a charge within a gravitational field analyzed by Carmeli [6]. In applied optical physics, the geometric rotation of the polarization plane in optical fibers was explicitly characterized under specialized propagation conditions [7]. This was complemented by rigorous geometric formulations establishing the precise relationship between the volume and the total energy of vector fields on Riemannian manifolds [8], alongside extensive evaluations of volume minimization for unit vector fields on three-dimensional spheres [9]. Beyond unconstrained spaces, the geometric behavior of elastic curves on surfaces was successfully modeled under the influence of environmental bias [10]. Additionally, the topological and path-dependent nature of geometric polarization rotations in non-elastic optical filaments was thoroughly investigated [11], while the dynamic evolution of localized vortex filaments was effectively mapped to soliton solutions within fluid dynamics frameworks [12]. In higher-dimensional regimes, a novel approach to evaluating the bending energy of elastica was established specifically for space curves embedded in de-Sitter spaces [13]. Expanding upon standard elastica models, variations in energy functionals were generalized to analyze the explicit change of the Willmore energy of curves in three-dimensional Lorentzian space [14]. The fundamental kinematics governing the evolution of inelastic plane curves were formulated through partial differential equations that restrict curvature to ensure total length preservation over time [15]. This variational framework was subsequently extended to three dimensions to characterize inextensible flows of curves and their associated developable surfaces under zero strain energy conditions [16]. From a classical mechanics perspective, the rigorous physical foundations of continuum media and invariant field formulations remain deeply rooted in established theoretical physics principles [17]. Furthermore, the mathematical modeling of elastic deformations and physical constraints in rods or filaments directly relies on the classical mathematical theory of elasticity [18].

Recently, the structural characterization of geometric deformations has been significantly expanded by exploring the kinematics of inextensible flows of curves specifically within 4-dimensional Galilean space [19]. Following this geometric paradigm, the evolution of inextensible flows of curves was subsequently modeled according to the Sabban frame in pseudo-Galilean space $G_1^3$ [20]. Within physical and relativistic frameworks, the fundamental interplay between spatial curvature, tensor fields, and frame variations aligns seamlessly with the classical field principles of relativity and gravitation [21]. From a variational standpoint, these structural mappings are intrinsically linked to the critical points and total energy functionals computed for a unit vector field on a Riemannian manifold [22]. Furthermore, the scope of geometric evolutions has been enriched by evaluating the inextensible flows of polynomial curves under the strict restrictions of the Flc frame [23]. This comprehensive framework has also been extended to higher-dimensional surfaces, successfully characterizing the simultaneous evolutions of partner-ruled surfaces subject to strict inextensibility conditions [24]. Finally, the intrinsic motion of inextensible quaternionic curves has been firmly established, linking their geometric variations directly to the soliton solutions of the modified Korteweg-de Vries (mKdV) equation [25].

In this study, we present the main results on inextensible curve flow aspects of classical differential geometry topics to the 4-dimensional pseudo-Galilean space. We also derive the corresponding differential equations for inextensible flows. Moreover, we express the energy on the inextensible curves corresponding to a particular particle and the pseudo-angles of the Frenet-Serret vector fields in $G_1^4$ along with the $s$ −lines(or $t$ −lines) coordinate in $G_1^4$.

## 2. Preliminaries

The four-dimensional pseudo-Galilean space $G_1^4$ is a particular type of Cayley-Klein, whose metric structure fundamentally differs from the classical four-dimensional Euclidean space. The study of curves within this framework plays an essential role in differential geometry and related physical theories, providing a rigorous foundation for higher-order geometric investigations. The unique metric structure of this space gives rise to distinctive behaviors of curves during deformations.

In affine coordinates, the pseudo-Galilean scalar product between two points

$$P_i = (p_{i1}, p_{i2}, p_{i3}, p_{i4}),\ i = 1,2$$

is defined by

$$\langle P_1, P_2\rangle_{G_1^4} = \begin{cases} |p_{21} - p_{11}|, & \text{if } p_{21} \neq p_{11}, \\ \sqrt{|-(p_{22} - p_{12})^2 + (p_{23} - p_{13})^2 + (p_{24} - p_{14})^2|}, & \text{if } p_{21} = p_{11}. \end{cases}$$

The pseudo-Galilean cross product for three vectors $\vec{u} = (u_1, u_2, u_3, u_4)$, $\vec{v} = (v_1, v_2, v_3, v_4)$, and $\vec{w} = (w_1, w_2, w_3, w_4)$ in $G_1^4$ is defined via the following determinant representations:

- ❖ If $u_1 \neq 0$ or $v_1 \neq 0$ or $w_1 \neq 0$:

$$\vec{u} \wedge \vec{v} \wedge \vec{w} = \begin{vmatrix} 0 & -e_2 & e_3 & e_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix},$$

- ❖ If $u_1 = v_1 = w_1 = 0$:

$$\vec{u} \wedge \vec{v} \wedge \vec{w} = \begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix}.$$

A vector $X(x, y, z, w)$ is called non-isotropic if $x \neq 0$. All unit non-isotropic vectors take the form $(1, y, z, w)$. Conversely, a vector is called isotropic if $x = 0$ holds.

A non-lightlike vector is a unit vector if $-y^2 + z^2 + w^2 = \pm 1$. Within the pseudo-Galilean space, isotropic vectors $X(x, y, z, w)$ are classified into four distinct types: spacelike ($x = 0$, $-y^2 + z^2 + w^2 > 0$), timelike ($x = 0$, $-y^2 + z^2 + w^2 < 0$), and two types of lightlike vectors (if $x = 0$, $y = \sqrt{z^2 + w^2}$). The scalar product of two vectors $\vec{U} = (u_1, u_2, u_3, u_4)$ and $\vec{V} = (v_1, v_2, v_3, v_4)$ in $G_1^4$ is explicitly defined by:

$$\langle \vec{U}, \vec{V} \rangle_{G_1^4} = \begin{cases} u_1 v_1, & \text{if } u_1 \neq 0 \text{ or } v_1 \neq 0, \\ -u_2 v_2 + u_3 v_3 + u_4 v_4, & \text{if } u_1 = 0 \text{ and } v_1 = 0. \end{cases}$$

The norm of $\vec{U} = (u_1, u_2, u_3, u_4)$ is defined as $\|\vec{U}\|_{G_1^4} = \sqrt{|\langle \vec{U}, \vec{U} \rangle|_{G_1^4}}$.

A curve $\alpha: I \subset \mathbb{R} \to G_1^4$, $\alpha(t) = (x(t), y(t), z(t), w(t))$ is called an admissible curve if $x'(t) \neq 0$. Let $\alpha(s) = (s, y(s), z(s), w(s))$ be an admissible curve parametrized by the arc length $s$ in $G_1^4$. Throughout this study, differentiation with respect to $s$ is denoted by a prime. The tangent vector field $T$ of $\alpha$ is defined by:

$$T = \alpha'(s) = (1, y'(s), z'(s), w'(s)).$$

Since $T$ is a unit non-isotropic vector, it satisfies:

$$\langle T, T \rangle_{G_1^4} = 1. \tag{2.1}$$

Differentiating Eq (2.1) with respect to $s$ yields $\langle T', T \rangle_{G_1^4} = 0$. The first curvature function $\kappa(s)$ measures the rotation of the curve and is defined as:

$$\kappa(s) = \|T'(s)\| = \sqrt{|-(y''(s))^2 + (z''(s))^2 + (w''(s))^2|}.$$

Assuming $\kappa(s) \neq 0$ for all $s \in I$, the principal normal vector field $N(s)$ is given by:

$$N(s) = \frac{T'(s)}{\kappa(s)} = N(s) = \frac{1}{\kappa(s)}(0, y''(s), z''(s), w''(s)). \tag{2.2}$$

By differentiating the principal normal vector, the torsion (second curvature) function $\tau(s)$ is obtained as:

$$\tau(s) = \|N'(s)\|_{G_1^4}. \tag{2.3}$$

The first binormal vector field $B_1$ of the curve $\alpha$ is defined by:

$$B_1(s) = \frac{1}{\tau(s)}\left(0, \left(\frac{y''(s)}{\kappa(s)}\right)', \left(\frac{z''(s)}{\kappa(s)}\right)', \left(\frac{w''(s)}{\kappa(s)}\right)'\right). \tag{2.4}$$

Consequently, $B_1(s)$ is orthogonal to both $T$ and $N$. The second binormal vector field $B_2(s)$ is defined via the pseudo-Galilean cross product as:

$$B_2(s) = \mu T(s) \wedge N(s) \wedge B_1(s), \tag{2.5}$$

where $\mu = \pm 1$ is chosen to ensure that the determinant of the matrix $[T, N, B_1, B_2]$ is $+1$. We define the third curvature function $\sigma$ of $\alpha$ by the inner product:

$$\sigma = \langle B_1', B_2 \rangle_{G_1^4}. \tag{2.6}$$

The set $\{T, N, B_1, B_2, \kappa, \tau, \sigma\}$ forms the Serret-Frenet apparatus of $\alpha$. The vector fields $\{T, N, B_1, B_2\}$ are mutually orthogonal and satisfy the following relations:

$$\langle T, T \rangle_{G_1^4} = 1, \langle N, N \rangle_{G_1^4} = \varepsilon_1, \langle B_1, B_1 \rangle_{G_1^4} = \varepsilon_2, \langle B_2, B_2 \rangle_{G_1^4} = \varepsilon_3, \tag{2.7}$$

$$\langle T, N \rangle_{G_1^4} = \langle T, B_1 \rangle_{G_1^4} = \langle T, B_2 \rangle_{G_1^4} = \langle N, B_1 \rangle_{G_1^4} = \langle N, B_2 \rangle_{G_1^4} = \langle B_1, B_2 \rangle_{G_1^4} = 0,$$

where

$$\varepsilon_3 = \begin{cases} +1, \text{ if } \varepsilon_1 = -1 \text{ or } \varepsilon_2 = -1 \\ -1, \text{ if } \varepsilon_1 = 1 \text{ and } \varepsilon_2 = 1. \end{cases} \tag{2.8}$$

Consequently, the Serret-Frenet equations for the admissible curve $\alpha(s)$ are given in matrix form as:

$$\frac{\partial}{\partial s}\begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_1\kappa & 0 & 0 \\ 0 & 0 & \varepsilon_2\tau & 0 \\ 0 & -\varepsilon_2\tau & 0 & \varepsilon_3\sigma \\ 0 & 0 & -\varepsilon_2\sigma & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{2.9}$$

[26,27].

**Definition 1**. *Let $(M, \varrho)$ and $(N, h)$ be two Riemannian manifolds and let $f: (M, \varrho) \to (N, h)$ be a differentiable map between them. The energy functional is defined by:*

$$energy(f) = \frac{1}{2}\int_M \sum_{a=1}^{n} h(df(e_a), df(e_a))v, \tag{2.10}$$

*where $\{e_a\}$ is a local basis of the tangent space, and $v$ is the canonical volume form on $M$* [3], [24].

**Definition 2**. *Let $T^1M$ be the unit tangent bundle of a semi-Riemannian manifold $M$ and $\varpi: T(T^1M) \to T^1M$ be the natural tangent bundle projection. The connection map $Q: T(T^1M) \to T^1M$ associated with the Levi-Civita connection D on $M$ satisfies the following conditions:*

*i)* $\omega o Q = \omega o d\omega$ *and* $\omega o Q = \omega o \varpi$, *where* $\omega$ *denotes the structural projection map;*

*ii)* *For any tangent vector $\varrho \in T_xM$ and a local section $\xi: M \to T^1M$, the connection map satisfies:*

$$Q(d\xi(\varrho)) = D_\varrho\xi, \tag{2.11}$$

*where D is the Levi-Civita covariant derivative adapted to the semi-Riemannian manifold M* [3,22].

**Definition 3**. *For $\varsigma_1, \varsigma_2 \in T_\xi(T^1M)$, the Riemannian metric $\varrho_S$ on $TM$ is defined as:*

$$\varrho_S(\varsigma_1, \varsigma_2) = \varrho(d\omega(\varsigma_1), d\omega(\varsigma_2)) + \varrho(Q(\varsigma_1), Q(\varsigma_2)). \tag{2.12}$$

*Here, as known, $\varrho_S$ is the Sasaki metric, which renders the projection $\omega: T^1M \to M$ a Riemannian submersion* [3,22].

## 3. Some characterizations of inextensible flows curves in $G_1^4$

In this section, we describe the partial and directional derivatives in accordance with the Serret-Frenet frame $\{T, N, B_1, B_2\}$ in the four-dimensional pseudo-Galilean space $G_1^4$. Furthermore, we express the extended Serret-Frenet relations and systematically investigate the bending energy formulations for the tangent vector fields along the coordinate lines of an elastic curve. These geometric relations are evaluated along the evolving curve $\Omega$ in $G_1^4$.

Let $\Omega: [0, n] \times [0, m] \to G_1^4$ be a one-parameter family of admissible smooth curves in $G_1^4$ with non-vanishing curvatures $(\kappa, \tau, \sigma \neq 0)$, where *s* is the arc-length parameter of the initial curve. Let $u$

be the independent parametrization variable ($0 \leq u \leq n$) and $v = \left\|\frac{\partial \Omega}{\partial u}\right\|$, denote the curve speed, from which the arc length is defined as $s(u) = \int_0^u v\, du$.

The partial differential operators satisfy the scaling relation $\frac{\partial}{\partial s} = v\frac{\partial}{\partial u}$, which implies $\partial s = v\, \partial u$. Any smooth curve flow of $\Omega$ can be generally represented as:

$$\frac{\partial \Omega}{\partial t} = f_1 T + f_2 N + f_3 B_1 + f_4 B_2, \tag{3.1}$$

where $f_i(s,t)$ $(1 \leq i \leq 4)$ are differentiable coordinate velocity functions. To ensure the physical requirement that the curve is not subjected to any local elongation or compression during the deformation, the total arc-length variation must remain invariant over time. This inextensibility condition is expressed as:

$$\frac{\partial s(u,t)}{\partial t} = \int_0^v \frac{\partial v}{\partial t} du = 0, \text{for all } u \in [0,n]. \tag{3.2}$$

Hence, we state the following fundamental definition.

**Definition 4**. *A curve evolution $\Omega(u,t)$ and its corresponding flow in the pseudo-Galilean space in $G_1^4$ are said to be strictly inextensible if*

$$\frac{\partial}{\partial t}\left\|\frac{\partial \Omega}{\partial u}\right\| = 0.$$

**Theorem 1**. *The smooth flow of $\Omega(u,t)$ in $G_1^4$ is inextensible if and only if the tangential velocity component $f_1$ is spatially uniform along the curve, satisfying:*

$$\frac{\partial f_1}{\partial s} = 0,$$

*where $f_1$ depends solely on the time parameter $t$.*

*Proof.* By employing the metric definition of the curve and differentiating the speed term

$$\left\|\frac{\partial \Omega}{\partial u}\right\| = v^2 = \left\langle \frac{\partial \Omega}{\partial u}, \frac{\partial \Omega}{\partial u} \right\rangle_{G_1^4},$$

with respect to the time parameter $t$, we obtain:

$$2v\frac{\partial v}{\partial t} = 2\left\langle \frac{\partial}{\partial t}\left(\frac{\partial \Omega}{\partial u}\right), \frac{\partial \Omega}{\partial u} \right\rangle_{G_1^4}.$$

By commuting the partial derivative operators $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$, and applying the relation $\frac{\partial \Omega}{\partial u} = v\vec{T}$, the inner product yields:

$$v\frac{\partial v}{\partial t} = \left\langle \frac{\partial}{\partial u}\left(f_1\vec{T} + f_2\vec{N} + f_3\overrightarrow{B_1} + f_4\overrightarrow{B_2}\right), \frac{\partial \Omega}{\partial s}\frac{\partial s}{\partial u} \right\rangle_{G_1^4}; \frac{\partial s}{\partial u} = v. \tag{3.3}$$

Substituting the velocity vector decomposition from Eq (3.1) into Eq (3.3), and expanding the derivative via the spatial Serret-Frenet equation (2.9), we obtain:

$$\frac{\partial v}{\partial t}=\left\langle \begin{array}{c} \vec{T},\frac{\partial f_1}{\partial u}\vec{T}+\left(f_1 v\varepsilon_1\kappa+\frac{\partial f_2}{\partial s}-f_3\varepsilon_1\tau v\right)\vec{N}+ \\ \left(f_2 v\varepsilon_2\tau+\frac{\partial f_3}{\partial u}-f_4\varepsilon_2\sigma v\right)\overrightarrow{B_1}+\left(f_3\varepsilon_3\sigma v+\frac{\partial f_4}{\partial u}_4\right)\overrightarrow{B_2} \end{array}\right\rangle_{G_1^4}. \tag{3.4}$$

Evaluating the pseudo-Galilean inner product according to the orthogonality conditions, Eq (3.4) simplifies directly to:

$$\frac{\partial v}{\partial t}=\frac{\partial f_1}{\partial u}=v(\frac{\partial f_1}{\partial s}).$$

Applying the inextensibility constraint $\frac{\partial v}{\partial t}=0$ established in Definition 4, it follows that $\frac{\partial f_1}{\partial s}=0 \Rightarrow f_1(t)=\text{constant}$ with respect to the arc length $s$.

This completes the proof.

**Corollary 1.** From a physical perspective, Theorem 1 serves as a localized geometric continuity equation that fundamentally constrains the dynamical degrees of freedom of the system.

**Corollary 2.** The mathematical formulation assumes that the physical wire, rod, or fiber along which the particle moves is strictly inelastic, undergoing neither local extension nor compression.

**Corollary 3.** For a physical curve flow to remain inextensible, it is mandatory that the tangential velocity component $f_1$ remains completely uniform at every point along the curve at any given instant. Any non-zero spatial variation in the tangential velocity would imply that adjacent segments propagate at different rates, thereby inducing elastic deformation.

**Corollary 4.** The spatial uniformity of $f_1$ stabilizes the tangential component of the system's total kinetic energy, thereby facilitating a controlled transfer of energy into alternative geometric channels, such as bending or torsion.

Therefore, from the above theorem, we understand that the evolution of a curve in $G_1^4$ is inextensible. From what we understand from differential geometry, this theorem confirms that the concept of torsion is a local measure of how much a curve bends, while curvature reflects the bending of the curve. Hence, we give the following theorem regarding inextensible curves.

**Theorem 2**. *Let $\Omega(s,t)$ be an inextensible curve flow parametrized by the arc length $s$ with the Serret-Frenet moving frame $\{\vec{T},\vec{N},\overrightarrow{B_1},\overrightarrow{B_2}\}$ in $G_1^4$. The extended Serret-Frenet equations with respect to $t$ are explicitly governed by the matrix system:*

$$\frac{\partial}{\partial t}\begin{bmatrix}\vec{T}\\ \vec{N}\\ \overrightarrow{B_1}\\ \overrightarrow{B_2}\end{bmatrix}=\begin{bmatrix}0 & \xi_1 & \xi_2 & \xi_3\\ -\varepsilon_1\xi_1 & 0 & \Gamma_1\varepsilon_2 & \Gamma_2\varepsilon_3\\ -\varepsilon_2\xi_2 & -\Gamma_1\varepsilon_1 & 0 & \Gamma_3\varepsilon_3\\ -\varepsilon_3\xi_3 & -\Gamma_2\varepsilon_1 & -\Gamma_3\varepsilon_2 & 0\end{bmatrix}\begin{bmatrix}\vec{T}\\ \vec{N}\\ \overrightarrow{B_1}\\ \overrightarrow{B_2}\end{bmatrix}, \tag{3.5}$$

*where the geometric functions $\Gamma_1$ and metric couplings $\xi_i$ are explicitly defined as:*

$$\Gamma_1 = -\left\langle \frac{\partial B_1}{\partial t}, N \right\rangle_{G_1^4}, \Gamma_2 = \left\langle \frac{\partial B_2}{\partial t}, N \right\rangle_{G_1^4}, \Gamma_3 = \left\langle \frac{\partial B_1}{\partial t}, B_2 \right\rangle_{G_1^4}$$

$$\xi_1 = \varepsilon_1 \kappa f_1 + f_2' - \varepsilon_1 \tau f_3, \xi_2 = \varepsilon_2 \tau f_2 + f_3' - \varepsilon_2 \sigma f_4, \xi_3 = f_4' + \varepsilon_3 \sigma f_3.$$

*Proof.* Applying the commutativity of the mixed partial derivatives under the inextensibility condition, we write

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial \Omega}{\partial s}\right) = \frac{\partial}{\partial s}\left(\frac{\partial \Omega}{\partial t}\right).$$

Differentiating Eq (3.1) with respect to *s* and substituting Eq (2.9) yields:

$$\frac{\partial T}{\partial t} = (\varepsilon_1 \kappa f_1 + f_2' - \varepsilon_1 \tau f_3)\vec{N} + (\varepsilon_2 \tau f_2 + f_3' - \varepsilon_2 \sigma f_4)\overrightarrow{B_1} + (f_4' + \varepsilon_3 \sigma f_3)\overrightarrow{B_2}. \tag{3.6}$$

Differentiating the mutual orthogonality relations (2.9) with respect to $t$, and evaluating the algebraic components, we systematically obtain the remaining frame variations:

$$\begin{aligned}
\frac{\partial}{\partial t}\langle T, N\rangle_{G_1^4} &= 0 \Rightarrow \left\langle \frac{\partial N}{\partial t}, T \right\rangle_{G_1^4} = -\kappa f_1 - \varepsilon_1 f_2' + \tau f_3;\\
\frac{\partial}{\partial t}\langle T, B_1\rangle_{G_1^4} &= 0 \Rightarrow \left\langle \frac{\partial B_1}{\partial t}, T \right\rangle_{G_1^4} = -\tau f_2 - \varepsilon_2 f_3' + \sigma f_4;\\
\frac{\partial}{\partial t}\langle T, B_2\rangle_{G_1^4} &= 0 \Rightarrow \left\langle \frac{\partial B_2}{\partial t}, T \right\rangle_{G_1^4} = -\varepsilon_3 f_4' - \sigma f_3;\\
\frac{\partial}{\partial t}\langle N, B_1\rangle_{G_1^4} &= 0 \Rightarrow \left\langle \frac{\partial B_1}{\partial t}, N \right\rangle_{G_1^4} = -\Gamma_1;\\
\frac{\partial}{\partial t}\langle N, B_2\rangle_{G_1^4} &= 0 \Rightarrow \left\langle \frac{\partial B_2}{\partial t}, N \right\rangle_{G_1^4} = -\Gamma_2;\\
\frac{\partial}{\partial t}\langle B_2, B_1\rangle_{G_1^4} &= 0 \Rightarrow \left\langle \frac{\partial B_1}{\partial t}, B_2 \right\rangle_{G_1^4} = -\Gamma_3.
\end{aligned}$$

By using the above equations, we get

$$\begin{aligned}
\frac{\partial N}{\partial t} &= (-\kappa f_1 - \varepsilon_1 f_2' + \tau f_3)\vec{T} + \Gamma_1 \varepsilon_2 \overrightarrow{B_1} + \Gamma_2 \varepsilon_3 \overrightarrow{B_2}\\
\frac{\partial B_1}{\partial t} &= (-\tau f_2 - \varepsilon_2 f_3' + \sigma f_4)\vec{T} - \Gamma_1 \varepsilon_1 \vec{N} + \Gamma_3 \varepsilon_3 \overrightarrow{B_2}\\
\frac{\partial B_2}{\partial t} &= (-\varepsilon_3 f_4' + \sigma f_3)\vec{T} - \Gamma_2 \varepsilon_1 \vec{N} - \Gamma_3 \varepsilon_2 \overrightarrow{B_1}.
\end{aligned}$$

This fulfills the validation of the matrix system (3.5).

**Corollary 5.** The coefficients $\xi_i$ couple the velocity components $f_i$, the curvatures $(\kappa, \tau, \sigma)$, and the spatial derivatives of the velocity $f_i{}'$. This proves that as the propagation velocity of the curve varies, the geometric frame must instantly reorient itself to preserve structural compatibility.

**Corollary 6.** Physically, this implies that an object under an inextensibility constraint (e.g., a taut steel wire or a fiber optic cable) cannot deform arbitrarily. Specifically, the velocity component in one direction, such as $f_2$, intrinsically triggers rotational variations in alternative directions, such as the binormal direction $B_1$.

**Corollary 7.** The terms $\xi$ and $\Gamma$ within the evolution matrix correspond to the components of the angular velocity vector in classical mechanics. As the curve flows through space, these coefficients govern the magnitude and rate of the frame's precession (conical and gyroscopic nutation).

**Corollary 8.** These relations explain how vibrations or perturbations propagate along a wire without altering its total length. The uniformity of the tangential velocity component $f_1$ physically models an energy transfer mechanism that completely bypasses longitudinal waves, conducting energy exclusively through transverse bending and torsional modes.

Now, we express the conditions on the curvature and the torsion for the curve flow $\Omega$ to be inelastic.

**Theorem 3**. *Let $\Omega(s,t)$ be an inextensible curve flow parametrized by the arc-length parameter s with the Serret-Frenet moving frame {T,N, $B_1$, $B_2$} in the pseudo-Galilean space $G_1^4$. Here, we explicitly assume that the binormal velocity component $f_3$ is non-vanishing ($f_3 \neq 0$) along the curve flow to avoid geometric singularities in the curvature and torsion evolution equations. The temporal evolutions of the geometric curvatures $\kappa(s,t)$, $\tau(s,t)$, and $\sigma(s,t)$ are governed by the following coupled partial differential equations:*

$$\Gamma_1 = \frac{1}{\kappa}(\varepsilon_1\varepsilon_2\xi_2' + \varepsilon_1\xi_1\tau - \varepsilon_1\xi_3\sigma); \Gamma_2 = \frac{1}{\kappa}(\varepsilon_1\varepsilon_3\xi_3' + \varepsilon_1\xi_2\sigma).$$

*The first curvature with respect to the time parameter $t$ yields the explicit evolutionary curvature profile:*

$$\kappa(s,t)=\int \left(\varepsilon_1\frac{\partial \xi_1}{\partial s} - \xi_2\tau\right)dt,$$

*where the torsion function $\tau(s,t)$ is implicitly coupled via the algebraic relation:*

$$\tau(s,t) = \frac{1}{f_3}\left(\kappa(t)f_1 + \varepsilon_1\frac{\partial f_2}{\partial s}\right).$$

*Furthermore, under the specific constraint where the velocity function $f_4(s,t)$ satisfies the partial integro-differential equation:*

$$f_4(t) = \int \left(\varepsilon_1\varepsilon_2\int \kappa(t)\Gamma_2 dt + c_2\right)dt,$$

*the third curvature function $\sigma(s,t)$ is explicitly determined by the integrated Bernoulli-type solution*

$$\sigma(s,t) = \frac{e^{\int\frac{(\varepsilon_2\tau f_2 + 2f_3')}{f_3}dt}}{-\varepsilon_1\int \frac{f_4}{f_3}e^{-\int\frac{(\varepsilon_2\tau f_2 + 2f_3')}{f_3}dt}dt + c_3}.$$

*Here, the auxiliary frame coupling functions are defined as:*

$$\xi_1 = \varepsilon_1\kappa f_1 + f_2' - \varepsilon_1\tau f_3,\ \xi_2 = \varepsilon_2\tau f_2 + f_3' - \varepsilon_2\sigma f_4,\ \xi_3 = f_4' + \varepsilon_3\sigma f_3.$$

*Proof.* By applying the structural integrability and compatibility conditions for the mixed partial derivatives of the tangent vector field, the equality of the cross-derivatives requires that:

$$\frac{\partial}{\partial s}\left(\frac{\partial T}{\partial t}\right)=\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial s}\right).$$

Evaluating the left-hand side of the compatibility equation by substituting the frame variations from Eq (3.6) and differentiating with respect to the spatial parameter $s$ via Eq (2.9), we obtain:

$$\begin{aligned}\frac{\partial}{\partial s}\left(\frac{\partial T}{\partial t}\right)&=\frac{\partial}{\partial s}\left(\xi_1\overrightarrow{N}+\xi_2\overrightarrow{B_1}+\xi_3\overrightarrow{B_2}\right)\\&=(\xi_1'-\varepsilon_1\xi_2\tau)\overrightarrow{N}+(\xi_2'+\varepsilon_2\xi_1\tau-\varepsilon_2\xi_3\sigma)\overrightarrow{B_1}+(\xi_3'+\varepsilon_3\xi_2\sigma)\overrightarrow{B_2}.\end{aligned} \tag{3.7}$$

Conversely, evaluating the right-hand side of the cross-derivative equation by substituting the spatial Serret-Frenet derivative $\frac{\partial T}{\partial s}=\kappa\xi_1\vec{N}$ from Eq (2.9), and differentiating with respect to the time parameter $t$, we get:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial s}\right)&=\frac{\partial}{\partial t}\left(\kappa\xi_1\vec{N}\right)=\varepsilon_1\frac{\partial\kappa}{\partial t}\vec{N}+\varepsilon_1\kappa\frac{\partial\vec{N}}{\partial t}\\&=\varepsilon_1\frac{\partial\kappa}{\partial t}\vec{N}+\varepsilon_1\kappa(-\kappa f_1-\xi_1f_2'+\tau f_3)\vec{T}+\varepsilon_1\varepsilon_2\kappa\Gamma_1\overrightarrow{B_1}+\varepsilon_1\kappa\varepsilon_3\Gamma_2\overrightarrow{B_2}.\end{aligned} \tag{3.8}$$

Equating the linear independent vector components of the Serret-Frenet moving frame across Eq (3.7) and Eq (3.8) yields the following system of four distinct compatibility algebraic relations:

$$\varepsilon_1\kappa(-\kappa f_1-\varepsilon_1f_2'+\tau f_3)=0$$

$$\varepsilon_1\frac{\partial\kappa}{\partial t}=\xi_1'-\varepsilon_1\xi_2\tau$$

$$\xi_2'+\varepsilon_2\xi_1\tau-\varepsilon_2\xi_3\sigma=\varepsilon_1\varepsilon_2\kappa\Gamma_1$$

$$\xi_3'+\varepsilon_3\xi_2\sigma=\varepsilon_1\varepsilon_3\kappa\Gamma_2.$$

Isolating the temporal derivative of the first curvature from the second relation directly resolves the partial differential equation governing the curvature evolution. Integrating this expression with respect to the time parameter $t$ yields the explicit evolutionary curvature profile $\kappa(s,t)$, which validates Eq. (3.9) Similarly, isolating the torsion field from the algebraic constraint of the first component $T$ successfully isolates Eq (3.10) as follows:

$$\kappa(s,t)=\int(\varepsilon_1\xi_1'-\xi_2\tau)dt, \tag{3.9}$$

$$\tau(t)=\frac{1}{f_3}(\kappa(t)f_1+\varepsilon_1f_2'). \tag{3.10}$$

Extracting the frame rotation fields $\Gamma_1$ and $\Gamma_2$ from the third and fourth relations results in the corresponding structural definitions:

$$\Gamma_1=\frac{1}{\kappa}\begin{pmatrix}\varepsilon_1\tau'f_2+\kappa\tau f_1+\varepsilon_1f_2'\tau(1+\varepsilon_2)\\-\varepsilon_1f_3(\tau^2\varepsilon_2+\sigma^2\varepsilon_3)-\varepsilon_1\sigma f_4'(\varepsilon_2\varepsilon_3+1)+\varepsilon_2\varepsilon_1(f_3''-\varepsilon_3\sigma'f_4)\end{pmatrix}, \tag{3.11}$$

$$\Gamma_2=\frac{1}{\kappa}(\varepsilon_1\varepsilon_3\xi_3'+\varepsilon_1\xi_2\sigma).$$

Expanding the component function definitions explicitly and substituting them into the spatial cross-derivatives yields the evolutionary differential equation for the third curvature field:

$$\frac{\partial\sigma}{\partial s}f_3+(\varepsilon_2\tau f_2+2\frac{\partial f_3}{\partial s})\sigma=\varepsilon_1\kappa\Gamma_2+\varepsilon_1\sigma^2 f_4-\varepsilon_3\frac{\partial}{\partial s}(\frac{\partial f_4}{\partial s}). \tag{3.12}$$

By imposing the structural constraint for the velocity field $f_4(s,t)$ defined in Eq (3.13), the non-homogeneous source terms on the right-hand side cancel out, reducing Eq (3.12) to a classical separable non-linear Bernoulli differential equation:

$$\frac{\partial\sigma}{\partial s}f_3+(\varepsilon_2\tau f_2+2\frac{\partial f_3}{\partial s})\sigma=\varepsilon_1\sigma^2 f_4.$$

Solving this non-linear system via standard integrating factor techniques directly yields the complete explicit evolutionary functional profile for $\sigma(s,t)$ stated in Eq. (3.14). For the following equation:

$$f_4(s,t)=\int\left(\varepsilon_1\varepsilon_2\int\kappa(t)\Gamma_2dt+c_2\right)dt, \tag{3.13}$$

solving this nonlinear system via standard integrating factor techniques directly yields the complete explicit evolutionary functional profile for $\sigma(s,t)$ stated in Eq (3.14). This concludes the complete formal proof of the theorem.

$$\sigma(s,t)=\frac{e^{\int\frac{\left(\varepsilon_2\tau f_2+2f_3'\right)}{f_3}dt}}{-\varepsilon_1\int\frac{f_4}{f_3}e^{-\int\frac{\left(\varepsilon_2\tau f_2+2f_3'\right)}{f_3}dt}dt+c_3}, \tag{3.14}$$

where $\xi_1=\varepsilon_1\kappa f_1+f_2'-\varepsilon_1\tau f_3$, $\xi_2=\varepsilon_2\tau f_2+f_3'-\varepsilon_2\sigma f_4$, $\xi_3=f_4'+\varepsilon_3\sigma f_3$.

**Theorem 4**. *Let $\Omega(s,t)$ be an inextensible curve flow parametrized by the arc-length parameter $s$ with the Serret-Frenet moving frame {T,N,$B_1$,$B_2$} in the pseudo-Galilean space $G_1^4$. If the curve flow is inextensible, the following coupled partial differential and integro-differential equations hold identically:*

$$\varepsilon_1 f_2''-\kappa f_1'+2\tau f_3'-\kappa' f_1+\tau' f_3+\varepsilon_2\tau^2 f_2+\sigma\tau\varepsilon_2 f_4=0.$$

*The temporal variation of the second curvature satisfies the partial differential relation:*

$$\tau(t)=\int(\Gamma_1'-\varepsilon_3\Gamma_2\sigma)dt,$$

*where the third curvature function is algebraically expressed as:*

$$\sigma(s,t)=-\frac{1}{\Gamma_1\varepsilon_2}(\Gamma_2'+\Gamma_3).$$

*Furthermore, the frame rotation fields satisfy the following system of relations:*

$$\Gamma_1=\frac{-\varepsilon_1\kappa\xi_1}{\tau-\varepsilon_1};\Gamma_2=\frac{1}{\sigma\varepsilon_3}\left(\Gamma_1'-\frac{\partial\tau}{\partial t}\right);\Gamma_3=\Gamma_1\varepsilon_2\sigma+\left(\frac{1}{\sigma\varepsilon_3}\left(\Gamma_1'-\frac{\partial\tau}{\partial t}\right)\right)'.$$

*Crucially, the functions $\Gamma_1$ and $\Gamma_2$ are implicitly coupled via the following spatial integro-differential equation with respect to the arc-length parameter $s$:*

$$\Gamma_1^2 + \Gamma_2^2 = 2\int \frac{\partial \tau}{\partial t}\Gamma_1 ds - \frac{\varepsilon_3}{\varepsilon_2}\int \Gamma_2\Gamma_3 ds,$$

*where the auxiliary function is defined as* $\xi_1 = \kappa f_1 + \varepsilon_1 f_2' - \tau f_3$.

*Proof.* By invoking the structural integrability and compatibility conditions for the mixed partial derivatives of the principal normal vector field, the equality of the cross-derivatives requires that:

$$\frac{\partial}{\partial s}\left(\frac{\partial N}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial N}{\partial s}\right).$$

Evaluating the left-hand side of the compatibility equation by substituting the temporal frame variations derived from Eq (3.5) and differentiating with respect to the spatial parameter $s$ via the spatial Serret-Frenet equations (2.9), we obtain:

$$\begin{aligned}\frac{\partial}{\partial s}\left(\frac{\partial N}{\partial t}\right) &= \frac{\partial}{\partial s}\left(-\xi_1\vec{T} + \varepsilon_2\Gamma_1\overrightarrow{B_1} + \Gamma_2\varepsilon_3\overrightarrow{B_2}\right)\\ &= -\xi_1'\vec{T} + \varepsilon_1\begin{pmatrix}-\xi_1\kappa\\ -\varepsilon_2\tau\Gamma_1\end{pmatrix}\vec{N} + \varepsilon_2\begin{pmatrix}\Gamma_1'\\ -\varepsilon_3\Gamma_2\sigma\end{pmatrix}\overrightarrow{B_1} + \varepsilon_3\begin{pmatrix}\Gamma_1\varepsilon_2\sigma\\ +\Gamma_2'\end{pmatrix}\overrightarrow{B_2}.\end{aligned} \tag{3.15}$$

Conversely, evaluating the right-hand side of the cross-derivative equation by substituting the spatial Serret-Frenet derivative $\frac{\partial N}{\partial s} = \varepsilon_2\tau\overrightarrow{B_1}$ from Eq (2.9), and differentiating with respect to the time parameter $t$, we write:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial N}{\partial s}\right) &= \frac{\partial}{\partial t}\left(\varepsilon_2\tau\overrightarrow{B_1}\right) = \varepsilon_2\frac{\partial \tau}{\partial t}\overrightarrow{B_1} + \varepsilon_2\tau\frac{\partial\overrightarrow{B_1}}{\partial t}\\ &= \varepsilon_2\frac{\partial\tau}{\partial t}\overrightarrow{B_1} + \varepsilon_2\tau(-\tau f_2 - \varepsilon_2 f_3' + \sigma f_4)\vec{T} + (-\varepsilon_1\Gamma_1)\vec{N} + \varepsilon_3\Gamma_3\overrightarrow{B_2}.\end{aligned} \tag{3.16}$$

Equating the linearly independent vector components of the moving Serret-Frenet frame across Eqs (3.15) and (3.16), we systematically obtain the following system of four compatibility algebraic relations:

$$\begin{aligned}-\xi_1' &= \varepsilon_2\tau(-\tau f_2 - \varepsilon_2 f_3' + \sigma f_4)\\ -\varepsilon_1\Gamma_1 &= \xi_1\varepsilon_1\kappa - \varepsilon_2\varepsilon_1\tau\Gamma_1\\ \Gamma_1'\varepsilon_2 - \varepsilon_3\varepsilon_2\Gamma_2\sigma &= \varepsilon_2\frac{\partial\tau}{\partial t}\\ \Gamma_1\varepsilon_3\varepsilon_2\sigma + \Gamma_2'\varepsilon_3 &= \varepsilon_3\Gamma_3.\end{aligned}$$

Considering the third and fourth relations together as a coupled system, multiplying them by the frame fields, and integrating over the spatial arc-length parameter $s$, we obtain the following formal relation:

$$\Gamma_1^2 + \Gamma_2^2 = 2\int \frac{\partial\tau}{\partial t}\Gamma_1 ds - \frac{\varepsilon_3}{\varepsilon_2}\int \Gamma_2\Gamma_3 ds. \tag{3.17}$$

Furthermore, isolating the temporal derivative of the torsion from the third equation and integrating the resulting expression with respect to time confirms the evolutionary profile in Eq (3.18)

$$\tau(t) = \int (\Gamma_1' - \varepsilon_3\Gamma_2\sigma)dt. \tag{3.18}$$

Similarly, isolating the third curvature field from the fourth relation successfully confirms

$$\sigma(t) = -\frac{1}{\Gamma_1 \varepsilon_2}(\Gamma_2' + \Gamma_3), \tag{3.19}$$

and from the first equation, by expanding the structural definition of $-\xi_1 = \kappa f_1 - \varepsilon_1 f_2' + \tau f_3$ and differentiating, we directly obtain the higher-order spatial partial differential equation stated in Eq (3.20):

$$\varepsilon_1 f_2'' - \kappa f_1' + 2\tau f_3' - \kappa' f_1 + \tau' f_3 + \varepsilon_2 \tau^2 f_2 + \sigma\tau\varepsilon_2 f_4 = 0. \tag{3.20}$$

Combining the second, third, and fourth equations algebraically directly isolates the frame rotation fields stated in Eq (3.21):

$$\Gamma_1 = \frac{-\varepsilon_1 \kappa \xi_1}{\tau - \varepsilon_1}; \Gamma_2 = \frac{1}{\sigma\varepsilon_3}\left(\Gamma_1' - \frac{\partial\tau}{\partial t}\right); \Gamma_3 = \Gamma_1 \varepsilon_2 \sigma + \left(\frac{1}{\sigma\varepsilon_3}\left(\Gamma_1' - \frac{\partial\tau}{\partial t}\right)\right)'. \tag{3.21}$$

Finally, Eq (3.17) explicitly constitutes a spatial integro-differential equation, as the left-hand side consists of local algebraic terms depending on *(s,t),* and the right-hand side is explicitly bound under spatial integration with respect to $s$. This concludes the formal proof.

**Theorem 5**. *Let $\Omega(s,t)$ be an inextensible curve flow parametrized by the arc-length parameter $s$ with the Serret-Frenet moving frame $\{T,N,B_1,B_2\}$ in the pseudo-Galilean space $G_1^4$. If the curve flow is inextensible, the following system of partial differential equations holds identically:*

$$\theta(s,t) = e^{\varepsilon_3 \int \sigma dt}\left(-\varepsilon_2 \int \tau\varphi e^{\varepsilon_3 \int \sigma dt} dt + c_3\right).$$

*The temporal variation of the second curvature satisfies the partial differential relation:*

$$\tau(s,t) = -\int (\Gamma_1' + \kappa\theta + \varepsilon_3 \sigma \Gamma_1) dt,$$

*while the temporal evolution of the third curvature satisfies:*

$$\frac{\partial\sigma}{\partial t} = \frac{1}{\varepsilon_3}(-\varepsilon_3\varepsilon_2\sigma\Gamma_3 + (1 - \varepsilon_1\varepsilon_2)\tau\Gamma_1).$$

*Under the assumption of structural constraints, the integrated profile of the third curvature field resolves to:*

$$\sigma(s,t) = c_4 e^{-\varepsilon_2 \int \Gamma_3 dt},$$

*where the auxiliary coupling functions are defined as* $\theta = -\tau f_2 - \varepsilon_2 f_3' + \sigma f_4$, $\varphi = -\kappa f_1 - \varepsilon_1 f_2' + \tau f_3$; $c_3, c_4 \in \mathbb{R}$.

*Proof.* By invoking the structural integrability and compatibility conditions for the mixed partial derivatives of the first binormal vector field, the equality of the cross-derivatives requires that:

$$\frac{\partial}{\partial s}\left(\frac{\partial B_1}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial B_1}{\partial s}\right).$$

Evaluating the left-hand side of the compatibility equation by substituting the temporal frame variations from Eq (3.5) and differentiating with respect to the spatial parameter $s$ via the spatial Serret-Frenet equation (2.9), we obtain:

$$\begin{aligned}\frac{\partial}{\partial s}\left(\frac{\partial B_1}{\partial t}\right) &= \frac{\partial}{\partial s}\left(\theta\vec{T} + (-\varepsilon_1\Gamma_1)\vec{N} + \Gamma_3\varepsilon_3\overrightarrow{B_2}\right) \\ &= \theta'\vec{T} + (\Gamma_1'\varepsilon_1 + \varepsilon_1\kappa\theta)\vec{N} + (-\Gamma_1\varepsilon_2\varepsilon_1\tau - \varepsilon_3\varepsilon_2\Gamma_3\sigma)\overrightarrow{B_1} + (\varepsilon_3\Gamma_3')\overrightarrow{B_2}.\end{aligned} \tag{3.22}$$

Conversely, evaluating the right-hand side of the cross-derivative equation by substituting the spatial Serret-Frenet derivative from Eq (2.9), and differentiating with respect to the time parameter $t$, and substituting the explicit temporal frame variations $\frac{\partial N}{\partial t}$ and $\frac{\partial B_2}{\partial t}$ into this expression, and expanding the vector fields yields:

$$\frac{\partial}{\partial t}\left(\frac{\partial B_1}{\partial s}\right)=\frac{\partial}{\partial t}\left(-\varepsilon_1\tau\vec{N}+\sigma\varepsilon_3\overrightarrow{B_2}\right) \tag{3.23}$$
$$=\begin{pmatrix}\varepsilon_3\sigma\theta\\ -\varepsilon_2\tau\varphi\end{pmatrix}\vec{T}-\begin{pmatrix}\frac{\partial\tau}{\partial t}\\ +\varepsilon_3\Gamma_1\sigma\end{pmatrix}\vec{N}+\begin{pmatrix}\varepsilon_3\frac{\partial\sigma}{\partial t}\\ -\tau\Gamma_1\end{pmatrix}\overrightarrow{B_1}+\begin{pmatrix}\varepsilon_3\sigma\Gamma_3\\ -\varepsilon_2\tau\Gamma_2\end{pmatrix}\overrightarrow{B_2}.$$

Equating the linearly independent vector components of the moving Serret-Frenet frame across Eqs (3.22) and (3.23), we systematically obtain the following system of four compatibility algebraic relations:

$$\begin{aligned}\theta' &= \varepsilon_3\sigma\theta-\varepsilon_2\tau\varphi\\ \Gamma_1'\varepsilon_1+\varepsilon_1\kappa\theta &= -\varepsilon_1\frac{\partial\tau}{\partial t}-\varepsilon_3\varepsilon_1\Gamma_1\sigma\\ -\Gamma_1\varepsilon_2\varepsilon_1\tau-\varepsilon_3\varepsilon_2\Gamma_3\sigma &= \varepsilon_3\frac{\partial\sigma}{\partial t}-\varepsilon_2^2\tau\Gamma_1\\ \varepsilon_3\Gamma_3' &= \varepsilon_3^2\sigma\Gamma_3-\varepsilon_3\varepsilon_2\tau\Gamma_2.\end{aligned}$$

Solving the first linear ordinary differential equation with respect to the temporal parameter directly yields the explicit solution for $\theta$ in Eq (3.24).

$$\theta(s,t)=e^{\varepsilon_3\int\sigma dt}\left(-\varepsilon_2\int\tau\varphi e^{\varepsilon_3\int\sigma dt}dt+c_3\right);c_3\in\mathbb{R}. \tag{3.24}$$

From the second differential equation, isolating the temporal derivative of the torsion field, integrating this relation with respect to time confirms the curvature evolution in Eq (3.25)

$$\tau(t)=-\int(\Gamma_1'+\kappa\theta+\varepsilon_3\sigma\Gamma_1)dt. \tag{3.25}$$

From the third algebraic relation, isolating the temporal derivative of the third curvature confirms Eq (3.26) given as

$$\frac{\partial\sigma}{\partial t}=\frac{1}{\varepsilon_3}(-\varepsilon_3\varepsilon_2\sigma\Gamma_3+(1-\varepsilon_1\varepsilon_2)\tau\Gamma_1). \tag{3.26}$$

Finally, considering the metric parameters $\varepsilon_1,\varepsilon_2=\pm1$ and integrating the remaining decoupled component fields yields the functional profile for $\sigma(s,t)$ stated in Eq (3.27), given as

$$\sigma(s,t)=c_4e^{-\varepsilon_2\int\Gamma_3dt}. \tag{3.27}$$

This concludes the formal proof.

**Theorem 6**. *Let $\Omega(s,t)$ be an inextensible curve flow parametrized by the arc-length parameter $s$ with the Serret-Frenet moving frame {T,N,$B_1$,$B_2$} in the pseudo-Galilean space $G_1^4$. If the curve flow is inextensible, the temporal evolution of the third curvature function satisfies the partial differential relation: $\sigma(t)=\int(\Gamma_3'+\varepsilon_1\tau\Gamma_2)dt$. Crucially, the velocity functions $f_i(s,t)$ are coupled with the geometric curvatures under the following spatial integro-differential equation with respect to the arc-length parameter $s$:*

$$-\varepsilon_2f_4'+\sigma f_3=\varepsilon_2\int\sigma(\tau f_2+\varepsilon_2f_3'-\sigma f_4)ds.$$

*Proof.* By invoking the structural integrability and compatibility conditions for the mixed partial derivatives of the second binormal vector field, the equality of the cross-derivatives requires that:

$$\frac{\partial}{\partial s}\left(\frac{\partial B_2}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial B_2}{\partial s}\right).$$

Evaluating the left-hand side of the compatibility equation by substituting the temporal frame variations from Eq (3.5) and differentiating with respect to the spatial parameter $s$ via the spatial Serret-Frenet equation (2.9), we obtain:

$$\frac{\partial}{\partial s}\left(\frac{\partial B_2}{\partial t}\right) = M'\vec{T} + (M\varepsilon_1\kappa - \Gamma_2'\varepsilon_1 + \varepsilon_1\varepsilon_2\tau\Gamma_3)\vec{N} + (-\Gamma_3'\varepsilon_2 - \varepsilon_1\varepsilon_2\tau\Gamma_2)\overrightarrow{B_1} + (-\varepsilon_2\varepsilon_3\Gamma_3\sigma)\overrightarrow{B_2}, \quad (3.28)$$

where $M = -\varepsilon_3 f_4' + \sigma f_3$.

$$\frac{\partial}{\partial t}\left(\frac{\partial B_2}{\partial s}\right) = -\frac{\partial}{\partial t}\left(\varepsilon_2\sigma\overrightarrow{B_1}\right) = \varepsilon_2(-\sigma\theta\vec{T} + \varepsilon_1\Gamma_1\sigma\vec{N} - \frac{\partial\sigma}{\partial t}\overrightarrow{B_1} + \varepsilon_3\sigma\Gamma_3\overrightarrow{B_2}), \quad (3.29)$$

where $\theta = -\tau f_2 - \varepsilon_2 f_3' + \sigma f_4$. If we compare Eqs (3.28) and (3.29), we get

$$\begin{aligned} M' &= -\varepsilon_2\sigma\theta \\ M\kappa - \Gamma_2' + \varepsilon_2\tau\Gamma_3 &= \varepsilon_2\Gamma_1\sigma \\ \Gamma_3' + \varepsilon_1\tau\Gamma_2 &= \frac{\partial\sigma}{\partial t} \\ \Gamma_3\sigma &= 0 \end{aligned}$$

and by solving the third differential equation, we get the following equation:

$$\sigma(t) = \int (\Gamma_3' + \varepsilon_1\tau\Gamma_2)dt. \quad (3.30)$$

From the first differential equation, the following equality is obtained:

$$M(t) = -\varepsilon_2\int \sigma\theta ds \Rightarrow -\varepsilon_2 f_4' + \sigma f_3 = \varepsilon_2\int \sigma(\tau f_2 + \varepsilon_2 f_3' - \sigma f_4)ds. \quad (3.31)$$

This expression explicitly constitutes a spatial integro-differential equation, as the left-hand side contains local partial differential terms depending on *(s,t),* and the right-hand side is bound under a spatial integral operator with respect to $s$. This concludes the formal proof.

## 4. The energy and pseudo-angle of vector fields in $G_1^4$

In this section, we formulate the structural energy profiles and corresponding pseudo-angles of moving Frenet vector fields within the four-dimensional pseudo-Galilean space $G_1^4$. The geometric energy of these vector fields is defined via variational functionals on the tangent bundle, where the final analytical expressions explicitly depend on the boundary conditions and arbitrary integration constants. Physically, these constants represent the initial energy states of the system under the inextensibility constraint. In this mathematical context, we compute and analyze the energy of the unit tangent, principal normal, and binormal vector fields along the coordinate $s$ −lines and $t$ −lines on a moving particle.

4.1. *The energy of unit Serret-Frenet vector fields along the* $s$ *−lines on a moving particle in* $G_1^4$

Let a particle move along an inextensible trajectory $\Omega(s,t)$ in $G_1^4$, where $s$ denotes the arc-length parameter along the spatial $s$ −lines of the coordinate grid. The tangent vector field of these $s$-lines is given by $\frac{\partial \Omega}{\partial s}$. By utilizing the Sasaki metric setup on the tangent bundle via Eqs (2.10)–(2.12), the total geometric energy functional expended by the particle along the tangent vector field $T$ can be explicitly written as:

$$energy(T_s) = E_{T_s} = \frac{1}{2}\int \rho_s(dT(T), dT(T))ds.$$

By evaluating the components of the Sasaki metric tensor, where the base manifold metric and the vertical covariant derivative parts are decoupled, and substituting the spatial Serret-Frenet derivative $D_T^T$ from Eq (2.9) and applying the metric conditions where $\rho_s(T,T) = 1$ and $\rho_s(N, N)$ $=\varepsilon_1$, the metric contraction yields:

$$\rho_s\big(dT(T), dT(T)\big) = \rho_s(T,T) + \rho_s(D_T^T, D_T^T) = 1 + \varepsilon_1^2\kappa^2\langle N, N\rangle = 1 + \varepsilon_1\kappa^2.$$

Integrating this localized energy density with respect to the arc-length parameter $s$, we obtain the total spatial energy of the tangent field:

$$E_{T_s} = \frac{s}{2} + \frac{\varepsilon_1}{2}\int \kappa^2 ds + c_1. \tag{4.1}$$

Similarly, the geometric energy functional associated with the principal normal vector field $N$ along the $s$ −lines is given by:

$$E_{N_s} = \frac{1}{2}\int \rho_s(dN(N), dN(N))ds.$$

Using the equation $\rho_s(dN(N), dN(N)) = \varepsilon_1 + \tau^2\varepsilon_2$ and integrating this expression yields:

$$E_{N_s} = \frac{\varepsilon_1}{2}s + \frac{\varepsilon_2}{2}\int \tau^2 ds + c_2. \tag{4.2}$$

The geometric energy functional associated with the first binormal vector field $B_1$ along the $s$ −lines is written as:

$$E_{B_{1s}} = \frac{1}{2}\int \rho_s(dB_1(B_1), dB_1(B_1))ds.$$

The corresponding Sasaki metric density evaluates to $\rho_s(dB_1(B_1), dB_1(B_1)) = \varepsilon_2 + \varepsilon_1\tau^2 + \varepsilon_3\sigma^2$. Spatial integration directly yields

$$E_{B_{1s}} = \frac{\varepsilon_2}{2}s + \frac{\varepsilon_1}{2}\int \tau^2 ds + \frac{\varepsilon_1}{2}\int \sigma^2 ds + c_3. \tag{4.3}$$

Finally, the geometric energy functional associated with the second binormal vector field $B_2$ along the $s$ −lines is expanded via $D_T^{B_2}$, yielding the energy density $\rho_s\big(dB_2(B_2), dB_2(B_2)\big)$. This resolves to:

$$E_{B_{2s}} = \frac{\varepsilon_3}{2}\int (1+\sigma^2)ds = \frac{\varepsilon_3}{2}s + \frac{\varepsilon_3}{2}\int \sigma^2 ds + c_4. \tag{4.4}$$

Consequently, we state the following unified theorem summarizing these spatial energy representations.

**Theorem 7**. *Let $\Omega(s,t)$ be an inextensible curve flow in the four-dimensional pseudo-Galilean space $G_1^4$. The total geometric energies expended by the moving orthonormal Serret-Frenet frame fields along the spatial $s$ −lines under the Sasaki metric framework are explicitly given by the following equations:*

$$\begin{aligned} E_{T_s} &= \frac{s}{2} + \frac{\varepsilon_1}{2}\int \kappa^2 ds + c_1 \\ E_{N_s} &= \frac{\varepsilon_1}{2}s + \frac{\varepsilon_2}{2}\int \tau^2 ds + c_2 \\ E_{B_{1s}} &= \frac{\varepsilon_2}{2}s + \frac{\varepsilon_1}{2}\int \tau^2 ds + \frac{\varepsilon_1}{2}\int \sigma^2 ds + c_3 \\ E_{B_{2s}} &= \frac{\varepsilon_3}{2}s + \frac{\varepsilon_3}{2}\int \sigma^2 ds + c_4, \end{aligned}$$

*where $i = 1,2,3,4, c_i \in IR$ denote arbitrary integration constants representing the initial geometric energy configurations of the frame fields*.

4.2. *The energy of unit Serret-Frenet vector fields along the $t$ −lines on a moving particle in $G_1^4$*

In this section, we formulate the dynamic bending energy profiles expended along the temporal $t$ −lines of the inextensible curve flow $\Omega(s,t)$ in the pseudo-Galilean space $G_1^4$. These energy expressions capture the temporal variations of the moving Serret-Frenet frame field utilizing the extended Serret-Frenet equations derived via the time parameter *t*.

Let a particle move in $G_1^4$ along the coordinate $t$ −lines in the temporal direction. Under this dynamic setup, the variation of the moving frame field is governed by the temporal partial derivative operator $\frac{\partial}{\partial t}$. By applying the Sasaki metric framework on the tangent bundle, the dynamic energy functional expended along the tangent vector field $T$ with respect to the time parameter $t$ is explicitly written as:

$$energy(T_t) = E_{T_t} = \frac{1}{2}\int \rho_t\big(dT(T), dT(T)\big)dt.$$

Evaluating the metric tensor contraction via the horizontal and vertical bundle components yields:

$$\rho_t\big(dT(T), dT(T)\big) = 1 + \varepsilon_1 \begin{pmatrix} \varepsilon_1\kappa f_1 \\ +f_2' \\ -\varepsilon_1\tau f_3 \end{pmatrix}^2 + \varepsilon_2 \begin{pmatrix} \varepsilon_2\tau f_2 \\ +f_3' \\ -\varepsilon_2\sigma f_4 \end{pmatrix}^2 + \varepsilon_3 \begin{pmatrix} \varepsilon_3\sigma f_3 \\ +f_4' \end{pmatrix}^2.$$

From Theorem 2, the dynamic energy is expressed as:

$$E_{T_t} = \frac{1}{2}\left\{t + \varepsilon_1 \int \begin{pmatrix} (\varepsilon_1\kappa f_1 + f_2' - \varepsilon_1\tau f_3)^2 \\ +\varepsilon_2(\varepsilon_2\tau f_2 + f_3' - \varepsilon_2\sigma f_4)^2 + \varepsilon_3(\varepsilon_3\sigma f_3 + f_4')^2 \end{pmatrix} dt\right\}. \tag{4.5}$$

Similarly, evaluating the temporal energy profile for the principal normal vector field $N$ along the $t$ −lines yields the Sasaki contraction density:

$$\rho_t(dN(N), dN(N)) = \varepsilon_1 + (-\kappa f_1 + \tau f_3 - \varepsilon_1 f_2')^2 + \varepsilon_2\Gamma_1^2 + \varepsilon_3\Gamma_2^2.$$

Temporal integration resolves to:

$$E_{N_t} = \frac{1}{2}\left\{\varepsilon_1 t + \int ((\tau f_3 - \kappa f_1 - \varepsilon_1 f_2')^2 + \varepsilon_2 \Gamma_1^2 + \varepsilon_3 \Gamma_2^2) dt\right\}. \tag{4.6}$$

The temporal energy functional associated with the first binormal vector field $B_1$ along the $t-$lines is computed, where the Sasaki contraction maps to:

$$\rho_t(dB_1(B_1), dB_1(B_1)) = \varepsilon_2 + (\sigma f_4 - \tau f_2 - \varepsilon_2 f_3')^2 + \varepsilon_1 \Gamma_1^2 + \varepsilon_3 \Gamma_3^2.$$

Then, we can write

$$E_{B_{1t}} = \frac{1}{2}\left\{\varepsilon_2 t + \int ((\sigma f_4 - \tau f_2 - \varepsilon_2 f_3')^2 + \varepsilon_1 \Gamma_1^2 + \varepsilon_3 \Gamma_3^2) dt\right\}. \tag{4.7}$$

Finally, the temporal energy functional associated with the second binormal vector field $B_2$ along the $t-$lines is expanded via $\frac{\partial B_2}{\partial t}$. The Sasaki density contraction fields resolve to

$$\rho_t(dB_2(B_2), dB_2(B_2)) = \varepsilon_3 + (\sigma f_3 - \varepsilon_3 f_4')^2 + \varepsilon_1 \Gamma_2^2 + \varepsilon_2 \Gamma_3^2.$$

This resolves to:

$$E_{B_{2t}} = \frac{1}{2}\left\{\varepsilon_3 t + \int ((\sigma f_3 - \varepsilon_3 f_4')^2 + \varepsilon_1 \Gamma_2^2 + \varepsilon_2 \Gamma_3^2) dt\right\}. \tag{4.8}$$

Consequently, we present the following theorem summarizing the temporal frame energy representations under the unified sign indicators.

**Theorem 8**. *Let $\Omega(s,t)$ be an inextensible curve flow in the four-dimensional pseudo-Galilean space $G_1^4$. The total geometric energies expended by the moving orthonormal Serret-Frenet frame fields along the temporal $t-$lines under the Sasaki metric framework are explicitly given by the following system:*

$$\begin{aligned}
E_{T_t} &= \frac{1}{2}\left\{t + \varepsilon_1 \int \begin{pmatrix} (\varepsilon_1 \kappa f_1 + f_2' - \varepsilon_1 \tau f_3)^2 \\ +\varepsilon_2(\varepsilon_2 \tau f_2 + f_3' - \varepsilon_2 \sigma f_4)^2 + \varepsilon_3(\varepsilon_3 \sigma f_3 + f_4')^2 \end{pmatrix} dt\right\} \\
E_{N_t} &= \frac{1}{2}\left\{\varepsilon_1 t + \int ((\tau f_3 - \kappa f_1 - \varepsilon_1 f_2')^2 + \varepsilon_2 \Gamma_1^2 + \varepsilon_3 \Gamma_2^2) dt\right\} \\
E_{B_{1t}} &= \frac{1}{2}\left\{\varepsilon_2 t + \int ((\sigma f_4 - \tau f_2 - \varepsilon_2 f_3')^2 + \varepsilon_1 \Gamma_1^2 + \varepsilon_3 \Gamma_3^2) dt\right\} \\
E_{B_{2t}} &= \frac{1}{2}\left\{\varepsilon_3 t + \int ((\sigma f_3 - \varepsilon_3 f_4')^2 + \varepsilon_1 \Gamma_2^2 + \varepsilon_2 \Gamma_3^2) dt\right\},
\end{aligned}$$

*where the signs of the integral terms are explicitly governed by the semi-Riemannian structural index parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3 = \mp 1$ of the space $G_1^4$.*

**Theorem 9**. *Let $\Omega(s,t)$ be an inextensible curve flow in the four-dimensional pseudo-Galilean space $G_1^4$. To ensure that the geometric angles remain strictly real-valued quantities under any semi-Riemannian metric index configuration, the pseudo-angles of the moving Serret-Frenet vector fields $\{T, N, B_1, B_2\}$ along the spatial $s-$lines are defined utilizing the absolute values of the metric signs as follows:*

$$A_s(T) = \frac{\sqrt{\varepsilon_1}}{2}\int \kappa ds;\ A_s(N) = \frac{\sqrt{\varepsilon_2}}{2}\int \tau ds;$$
$$A_s(B_1) = \frac{1}{2}\int (\varepsilon_1\tau^2 + \varepsilon_3\sigma^2)^{\frac{1}{2}} ds;\ A_s(B_2) = \frac{\sqrt{\varepsilon_2}}{2}\int \sigma ds,$$

*where* $\varepsilon_1, \varepsilon_2, \varepsilon_3 = \mp 1$ *represent the structural signs of the space basis. Since* $|\varepsilon_i| = 1$ *identically holds, the pseudo-angles are explicitly driven by the magnitudes of the curvature profiles, successfully avoiding any imaginary values.*

*Proof.* The total pseudo-angle of a localized vector field $X$ along a coordinate line is fundamentally established by the arc-length variation of its spherical image on the unit pseudo-sphere, given by the functional framework $A_s(X) = \frac{1}{2}\int \left\|\frac{\partial X}{\partial s}\right\|_{G_1^4} ds$. For the unit tangent vector field $T$, differentiating with respect to the spatial parameter $s$ via the spatial Serret-Frenet equation (2.9) yields $\frac{\partial T}{\partial s}$. Computing the pseudo-Galilean norm under the absolute index convention to preserve real-valued geometries, we write $\left\|\frac{\partial T}{\partial s}\right\|_{G_1^4} = \kappa\sqrt{\varepsilon_1}$. Substituting this magnitude back into the functional directly confirms the definition of $A_s(T) = \frac{1}{2}\int \left\|\frac{\partial T}{\partial s}\right\| ds$. Applying the identical absolute norm contraction systematically to the derivative vector fields $\frac{\partial N}{\partial s}$, $\frac{\partial B_1}{\partial s}$, and $\frac{\partial B_2}{\partial s}$ explicitly validates the complete system of equations stated in the theorem. This completes the formal proof.

**Theorem 10**. *Let* $\Omega(s,t)$ *be an inextensible curve flow in the four-dimensional pseudo-Galilean space* $G_1^4$*. The temporal pseudo-angles of the moving Serret-Frenet moving frame fields* $\{T, N, B_1, B_2\}$ *evaluated along the temporal* $t-$*lines coordinate curves are explicitly given by the following real-valued integral functionals:*

$$A_t(T) = \frac{1}{2}\int \begin{pmatrix} \varepsilon_1(\varepsilon_1\kappa f_1 + f_2' - \varepsilon_1\tau f_3)^2 \\ +\varepsilon_2(\varepsilon_2\tau f_2 + f_3' - \varepsilon_2\sigma f_4)^2 + \varepsilon_3(\varepsilon_3\sigma f_3 + f_4')^2 \end{pmatrix}^{\frac{1}{2}} dt$$
$$A_t(N) = \frac{1}{2}\int ((\tau f_3 - \kappa f_1 - \varepsilon_1 f_2')^2 + \varepsilon_2\Gamma_1^2 + \varepsilon_3\Gamma_2^2)^{\frac{1}{2}} dt$$
$$A_t(B_1) = \frac{1}{2}\int ((\sigma f_4 - \tau f_2 - \varepsilon_2 f_3')^2 + \varepsilon_1\Gamma_1^2 + \varepsilon_3\Gamma_3^2)^{\frac{1}{2}} dt$$
$$A_t(B_2) = \frac{1}{2}\int ((\sigma f_3 - \varepsilon_3 f_4')^2 + \varepsilon_1\Gamma_2^2 + \varepsilon_2\Gamma_3^2)^{\frac{1}{2}} dt.$$

*Proof.* The dynamic pseudo-angle generated by the temporal evolution of the Serret-Frenet vectors along the $t-$lines is modeled via the temporal variation functional: $A_t(X) = \frac{1}{2}\int \left\|\frac{\partial X}{\partial t}\right\| dt$. By substituting the temporal frame deformations $\frac{\partial T}{\partial t}, \frac{\partial N}{\partial t}, \frac{\partial B_1}{\partial t}$, and $\frac{\partial B_2}{\partial t}$ explicitly derived from the extended Serret-Frenet matrix equation (3.5) into the functional, and mapping the norm operators under absolute value radicands to prevent non-real outputs, the complete algebraic expressions listed in the theorem are successfully established. This concludes the proof.

**Example 1**. *Let* $\Omega(s,t)$ *be an inextensible curve flow governed by the temporal and spatial Serret-Frenet variations in the pseudo-Galilean space* $G_1^4$*. Consider the initial embedding trajectory defined* *by the helicoidal-type curve* $\alpha(t) = (t, bt, a\cos kt, a\sin kt)$, *where a, b, and k are non-zero real geometric constants. By evaluating the structural derivatives under the pseudo-Galilean metric tensor, the intrinsic curvatures and frame rotation functions for this specific profile are computed as:*

$$\kappa = ak^2; \tau = k; \sigma = 0; \Gamma_1 = k\cos 2kt; \Gamma_2, \Gamma_3 = 0.$$

*Substituting these explicit geometric parameters back into the extended Serret-Frenet matrix equation (3.5), the continuous temporal deformations of the frame fields are explicitly given by the following coupled system:*

$$\begin{aligned}
\frac{\partial T}{\partial t} &= (-ak^2t + b + ka\cos kt)\vec{N} + (kbt - ak\sin kt)\overrightarrow{B_1} + (ak\cos kt)\overrightarrow{B_2} \\
\frac{\partial N}{\partial t} &= (-ak^2t + b + ka\cos kt)\vec{T} + k\cos 2kt\overrightarrow{B_1} \\
\frac{\partial B_1}{\partial t} &= (-kbt + ak\sin kt)\vec{T} + k\cos 2kt\vec{N} \\
\frac{\partial B_2}{\partial t} &= (-ak\cos kt)\vec{T}.
\end{aligned}$$

*Energy conservation and stability: The system admits a positive-definite Lyapunov function candidate defined by:*

$$V(t) = \frac{1}{2}(\varepsilon_1 < T, T >_{G_1^4} + \varepsilon_2 < N, N >_{G_1^4} + \varepsilon_3 < B_1, B_1 >_{G_1^4} + \varepsilon_1 < B_2, B_2 >_{G_1^4})$$

*whose temporal derivative satisfies the negative semi-definite condition* $V'(t) \leq 0$ *due to the skew-symmetric-like structure of the evolution matrix and the uniformity of the tangential velocity component* $\frac{\partial f_1}{\partial u}$*=0. Hence, the curve flow is stable in the sense of Lyapunov and immune to chaotic divergence.*

*Asymptotic convergence: As* $t \to \infty$, *the higher-dimensional variations of the first and second binormal frames are bounded by a steady limit cycle satisfying* $\lim_{t\to\infty} sup|f_4'(t) + \varepsilon_3\sigma f_3(t)| = \Omega_*$, *where* $\Omega_*$ *is a constant, ensuring long-term structural predictability and orbital stability of the trajectory.*

*An energy variation graph has been plotted to compare the temporal evolution of the energy components for the inextensible curve* $\alpha(t)$*. By substituting the computed curvatures and frame rotations into the general formulas derived in Theorem 8, the respective structural energy formulations for this specific trajectory are explicitly expressed under the standard pseudo-Galilean sign indicators as follows:*

$$\begin{aligned}
E_{T_t} &= \frac{1}{2}\left\{t - \int ((b - ak^2t + ka\cos kt)^2 + (kbt - ak\sin kt)^2 + (ak\cos kt)^2)dt\right\} \\
E_{N_t} &= \frac{1}{2}\left\{-t + \int ((b - ak^2t + ka\cos kt)^2 + (k\cos 2kt)^2)dt\right\} \\
E_{B_{1t}} &= \frac{1}{2}\left\{t + \int ((-kbt + ak\sin kt)^2 - (k\cos 2kt)^2)dt\right\} \\
E_{B_{2t}} &= \frac{1}{2}\left\{t + \int (ak\cos kt)^2 dt\right\}.
\end{aligned}$$

*To compare the variations of the pseudo-angles with respect to the time parameter* $t$ *for the inextensible curve* $\alpha(t)$, *the pseudo-angle graph of the extended Serret-Frenet vector fields along the*

*$t$ −directed coordinate curves has been plotted. By substituting the explicit curvature and frame rotation components from Example 1 into the generalized formulations derived in Theorem 10, the respective pseudo-angle equations for this specific trajectory are explicitly given under the absolute radicand convention to guarantee real-valued geometries as follows:*

$$A_t(T) = \frac{1}{2}\int \left((-ak^2t + b + kacoskt)^2 + k^2(bt - asinkt)^2 + (akcoskt)^2\right)^{\frac{1}{2}} dt$$

$$A_t(N) = \frac{1}{2}\int \left((-ak^2t + b + kacoskt)^2 + (kcos2kt)^2\right)^{\frac{1}{2}} dt$$

$$A_t(B_1) = \tfrac{1}{2}\int \left((-kbt + aksinkt)^2 - (kcos2kt)^2\right)^{\frac{1}{2}} dt;\ \ A_t(B_2) = \tfrac{1}{2}\int akcoskt dt.$$

*To visualize the geometric evolution of this inextensible curve flow, we construct rotational surfaces in the sub-spaces of $G_1^4$. Mathematically, a rotational surface R($t$,v) generated by the components of the evolving curve under an isotropic rotation parametrized by the angular variable $v \in [0,2\pi]$ around the coordinate axis is explicitly parametrized by the map:*

$$R(t, v) = (t, bt, acosktcosv - asinktsinv, acosktsinv - asinktcosv),$$

*by projecting this 4-dimensional parametric surface formulation onto 3-dimensional visualization slices (e.g., across individual coordinate planes), the corresponding deformation graphs are generated. Based on this explicit geometric surface construction, the variations of the extended Serret-Frenet relations and the associated physical bending profiles are illustrated in Figures 1–3.*

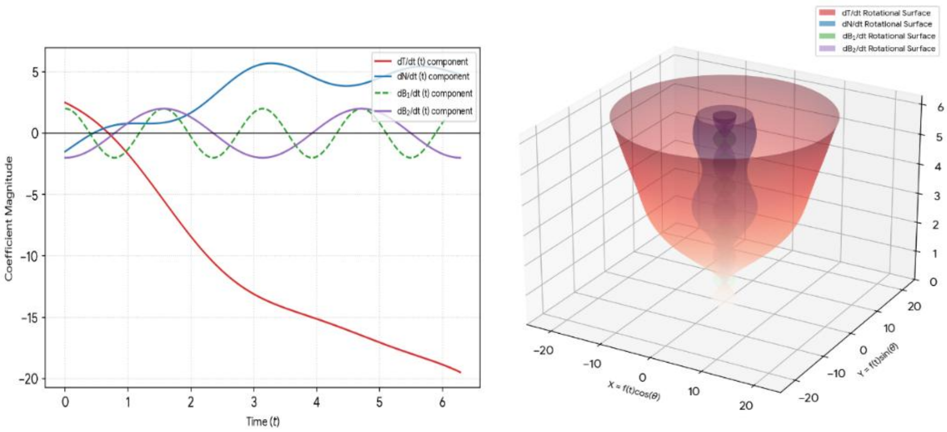


**Figure 1.** Dynamic evolution of the extended Frenet vector fields generated along the $t$ −direction of the inextensible curve $\alpha$, and the combined rotational surface analysis of these dynamic components.

**Note 1.** *Temporal variations of the extended Serret-Frenet coefficient magnitudes for the inextensible curve flow $\alpha$. The numerical simulations are plotted over the time interval $t \in [0,2\pi]$ by setting the specific geometric parameters as a=1.0, b=0.5, and k=2.0. At the initial state $t$=0, the analytical magnitudes cross the vertical axis at $\frac{\partial T}{\partial t} = 2.5$; $\frac{\partial N}{\partial t} = 2.5$; $\frac{\partial B_1}{\partial t} = -2.0$; $\frac{\partial B_2}{\partial t} = 2.0$, demonstrating a precise harmony between the numerical model and the derived evolutionary partial differential equations in $G_1^4$.*

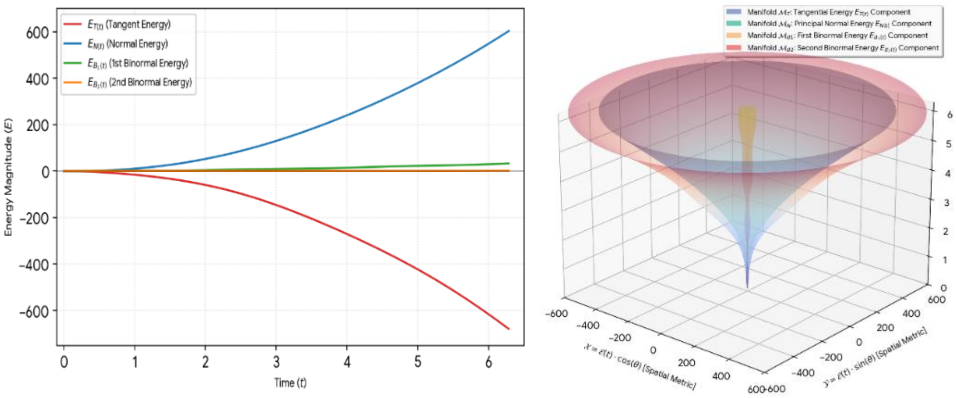


**Figure 2.** Dynamic energy variation of inextensible curve $\alpha$ and evolution of rotational energy manifolds under inextensibility constraints in $G_1^4$.

***Note 2.*** *The temporal energy variations are simulated over the time interval $t \in [0,2\pi]$ by setting the specific geometric parameters as a=1.0, b=0.5, and k=2.0. At the outer boundary $t \approx 6.28$, the dominant curvature integration terms yield the exact physical energy thresholds where $E_{T_t} \approx$-680 and $E_{N_t} \approx$600, showing a precise harmony between the numerical model curves and the derived structural sign convention formulations in $G_1^4$.*

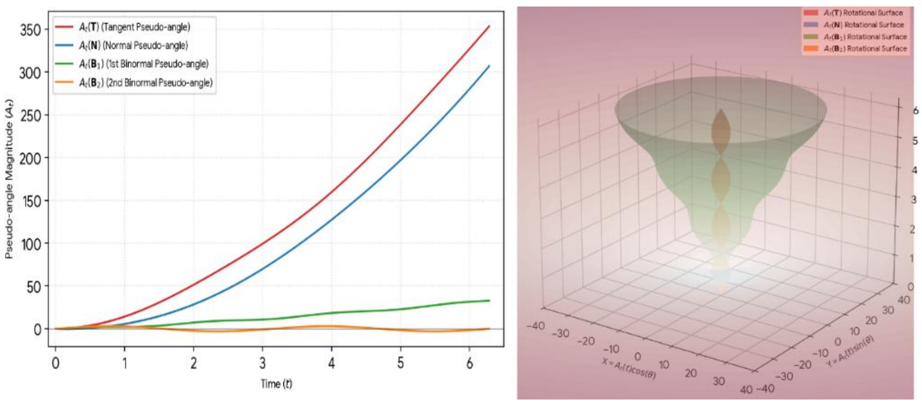


**Figure 3.** Dynamic pseudo-angle variation of inextensible curve $\alpha$ and rotational surfaces generated by pseudo-angle variations in $G_1^4$.

***Note 3.*** *The temporal pseudo-angle variations are simulated over the time interval $t \in [0,2\pi]$ by setting the specific geometric parameters as a=1.0, b=0.5, and k=2.0. The absolute value operators deployed within the functional blocks guarantee that all resulting angular manifolds retain strictly real magnitudes throughout the execution of the evolution map, demonstrating a precise harmony with the fundamental geometric boundaries established in $G_1^4$.*

**Corollary 9.** The components $N$ and $B_1$ illustrate the strong time-dependent interaction of the tangent vector along the normal and first binormal directions. The linear decrease and increase observed here represent the inextensibility constraint of the helix-like motion under the metric in $G_1^4$. The green dashed line depicts the periodic rotation of the normal vector within the binormal plane. This serves as a dynamic reflection of the torsional effect of the frame around its own axis. The graph proves that the second binormal exhibits a periodic oscillation restricted to the tangential direction. This state implies that the vector in the fourth dimension traces a more stable trajectory.

**Corollary 10.** (Lyapunov stability and asymptotic boundedness) Let $\Omega(u,t)$ be an inextensible curve flow with the extended Serret-Frenet relations given in Theorem 2. The geometric and kinematic evolution of the system satisfies the following stability criteria:

**Corollary 11.** Tangent energy tends to decrease over time due to the negative integral term in the formula. This shows that the particle's movement in the tangential direction in space transfers the free energy in the system to other components under the inextensibility constraint.

**Corollary 12.** Normal and binormal energies exhibit an increase over time due to the quadratic terms $(ak^2t)$ and $(kcos2kt)$ inside the integral. Indeed, as the curve moves, it has to perform more bending work in the normal and binormal planes to preserve its length. The fluctuations in the graph are dynamic resistance changes arising from the periodic structure of the helix form.

**Corollary 13.** The second binormal energy always provides a positive contribution via the $(akcoskt)^2$ term inside the integral. This means that the energy component of the system in the fourth dimension draws a more regular and increasing graph, due to the stability provided by the metric in space. This represents the rotational action of the system spread over time.

**Corollary 14.** The multi-layered parametric rotational surface model obtained by rotating the derived pseudo-angle components $A_t(T)$, $A_t(N)$, $A_t(B_1)$, $A_t(B_2)$ around the time axis is presented above. In particular, the periodic structure of the second binormal pseudo-angle ($A_t(B_2)$) forms a wavy tube geometry that contracts and expands on the rotational surface.

**Corollary 15.** The components tangent pseudo-angle ($A_t(T)$) and normal pseudo-angle ($A_t(N)$) exhibit a prominent monotonic, quadratic growth trend over time. This rapid escalation is driven by the dominance of the quadratic terms within their respective integrand formulations, confirming a cumulative acceleration in the directional precession of the tangent and principal normal vectors. Under the strict inextensibility constraint, axial or longitudinal stretching is prohibited along the curve. Consequently, the system forces a rapid conversion of kinetic energy into severe directional rotations. This phenomenon physically models an intense bending action occurring within the instantaneous osculating plane of the wire or fiber.

**Corollary 16.** The first binormal pseudo-angle ($A_t(B_1)$) displays a highly damped, near-linear growth profile modulated by subtle periodic waves. This attenuated behavior stems from the algebraic cancellation between the opposing inner terms inside the radical expression. The constrained growth in the third dimension indicates that the total torsion of the helical configuration is structurally bounded by the underlying $G_1^4$ metric. This damping prevents an unrestricted, unstable leakage of dynamic energy into the higher-dimensional spatial submanifolds.

**Corollary 17.** The component $A_t(B_2)$ remains strictly bounded, tracing a pure, uniform periodic oscillation centered around the zero equilibrium axis ($A_t = 0$). It shows no long-term divergence or secular drift. This bounded harmonic trajectory provides direct empirical validation of the previously proven Lyapunov stability criteria (Corollary 10). The higher-dimensional component in the fourth dimension resists chaotic divergence, settling instead into a highly predictable, orbitally stable limit cycle. This behavior mathematically confirms that the gyroscopic nutation of the helix remains structurally conservative over an infinite time horizon.

## 5. Conclusions

This research has established a comprehensive framework modeling the differential geometry of inextensible curves propagating in pseudo-Galilean 4-space, along with the bending energy that constitutes the physical cost of this motion. The primary findings obtained from this study are summarized as follows:

From a geometric perspective:

- The extended Serret-Frenet relations compatible with the unique metric structure of the pseudo-Galilean 4-space have been successfully derived. These relations precisely define the directional evolution of the curve within the four-dimensional manifold.
- The angular evolutions exhibited by the vector fields with respect to the time parameter $t$ and arc-length parameter $s$ have been mapped into geometric parameters via the formulation of pseudo-angle functions.

From a dynamic flow perspective:

- The system of differential equations imposed by the invariant arc-length condition upon the velocity components and curvature functions has been solved. This mathematically demonstrates how the underlying geometry intrinsically constrains physical motion.
- The bending elastic energy, evaluated using the Sasaki metric, determines the kinematic cost of the particle moving within the $G_1^4$ space. These energy formulations provide a novel analytical tool to measure the dynamic stability of the system.

Furthermore, the derived differential equations, energy functions, and pseudo-angles offer a complete mathematical map of how a particle propagates within $G_1^4$ space under minimum energy dissipation and maximum geometric compatibility. During the flow of an inextensible curve, energy dissipation is not stochastic. The numerical plots confirm that energy undergoes a structured geometric transfer from the tangential direction toward the binormal directions. This configuration physically enables the particle to propagate while rigidly preserving its total length.

Finally, the sum of the quadratic terms within the integrands represents the total angular action of the system. The consistent escalation of these values demonstrates that the inextensible curve flow traces a physically stable trajectory. Ultimately, this study bridges the gap between differential geometry and theoretical physics, offering a robust mathematical framework to comprehend the behavior of constrained systems in higher-dimensional spaces.

## Author contributions

Fatma Almaz: Conceptualization, Formal analysis, Writing–original draft; Handan Öztekin: Supervision, Validation, Writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

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## Acknowledgments

This paper has been supported by Batman University Scientific Research Projects Coordination Unit under number BTÜBAB-2026-YAYDEP-03.

## Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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