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Research article

Mathematical analysis of a fractional model for a scheduling problem with precedence constraints and application to ant colony optimization

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Abstract: We proposed an original hybrid approach that combined a continuous modeling framework based on Caputo fractional differential equations with an Earliest Deadline First–Ant Colony Optimization (EDF–ACO) algorithm for solving the parallel machine scheduling problem with precedence constraints. Unlike most existing works, where the fractional order is usually fixed at its classical value, we investigated the influence of the fractional order α in the interval $(0, 1]$ and analyzed its impact on the optimization process. The results showed that intermediate values of α allowed the effective incorporation of memory effects, leading to improved numerical stability, smoother convergence, and a reduction of oscillatory behavior. The fractional evaluation mechanism was coupled with the pheromone update strategy of the EDF–ACO algorithm, providing a more stable guidance for the search process. The existence and uniqueness of the solution to the fractional model were established using Banach's fixed point theorem, ensuring the consistency of the proposed continuous evaluation framework. Numerical experiments confirmed the effectiveness of the approach in terms of solution quality and convergence stability across different scheduling configurations.

Keywords: fractional derivative; fixed point; scheduling; fractional modeling; ant colony optimization; hybrid optimization

Mathematics Subject Classification: 26A33, 34A08, 47H10, 68T20, 90B35, 90B36

1. Introduction

Scheduling problems constitute a fundamental class of problems in operations research, due to their central role in many complex systems such as industrial production, logistics networks, parallel computing environments, and service systems [12]. Among them, the identical parallel machine scheduling problem with precedence constraints aims to determine both the assignment of tasks to machines and their sequencing, while minimizing the total completion time and respecting given precedence relations.

This problem is well known for its computational difficulty. It is nondeterministic polynomial time (NP)-hard, which makes exact methods unsuitable for medium- and large-scale instances [8]. This complexity has motivated the development of a wide range of approximate methods, including list heuristics and metaheuristics, which are able to produce high-quality solutions within reasonable computational times [4]. Among the most widely studied metaheuristics, ant colony optimization (ACO) algorithms are particularly attractive due to their ability to efficiently explore complex combinatorial solution spaces through a cooperative mechanism based on collective memory [7]. These algorithms have been successfully applied to various scheduling problems, including settings involving precedence constraints and parallel machines [3]. However, in most existing approaches, the evaluation of the solutions constructed by ants relies on classical discrete criteria, in particular, the makespan, which may lead to unstable search dynamics and convergence oscillations.

In parallel, fractional differential equations have attracted increasing attention in the modeling of complex dynamical systems, due to their ability to incorporate memory and hereditary effects that are absent in classical differential models [5, 10, 13]. These properties have been successfully exploited in many application domains. However, their use in combinatorial optimization problems, and, in particular, in scheduling, remains relatively limited.

Recent advances in numerical methods for fractional differential equations have led to the development of efficient computational schemes, including predictor–corrector approaches, high-order numerical methods, and optimal-control-based techniques. Such methods have been successfully applied to a wide range of fractional dynamical systems and control problems. A representative contribution is the predictor–corrector approach proposed by Diethelm. [6], which has become a widely used numerical framework for the solution of fractional differential equations. These developments further demonstrate the richness of the fractional-calculus framework and provide additional motivation for exploring fractional modeling in scheduling and combinatorial optimization problems.

More recently, significant advances have been achieved in the study of fractional and fractal dynamical systems, including fractional oscillators, pendulum models, nonlinear fractional wave phenomena, and fractional quantum mechanical systems. Recent investigations on fractional rotating pendulum oscillators and fractal oscillators with rolling-wheel dynamics have demonstrated the ability of nonlocal operators to capture complex dynamical behaviors that cannot be described by classical integer-order models [9, 14]. In addition, recent studies on nonlinear spatiotemporal fractional quantum mechanics equations and fractional coupled Higgs systems have further highlighted the effectiveness of fractional operators for describing complex wave propagation phenomena and nonlinear dynamical behaviors [1]. These developments further emphasize the versatility of fractional calculus and motivate the exploration of fractional modeling techniques in new application areas, including combinatorial optimization and scheduling.

In this work, we propose an original hybrid approach that combines a continuous modeling framework based on fractional differential equations with an ant colony optimization algorithm to solve the parallel machine scheduling problem with precedence constraints.

The Caputo fractional derivative is adopted in the proposed framework because it allows the use of classical initial conditions while preserving memory effects. Compared with other fractional operators, such as the Riemann–Liouville and Atangana–Baleanu derivatives, the Caputo formulation provides a convenient analytical setting for the theoretical investigation of existence and uniqueness properties, which is particularly suitable for the mathematical analysis developed in this work.

Unlike existing works, where the fractional order is generally fixed at its classical value, we consider the fractional order as a full algorithmic parameter, directly influencing the convergence dynamics and the stability of the optimization process.

Motivated by these observations, the present work investigates whether the memory properties provided by fractional calculus can be exploited to enhance the evaluation and search mechanisms of an ACO-based scheduling algorithm.

The main contributions of this paper are threefold. First, we introduce a continuous fractional model for the evaluation of discrete schedules, which allows memory effects to be incorporated into the assessment of schedule quality. Second, we rigorously establish the existence and uniqueness of the solution of the fractional model using Banach's fixed point theorem, thus guaranteeing the consistency and robustness of the evaluation process. Third, we propose a hybrid EDF–ACO algorithm in which the fractional evaluation framework is coupled with the pheromone update mechanism, thereby influencing the search dynamics and the convergence behavior of the optimization process.

In addition, a convergence analysis based on the relative performance criterion R is performed in order to evaluate the influence of the fractional order and the stability of the optimization process under different scheduling configurations.

To the best of our knowledge, very few studies have investigated the algorithmic role of the fractional order within an ACO-based framework for scheduling problems with precedence constraints. In this study, the fractional order is not viewed as a mere modeling parameter, but as a genuine algorithmic variable that directly affects the behavior of the optimization process.

The remainder of the paper is organized as follows. Section 2 introduces the scheduling problem and the proposed fractional formulation. Section 3 presents the mathematical analysis of the model and establishes the main existence and uniqueness result. Section 4 describes the hybrid EDF–ACO algorithm and its implementation. Section 5 introduces the relative performance criterion adopted in the numerical study. Section 6 reports and discusses the numerical results obtained under different scheduling configurations. Section 7 presents the convergence analysis based on the relative performance criterion. Section 8 discusses the limitations of the proposed framework and outlines future research directions. Finally, Section 9 concludes the paper.

2. Problem formulation

This section presents the scheduling problem under consideration and introduces the fractional continuous framework used to evaluate candidate schedules generated during the optimization process.

2.1. Classical scheduling problem

Let

$$T = \{1, 2, \ldots, n\}$$

be a finite set of tasks to be processed on a set of identical parallel machines. Each task $i \in T$ is associated with a positive processing time $p_i > 0$.

The precedence relations between tasks are represented by a directed acyclic graph (DAG)

$$G = (T, P),$$

where an arc $(i, j) \in P$ indicates that task i must be completed before task j can start.

A feasible schedule specifies, for each task $i \in T$, a starting time S_i, a completion time

$$C_i = S_i + p_i,$$

and an assignment to one of the available machines. The schedule must satisfy both machine-capacity constraints and precedence requirements.

The objective is to minimize the makespan

$$C_{\max} = \max_{i \in T} C_i.$$

The resulting parallel-machine scheduling problem with precedence constraints is known to be NP-hard [8, 12]. Consequently, exact combinatorial approaches rapidly become computationally expensive for large-scale instances, motivating the development of heuristic optimization procedures and complementary analytical evaluation tools.

2.2. Continuous evaluation framework

The optimization process explored in this work generates a large number of candidate schedules. Evaluating these schedules exclusively through discrete combinatorial measures may become computationally demanding when the search space is large.

To provide an analytical description of the evaluation process, we introduce a continuous state vector

$$C(t) = \big(C_1(t), C_2(t), \ldots, C_n(t)\big) \in \mathbb{R}^n,$$

where the component $C_i(t)$ represents the continuous evaluation associated with task i, for $i = 1, \ldots, n$. The vector $C(t)$ provides a smoothed description of the global scheduling dynamics over time.

This continuous representation does not replace the original scheduling problem. Instead, it serves as an auxiliary framework that captures the global evolution of schedule quality during the optimization process.

The evolution of the evaluation state is governed by a vector-valued operator

$$F : \mathbb{R}^n \to \mathbb{R}^n,$$

defined by

$$F(C) = (F_1(C), F_2(C), \ldots, F_n(C)),$$

where each component describes the contribution of the corresponding task to the continuous evaluation dynamics.

2.3. *Fractional dynamic model*

Fractional calculus provides a natural framework for incorporating memory effects through nonlocal operators. Unlike classical differential operators, fractional derivatives depend on the entire history of the state variable.

In scheduling environments, the quality of a candidate solution may depend not only on the current configuration but also on previously explored configurations. Consequently, fractional operators offer a suitable mechanism for introducing memory-dependent evaluation dynamics.

Let $\alpha \in (0, 1]$ denote the fractional order. The proposed evaluation process is described by the Caputo fractional differential system

$$^{C}D_t^{\alpha} C(t) = F(C(t)), \qquad t \in [0, T], \tag{2.1}$$

subject to the initial condition

$$C(0) = C_0, \qquad C_0 \in \mathbb{R}^n, \tag{2.2}$$

where $^{C}D_t^{\alpha}$ denotes the Caputo fractional derivative of order α applied componentwise.

The parameter α controls the influence of past states on the current dynamics. When $\alpha = 1$, the model reduces to the classical first-order dynamical system, whereas smaller values of α increase the contribution of memory effects.

2.4. *Assumptions on the evaluation operator*

Throughout this work, the evaluation operator F is assumed to satisfy the following conditions.

(H1) The operator

$$F : \mathbb{R}^n \to \mathbb{R}^n$$

is continuous.

(H2) There exists a constant $L > 0$ such that

$$\|F(x) - F(y)\| \leq L\|x - y\|, \qquad \forall x, y \in \mathbb{R}^n, \tag{2.3}$$

where $\| \cdot \|$ denotes a norm on $\mathbb{R}^n$.

Assumption (H2) states that F satisfies a global Lipschitz condition. These hypotheses are standard in the theory of fractional differential systems and will be used to establish the existence and uniqueness of solutions.

2.5. Equivalent integral formulation

Applying the fractional integral operator of order α associated with the Caputo derivative to both sides of (2.1), we obtain that the fractional differential system can be rewritten in the equivalent Volterra integral form

$$C(t) = C_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(C(s))\, ds, \tag{2.4}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

The integral representation (2.4) provides the basis for the theoretical analysis developed in the next section. In particular, it enables the application of fixed-point techniques to establish the well-posedness of the proposed fractional framework and guarantees the mathematical consistency of the continuous evaluation process used in the EDF–ACO algorithm.

3. Mathematical analysis: Existence and uniqueness

In this section, we establish the existence and uniqueness of the solution of the fractional dynamic model introduced in Section 2. These results ensure that the continuous evaluation process employed within the EDF–ACO framework is mathematically well posed.

The Caputo fractional derivative is adopted because it allows the use of classical initial conditions while preserving memory effects. Compared with other fractional operators, such as the Riemann–Liouville and Atangana–Baleanu derivatives, the Caputo formulation provides a convenient analytical framework for establishing existence and uniqueness results through fixed-point techniques.

3.1. Preliminary results

Definition 3.1. *Let $0 < \alpha \leq 1$ and let $f \in C^1([0,T])$. The Caputo fractional derivative of order α is defined by*

$${}^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)\, ds.$$

Remark 3.1. *For any constant function $c \in \mathbb{R}$ and any fractional order $\alpha \in (0,1)$, the Caputo fractional derivative satisfies*

$${}^C D_t^\alpha c = 0.$$

Indeed,

$${}^C D_t^\alpha c = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{d}{ds}(c)\, ds = 0.$$

This property is consistent with the classical integer-order derivative and constitutes one of the main advantages of the Caputo formulation when classical initial conditions are prescribed.

3.2. Problem setting

For the theoretical analysis, the operator F introduced in Section 2 is considered through its induced action on the Banach space $C([0,T],\mathbb{R}^n)$. In this functional framework, F is assumed to be a bounded linear operator, which allows the application of Banach's fixed-point theorem.

We consider the fractional initial value problem

$$^{C}D_t^{\alpha}C(t) = F(C(t)), \qquad t \in [0, T], \tag{3.1}$$

subject to

$$C(0) = C_0. \tag{3.2}$$

The solution is sought in the form

$$C : [0, T] \longrightarrow \mathbb{R}^n,$$

defined by

$$C(t) = (C_1(t), C_2(t), \dots, C_n(t)),$$

where

$$C_i \in C([0, T], \mathbb{R}), \qquad i = 1, \dots, n.$$

Consequently,

$$C \in C([0, T], \mathbb{R}^n).$$

The operator

$$F = (F_1, F_2, \dots, F_n)$$

is defined componentwise, where

$$F_i : C([0, T], \mathbb{R}) \longrightarrow C([0, T], \mathbb{R}), \qquad i = 1, \dots, n,$$

are bounded linear operators.

Remark 3.2. *The operator F should not be interpreted as a complete representation of the underlying combinatorial scheduling problem. Instead, it acts as a continuous evaluation operator associated with the proposed fractional framework. The bounded linearity assumption is introduced to establish rigorous existence and uniqueness results and to guarantee the mathematical consistency of the continuous evaluation process.*

Lemma 3.1. *Let*

$$h_i \in C([0, T], \mathbb{R}).$$

Then, the solution of

$$^{C}D_t^{\alpha}C_i(t) = h_i(t), \qquad C_i(0) \geq 0,$$

is given by

$$C_i(t) = C_i(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h_i(s)\, ds.$$

Proof. Applying the fractional integral operator of order α to both sides yields

$$I^{\alpha}\Big(^{C}D_t^{\alpha}C_i(t)\Big) = I^{\alpha}h_i(t).$$

Using the classical property of the Caputo derivative,

$$I^{\alpha}\Big(^{C}D_t^{\alpha}C_i(t)\Big) = C_i(t) - C_i(0),$$

we obtain
$$C_i(t) - C_i(0) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h_i(s)\, ds.$$
Hence,
$$C_i(t) = C_i(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h_i(s)\, ds.$$
□

3.3. Main result

Assume that:

(H1) For each $i = 1, \ldots, n$, there exists $M_i > 0$ such that
$$\|F_i\| \leq M_i.$$

(H2)
$$0 < \frac{T^\alpha M}{\Gamma(\alpha+1)} < 1,$$
where
$$M = \max_{1\leq i\leq n} M_i.$$

Theorem 1. *Under Assumptions* (H1) *and* (H2)*, the fractional problem* (3.1) *and* (3.2) *admits a unique solution in*
$$C([0,T], \mathbb{R}^n).$$

Proof. Consider the Banach space
$$X = C([0,T], \mathbb{R}^n)$$
endowed with the norm
$$\|C\|_X = \max_{1\leq i\leq n} \|C_i\|_\infty.$$
By the previous lemma, define
$$\Psi : X \to X$$
by
$$\Psi C(t) = C(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(C(s))\, ds.$$

Step 1: The operator Ψ maps X into itself.

Let
$$C \in X = C([0,T], \mathbb{R}^n).$$
We show that
$$\Psi C \in C([0,T], \mathbb{R}^n).$$
Let $i \in \{1, \ldots, n\}$ and let $(t_n)_{n\geq 1} \subset [0,T]$ be a sequence such that
$$t_n \longrightarrow t_0, \qquad n \to \infty.$$

We prove that

$$\Psi_i C_i(t_n) \longrightarrow \Psi_i C_i(t_0).$$

Using the definition of Ψ, we have

$$\Psi_i C_i(t) = C_i(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F_i(C_i)(s)\, ds.$$

Therefore,

$$\begin{aligned} &|\Psi_i C_i(t_n) - \Psi_i C_i(t_0)| \\ &= \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_n} (t_n - s)^{\alpha-1} F_i(C_i)(s)\, ds - \int_0^{t_0} (t_0 - s)^{\alpha-1} F_i(C_i)(s)\, ds \right|. \end{aligned}$$

Adding and subtracting

$$\int_0^{t_0} (t_n - s)^{\alpha-1} F_i(C_i)(s)\, ds,$$

and using the triangle inequality, we obtain

$$\begin{aligned} &|\Psi_i C_i(t_n) - \Psi_i C_i(t_0)| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_0} \left| (t_n - s)^{\alpha-1} - (t_0 - s)^{\alpha-1} \right| |F_i(C_i)(s)|\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_n} (t_n - s)^{\alpha-1} |F_i(C_i)(s)|\, ds. \end{aligned}$$

Since $F_i(C_i)$ is continuous on the compact interval $[0, T]$, it is bounded. Hence, there exists a constant $K_i > 0$ such that

$$|F_i(C_i)(s)| \leq K_i, \qquad s \in [0, T].$$

Consequently,

$$\begin{aligned} &|\Psi_i C_i(t_n) - \Psi_i C_i(t_0)| \\ &\leq \frac{K_i}{\Gamma(\alpha)} \int_0^{t_0} \left| (t_n - s)^{\alpha-1} - (t_0 - s)^{\alpha-1} \right| ds \\ &\quad + \frac{K_i}{\Gamma(\alpha)} \int_{t_0}^{t_n} (t_n - s)^{\alpha-1} ds. \end{aligned}$$

The second integral tends to zero as $n \to \infty$. Moreover,

$$(t_n - s)^{\alpha-1} \longrightarrow (t_0 - s)^{\alpha-1},$$

for every $s \in [0, t_0)$. By the dominated convergence theorem,

$$\int_0^{t_0} \left| (t_n - s)^{\alpha-1} - (t_0 - s)^{\alpha-1} \right| ds \longrightarrow 0.$$

Hence,

$$\Psi_i C_i(t_n) \longrightarrow \Psi_i C_i(t_0).$$

Therefore,

$$\Psi_i C_i \in C([0, T], \mathbb{R}), \qquad i = 1, \ldots, n.$$

Consequently,

$$\Psi C \in C([0,T],\mathbb{R}^n).$$

Thus,

$$\Psi(X) \subset X.$$

Step 2: Invariance of a closed ball.

Let

$$C_0^* = \max_{1\le i\le n} |C_i(0)|.$$

Choose $\rho > 0$ such that

$$\rho \geq \frac{-C_0^*\Gamma(\alpha+1)}{T^\alpha M - \Gamma(\alpha+1)}.$$

Define

$$B_\rho = \{C \in X : \|C\|_X \leq \rho\}.$$

Let $C \in B_\rho$. For any $i = 1, \ldots, n$, we have

$$|\Psi_i C_i(t)| \leq |C_i(0)| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |F_i(C_i)(s)|\, ds.$$

Since F_i is a bounded linear operator,

$$|F_i(C_i)(s)| \leq \|F_i(C_i)\|_\infty \leq \|F_i\|\, \|C_i\|_\infty \leq M_i \|C_i\|_\infty.$$

Therefore,

$$|\Psi_i C_i(t)| \leq |C_i(0)| + \frac{M_i}{\Gamma(\alpha)} \|C_i\|_\infty \int_0^t (t-s)^{\alpha-1} ds.$$

Using

$$\int_0^t (t-s)^{\alpha-1} ds = \frac{t^\alpha}{\alpha},$$

we obtain

$$|\Psi_i C_i(t)| \leq |C_i(0)| + \frac{t^\alpha M_i}{\Gamma(\alpha+1)} \|C_i\|_\infty.$$

Since $t \leq T$,

$$|\Psi_i C_i(t)| \leq |C_i(0)| + \frac{T^\alpha M_i}{\Gamma(\alpha+1)} \|C_i\|_\infty.$$

Taking the supremum over $t \in [0,T]$ gives

$$\|\Psi_i C_i\|_\infty \leq |C_i(0)| + \frac{T^\alpha M_i}{\Gamma(\alpha+1)} \|C_i\|_\infty.$$

Consequently,

$$\|\Psi C\|_X \leq C_0^* + \frac{T^\alpha M}{\Gamma(\alpha+1)} \|C\|_X.$$

Since $C \in B_\rho$,

$$\|C\|_X \leq \rho,$$

and therefore

$$\|\Psi C\|_X \le C_0^* + \frac{T^\alpha M}{\Gamma(\alpha+1)}\rho.$$

By the choice of ρ, we obtain

$$\|\Psi C\|_X \le \rho.$$

Since $\|C\|_X \le \rho$,

$$\|\Psi C\|_X \le C_0^* + \frac{MT^\alpha}{\Gamma(\alpha+1)}\rho.$$

Hence,

$$\Psi(B_\rho) \subset B_\rho.$$

Step 3: Contraction property.

Let

$$C, \widetilde{C} \in B_\rho.$$

For each $i = 1, \ldots, n$, we have

$$\Psi_i C_i(t) - \Psi_i \widetilde{C}_i(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(F_i(C_i)(s) - F_i(\widetilde{C}_i)(s) \Big) ds.$$

Taking the absolute value yields

$$\left| \Psi_i C_i(t) - \Psi_i \widetilde{C}_i(t) \right| \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| F_i(C_i)(s) - F_i(\widetilde{C}_i)(s) \right| ds.$$

Since F_i is a bounded linear operator, we obtain

$$F_i(C_i)(s) - F_i(\widetilde{C}_i)(s) = F_i(C_i - \widetilde{C}_i)(s),$$

and therefore

$$\left| F_i(C_i)(s) - F_i(\widetilde{C}_i)(s) \right| = \left| F_i(C_i - \widetilde{C}_i)(s) \right|.$$

Using the boundedness of F_i, we have

$$\left| F_i(C_i - \widetilde{C}_i)(s) \right| \le \|F_i\| \, \|C_i - \widetilde{C}_i\|_\infty \le M_i \|C_i - \widetilde{C}_i\|_\infty.$$

Substituting this estimate into the previous inequality gives

$$\left| \Psi_i C_i(t) - \Psi_i \widetilde{C}_i(t) \right| \le \frac{M_i}{\Gamma(\alpha)} \|C_i - \widetilde{C}_i\|_\infty \int_0^t (t-s)^{\alpha-1} ds.$$

Since

$$\int_0^t (t-s)^{\alpha-1} ds = \frac{t^\alpha}{\alpha},$$

we obtain

$$\left| \Psi_i C_i(t) - \Psi_i \widetilde{C}_i(t) \right| \le \frac{t^\alpha M_i}{\Gamma(\alpha+1)} \|C_i - \widetilde{C}_i\|_\infty.$$

Since $t \le T$,

$$\left|\Psi_i C_i(t) - \Psi_i \widetilde{C}_i(t)\right| \le \frac{T^\alpha M_i}{\Gamma(\alpha+1)} \|C_i - \widetilde{C}_i\|_\infty.$$

Taking the supremum over $t \in [0, T]$, we obtain

$$\|\Psi_i C_i - \Psi_i \widetilde{C}_i\|_\infty \le \frac{T^\alpha M_i}{\Gamma(\alpha+1)} \|C_i - \widetilde{C}_i\|_\infty.$$

Taking the maximum over $i = 1, \ldots, n$ yields

$$\|\Psi C - \Psi \widetilde{C}\|_X \le \frac{T^\alpha M}{\Gamma(\alpha+1)} \|C - \widetilde{C}\|_X,$$

where

$$M = \max_{1 \le i \le n} M_i.$$

Let

$$k = \frac{T^\alpha M}{\Gamma(\alpha+1)}.$$

By Assumption (H2),

$$0 < k < 1.$$

Therefore, Ψ is a contraction on B_ρ.

Step 4: Application of Banach's fixed-point theorem.

Since B_ρ is a closed subset of the Banach space X and Ψ is a contraction on B_ρ, Banach's fixed-point theorem guarantees the existence of a unique fixed point

$$C^* \in B_\rho$$

such that

$$\Psi(C^*) = C^*.$$

Therefore,

$$C^*(t) = C(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(C^*(s))\, ds.$$

By Lemma 3.1, the previous integral equation is equivalent to the fractional differential problem

$${}^C D_t^\alpha C(t) = F(C(t)), \qquad C(0) = C_0.$$

Hence, C^* is the unique solution of the fractional problem (3.1) and (3.2). This completes the proof. □

Remark 3.3. *The previous theorem guarantees that the proposed fractional dynamic model admits a unique solution under Assumptions (H1) and (H2). Consequently, the continuous evaluation process employed in the EDF–ACO framework is mathematically consistent and can be used reliably throughout the optimization procedure.*

4. Hybrid EDF–ACO algorithm

This section presents the proposed hybrid EDF–ACO algorithm for solving the parallel machine scheduling problem with precedence constraints. The originality of the approach lies in the coupling between an ant colony optimization metaheuristic and a dynamic evaluation mechanism based on the fractional modeling introduced in the previous section.

4.1. General principle

ACO, introduced by Dorigo and Stützle [7], is a metaheuristic inspired by the collective behavior of ants during food foraging. A population of artificial agents cooperatively constructs solutions by exploiting pheromone trails and heuristic information.

The EDF–ACO algorithm combines the EDF scheduling rule with the exploration capabilities of ACO. The EDF rule is used to prioritize admissible tasks according to their deadlines, while the ACO mechanism explores alternative scheduling configurations through pheromone-guided decisions. The fractional model is employed as an evaluation tool, allowing memory effects to be incorporated into the assessment of candidate schedules.

In the proposed EDF–ACO algorithm, each ant incrementally builds a feasible schedule that satisfies precedence constraints and machine capacity constraints. Unlike classical ACO approaches, the quality of each constructed schedule is evaluated using the fractional model, which provides a continuous and more informative estimation of the makespan.

To highlight the main characteristics of the proposed approach, Table 1 summarizes the principal differences between the classical ACO algorithm and the proposed EDF–ACO framework.

Table 1. Comparison between classical ACO and the proposed EDF–ACO approach.

Feature	Classical ACO	Proposed EDF–ACO
Task selection	Heuristic-based	EDF-guided + heuristic
Evaluation	Discrete makespan	Fractional evaluation
Memory effects	Indirect	Explicit through α
Pheromone update	Makespan-based	Fractional-evaluation-based
Convergence analysis	Usually absent	Supported by fractional framework

As shown in Table 1, the proposed EDF–ACO approach differs from the classical ACO algorithm through the integration of the EDF dispatching rule and the fractional evaluation mechanism. These additional components provide a more informative assessment of candidate schedules and contribute to improving the stability of the search process.

4.2. Solution construction

At each iteration, each ant selects a task from the set of available tasks, that is, tasks whose precedence constraints are satisfied. The probability of selecting a task i is given by

$$P_i = \frac{\tau_i^{\alpha_1}\eta_i^{\beta_1}}{\sum_{j\in\mathcal{A}}\tau_j^{\alpha_1}\eta_j^{\beta_1}}. \tag{4.1}$$

The parameters α_1 and β_1 correspond to the classical ACO parameters and should not be confused with the fractional order α introduced in the mathematical model. The former controls the balance between pheromone information and heuristic information, whereas the latter governs the memory effect in the fractional evaluation process. The parameters are defined as follows: $\mathcal{A}$ denotes the set of admissible tasks,

τ_i represents the pheromone value associated with task i,

η_i is a heuristic information, generally related to the processing time or the priority of task i,

α_1 and β_1 control the relative influence of pheromone trails and heuristic information.

Once a task is selected, it is assigned to the machine that becomes available at the earliest time, in order to satisfy machine capacity constraints.

4.3. Pheromone update and fractional evaluation

In the present work, the fractional differential model is primarily used as a theoretical and analytical framework for the continuous evaluation of schedules. The EDF–ACO algorithm does not rely on the direct numerical integration of the fractional differential equation during the optimization process. Instead, the fractional formulation provides a continuous quality indicator used in the evaluation and pheromone update mechanisms. The development of dedicated numerical schemes for the fractional model constitutes an interesting direction for future research. Once a complete schedule has been generated, its quality is evaluated through the fractional makespan model introduced in Section 3. Let $C_{\max}$ denote the classical makespan of the generated schedule. Motivated by the integral representation of the fractional model established in Section 3, a fractional evaluation indicator is associated with the schedule. This indicator is intended to provide a memory-dependent assessment of schedule quality and should be interpreted as a continuous surrogate measure rather than as the exact numerical solution of the fractional differential system. The resulting quantity is denoted by $C_{\max}^{(\alpha)}$ and is used to guide the pheromone update mechanism. The fractional evaluation indicator is defined by

$$C_{\max}^{(\alpha)} = C(0) + \frac{1}{\Gamma(\alpha)} \int_0^{C_{\max}} (C_{\max} - s)^{\alpha-1} F(C(s))\, ds, \tag{4.2}$$

which is inspired by the integral representation of the fractional model established in Section 3. The quantity $C_{\max}^{(\alpha)}$ provides a memory-dependent continuous assessment of the generated schedule and is subsequently used in the pheromone update mechanism.

The pheromone update rule is given by

$$\tau_i \leftarrow (1-\rho)\tau_i + \Delta\tau_i \tag{4.3}$$

where $\rho \in (0, 1)$ denotes the evaporation coefficient.

The reinforcement term is computed according to

$$\Delta\tau_i = \frac{1-\rho}{C_{\max}^{(\alpha)}}. \tag{4.4}$$

Consequently, schedules associated with smaller fractional makespan values receive stronger pheromone reinforcement and are more likely to influence future search iterations.

The fractional makespan $C_{\max}^{(\alpha)}$ acts as a continuous quality indicator of the generated schedule. Better schedules correspond to smaller values of $C_{\max}^{(\alpha)}$ and therefore receive stronger pheromone reinforcement. This mechanism establishes a direct coupling between the fractional evaluation model and the exploration process of the EDF–ACO algorithm.

For the numerical experiments reported in this paper, the final performance assessment is based on the classical makespan of the schedules generated by the EDF–ACO algorithm. The fractional formulation is used to provide a theoretical evaluation framework and to guide the search process through the pheromone update mechanism. Consequently, the relative performance criterion introduced in the next section is computed from the classical makespan of the final schedule, which facilitates comparisons with existing scheduling studies.

4.4. Algorithmic parameters

The main parameters used in the EDF–ACO algorithm are summarized in Table 2 [11].

Table 2. EDF–ACO algorithm parameters.

Parameter	Value
Pheromone influence parameter (α_1)	1
Heuristic influence parameter (β_1)	2
Evaporation coefficient (ρ)	0.5
Pseudo-random factor (q_0)	0.8
Number of ants (N_a)	20
Maximum iterations ($N_{\max}$)	500
Fractional order (α)	Variable

The stopping criterion is reached when either the maximum number of iterations is attained or when the relative improvement of the best solution remains below a prescribed threshold during a fixed number of consecutive iterations.

Although the fractional evaluation introduces an additional computational effort compared with a purely discrete makespan evaluation, the observed increase in Central Processing Unit (CPU) time remains moderate. The improvement in convergence stability and the reduction of oscillatory behavior compensate for this additional cost. The final solution corresponds to the schedule associated with the best evaluation value observed during the search process and is ultimately assessed through its classical makespan.

5. Relative performance criterion

A relative performance criterion, denoted by R, is introduced in order to provide a scale-independent measure of performance. Unlike the raw makespan, R allows a fair comparison between instances of different sizes and densities by normalizing the results with respect to a reference bound B.

The reference value B is defined as the maximum of the classical lower bound $\frac{1}{m}\sum_{i=1}^{n} p_i$ and the makespan obtained by a simple list heuristic.

Thus, the relative performance criterion is defined as

$$R = \frac{C_{\max}^{EDF-ACO}}{\max\left(C_{\max}^{list}, \frac{1}{m}\sum_{i=1}^{n} p_i\right)}. \tag{5.1}$$

For the numerical experiments reported in this paper, $C_{\max}^{EDF-ACO}$ denotes the classical makespan of the final schedule generated by the EDF–ACO algorithm. Although the fractional model contributes to the evaluation process and influences the search dynamics through the pheromone update mechanism, the reported values of R are computed from the classical makespan in order to facilitate comparisons with existing scheduling studies and benchmark results.

A value of R close to 1 indicates that the obtained schedule is near the reference bound and therefore exhibits a high solution quality. Larger values of R correspond to less efficient schedules. Consequently, the objective of the proposed EDF–ACO algorithm is to generate solutions for which R remains as close as possible to unity.

In the numerical experiments, the criterion R is used to investigate the influence of the fractional order α on the quality of the final schedules generated by the EDF–ACO algorithm. The evolution of R with respect to α provides useful information regarding the influence of the fractional evaluation mechanism on the quality of the generated schedules.

Consequently, the criterion (5.1) will be used in the next section to analyze the performance of the proposed EDF–ACO algorithm under different values of the fractional order α and various scheduling instances.

6. Numerical results

All numerical results reported in this section are obtained from the final schedules generated by the EDF–ACO algorithm. The relative performance criterion R is computed using the classical makespan of the final schedule, while the fractional model contributes indirectly by guiding the search process through the evaluation mechanism incorporated into the optimization framework. All numerical experiments were carried out using an implementation of the EDF–ACO algorithm developed in Delphi 7. For each parameter configuration, a set of randomly generated scheduling instances was considered. Processing times were generated within predefined intervals and precedence relations were represented by directed acyclic graphs with varying densities. The reported values correspond to average results obtained over a set of independently generated scheduling instances. For each configuration, several executions of the EDF–ACO algorithm were performed and the mean values of the performance indicators were recorded.

6.1. Influence of the precedence graph density

The results reported in Table 3 suggest that increasing the density of the precedence graph does not lead to a significant degradation of the relative performance criterion R. This observation suggests the EDF–ACO algorithm is able to effectively handle stricter precedence constraints without a noticeable loss in solution quality. This behavior can be explained by the intrinsic nature of ACO, which progressively favors sequences compatible with precedence constraints, as well as by the fractional evaluation, which smooths the local impact of discrete scheduling decisions. The combination of both mechanisms leads to a stable search process, even in highly constrained environments.

The results show that the value of R remains close to 1 for all densities, which confirms the robustness of the EDF–ACO algorithm with respect to precedence constraints. The CPU time slightly increases with the density due to the higher number of constraint checks, but this increase remains moderate.

Table 3. Influence of the precedence graph density on R and CPU time.

Density (%)	R	CPU time (s)
25	0.98	4.85
50	1.01	5.10
75	0.97	5.42
90	0.99	5.88

6.2. Influence of the number of tasks

When the number of tasks increases, the relative performance criterion R remains globally close to 1, which indicates that the relative quality of the solutions is preserved as shown in Table 4. This observation suggests a satisfactory scalability of the EDF–ACO algorithm over the considered range of problem sizes. The moderate growth of CPU time confirms that the computational complexity of the method remains compatible with large-scale instances. This good scalability is due to the distributed nature of ACO and to the fractional evaluation mechanism, which contributes to a smoother exploration of the search space during the optimization process.

Table 4. Influence of the number of tasks on R and CPU time.

n	R	CPU time (s)
50	0.95	0.12
100	1.02	0.48
200	1.01	1.84
400	1.02	3.91
800	1.01	6.90

6.3. Influence of the number of machines

Increasing the number of machines enlarges the assignment possibilities and, consequently, the size of the search space as shown in Table 5. Nevertheless, the results show that the relative performance criterion R remains stable. These results suggest that the EDF–ACO algorithm efficiently exploits the additional parallelism offered by extra machines without being adversely affected by the increased combinatorial complexity. This property is essential for practical applications where the number of available resources may vary.

A slight degradation of R is observed as the number of machines increases, which can be explained by the growth of the combinatorial assignment space. However, the CPU time remains reasonable, confirming the computational efficiency of the EDF–ACO algorithm.

Table 5. Influence of the number of machines on R and CPU time.

m	R	CPU time (s)
2	0.97	5.20
5	1.00	5.85
8	1.03	6.40
10	1.05	6.95

6.4. *Influence of the fractional order*

The fractional order α plays a key role in balancing memory effects and reactivity in the model as shown in Table 6. In the present study, the numerical investigation was restricted to the interval [0.6, 1], which provided stable and practically relevant results for the considered scheduling instances. The reported values of R are computed from the classical makespan of the final schedules, whereas the influence of the fractional order is reflected through its impact on the search dynamics and the quality of the generated solutions. A more extensive investigation involving smaller fractional orders will be considered in future work.

Table 6. Influence of the fractional order α on R and CPU time.

α	R	CPU time (s)
0.6	1.07	8.42
0.8	1.03	7.35
0.9	1.01	6.98
1.0	1.00	6.90

The results indicate that the best values of the relative performance criterion are obtained for fractional orders close to one. As the fractional order decreases, the value of R slightly increases, which suggests that stronger memory effects may reduce the responsiveness of the search process for the considered scheduling instances. Nevertheless, the degradation remains moderate, which confirms that the proposed EDF–ACO framework is relatively robust with respect to variations of the fractional order. These observations highlight the influence of the fractional evaluation mechanism on the search dynamics and justify the inclusion of fractional concepts within the optimization framework. Overall, the numerical experiments suggest that the proposed EDF–ACO algorithm maintains a stable level of performance with respect to variations in graph density, problem size, and machine availability. Moreover, the fractional evaluation mechanism appears to influence the search dynamics through the parameter α, while preserving the overall robustness of the optimization procedure. These results provide preliminary evidence supporting the effectiveness of the proposed hybrid framework for parallel machine scheduling problems with precedence constraints.

7. Convergence analysis based on the relative performance criterion

Unlike classical convergence analyses based solely on the makespan, we study here the convergence behavior of the EDF–ACO algorithm using the relative performance criterion R.

Since R is computed from the classical makespan of the final schedules, the convergence analysis reflects the evolution of the actual scheduling performance throughout the optimization process. The fractional model contributes indirectly by influencing the search dynamics and the evaluation mechanism used within the EDF–ACO framework. This choice enables a normalized analysis that is independent of instance size and directly comparable across different configurations. Figure 1 illustrates the evolution of the relative performance criterion R over the iterations.

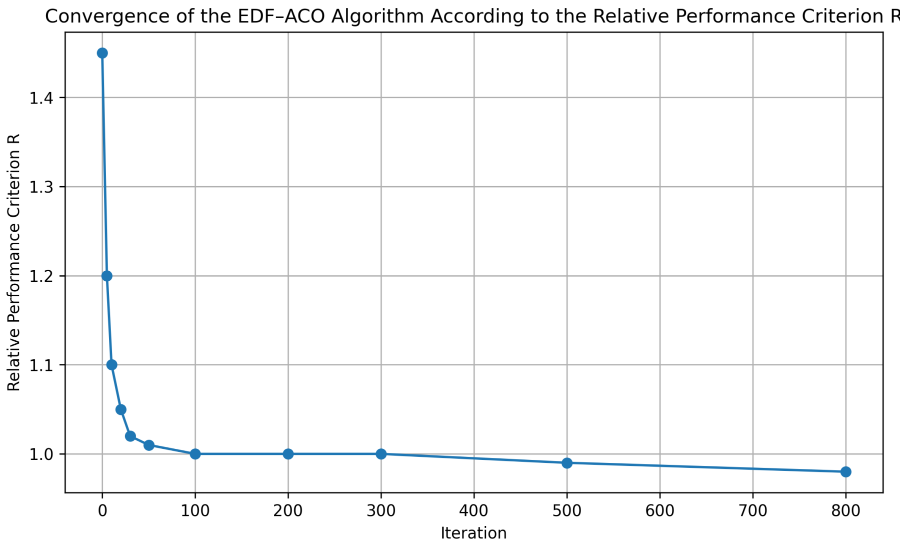

Figure 1. Evolution of the relative performance criterion R over the iterations.

A rapid decrease of R is observed during the first iterations, corresponding to a significant improvement in solution quality. This phase is followed by a stabilization around values close to 1, suggesting convergence toward high-quality solutions. The stabilization of R observed in Figure 1 is consistent with the stopping criterion adopted in the EDF–ACO algorithm and confirms that no significant improvement is obtained after a certain number of iterations.

The convergence behavior is also influenced by the fractional order α. In particular, values of α close to one were observed to produce a more stable decrease of the performance criterion R and faster convergence toward its stationary regime.This observation is consistent with the numerical results reported in Section 6, where larger values of α yielded better relative performance. Numerical experiments indicate that values of α close to 1 provide a favorable balance between memory effects and reactivity, resulting in faster stabilization of the performance criterion R.

This analysis confirms the effectiveness of the coupling between fractional modeling and ACO, both in terms of convergence speed and stability of the obtained solutions. The observed behavior also supports the relevance of the relative performance criterion as a robust indicator for monitoring the optimization process across scheduling instances of different sizes and characteristics.

8. Limitations and future research

The proposed EDF–ACO framework presents several limitations that should be acknowledged. First, the theoretical analysis relies on a bounded linear evaluation operator, which constitutes a simplified representation of the underlying scheduling process. As such, the current fractional model serves primarily as an analytical tool rather than an exact numerical representation of the scheduling dynamics. Second, the computational cost of the algorithm increases for very small values of the fractional order α due to stronger memory effects, which require the evaluation of a longer history of past states. Third, the present study focuses exclusively on identical parallel machines with precedence constraints. Extending the proposed approach to heterogeneous machine environments and more complex scheduling settings remains an interesting avenue for future research. Fourth, although several comparative studies involving metaheuristics such as genetic algorithms, simulated annealing, and tabu search have been previously conducted by the authors in related scheduling frameworks, the present work does not provide a direct experimental comparison. A comprehensive evaluation of the fractional EDF–ACO algorithm against other metaheuristics under identical experimental conditions constitutes an important direction for future investigation. Fifth, the numerical investigation of the fractional order α in this study was restricted to the interval $[0.6, 1]$. Exploring the behavior of the algorithm for smaller fractional orders will be considered in future studies. Finally, no dedicated numerical scheme for the explicit solution of the fractional differential model has been developed in this work. Although efficient methods for fractional differential equations exist in the literature, their adaptation to the scheduling framework considered here remains an open research problem that will be addressed in subsequent research.

9. Conclusions

In this paper, we proposed a hybrid approach for solving the parallel machine scheduling problem with precedence constraints by integrating a continuous fractional modeling framework with an EDF–ACO optimization algorithm.

From a mathematical standpoint, the existence and uniqueness of the solution to the fractional model were established using Banach's fixed point theorem, ensuring a well-defined and consistent continuous evaluation process. The use of the Caputo fractional derivative allows the incorporation of memory effects while maintaining an analytically tractable framework.

Algorithmically, the fractional model was incorporated into the EDF–ACO procedure through a continuous evaluation mechanism and a pheromone update strategy guided by the fractional performance criterion. Numerical experiments demonstrated that the proposed approach achieves stable convergence and high-quality solutions across a variety of scheduling configurations. In particular, the relative performance criterion remained close to unity across different precedence graph densities, numbers of tasks, and numbers of machines, confirming the robustness of the framework.

The analysis of the fractional order α indicated that values within the interval $[0.6, 1]$ provide a favorable balance between memory effects, convergence stability, and computational efficiency. This highlights the interpretation of the fractional order as an algorithmic parameter rather than a purely modeling coefficient.

Future research directions include the dynamic adaptation of the fractional order during the optimization process, exploration of alternative fractional operators, extension to heterogeneous and multi-objective scheduling environments, comprehensive comparative studies with other metaheuristics under identical experimental conditions, and the development of dedicated numerical schemes for the fractional scheduling model.

Author contributions

Each author made an equal and substantial contribution to each part and section of the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare they have no conflicts of interest in this paper.

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