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Research article

Flag-transitive symmetric designs and double quasi-symmetric designs with 8-dimensional projective orthogonal group $P\Omega_8^{\pm}(3)$ as socle

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Abstract: This paper contributes to the study of flag-transitive symmetric 2-(v, k, λ) designs that admit primitive automorphism groups whose socle is the 8-dimensional orthogonal group $P\Omega_8^+(3)$ or $P\Omega_8^-(3)$. We prove that no such design exists. Subsequently, we study the related double quasi-symmetric designs and provide some specific examples. The block intersection numbers of these designs are all relatively large. This paper offers a new approach to the research of double quasi-symmetric designs and symmetric partially balanced incomplete block designs.

Keywords: symmetric design; double quasi-symmetric design; flag-transitive; simple group; projective orthogonal group; partially balanced incomplete block design
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1. Introduction

A *symmetric 2-(v, k, λ) design* (i.e. *symmetric balanced incomplete block design*, abbreviated as *SBIBD*) is a finite incidence structure $\mathcal{D}$=$(\mathcal{P}, \mathcal{B})$, where $\mathcal{P}$ is a set of v elements called points and $\mathcal{B}$ ($|\mathcal{B}| = b = v$) is a set of k-subsets of $\mathcal{P}$ called blocks such that any two distinct points are incident with exactly λ blocks and any point is contained in exactly r blocks. The symmetric design $\mathcal{D}$ is *nontrivial* if $2 < k < v - 1$. A *flag* in a design is an incident point-block pair.

An *automorphism* of $\mathcal{D}$ is an isomorphism from the design to itself. The set of all automorphisms of a design forms a group, called *the (full) automorphism group*, denoted by Aut($\mathcal{D}$). An *automorphism* *group* of the design is any subgroup of the full automorphism group [5]. For $G \leq \mathrm{Aut}(\mathcal{D})$, the design $\mathcal{D}$ is said to be *point-primitive* if G is primitive on $\mathcal{P}$ and *flag-transitive* if G is transitive on the set of flags. The *socle of a finite group* is the product of all its minimal normal subgroups. Additional standard notations and definitions are available, for instance, in [5, 12, 24].

Symmetric designs, which possess excellent symmetry, are a significant subject in design theory research. Many scholars have studied the classification problem of flag-transitive and point-primitive symmetric designs ([1, 17, 23]). As early as 1985, Kantor [9] provided a classification of 2-transitive symmetric 2-(v, k, λ) designs. Around 2011, Braić et al. [3] almost classified the point-primitive symmetric 2-(v, k, λ) designs with $v < 2500$. From 2005 to 2010, O'Reilly [15, 16] proved that when $\lambda \leq 4$, the automorphism group of a symmetric design must be of affine type or almost simple type. Tian and Zhou [19] extended this conclusion to the case where $\lambda \leq 100$.

In 2015, Tian and Zhou [20] completely classified all flag-transitive, point-primitive symmetric 2-(v, k, λ) designs with sporadic socle. In the following year, they [21] obtained two flag-transitive symmetric 2-(v, k, λ) designs admitting primitive automorphism groups of almost simple type with socle $T = \mathrm{PSL}(12, 2)$. The 8-dimensional projective orthogonal groups $T = P\Omega_8^{\pm}(3)$ are of particular interest due to their exceptional properties, including the phenomenon of triality, which often gives rise to unique combinatorial structures. Now we consider the case in which the automorphism group has the group $P\Omega_8^{+}(3)$ or $P\Omega_8^{-}(3)$ as its socle, and obtain the following conclusion.

Theorem 1. *There is no symmetric 2-(v, k, λ) design $\mathcal{D}$ admitting a flag-transitive, point-primitive automorphism group G of almost simple type with socle $T = P\Omega_8^{\pm}(3)$.*

The condition that "any two points are contained in exactly λ blocks" in symmetric designs is quite restrictive. Let's introduce another type of block design as a slight generalization of the symmetric 2-(v, k, λ) design.

A *double quasi-symmetric design* is an incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathbf{I})$, where $\mathcal{P}$ is a finite set of *points* and $\mathcal{B}$ is a finite set of *blocks*, satisfying the following axioms:

(1) All blocks have the same size k, i.e., for every block $B \in \mathcal{B}$, $|B| = k$, where $2 < k < v - 1$ and $v = |\mathcal{P}|$.

(2) Every point is contained in a constant number r of blocks. Furthermore, every pair of distinct points is incident with at least one block; i.e., for every $\{p, q\} \subset \mathcal{P}$ with $p \neq q$, there exists a block $B \in \mathcal{B}$ such that $p\,\mathbf{I}\,B$ and $q\,\mathbf{I}\,B$.

(3) The structure has at most two intersection numbers for both points and blocks, as follows:

(a) *Point Intersection Numbers*: For any two distinct points $p, q \in \mathcal{P}$, the number of blocks containing both p and q, denoted by $\lambda(p, q)$, takes at most two values, say λ_1 and λ_2.

(b) *Block Intersection Numbers*: For any two distinct blocks $B, B' \in \mathcal{B}$, the number of points in their intersection, denoted by $\mu(B, B')$, takes at most two values, say μ_1 and μ_2.

In fact, these two sets, point intersection numbers and block intersection numbers, are not assumed to be equal in general.

It is often assumed that the double quasi-symmetric design is *proper*, meaning that the point intersection number takes two distinct values $\lambda_1 \neq \lambda_2$ and the block intersection number takes two distinct values $\mu_1 \neq \mu_2$, each occurring for some pair of points and some pair of blocks, respectively. A quasi-symmetric design is a "non-proper" double quasi-symmetric design, in which the point intersection number is a fixed value λ. In 1991, Shrikhande and Sane [18] published a book entitled

Quasi-Symmetric Designs. Since then, many scholars have conducted research on quasi-symmetric designs [11,13,25]. Further studies on flag-transitive quasi-symmetric designs and related classification problems can be found in [14, 26].

In the process of studying the existence problem of symmetric designs in this paper, some double quasi-symmetric designs were obtained, and the conclusions are as follows.

Theorem 2. *Let G be a flag-transitive, point-primitive group with $P\Omega_8^+(3)$ as the socle that acts on 1080 points. Then there exist four double quasi-symmetric designs, the block sizes, the point intersection numbers $\{\lambda_1, \lambda_2\}$, and the block intersection numbers $\{\mu_1, \mu_2\}$ are listed as follows:*

(1) if $k = 351$, then both $\{\lambda_1, \lambda_2\}$ and $\{\mu_1, \mu_2\}$ are equal to {108,126};

(2) if $k = 728$, then both $\{\lambda_1, \lambda_2\}$ and $\{\mu_1, \mu_2\}$ are equal to {484,504};

(3) if $k = 378$, then both $\{\lambda_1, \lambda_2\}$ and $\{\mu_1, \mu_2\}$ are equal to {126,135};

(4) if $k = 702$, then both $\{\lambda_1, \lambda_2\}$ and $\{\mu_1, \mu_2\}$ are equal to {450,459}.

Theorem 3. *Let G be a flag-transitive, point-primitive group with $P\Omega_8^-(3)$ as the socle that acts on 1066 points. Then there exist two double quasi-symmetric designs, the block sizes, the point intersection numbers $\{\lambda_1, \lambda_2\}$, and the block intersection numbers $\{\mu_1, \mu_2\}$ are listed as follows:*

(1) if $k = 336$, then both $\{\lambda_1, \lambda_2\}$ and $\{\mu_1, \mu_2\}$ are equal to {92,112};

(2) if $k = 729$, then both $\{\lambda_1, \lambda_2\}$ and $\{\mu_1, \mu_2\}$ are equal to {486,504}.

The paper is organized as follows: In Section 2, we give some fundamental results on groups and designs. Sections 3–5 are the main part of the paper, where the theorems will be completely proved. In Section 6, we expand the research and construct several symmetric partially balanced incomplete block designs. Section 7 is a summary and presents several open questions.

2. Some preliminary results

In this section, we state several known facts that are essential to the proof of our main theorem.

Lemma 2.1. *([24, Theorem 2.3]) Let $x \in \Omega$, $|\Omega| > 1$. A transitive group G on Ω is primitive if and only if G_x is a maximal subgroup of G.*

Lemma 2.2. *([5]) The five parameters v, b, r, k, and λ of a 2-(v, k, λ) design satisfy the following three relationships:*

$$vr = bk,\ \lambda(v-1) = r(k-1),\ b \geq v\ (Fisher's\ inequality).$$

Lemma 2.3. *([7, Lemma 4.3.17]) Let $T = P\Omega_8^+(3)$, then Table 1 gives all groups $M \cap T$ such that M is maximal in an almost simple group G with socle T, together with the index of M in G.*

Remark 2.1. *For Table 1, this list of groups has been compared to the list of maximal subgroups in [6] (which is not claimed to be complete), and no discrepancy was found. The groups are presented in the structures given by [10]. The group $\hat{\ }H$ represents the image of $H \leq \Omega_8^+(3)$ in the group $P\Omega_8^+(3)$, and $\frac{1}{m}H$ is a subgroup of H of index m.*

Lemma 2.4. *([6]) Let $T = P\Omega_8^-(3)$, then Table 2 gives all groups $M \cap T$ such that M is maximal in an almost simple group G with socle T, together with the index of M in G.*

Table 1. Maximal subgroups M of groups G with socle $T := P\Omega_8^+(3)$.

	Structure $M \cap T$	Order $\lvert M \cap T\rvert$	Index $\lvert G : M\rvert$
Ordinary maximal subgroups	$\Omega_7(3)$	4585351680	1080
	$\hat{}[3^6] : (\frac{1}{2}\mathrm{GL}_4(3))$	4421589120	1120
	$\Omega_8^+(2)$	174182400	28431
	$\hat{}[3^9] : (\frac{1}{2}\mathrm{GL}_2(3) \times \Omega_4^+(3)).2$	136048896	36400
	$\hat{}(2 \times \Omega_6^-(3)).2^2$	26127360	189540
	$(L_2(3) \times S_4(3)).2$	622080	7960680
	$(\Omega_4^-(3) \times \Omega_4^-(3)).2^2$	518400	9552816
	$(\Omega_4^+(3) \circ \Omega_4^+(3)).2.2^2$	331776	14926275
Novelty maximal subgroups	$\hat{}\Omega_6^+(3).2^2$	12130560	408240
	$G_2(3)$	4245696	1166400
	$\hat{}\frac{1}{2}\mathrm{GL}_3(3).2^2$	11232	440899200
	$\hat{}[3^{11}] : \frac{1}{2}\mathrm{GL}_2(3).2^2$	8503056	582400
	$\hat{}[3^9] : \frac{1}{2}\mathrm{GL}_3(3).2$	110539728	44800
	$2^6 : A_8$	1290240	3838185
	$\hat{}2^4.2^3.2^3.S_4$	12288	403009425
	$[2^9] : \mathrm{L}_3(2)$	86016	57572775
	$\hat{}(2 \times \frac{1}{2}\mathrm{GU}_3(3)).2^2$	48384	102351600
	$(\mathrm{D}_{10} \times \mathrm{D}_{10}).2^2$	400	12380449536

Table 2. Maximal subgroups M of groups G with socle $T := P\Omega_8^-(3)$.

Structure $M \cap T$	Order $\lvert M \cap T\rvert$	Index $\lvert G : M\rvert$
$3^6 : 2U_4(3).2$	9523422720	1066
$O_7(3) : 2_1$	9170703360	1107
$O_7(3) : 2_2$	9170703360	1107
$3^{3+6} : (L_3(3) \times 2^2)$	442158912	22960
$3_+^{1+8} : (2S_4 \times A_6)$	340122240	29848
$(4 \times L_4(3)) : 2$	48522240	209223
$S_4 \times U_4(2) : 2_1$	1244160	8159697
$S_4 \times U_4(2) : 2_2$	1244160	8159697
$L_2(81) : 2$	531360	19105632
$(A_6 \times 2(A_4 \times A_4).2).2$	414720	24479091

We extend a well-known conclusion of the 2-(v, k, λ) design as follows.

Lemma 2.5. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be an incidence structure, where $\mathcal{P}$ is the point set, $\mathcal{B}$ is the block set, $|\mathcal{P}| = v$, $|\mathcal{B}| = b$, and any point is contained in at least one block. All blocks have the same size k with $2 < k < v - 1$. A flag is a pair (x, B), where $x \in B \in \mathcal{B}$. Let G be an automorphism group of $\mathcal{D}$. Then the following three statements are equivalent:*

(i) *G is flag-transitive on $\mathcal{D}$;*
(ii) *G is transitive on points, and for some (hence any) point x, the point stabiliser G_x is transitive on the set $B(x)$ of all blocks containing x;*
(iii) *G is transitive on blocks, and for some (hence any) block B, the block stabiliser G_B is transitive on the points of B.*

Proof. Assume (*i*) holds. Then G is flag-transitive on $\mathcal{D}$. For any two points $y, z \in \mathcal{P}$, there exists $g \in G$ mapping the flag (y, B_1) to (z, B_2), and thus g maps y to z. Hence, G is transitive on P.

For the point stabiliser G_x, take any two blocks $B_1, B_2 \in B(x)$. Since G is flag-transitive, there exists $g \in G$ mapping (x, B_1) to (x, B_2). Here, $g \in G_x$, so G_x is transitive on $B(x)$, and (*ii*) holds.

Assume (*ii*) holds. Then G is transitive on $\mathcal{P}$, and G_x is transitive on $B(x)$. For any two blocks $B_1, B_2 \in \mathcal{B}$, take $y \in B_1$ and $z \in B_2$. Since G is transitive on $\mathcal{P}$, there exists $g \in G$ mapping y to z. Let $B_0 = (B_1)^g$, then $z \in B_0$. Since G_z is transitive on $B(z)$, there exists $h \in G_z$ mapping B_0 to B_2. Thus, gh maps B_1 to B_2, so G is transitive on $\mathcal{B}$.

For the block stabiliser G_B, take any $y, z \in B$. Since G is transitive on $\mathcal{P}$, there exists $g \in G$ mapping y to z. Let $B_0 = B^g$ and $z \in B_0$. Since G_z is transitive on $B(z)$, there exists $h \in G_z$ mapping B_0 to B. Then $gh \in G_B$ maps y to z, so G_B is transitive on the points of B, and (*iii*) holds.

Assume (*iii*) holds. Then G is transitive on $\mathcal{B}$, and G_B is transitive on the points of B. Take any two flags (x_1, B_1) and (x_2, B_2). Since G is transitive on $\mathcal{B}$, there exists $g \in G$ mapping B_1 to B_2. Let $x_0 = (x_1)^g \in B_2$. Since G_{B_2} is transitive on B_2, there exists $h \in G_{B_2}$ mapping x_0 to x_2. Thus, gh maps (x_1, B_1) to (x_2, B_2), so G is flag-transitive, and (*i*) holds. □

Lemma 2.6. *Let $\mathcal{D}$ be a symmetric 2-(v, k, λ) design admitting a flag-transitive, point-primitive automorphism group G. Then the following hold:*

(i) $k(k-1) = \lambda(v-1)$*;*
(ii) $k \mid |G_x|$*, where G_x is any point-stabiliser of G.*

Proof. By Lemma 2.2, the result of part (*i*) is derived from the equation $vr = bk$, $\lambda(v-1) = r(k-1)$, combined with the condition $b = v$.

For part (*ii*), by Lemma 2.5, the flag-transitivity of G implies that $r \mid |G_x|$, and since $k = r$, it follows that $k \mid |G_x|$. □

3. Proof of Theorem 1

Suppose G is a flag-transitive, point-primitive automorphism group of a symmetric 2-(v, k, λ) design $\mathcal{D}$, with socle $T = P\Omega_8^{\pm}(3)$. It is known that the order of $T = P\Omega_8^{+}(3)$ is 4,952,179,814,400 while the order of $T = P\Omega_8^{-}(3)$ is 10,151,968,619,520, and $G = P\Omega_8^{\pm}(3)$, $2.P\Omega_8^{\pm}(3)_1$, $2.P\Omega_8^{\pm}(3)_2$, $2.P\Omega_8^{\pm}(3)_3$, or $2^2.P\Omega_8^{\pm}(3)$, where the notation "$2.P\Omega_8^{\pm}(3)_i$" ($i = 1, 2, 3$) means that there are three non-isomorphic groups corresponding to the symbol "$2.P\Omega_8^{\pm}(3)$".

Lemma 3.1. *From Tables 1 and 2, the possible values of the parameter v for the existing 2-designs can be determined.*

Proof. By Lemma 2.1, since G is primitive, G_x, the stabiliser of G for some $x \in P$, is a maximal subgroup of G. Hence, we consider each of the maximal subgroups of G as G_x to search for possible symmetric designs. Tables 1 and 2, given by Lemmas 2.3 and 2.4, list all the potential groups $G_x \cap T$ such that G_x is maximal in the group G.

Since $T \trianglelefteq G$ and G_x is maximal in G, we get $G/T = TG_x/T \cong G_x/(G_x \cap T)$, which implies $|G : G_x| = |T/(G_x \cap T)|$. So, it is not hard to get the index of G_x in G, which is listed in the last column of Tables 1 and 2, and they are the possible values of the parameter v for the existing 2-designs. □

We now present Algorithm 1 to search for potential parameter sets. The output of the algorithm is a list Designs of parameter sequences (v, k, λ) of potential symmetric designs.

Algorithm 1 Search for potential parameter sets.

```
Input: |G_x|, v.
Output: The list Designs := S.
Set S := an empty list;
for each k such that k | |G_x| and 2 < k < v − 1
    λ := k ∗ (k − 1)/(v − 1);
    if λ is an integer
        Add (v, k, λ) to the list S;
return S.
```

3.1. *Proof of Theorem 1 when $T = P\Omega_8^+(3)$*

Algorithm 1 checks all possibilities for any given $\{|G_x|, v\}$ pairs coming from Table 1. We get only one potential parameter set $(v, k, \lambda) = (1080, 416, 160)$.

In this case, Table 1 shows that $G_x \cap T = \Omega_7(3)$. We can get the primitive permutation representations of G acting on the set $\mathcal{P} = \{1, 2, ..., 1080\}$ by using Magma [4] command PrimitiveGroup(1080, i), where $i = 5, 6, 7, 8, 9$.

Lemma 3.2. *In a flag-transitive 2-(v, k, λ) design $\mathcal{D}$, for every block $B \in \mathcal{B}$, there exists at least one orbit O of G_B with size k and $|O^G| = b$.*

Proof. By Lemma 2.5(*iii*), for every block $B \in \mathcal{B}$, the flag-transitivity of G implies that block-stabiliser G_B is transitive on the k points of B. Thus, there exists at least one orbit O of G_B with size k and $|O^G| = b$. □

If $G = P\Omega_8^+(3)$, then G acts as a 1-transitive permutation group on the set $\mathcal{P}$ of 1080 points by GAP [8]. There are six conjugacy classes of subgroups of index $v = 1080$, but none of them has an orbit with size k (= 416) (see Table 3). So there is no such symmetric 2-(v, k, λ) design $\mathcal{D}$ admitting a flag-transitive, point-primitive automorphism group $G = P\Omega_8^+(3)$.

If $G = 2.P\Omega_8^+(3)_1$, $2.P\Omega_8^+(3)_2$, $2.P\Omega_8^+(3)_3$, or $2^2.P\Omega_8^+(3)$, then G acts as a 1-transitive permutation group on the set $\mathcal{P}$ of 1080 points. There are three or four conjugacy classes of subgroups of index $v = 1080$, but none of them has an orbit with size k (see Table 4). So there is no such symmetric 2-(v, k, λ) design $\mathcal{D}$ admitting a flag-transitive, point-primitive automorphism group $G = 2.P\Omega_8^+(3)_1$, $2.P\Omega_8^+(3)_2$, $2.P\Omega_8^+(3)_3$ or $2^2.P\Omega_8^+(3)$.

Table 3. Lengths of orbits of six conjugacy classes.

No.	Lengths of orbits	Reference
1	1, 351, and 728	This subgroup is isomorphic to G_x
2	378 and 702	
3	1080	This subgroup is transitive on 1080 points
4	1080	This subgroup is transitive on 1080 points
5	1080	This subgroup is transitive on 1080 points
6	1080	This subgroup is transitive on 1080 points

Table 4. Lengths of orbits of 3 or 4 conjugacy classes.

No.	G	Lengths of orbits	Reference
1	All	1, 351, and 728	This subgroup is isomorphic to G_x
2	All	378 and 702	
3	All	1080	This subgroup is transitive on 1080 points
4	Except for $2^2.P\Omega_8^+(3)$	1080	This subgroup is transitive on 1080 points

3.2. *Proof of Theorem 1 when $T = P\Omega_8^-(3)$*

Algorithm 1 checks all possibilities for any given $\{|G_x|, v\}$ pair coming from Table 2. We get only one potential parameter set $(v, k, \lambda) = (1066, 640, 384)$.

In this case, Table 2 shows that $G_x \cap T = 3^6{:}2U_4(3).2$. We can get the primitive permutation representations of G acting on the set $\mathcal{P} = \{1, 2, ..., 1066\}$ by using Magma command PrimitiveGroup$(1066, i)$, where $i = 1, 2, 3, 4, 5$.

By Lemmas 2.5 and 3.2, for any block $B \in \mathcal{B}$, the flag-transitivity of G implies that G_B is transitive on the k $(= 640)$ points of B, and there must exist at least one orbit O of G_B with size k and $|O^G| = b$.

If $G = P\Omega_8^-(3)$, then G acts as a 1-transitive permutation group on the set $\mathcal{P}$ of 1066 points by GAP. There is only one conjugacy class of subgroups of index $v = 1066$, but it does not have an orbit with size k $(= 640)$ (lengths of orbits are 1,336 and 729). So there is no such symmetric 2-(v, k, λ) design $\mathcal{D}$ admitting a flag-transitive, point-primitive automorphism group $G = P\Omega_8^-(3)$.

If $G = 2.P\Omega_8^-(3)_1$, $2.P\Omega_8^-(3)_2$, $2.P\Omega_8^-(3)_3$ or $2^2.P\Omega_8^-(3)$, then G acts as a 1-transitive permutation group on the set $\mathcal{P}$ of 1066 points. There are one or three conjugacy classes of subgroups of index v=1066, but none of them has an orbit with size k (lengths of orbits are always 1,336 and 729; see Table 5). So, there is no such symmetric 2-(v, k, λ) design $\mathcal{D}$ admitting a flag-transitive, point-primitive automorphism group $G = 2.P\Omega_8^-(3)_1$, $2.P\Omega_8^-(3)_2$, $2.P\Omega_8^-(3)_3$, or $2^2.P\Omega_8^-(3)$.

Table 5. Lengths of orbits of G_B.

No.	G	Number of conjugacy classes	Lengths of orbits
1	$P\Omega_8^-(3)$	1	1, 336, and 729
2	$2.P\Omega_8^-(3)_1$	3	1, 336, and 729
3	$2.P\Omega_8^-(3)_2$	1	1, 336, and 729
4	$2.P\Omega_8^-(3)_3$	3	1, 336, and 729
5	$2^2.P\Omega_8^-(3)$	3	1, 336, and 729

This completes the proof of Theorem 1.

In the proof of Theorem 1, we further investigated several parameters. Based on the orbital lengths of the block stabiliser G_B, we explore the existence of the 2-block design with nontrivial orbital lengths as block sizes. Given that the number of blocks containing any two distinct points is not constant, the question arises as to how many distinct values this number may assume. Can some double quasi-symmetric designs be constructed?

The following analyzes the cases where the socles of the automorphism group are $P\Omega_8^+(3)$ and $P\Omega_8^-(3)$ in sequence and provides the proofs of Theorems 2 and 3.

4. Proof of Theorem 2

Let G be the primitive group with socle $P\Omega_8^+(3)$ that acts on 1080 points. Taking the primitive group $G = P\Omega_8^+(3)$ as an example, we explore the construction problem of double quasi-symmetric designs.

Lemma 4.1. *When $G = P\Omega_8^+(3)$ acts on 1080 points and the subgroup comes from the first case in Table 3, there are two double quasi-symmetric designs as follows:*

(1) $\mathcal{D}_1 = (\mathcal{P},\mathcal{B}_1)$ with $k_1 = 351$, and both the point intersection numbers and the block intersection numbers are equal to $\{108,126\}$.

(2) $\mathcal{D}_2 = (\mathcal{P},\mathcal{B}_2)$ with $k_2 = 728$, and both the point intersection numbers and the block intersection numbers are equal to $\{484,504\}$.

Proof. There are six conjugacy classes of subgroups of index $v = 1080$, among which one conjugacy class, denoted by H, has orbital lengths of 1,351 and 728 (see Table 3).

The orbits of H acting on $\mathcal{P} = \{1, 2, \ldots, 1080\}$ are

O_1={ 115 };

O_2={ 1, 4, 8, 9, 10, 11, 14, 15, 16, 17, 21, 22, 28, 29, 30, 36, 37, 39, 40, 41, 43, 49, 51, 54, 58, 63, 64, 68, 69, 74, 81, 82, 84, 88, 89, 94, 96, 97, 100, 101, 105, 106, 107, 114, 116, 121, 122, 123, 124, 125, 126, 127, 132, 136, 139, 142, 143, 144, 160, 161, 165, 167, 168, 169, 171, 173, 177, 178, 181, 182, 183, 184, 187, 189, 193, 194, 196, 203, 210, 211, 213, 215, 232, 233, 237, 241, 247, 255, 261, 262, 266, 267, 269, 273, 274, 275, 279, 281, 282, 286, 290, 291, 297, 299, 300, 315, 317, 318, 319, 323, 324, 325, 330, 334, 337, 341, 345, 346, 350, 352, 353, 357, 359, 363, 364, 367, 374, 377, 383, 385, 390, 392, 398, 399, 400, 403, 404, 405, 411, 415, 416, 418, 419, 425, 434, 453, 455, 457, 462, 465, 480, 483, 484, 485, 486, 489, 490, 495, 496, 499, 500, 504, 509, 510, 512, 522, 532, 533, 535, 537, 539, 542, 545, 546, 548, 549, 553, 561, 567, 570, 572, 575, 578, 583, 584, 586, 587, 588, 590, 591, 594, 598, 599, 608, 614, 618, 621, 625, 627, 631, 632, 633, 635, 637, 638, 640, 641, 642, 649, 652, 653, 655, 658, 659, 661, 663, 664, 666, 667, 668, 670, 673, 676, 679, 686, 689, 690, 695, 696, 698, 699, 702, 703, 706, 709, 711, 716, 719, 720, 721, 725, 727, 729, 736, 740, 744, 745, 746, 747, 749, 750, 752, 753, 754, 757, 760, 761, 770, 773, 774, 776, 777, 778, 779, 780, 782, 783, 787, 788, 793, 795, 796, 797, 798, 800, 801, 807, 810, 811, 815, 816, 818, 821, 828, 829, 831, 839, 841, 843, 846, 853, 856, 862, 865, 868, 872, 877, 891, 896, 897, 908, 910, 911, 915, 916, 918, 926, 932, 933, 935, 938, 939, 945, 951, 956, 958, 959, 961, 968, 970, 971, 976, 981, 983, 992, 993, 996, 997, 1002, 1004, 1018, 1019, 1025, 1026, 1027, 1029, 1034, 1037, 1038, 1041, 1042, 1043, 1045, 1048, 1051, 1052, 1053, 1065, 1071, 1074, 1078};

$O_3 = \mathcal{P} - O_1 - O_2$.

By calculation, $|O_2| = 351 = k_1$, $|O_2^G| = 1080 = b_1$, and each 2-subset $\{j,k\}(1 \le j < k \le 1080)$ is incident with 108 or 126 blocks in $\mathcal{B}_1 = O_2^G$, while the number of intersection points between any two different blocks is 108 or 126. In other words, both the point intersection numbers and the block intersection numbers are equal to {108,126}. Each point is contained in exactly 351 of these blocks, so $r_1 = 351$.

Similarly, $|O_3| = 1080 - 1 - 351 = 728 = k_2$, $|O_3^G| = 1080 = b_2$, $r_2 = 728$, and both the point intersection numbers and the block intersection numbers are equal to {484,504}.

Since H acts transitively on the points of O_2 and O_3, respectively, by Lemma 2.5, G acts flag-transitively on $\mathcal{D}_1$ and $\mathcal{D}_2$. □

Lemma 4.2. *When $G = P\Omega_8^+(3)$ acts on 1080 points and the subgroup comes from the second case in Table 3, there are two double quasi-symmetric designs as follows:*

(1) $\mathcal{D}_3 = (\mathcal{P},\mathcal{B}_3)$ with $k_3 = 378$, and both the point intersection numbers and the block intersection numbers are equal to {126,135}.

(2) $\mathcal{D}_4 = (\mathcal{P},\mathcal{B}_4)$ with $k_4 = 702$, and both the point intersection numbers and the block intersection numbers are equal to {450,459}.

Proof. There is one conjugacy class of subgroups with index $v = 1080$, denoted by K, that has orbital lengths of 378 and 702 (See Table 3). The orbits of K acting on $\mathcal{P} = \{1, 2, \ldots, 1080\}$ are

O_1={ 2, 5, 7, 11, 15, 17, 20, 24, 25, 26, 32, 39, 40, 44, 45, 46, 48, 49, 57, 65, 68, 72, 73, 74, 77, 81, 85, 87, 91, 92, 94, 96, 99, 101, 104, 105, 107, 108, 110, 111, 112, 113, 116, 119, 120, 126, 129, 131, 135, 143, 144, 145, 149, 152, 154, 155, 166, 171, 172, 173, 174, 175, 177, 178, 179, 181, 186, 191, 193, 194, 196, 198, 199, 203, 205, 207, 210, 211, 218, 219, 221, 223, 232, 235, 239, 242, 243, 249, 250, 263, 265, 266, 269, 273, 275, 278, 279, 282, 285, 290, 292, 295, 296, 302, 304, 307, 308, 311, 313, 327, 330, 331, 333, 335, 338, 339, 341, 345, 346, 349, 351, 352, 356, 359, 360, 361, 363, 366, 367, 368, 371, 373, 375, 376, 382, 384, 385, 386, 389, 390, 400, 406, 412, 416, 423, 426, 430, 439, 440, 442, 443, 444, 445, 446, 450, 453, 460, 461, 462, 464, 466, 469, 470, 477, 478, 482, 487, 493, 496, 497, 498, 499, 502, 503, 504, 509, 510, 511, 519, 523, 526, 529, 534, 536, 537, 544, 546, 547, 554, 555, 559, 560, 561, 562, 563, 564, 568, 572, 575, 576, 578, 584, 585, 587, 594, 595, 599, 605, 608, 611, 616, 619, 621, 627, 629, 633, 635, 637, 639, 646, 647, 651, 660, 669, 671, 672, 676, 677, 678, 682, 683, 685, 686, 690, 694, 695, 696, 700, 702, 703, 705, 709, 711, 712, 717, 719, 721, 723, 728, 731, 733, 741, 744, 747, 748, 749, 750, 758, 759, 762, 764, 765, 767, 770, 771, 774, 775, 778, 781, 782, 788, 801, 803, 805, 807, 809, 811, 812, 815, 816, 818, 819, 824, 831, 833, 836, 837, 840, 841, 842, 843, 844, 850, 851, 852, 854, 856, 862, 863, 870, 874, 875, 876, 883, 884, 887, 892, 896, 900, 902, 905, 909, 911, 912, 916, 920, 921, 922, 923, 924, 931, 932, 933, 938, 941, 942, 945, 946, 947, 949, 955, 957, 958, 963, 965, 966, 968, 971, 973, 983, 985, 996, 1006, 1008, 1010, 1011, 1015, 1016, 1019, 1021, 1024, 1026, 1027, 1030, 1033, 1034, 1036, 1037, 1038, 1039, 1041, 1043, 1046, 1049, 1051, 1052, 1054, 1057, 1059, 1060, 1062, 1064, 1071, 1072, 1073, 1075, 1077, 1079 };

$O_2 = \mathcal{P} - O_1$.

Through calculation, we can get that $|O_1| = 378 = k_3$, $|O_2| = 702 = k_4$, $|O_1^G| = |O_2^G| = 1080 = b$, $r_3 = 378$, and $r_4 = 702$. Both the point intersection numbers and the block intersection numbers are equal to {126,135} and {450,459}, respectively.

Since K acts transitively on the points of O_1 and O_2, respectively, by Lemma 2.5, G acts flag-transitively on $\mathcal{D}_3$ and $\mathcal{D}_4$. □

Through a similar solution and verification process, we can get that the double quasi-symmetric designs with $G = 2.P\Omega_8^+(3)_1$, $2.P\Omega_8^+(3)_2$, $2.P\Omega_8^+(3)_3$, or $2^2.P\Omega_8^+(3)$ acting on 1080 points also exist. Through comparison, it is found that the block sets of the obtained designs are exactly the same as those given in Lemmas 4.1 and 4.2.

This completes the proof of Theorem 2.

5. Proof of Theorem 3

Let G be the primitive group with socle $P\Omega_8^-(3)$ that acts on 1066 points $\mathcal{P} = \{1, 2, \ldots, 1066\}$.

The proof process is the same as that of Theorem 2 in Section 4, and we list only the base blocks of two double quasi-symmetric designs of this theorem.

O_1={ 1, 4, 6, 7, 8, 10, 14, 16, 17, 18, 20, 24, 26, 27, 28, 30, 32, 34, 35, 37, 38, 46, 49, 52, 59, 60, 61, 64, 66, 75, 77, 81, 82, 84, 85, 89, 97, 105, 109, 112, 118, 120, 122, 127, 129, 133, 138, 145, 149, 150, 158, 159, 161, 162, 163, 164, 165, 174, 179, 180, 181, 184, 186, 191, 192, 194, 195, 196, 197, 198, 200, 202, 209, 212, 214, 217, 221, 223, 225, 229, 234, 235, 237, 239, 240, 244, 245, 246, 247, 257, 263, 265, 267, 274, 276, 279, 282, 295, 298, 303, 305, 310, 312, 318, 322, 326, 332, 333, 334, 335, 336, 337, 344, 349, 350, 352, 354, 358, 359, 361, 362, 363, 364, 366, 367, 373, 376, 378, 380, 383, 385, 386, 390, 394, 395, 398, 399, 401, 402, 403, 404, 412, 416, 418, 420, 426, 428, 429, 432, 445, 450, 456, 462, 473, 474, 475, 476, 480, 485, 486, 488, 490, 492, 493, 494, 495, 501, 503, 505, 506, 508, 513, 514, 515, 516, 520, 525, 527, 534, 536, 538, 541, 543, 547, 552, 553, 556, 564, 566, 567, 568, 570, 574, 576, 579, 580, 582, 583, 585, 586, 587, 588, 590, 594, 596, 601, 602, 607, 608, 612, 613, 619, 621, 624, 628, 633, 634, 638, 651, 659, 661, 662, 668, 669, 672, 674, 679, 681, 682, 688, 690, 691, 693, 694, 700, 701, 705, 707, 708, 710, 717, 719, 723, 730, 732, 733, 736, 749, 750, 751, 752, 754, 765, 767, 778, 779, 781, 782, 785, 789, 791, 795, 801, 805, 807, 811, 813, 815, 819, 823, 825, 829, 831, 844, 846, 851, 854, 863, 872, 874, 877, 881, 883, 884, 890, 891, 893, 897, 902, 907, 909, 913, 914, 915, 919, 920, 924, 927, 929, 934, 937, 942, 945, 947, 951, 956, 958, 972, 974, 985, 986, 992, 993, 994, 997, 998, 1000, 1003, 1008, 1012, 1013, 1018, 1023, 1029, 1030, 1032, 1035, 1036, 1040, 1043, 1051, 1053, 1056, 1058, 1061, 1063};

$O_2 = \mathcal{P} - \{776\} - O_1$. Here, $|O_1| = 336$ and $|O_2| = 729$.

This completes the proof of Theorem 3.

6. Supplementary instruction

A survey of Delsarte's work on the algebraic theory of association schemes was presented by Goethals in [2]. Subsequently, the concept of a partially balanced incomplete block design gained prominence (see [5, 22]).

Definition 6.1. *([22, Definition 9.1]) Let X be a finite set of points, $|X| = v$, and $R_0, R_1, \ldots, R_d$ be nonempty subsets of $X \times X$. Assume the following conditions are satisfied:*

(i) $R_0 = \{(x, x) \mid x \in X\}$.

(ii) $R_i \cap R_j = \emptyset$ for all $i \neq j$ and $X \times X = R_0 \cup R_1 \cup \cdots \cup R_d$.

(iii) For any R_i $(1 \leq i \leq d)$, define $R_i^* = \{(y, x) \mid (x, y) \in R_i\}$. Then for any $i \in \{0, 1, \ldots, d\}$, there is an $i' \in \{0, 1, \ldots, d\}$ such that $R_i^* = R_{i'}$.

(iv) For any $i, j, k \in \{0, 1, \ldots, d\}$ and any pair $(x, y) \in R_k$, the number

$$p_{ij}^k = |\{z \in X \mid (x, z) \in R_i \text{ and } (z, y) \in R_j\}|$$

is independent of the pair $(x, y) \in R_k$.

Then such a structure $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ is called a d-class association scheme, and the i-th relation R_i $(0 \le i \le d)$ is called the i-th associate. In particular, 0-associates are equal.

If, furthermore, the association scheme $\mathcal{X}$ also satisfies the condition

(v) $p_{ij}^k = p_{ji}^k$ for all $i, j, k \in \{0, 1, \ldots, d\}$, then $\mathcal{X}$ is called a commutative association scheme.

If $\mathcal{X}$ satisfies

(vi) $i' = i$ for all $i = 0, 1, \ldots, d$, then $\mathcal{X}$ is called a symmetric association scheme.

Definition 6.2. *([22, Definition 9.2]) Let X be a finite set of points, $|X| = v$, and $\mathcal{B}$ be a finite collection of subsets of X, called blocks, $|\mathcal{B}| = b$. The pair $(X, \mathcal{B})$ is called a partially balanced incomplete block design or, simply, a PBIBD, if the following conditions hold:*

(i) A d-class symmetric association scheme $(X, \{R_i\}_{0 \le i \le d})$ is defined on X.

(ii) Every point is contained in r blocks.

(iii) $|B| = k$ for all $B \in \mathcal{B}$.

(iv) For any $i (i = 1, 2, \ldots, d)$ and any pair of points $(p, q) \in R_i$, there are exactly λ_i blocks containing both p and q.

The parameters $v, r, b, k, \lambda_i, n_i$ and the intersection matrices $P^k = (p_{ij}^k)$ $(i, j, k = 1, 2, \ldots, d)$ are called the parameters of the PBIBD.

When $d = 1$, a PBIBD reduces to a BIBD. When $v = b$, this design is called a symmetric partially balanced incomplete block design (SPBIBD), often denoted by SPBIBD$(v, k, \{\lambda_1, \lambda_2, \ldots, \lambda_d\})$.

Among the six double quasi-symmetric designs obtained in Theorems 2 and 3, four are SPBIBDs with two associate classes after calculating the relevant parameters. The results are stated in the following propositions.

Proposition 6.1. *Let G be a flag-transitive, point-primitive group with socle $P\Omega_8^+(3)$, acting on 1080 points. Then there exist the following two SPBIBDs with two associate classes:*

An SPBIBD$(1080, 351, \{108, 126\})$ and an SPBIBD$(1080, 728, \{484, 504\})$ with $n_1 = 728$, $n_2 = 351$, and the intersection matrices are $P^1 = \left(\begin{smallmatrix} 484 & 243 \\ 243 & 108 \end{smallmatrix}\right)$, $P^2 = \left(\begin{smallmatrix} 504 & 224 \\ 224 & 126 \end{smallmatrix}\right)$.

For the candidate parameters $(v, k, \{\lambda_1, \lambda_2\}) = (1080, 378, \{126, 135\})$ and $(1080, 702, \{450, 459\})$, we attempted to construct the corresponding SPBIBDs using the orbits of the relevant subgroup K as described in Section 4. However, the resulting incidence structures did not satisfy the required association scheme conditions.

Proposition 6.2. *Let G be a flag-transitive, point-primitive group with socle $P\Omega_8^+(3)$, acting on 1080 points. Then there is no SPBIBD with $k = 378$, $\lambda_1 = 126$, $\lambda_2 = 135$. There is also no such design with $k = 702$, $\lambda_1 = 450$, $\lambda_2 = 459$.*

Proposition 6.3. *Let G be a flag-transitive, point-primitive group with socle $P\Omega_8^-(3)$, acting on 1066 points. Then there exist the following two partially balanced incomplete block designs with two associate classes:*

An SPBIBD$(1066, 336, \{92, 112\})$ and an SPBIBD$(1066, 729, \{486, 504\})$ with $n_1 = 336$, $n_2 = 729$, and the intersection matrices are $P^1 = \left(\begin{smallmatrix} 92 & 243 \\ 243 & 486 \end{smallmatrix}\right)$, $P^2 = \left(\begin{smallmatrix} 112 & 224 \\ 224 & 504 \end{smallmatrix}\right)$.

7. Conclusions and open questions

In this paper, we studied the classification problem of flag-transitive, point-primitive symmetric 2-(v, k, λ) designs with 8-dimensional projective orthogonal group $P\Omega_8^{\pm}(3)$ as the socle. Subsequently, we explored the existence of double quasi-symmetric designs and obtained six pairwise non-isomorphic examples. Research on double quasi-symmetric designs is currently limited; this paper offers a new approach to their study.

Under the action of the primitive group G of the flag-transitive, point-primitive with $P\Omega_8^{\pm}(3)$ as the socle, the following open questions are raised. These problems are particularly difficult.

Problem 1. Can one determine all the non-symmetric designs admitting such a group G?

Problem 2. Can one determine all the quasi-symmetric designs admitting such a group G?

It is noteworthy that, in all six double quasi-symmetric designs obtained, the set of point intersection numbers coincides with the set of block intersection numbers. A natural question is:

Problem 3. Is there a situation where the point intersection numbers and the block intersection numbers of a proper double quasi-symmetric design are not equal?

With respect to the relationship between double quasi-symmetric designs and SPBIBDs, we pose the following question:

Problem 4. What is the precise relationship between double quasi-symmetric designs and SPBIBDs? In particular, under what conditions does a double quasi-symmetric design become an SPBIBD and vice versa?

Author contributions

Delu Tian: conceptualization, writing-original draft, software, formal analysis; Qianfen Liao: methodology, data curation, formal analysis; Zhilin Zhang: supervision, resources, reviewing and editing; Haiyan Guan: conceptualization, data curation, investigation, writing; Xingyu Chen: writing-original draft, resources, writing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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