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***Research article***

# Forced waves and gap formations for a three-species competitive-cooperative system with nonlocal dispersal and shifting habitats

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**Abstract:** This paper is concerned with the existence of forced waves and gap formations for a three-species competitive-cooperative system with nonlocal dispersal under habitat shifting at a constant speed, in which two species cooperate with each other while simultaneously competing with the third species together. First, the existence of forced waves is established by means of iterative techniques, where the speed lies within a specified interval. In the existence part, kernel functions are asymmetric, and the growth functions are allowed to be sign-preserving. Then, under the conditions that the kernel functions are symmetric and that the growth functions change sign, the gap formations are derived via a comparison principle.

**Keywords:** competitive-cooperative system; nonlocal dispersal; shifting habitat; forced waves; gap formations
**Mathematics Subject Classification:** 35K57, 35C07, 92D40

## 1. Introduction

In recent years, many authors have extensively studied diffusion in shifting habitats from various perspectives. For example, considerable interest has been devoted to the following equation:

$$u_t = d\Delta u + f(x - ct, u), \tag{1.1}$$

where $c$ denotes the speed of the shifting habitat edge, and the Laplacian operator $\Delta u$ represents the diffusion which occurs only between adjacent spatial locations. There are many results about (1.1) with various nonlinear terms; see [1,3,6]. A key question is whether a species can shift its range in step with the moving climate, or whether a pathogen can spread fast enough to keep up with its invading host. In reality, we often describe the propagation of species through nonlocal diffusion equations, where the nonlocal dispersal term is denoted by an integral operator

$$[J * u](t,x) - u(t,x) = \int_{\mathbb{R}} J(y)u(t,x-y)\mathrm{d}y - u(t,x).$$

In mathematical biology, nonlocal dispersal typically captures long-range effects, such as infectious diseases jumping between countries via air travel; see [9, 11, 22] and the references therein. In recent years, there have been many conclusions about combining nonlocal diffusion with shifting habitats to study dynamics. For the case of considering a species, Li, Wang, and Zhao [10] investigated the persistence criterion and existence of forced waves for the following population model

$$u_t(t,x) = d(J * u - u)(t,x) + u(t,x)[r(x-ct) - u(t,x)], \tag{1.2}$$

where the shifting speed $c > 0$, functions $J(\cdot)$ and $r(\cdot)$ satisfy the following conditions:

(J) $J \in C(\mathbb{R}, \mathbb{R}^+)$ is symmetric and compactly supported, and $\int_{\mathbb{R}} J(x)\mathrm{d}x = 1$;
(H) $r(\xi) \in C(\mathbb{R})$ is nondecreasing, $r(\pm\infty)$ are finite, and $r(-\infty) < 0 < r(\infty)$.

Under the conditions (J) and (H), Wang and Zhao [16] obtained the uniqueness by using the sliding technique and the stability of forced waves for (1.2) by applying the monotone semiflows approach. Later, Yan et al. [18] obtained the globally exponential stability of forced waves for (1.2) by applying the Fourier transform and weighted energy estimate. When under the following weaker condition,

($\mathrm{H}^*$) $r(\xi) \in C(\mathbb{R})$ is nondecreasing, $r(\pm\infty)$ are finite and $r(-\infty) \leq 0 < r(\infty)$,

by introducing a novel approach based on semigroup theory, Hu, Yi, and Zou [8] obtained some results on the spatial-temporal dynamics for (1.1).

As is well known, competition and cooperation between species are fundamental relationships in population biology. For mathematicians and ecologists, studying the dynamics of biological population systems has always been an interesting topic, and many achievements have been made in this area, such as [13, 19, 20] and their references. In this aspect, Wang and Li [14] obtained the existence and tail behaviors of forced waves for the following nonlocal dispersal Lotka-Volterra cooperation model under shifting habitat:

$$\begin{cases} u_t = d_1(J_1 * u - u) + u[r_1(x-ct) - u + a_1 v], \\ v_t = d_2(J_2 * v - v) + v[r_2(x-ct) + a_2 u - v], \end{cases} \tag{1.3}$$

where the growth functions $r_i (i = 1,2)$ satisfy (H), and the requirements that the kernel functions be compactly supported and symmetric have been removed. Subsequently, by constructing a pair of super- and sub-solutions and using the method of monotone iteration, Hu et al. [7] proved the existence of forced waves with any positive constant shifting speed for (1.3), where $J_i$ satisfies (J), and $r_i$ satisfies ($\mathrm{H}^*$), $i = 1,2$. For a Lotka-Volterra competition system under climate change, Wu et al. [17] identified certain ranges of the worsening speed $c$ by analyzing the spatial-temporal dynamics, where the kernel functions satisfy (J), and growth functions satisfy (H). Berestycki [2] proved the existence of a nontrivial forced wave and investigated the gap formation caused by the shifting habitats for a Lotka-Volterra competition model with Laplacian operator. Then, similar conclusions have been proved by Wang et al. [12] for a lattice Lotka-Volterra competition system under climate change. Later, Wang and Wu [15] extended these results to a Lotka-Volterra competition model with nonlocal dispersal and shifting habitats. Very recently, Zhou and Wu [23] investigated the spreading properties for a predator-prey system with nonlocal dispersal and climate change.

As far as we know, little work has been devoted to three species systems with nonlocal dispersal in shifting environments. Inspired by these previous works, in this paper, we consider the following three species competitive-cooperative system under climate change,

$$\begin{cases} u_t = d_1(J_1 * u - u) + b_1 u[r_1(x-ct) - u - a_{12}w_1 - a_{13}w_2], \\ w_{1_t} = d_2(J_2 * w_1 - w_1) + b_2 w_1[r_2(x-ct) - a_{21}u - w_1 + a_{23}w_2], \\ w_{2_t} = d_3(J_3 * w_2 - w_2) + b_3 w_2[r_3(x-ct) - a_{31}u + a_{32}w_1 - w_2], \end{cases} \tag{1.4}$$

where $t \geq 0, x \in \mathbb{R}$, $d_i, b_i, a_{ij}$ are positive constants for $i \neq j, i, j \in \{1,2,3\}$. Functions $r_i(x-ct)(i = 1,2,3)$ are growth rates, where $c \in \mathbb{R}$ is the speed of the shifting habitat edge. $w_1(t,x)$ and $w_2(t,x)$ represent the population densities of two species which live together in a mutually beneficial way, and both species compete with another species, whose density denoted by $u(t,x)$. Throughout this paper, we give some assumptions as follows.

(J$_1$) $J_i \in C(\mathbb{R}),\ J_i(x) \geq 0,\ x \in \mathbb{R},\ J_i(0) > 0,\ \int_{\mathbb{R}} J_i(x)\mathrm{d}x = 1$ and $\int_{\mathbb{R}} J_i(x)e^{\lambda x}\mathrm{d}x < \infty$ for any $\lambda > 0, i = 1,2,3$.

(H$_1$) $r_1(\cdot) \in C(\mathbb{R})$ is nonincreasing with $-\infty < r_1(\infty) \leq 0 < r_1(-\infty) < \infty$ and for $i = 2,3$, $r_i(\cdot) \in C(\mathbb{R})$ are nondecreasing with $-\infty < r_i(-\infty) \leq 0 < r_i(\infty) < \infty$. For convenience, denote $l_1 := r_1(-\infty),\ l_i := r_i(\infty),\ i = 2,3$.

(J$_2$) $c_1^* > \max\left\{-c_2^*, -c_3^*\right\}$, where

$$c_i^* = \inf_{\lambda>0} \frac{d_i\left(\int_{\mathbb{R}} J_i(y)e^{\lambda y}\mathrm{d}y - 1\right) + b_i l_i}{\lambda},\ i = 1,2,3.$$

(H$_2$) $0 < a_{ij} < 1$ for $i, j \in \{1,2,3\},\ i \neq j$.

**Remark 1.1.** *Let*

$$f_i(\lambda) = \frac{d_i\left(\int_{\mathbb{R}} J_i(y)e^{\lambda y}\mathrm{d}y - 1\right) + b_i l_i}{\lambda},\ i = 1,2,3.$$

*By direct calculations, we have $f_i(\lambda) \to \infty$ as $\lambda \to 0^+$, and if there exists a $\lambda_0 > 0$ such that $f_i'(\lambda_0) = 0$, then there is $f_i''(\lambda_0) > 0$. There are three cases as follows.*

(1) *If $J_i(y) \not\equiv 0$ in $\mathbb{R}^+$, $f_i(\lambda) \to \infty$ as $\lambda \to \infty$, which means there exists a unique $\lambda_i > 0$ such that*

$$f_i(\lambda_i) = \min_{\lambda>0} f_i(\lambda).$$

(2) *If $J_i(y) \equiv 0$ in $\mathbb{R}^+$, and $b_i l_i < d_i$, then $f_i(\lambda) \to 0^-$ as $\lambda \to \infty$ and $f_i'(\lambda) > 0$ for large enough $\lambda$, which means there exists a unique $\lambda_i > 0$ such that*

$$f_i(\lambda_i) = \min_{\lambda>0} f_i(\lambda).$$

(3) *If $J_i(y) \equiv 0$ in $\mathbb{R}^+$, and $b_i l_i \geq d_i$, then $f_i(\lambda) \to 0^+$ as $\lambda \to \infty$, and $f_i{}'(\lambda) < 0$ for any $\lambda > 0$, which means*

$$\lim_{\lambda\to\infty} f_i(\lambda) = 0;$$

*that is, $c_i^* = 0$. Moreover, the inverse function of $f_i(\lambda)$ exists.*

*Furthermore, it is worth noting that $c_i^* > 0$ whenever $J_i$ is symmetric. However, it is possible that $c_i^* \leq 0$ when $J_i$ is asymmetric; see [5]. Hence, if $J_i$ is symmetric, the condition $(J_2)$ can be removed, $i = 1, 2, 3$.*

**Remark 1.2.** *Let $d_1 = d_2 = d_3 = 1,\ b_1 = 0.5,\ b_2 = b_3 = 0.8,\ r_1(x-ct) = -\frac{4}{\pi}\arctan(x-ct),\ r_2(x-ct) = \frac{2}{\pi}\arctan(x-ct),\ r_3(x-ct) = \frac{3}{\pi}\arctan(x-ct),$*

$$J_1(y) = \frac{1}{\sqrt{2\pi}} e^{-\dfrac{(y-1)^2}{2}},\ y \in \mathbb{R},$$
$$J_2(y) = J_3(y) = \begin{cases} 2(y+1), & y \in [-1, 0], \\ 0, & \text{elsewhere.} \end{cases}$$

*By calculations, $c_1^* \approx 3.63,\ c_2^* \approx -0.00528,\ c_3^* \approx 0.$*

For (1.4), we are interested in a forced wave connecting $E_1$ and $E_2$, which is a solution of the form

$$(u(t,x), w_1(t,x), w_2(t,x)) = (\varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi)),\ \xi := x - ct$$

satisfying

$$\begin{cases} d_1\left[\int_{\mathbb{R}} J_1(y)\varphi_1(\xi-y)\mathrm{d}y - \varphi_1\right] + c\varphi_1' + b_1\varphi_1\left[r_1(\xi) - \varphi_1 - a_{12}\varphi_2 - a_{13}\varphi_3\right] = 0, \\ d_2\left[\int_{\mathbb{R}} J_2(y)\varphi_2(\xi-y)\mathrm{d}y - \varphi_2\right] + c\varphi_2' + b_2\varphi_2\left[r_2(\xi) - a_{21}\varphi_1 - \varphi_2 + a_{23}\varphi_3\right] = 0, \\ d_3\left[\int_{\mathbb{R}} J_3(y)\varphi_3(\xi-y)\mathrm{d}y - \varphi_3\right] + c\varphi_3' + b_3\varphi_3\left[r_3(\xi) - a_{31}\varphi_1 + a_{32}\varphi_2 - \varphi_3\right] = 0, \end{cases} \tag{1.5}$$

and

$$(\varphi_1, \varphi_2, \varphi_3)(-\infty) = E_1,\ (\varphi_1, \varphi_2, \varphi_3)(+\infty) = E_2, \tag{1.6}$$

where

$$E_1 := (k_1, 0, 0) = (l_1, 0, 0),$$
$$E_2 := (0, k_2, k_3) = \left(0, \frac{l_2 + a_{23}l_3}{1 - a_{23}a_{32}}, \frac{a_{32}l_2 + l_3}{1 - a_{23}a_{32}}\right).$$

In what follows we use bold letters to represent vectors in $\mathbb{R}^3$. For example, $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$ for any $a_i \in \mathbb{R}, i = 1, 2, 3$. Let $C$ be the set of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and $\mathcal{X}$ be the set of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}^3$. We equip $\mathcal{X}$ with the compact-open topology. For $\mathbf{\Phi} = (\phi_1, \phi_2, \phi_3), \mathbf{\Psi} = (\psi_1, \psi_2, \psi_3) \in \mathcal{X}$, we define $\mathbf{\Phi} \leq \mathbf{\Psi}$ to mean that $\phi_i(x) \leq \psi_i(x), i = 1, 2, 3,\ \forall x \in \mathbb{R}$. It is obvious that $\mathcal{X}_+ = \{\mathbf{\Phi} = (\phi_1, \phi_2, \phi_3) \in \mathcal{X} : \phi_i(x) \geq 0, i = 1, 2, 3\}$ is a closed cone of $\mathcal{X}$. Let $\boldsymbol{k} := (k_1, k_2, k_3)$, we define $\mathcal{X}_{\boldsymbol{k}} := \{\mathbf{u} \in \mathcal{X} : \mathbf{0} \leq \mathbf{u} \leq \boldsymbol{k}\}$.

The concept of gap formation was first introduced by Berestycki, Desvillettes, and Diekmann [2] in a two-species reaction-diffusion competition model under climate change. They showed that when the speed $c$ of the shifting habitat exceeds a critical value $c_0$ (the Fisher-KPP invasion speed of the advancing species), an expanding spatial region appears where both species are at very low densities, the retreating species has been suppressed while the advancing one has not yet established. This phase separation phenomenon is termed a gap. Subsequently, Wang et al. [15] extended this idea to a nonlocal dispersal Lotka-Volterra competition system with symmetric kernels, proving gap formation for $c > c_1^*$ or $c < -c_2^*$. In the present three-species cooperative-competitive system, we demonstrate that, under the assumptions of symmetric kernels and sign-changing growth functions, gaps also emerge.

Because individual dispersal can be shaped by so many different factors such as wind and landscape structure, it is natural to ask what happens when the dispersal kernel is asymmetric. In Section 2, we assume that the kernel functions are asymmetric and that the growth functions may be sign-preserving. By applying the iterative technique, we prove that there exists a forced wave of (1.4) when the speed $c$ lies within a specified interval. Moreover, we consider the case of $c \in \mathbb{R}$. In Section 3, under the conditions that the kernel functions are symmetric and that the growth functions change sign, gap formations are established. Therefore, when the growth functions do not change sign, new methods are needed to establish gap formations, which will be addressed in the future.

## 2. Forced waves

In this section, we mainly focus our attention on the existence of forced waves connecting $E_1$ and $E_2$ for (1.4).

**Theorem 2.1.** *For any $c < c_1^*$, the equation*

$$d_1\left[\int_{\mathbb{R}} J_1(y)\phi_1(\xi-y)\mathrm{d}y - \phi_1(\xi)\right] + c\phi_1'(\xi) + b_1\phi_1(\xi)\left[r_1(\xi) - \phi_1(\xi)\right] = 0, \tag{2.1}$$

*admits a nonincreasing solution $\phi_1(\xi)$ satisfying $\phi_1(-\infty) = l_1$ and $\phi_1(\infty) = 0$. Moreover, we can consider $\phi_1$ as a function of $r_1$, which is nondecreasing with respect to $r_1$.*

*Proof.* We begin by constructing a subsolution to (2.1). For any given small $\epsilon > 0$, consider the following equation

$$u_{i_t}(t,x) = d_i\left(\tilde{J}_i * u_i - u_i\right)(t,x) + f_{i\epsilon}(u_i(t,x)),\ i = 1,2,3, \tag{2.2}$$

where $\tilde{J}_1(x) = J_1(x), \tilde{J}_i(x) = J_i(-x), i = 2,3, x \in \mathbb{R}$, and

$$f_{i\epsilon}(u_i) = \begin{cases} b_i u_i(l_i - \epsilon - u_i), & \text{if } u_i \geq 0, \\ 0, & \text{if } -\epsilon \leq u_i < 0. \end{cases}$$

By [4], there exists a nonincreasing traveling wave solution $\phi_{i\epsilon}(\xi)(\xi = x - c_{i\epsilon}t)$ connecting $l_i - \epsilon$ to $-\epsilon$ with a speed $c_{i\epsilon} \in \mathbb{R}$ for (2.2), which satisfies

$$\begin{cases} d_i\left[\int_{\mathbb{R}} \tilde{J}_i(y)\phi_{i\epsilon}(\xi-y)\mathrm{d}y - \phi_{i\epsilon}(\xi)\right] + c_{i\epsilon}\phi_{i\epsilon}'(\xi) + f_{i\epsilon}\left(\phi_{i\epsilon}(\xi)\right) = 0,\ i = 1,2,3, \\ \phi_{i\epsilon}(-\infty) = l_i - \epsilon,\ \phi_{i\epsilon}(\infty) = -\epsilon. \end{cases} \tag{2.3}$$

By similar arguments in [10, 14], we can prove that the wave speed $c_{i\epsilon}$ in (2.3) satisfies

$$\lim_{\epsilon \to 0^+} c_{i\epsilon} = c_i^*, \tag{2.4}$$

where $c_i^*$ defined in $(J_2)$ is the minimal wave speed of the monotone traveling wave connecting $l_i$ to 0 of the following homogeneous equation ( [5]):

$$u_{i_t}(t,x) = d_i\left(\tilde{J}_i * u_i - u_i\right)(t,x) + b_i u_i(t,x)[l_i - u_i(t,x)],\ i = 1,2,3.$$

Set

$$\underline{\phi}_1(\xi) = \max\{\phi_{1\epsilon}(\xi), 0\},$$

where $\phi_{1\epsilon}(\xi)$ is the solution of (2.3) with $i = 1$. By fixing a proper small $\epsilon$, there exists a $\xi_1 \ll -1$ such that $r_1(\xi_1) \geq l_1 - \epsilon$. For this $\xi_1$, we can further assume that $\phi_{1\epsilon}(\xi_1) = 0$, because $\phi_{1\epsilon}(\xi)$ remains a solution to variable $\xi$ even after translation. Moreover, it follows from (2.4) that for sufficiently small $\epsilon > 0$, we obtain $c < c_{1\epsilon}$ if $c < c_1^*$. Hence, if $\xi < \xi_1$, $\underline{\phi}_1(\xi) = \phi_{1\epsilon}(\xi)$, we have

$$\begin{aligned}
&d_1\left[\int_{\mathbb{R}} J_1(y)\underline{\phi}_1(\xi - y)\mathrm{d}y - \underline{\phi}_1(\xi)\right] + c\underline{\phi}_1'(\xi) + b_1\underline{\phi}_1(\xi)\left[r_1(\xi) - \underline{\phi}_1(\xi)\right] \\
&\geq d_1\left[\int_{\mathbb{R}} J_1(y)\phi_{1\epsilon}(\xi - y)\mathrm{d}y - \phi_{1\epsilon}(\xi)\right] + c_{1\epsilon}\phi_{1\epsilon}'(\xi) + b_1\phi_{1\epsilon}(\xi)\left[l_1 - \epsilon - \phi_{1\epsilon}(\xi)\right] \\
&= 0.
\end{aligned}$$

If $\xi \geq \xi_1$, $\underline{\phi}_1(\xi) = 0$, then we have

$$d_1\left[\int_{\mathbb{R}} J_1(y)\underline{\phi}_1(\xi - y)\mathrm{d}y - \underline{\phi}_1(\xi)\right] + c\underline{\phi}_1'(\xi) + b_1\underline{\phi}_1(\xi)\left[r_1(\xi) - \underline{\phi}_1(\xi)\right] \geq 0.$$

Thus, $\underline{\phi}_1(\xi)$ is a subsolution of (2.1). Let $\bar{\phi}_1(\xi) \equiv l_1$, it is obvious that for any $c < c_1^*$, $\bar{\phi}_1(\xi)$ is a supersolution of (2.1), and $\underline{\phi}_1(\xi) \leq \bar{\phi}_1(\xi)$ for all $\xi \in \mathbb{R}$.

Let

$$\mathcal{A} := \left\{\phi_1 \in C : \underline{\phi}_1 \leq \phi_1 \leq \bar{\phi}_1\right\}.$$

Define $F_1 : \mathcal{A} \to C(\mathbb{R},\mathbb{R})$ by

$$F_1(\phi_1(\xi)) := \rho_1\phi_1(\xi) + d_1\left[\int_{\mathbb{R}} J_1(y)\phi_1(\xi - y)\mathrm{d}y - \phi_1(\xi)\right] + b_1\phi_1(\xi)\left[r_1(\xi) - \phi_1(\xi)\right],$$

where $\rho_1 > d_1 - b_1(r_1(\infty) - 2l_1)$. Then (2.1) can be written as

$$-c\phi_1'(\xi) = -\rho_1\phi_1(\xi) + F_1(\phi_1(\xi)). \tag{2.5}$$

Because $\phi_1 \in \mathcal{A}$, then $0 \leq \phi_1(\xi) \leq l_1$ for $\forall\xi \in \mathbb{R}$ and $F_1(\phi_1(\xi))$ is bounded and continuous with $\phi_1 \in \mathcal{A}$. For $\phi_1 \in \mathcal{A}$, define the operator

$$\mathcal{P}_1[\phi_1](\xi) = \begin{cases} \frac{1}{c}\int_\xi^\infty e^{\frac{\rho_1}{c}(\xi - s)}F_1(\phi_1(s))\mathrm{d}s, & \text{if } c > 0, \\ \frac{1}{\rho_1}F_1(\phi_1(\xi)), & \text{if } c = 0, \\ -\frac{1}{c}\int_{-\infty}^\xi e^{\frac{\rho_1}{c}(\xi - s)}F_1(\phi_1(s))\mathrm{d}s, & \text{if } c < 0, \end{cases}$$

which suggests that the solution $\phi_1$ of (2.1) connecting $l_1$ to 0 is a fixed point of the operator $\mathcal{P}_1$. It is easy to prove that $\mathcal{P}_1$ is a nondecreasing operator and maps $\mathcal{A}$ to $\mathcal{A}$. Moreover, when $\phi_1(\xi)$ is nonincreasing with $\xi$, then $\mathcal{P}_1[\phi_1](\xi)$ is also nonincreasing with $\xi$. Similar to the proof process of Theorem 4.5 in [10], the existence result can be obtained.

Finally, we prove that $\phi_1$ satisfies the boundary conditions. Because $\phi_1$ is monotone and bounded with $\xi$, then $\phi_1(\pm\infty)$ exist. Suppose $p_1^- := \phi_1(-\infty)$, then we obtain

$$b_1 p_1^-(l_1 - p_1^-) = 0,$$

which means $p_1^- = 0$ or $p_1^- = l_1$. If $p_1^- = 0$, by the monotonicity of $\phi_1$, there must be $\phi_1 \equiv 0$, which contradicts $\phi_1 \geq \underline{\phi}_1$. Hence, it holds that $\phi_1(-\infty) = l_1$. Similarly, we can prove $\phi_1(\infty) = 0$. □

Next, we introduce the existence of solutions for the following Lotka-Volterra cooperative system:

$$\begin{cases} d_2\left[\int_{\mathbb{R}} J_2(y)\phi_2(\xi-y)\mathrm{d}y - \phi_2\right] + c\phi_2' + b_2\phi_2\left[r_2(\xi) - a_{21}\phi_1 - \phi_2 + a_{23}\phi_3\right] = 0, \\ d_3\left[\int_{\mathbb{R}} J_3(y)\phi_3(\xi-y)\mathrm{d}y - \phi_3\right] + c\phi_3' + b_3\phi_3\left[r_3(\xi) - a_{31}\phi_1 + a_{32}\phi_2 - \phi_3\right] = 0, \end{cases} \tag{2.6}$$

where $\phi_1(\xi)$ is a known continuous and nonincreasing function satisfying $\phi_1(-\infty) = l_1$ and $\phi_1(\infty) = 0$.

**Lemma 2.2.** *Assume that* $(J_1), (H_1)$ *hold. Let* $c_0^* = \max\left\{-c_2^*, -c_3^*\right\}$*, and*

$$\underline{\chi}_i(\xi) = \max\left\{\phi_{i\epsilon}(\xi), 0\right\},\ i = 2, 3,$$

*where* $\phi_{i\epsilon}(\xi)$ *is the solution of* (2.3). *Define*

$$\underline{\phi}_i(\xi) = \underline{\chi}_i(-\xi),\ i = 2, 3.$$

*Then for any* $c > c_0^*$, $\left(\underline{\phi}_2(\xi), \underline{\phi}_3(\xi)\right)$ *is a subsolution of* (2.6).

*Proof.* Let $(\chi_2(\xi), \chi_3(\xi)) = (\phi_2(-\xi), \phi_3(-\xi)), \xi \in \mathbb{R}$. It follows from (2.6) that $(\chi_2(\xi), \chi_3(\xi))$ satisfies

$$\begin{cases} d_2\left[\int_{\mathbb{R}} J_2(-y)\chi_2(\xi-y)\mathrm{d}y - \chi_2\right] - c\chi_2' + b_2\chi_2\left[r_2(-\xi) - a_{21}\phi_1(-\xi) - \chi_2 + a_{23}\chi_3\right] = 0, \\ d_3\left[\int_{\mathbb{R}} J_3(-y)\chi_3(\xi-y)\mathrm{d}y - \chi_3\right] - c\chi_3' + b_3\chi_3\left[r_3(-\xi) - a_{31}\phi_1(-\xi) + a_{32}\chi_2 - \chi_3\right] = 0. \end{cases} \tag{2.7}$$

Now, we prove $\left(\underline{\chi}_2(\xi), \underline{\chi}_3(\xi)\right)$ is a subsolution of (2.7). For $i = 2, 3$, by fixing a proper small $\epsilon$, there exists a $\xi_i \ll -1$ such that $r_i(-\xi_i) \geq l_i - \epsilon/2$, and $\phi_1(-\xi_i) < \epsilon/2a_{i1}$. For this $\xi_i$, we can further assume that $\varphi_{i\epsilon}(\xi_i) = 0$, as $\phi_{i\epsilon}(\xi)$ remains a solution to variable $\xi$ even after translation. Moreover, it follows from (2.4) that for sufficiently small $\epsilon > 0$, we obtain $-c < c_{i\epsilon}$ if $c > -c_i^*$. Hence, if $\xi < \xi_i$, $\underline{\chi}_i(\xi) = \phi_{i\epsilon}(\xi)$, for $i, j \in \{2, 3\}, i \neq j$, it holds that

$$\begin{aligned} & d_i\left[\int_{\mathbb{R}} J_i(-y)\underline{\chi}_i(\xi-y)\mathrm{d}y - \underline{\chi}_i(\xi)\right] - c\underline{\chi}_i'(\xi) + b_i\underline{\chi}_i(\xi)\left[r_i(-\xi) - a_{i1}\phi_1(-\xi) - \underline{\chi}_i(\xi) + a_{ij}\underline{\chi}_j(\xi)\right] \\ & \geq d_i\left[\int_{\mathbb{R}} J_i(-y)\phi_{i\epsilon}(\xi-y)\mathrm{d}y - \phi_{i\epsilon}(\xi)\right] + c_{i\epsilon}\phi_{i\epsilon}'(\xi) + b_i\phi_{i\epsilon}(\xi)\left[l_i - \epsilon - \phi_{i\epsilon}(\xi)\right] \\ & = 0. \end{aligned}$$

If $\xi \geq \xi_i, \underline{\chi}_i(\xi) = 0$, then for $i, j \in \{2, 3\}, i \neq j$, we have

$$d_i\left[\int_{\mathbb{R}} J_i(-y)\underline{\chi}_i(\xi-y)\mathrm{d}y - \underline{\chi}_i(\xi)\right] - c\underline{\chi}_i'(\xi) + b_i\underline{\chi}_i(\xi)\left[r_i(-\xi) - a_{i1}\phi_1(-\xi) - \underline{\chi}_i(\xi) + a_{ij}\underline{\chi}_j(\xi)\right] \geq 0.$$

Thus, $\left(\underline{\chi}_2(\xi), \underline{\chi}_3(\xi)\right)$ is a subsolution of (2.7). Since $\underline{\phi}_i(\xi) = \underline{\chi}_i(-\xi),\ i = 2, 3$, we can verify that

$$\begin{cases} d_2\left[\int_{\mathbb{R}} J_2(y)\underline{\phi}_2(\xi - y)\mathrm{d}y - \underline{\phi}_2\right] + c\underline{\phi}_2' + b_2\underline{\phi}_2\left[r_2(\xi) - a_{21}\phi_1 - \underline{\phi}_2 + a_{23}\underline{\phi}_3\right] \geq 0,\ \xi \neq -\xi_2, \\ d_3\left[\int_{\mathbb{R}} J_3(y)\underline{\phi}_3(\xi - y)\mathrm{d}y - \underline{\phi}_3\right] + c\underline{\phi}_3' + b_3\underline{\phi}_3\left[r_3(\xi) - a_{31}\phi_1 + a_{32}\underline{\phi}_2 - \underline{\phi}_3\right] \geq 0,\ \xi \neq -\xi_3. \end{cases}$$

□

On the other hand, it is easy to verify that $(\bar{\phi}_2(\xi), \bar{\phi}_3(\xi)) := (k_2, k_3)$ is a supersolution of (2.6) and $\underline{\phi}_i \leq \bar{\phi}_i, i = 2, 3$. Let

$$\mathcal{B} := \left\{(\phi_2, \phi_3) : \underline{\phi}_i \leq \phi_i \leq \bar{\phi}_i,\ \phi_i \in C,\ i = 2, 3\right\}.$$

For $i, j \in \{2, 3\}, i \neq j$, define $F_i : \mathcal{B} \to C(\mathbb{R}, \mathbb{R})$ by

$$F_i(\phi_i, \phi_j)(\xi) := \mu_i\phi_i(\xi) + d_i\left[\int_{\mathbb{R}} J_i(y)\phi_i(\xi - y)\mathrm{d}y - \phi_i(\xi)\right] + b_i\phi_i(\xi)\left[r_i(\xi) - a_{i1}\phi_1(\xi) - \phi_i(\xi) + a_{ij}\phi_j(\xi)\right],$$

where $\mu_i > \max\{d_i - b_i[r_i(-\infty) - a_{i1}l_1 - 2k_i]\}$. Then (2.6) can be written as

$$-c\phi_i'(\xi) = -\mu_i\phi_i(\xi) + F_i(\phi_i, \phi_j)(\xi). \tag{2.8}$$

Noting that $F_i(\phi_i, \phi_j)$ is bounded and continuous with $\left(\phi_i, \phi_j\right) \in \mathcal{B}$. Then, for $(\phi_i, \phi_j) \in \mathcal{B}$ and $i, j \in \{2, 3\}, i \neq j$, define the operator

$$\mathcal{P}_i[\phi_i, \phi_j](\xi) = \begin{cases} \frac{1}{c}\int_{\xi}^{\infty} e^{\frac{\mu_i}{c}(\xi - s)}F_i(\phi_i, \phi_j)(s)\mathrm{d}s, & \text{if } c > 0, \\ \frac{1}{\mu_i}F_i(\phi_i, \phi_j)(\xi), & \text{if } c = 0, \\ -\frac{1}{c}\int_{-\infty}^{\xi} e^{\frac{\mu_i}{c}(\xi - s)}F_i(\phi_i, \phi_j)(s)\mathrm{d}s, & \text{if } c < 0, \end{cases} \tag{2.9}$$

which suggests that the solution $(\phi_2, \phi_3)$ of (2.6) connecting $(0, 0)$ to $(k_2, k_3)$ is a fixed point of the operator $\mathcal{P} := (\mathcal{P}_2, \mathcal{P}_3)$.

Now, we present some properties of operator $\mathcal{P}$.

**Lemma 2.3.** *$\mathcal{P}$ is nondecreasing with $(\phi_2, \phi_3) \in \mathcal{B}$ and maps $\mathcal{B}$ to $\mathcal{B}$. Further, if $(\phi_2, \phi_3) \in \mathcal{B}$ is nondecreasing with $\xi \in \mathbb{R}$, then $\mathcal{P}[\phi_2, \phi_3](\xi)$ is also nondecreasing with $\xi \in \mathbb{R}$.*

*Proof.* For any $(\phi_i, \phi_j), (\dot{\phi}_i, \dot{\phi}_j) \in \mathcal{B}$ with $\phi_i \geq \dot{\phi}_i, \phi_j \geq \dot{\phi}_j$, we get

$$\begin{aligned} F_i(\phi_i, \phi_j)(\xi) - F_i(\dot{\phi}_i, \dot{\phi}_j)(\xi) &= \left[\mu_i - d_i + b_i(r_i(\xi) - a_{i1}\phi_1 - (\phi_i + \dot{\phi}_i))\right]\left(\phi_i - \dot{\phi}_i\right) \\ + d_i\int_{\mathbb{R}} J_i(y)\left[\phi_i(\xi - y) - \dot{\phi}_i(\xi - y)\right]\mathrm{d}y &+ b_i a_{ij}\left[\phi_j\left(\phi_i - \dot{\phi}_i\right) + \dot{\phi}_i\left(\phi_j - \dot{\phi}_j\right)\right] \geq 0, \end{aligned}$$

which means $\mathcal{P}_i[\phi_i, \phi_j](\xi) \geq \mathcal{P}_i[\dot{\phi}_i, \dot{\phi}_j](\xi)$ for all $\xi \in \mathbb{R}$. Then for all $(\phi_2, \phi_3) \in \mathcal{B}$ and $\xi \in \mathbb{R}$, it holds that

$$\mathcal{P}[\underline{\phi}_2, \underline{\phi}_3](\xi) \leq \mathcal{P}[\phi_2, \phi_3](\xi) \leq \mathcal{P}[\bar{\phi}_2, \bar{\phi}_3](\xi). \tag{2.10}$$

Moreover, we investigate that

$$\mathcal{P}_i\left[\underline{\phi}_i, \underline{\phi}_j\right](\xi) = \frac{1}{c}\int_{\xi}^{\infty} e^{\frac{\mu_i}{c}(\xi - s)}F_i(\underline{\phi}_i, \underline{\phi}_j)(s)\mathrm{d}s$$

$$\geq \frac{1}{c}\left\{\int_{\xi}^{\infty} e^{\frac{\mu_i}{c}(\xi-s)}\left[-c\underline{\phi}_i'(s)+\mu_i\underline{\phi}_i(s)\right]\mathrm{d}s\right\}$$
$$= \underline{\phi}_i(\xi), \text{ for } c>0. \tag{2.11}$$

Equation (2.11) holds for $c \leq 0$. Similarly, we can prove

$$\mathcal{P}_i[\bar{\phi}_i, \bar{\phi}_j](\xi) \leq \bar{\phi}_i(\xi). \tag{2.12}$$

It follows from (2.10)–(2.12) that $\mathcal{P}(\mathcal{B}) \subseteq \mathcal{B}$.

If $(\phi_2, \phi_3)(\xi) \in \mathcal{B}$ is nondecreasing with $\xi \in \mathbb{R}$, then for any $\iota > 0$, we have

$$\begin{aligned}
&F_i(\phi_i,\phi_j)(\xi+\iota) - F_i(\phi_i,\phi_j)(\xi)\\
&= \left[\mu_i - d_i + b_i r_i(\xi+\iota) - b_i a_{i1}\phi_1(\xi+\iota) - b_i\left(\phi_i(\xi+\iota)+\phi_i(\xi)\right) + b_i a_{ij}\phi_j(\xi+\iota)\right]\left(\phi_i(\xi+\iota)-\phi_i(\xi)\right)\\
&\quad + d_i\int_{\mathbb{R}} J_i(y)\left[\phi_i(\xi+\iota-y)-\phi_i(\xi-y)\right]\mathrm{d}y + b_i\phi_i(\xi)\left[r_i(\xi+\iota)-\phi_i(\xi)\right]\\
&\quad - b_i a_{i1}\phi_i(\xi)[\phi_1(\xi+\iota)-\phi_1(\xi)] + b_i a_{ij}\phi_i(\xi)\left[\phi_j(\xi+\iota)-\phi_j(\xi)\right]\\
&\geq 0.
\end{aligned}$$

Thus it holds that

$$\begin{aligned}
&\mathcal{P}_i[\phi_i,\phi_j](\xi+\iota) - \mathcal{P}_i(\phi_i,\phi_j)(\xi)\\
&= \frac{1}{c}\int_{-\infty}^{0} e^{\frac{\mu_i}{c}s}\left[F_i(\phi_i,\phi_j)(\xi+\iota-s) - F_i(\phi_i,\phi_j)(\xi-s)\right]\mathrm{d}s\\
&\geq 0, \text{ for } c>0.
\end{aligned}$$

This conclusion is also true for $c \leq 0$. □

Next, we show the existence of solutions for (2.6).

**Theorem 2.4.** *Assume that* $(J_1), (H_1)-(H_2)$ *hold. For any* $c > c_0^*$, (2.6) *admits a nondecreasing solution* $(\phi_2(\xi), \phi_3(\xi))$ *connecting* $(0,0)$ *and* $(k_2,k_3)$. *Moreover,* $\phi_i(\xi) > 0$ *for all* $\xi \in \mathbb{R}$. *Further, we can consider* $\phi_i$ *as a function of* $r_i$, *which is nondecreasing with respect to* $r_i$, $i = 2, 3$.

*Proof.* The existence of the solution can be proven by combining the super- and sub-solutions constructed above with iterative techniques. Because the proof is similar to Theorem 2.5 in [14], it is omitted here. Moreover, we can get

$$(\phi_2(-\infty), \phi_3(-\infty)) = (0,0),\ (\phi_2(\infty), \phi_3(\infty)) = (k_2,k_3).$$

Finally, we claim that

$$\phi_i(\xi) > 0 \text{ for all } \xi \in \mathbb{R},\ i = 2, 3.$$

Here we will use the contradiction argument. Assume that there exists a $\bar{\xi}_1 \in \mathbb{R}$ such that $\phi_2(\bar{\xi}_1) = 0$. Obviously, $\phi_2'(\bar{\xi}_1) = 0$ because $0 \leq \phi_2 \leq k_2$. Then from the first equation of (2.6), $\phi_2(\bar{\xi}_1 - y) = 0$ for $y \in \{y \in \mathbb{R} | J_2(y) \neq 0\}$. By repeating this process, we obtain $\phi_2(\xi) \equiv 0$ for all $\xi \in \mathbb{R}$. This contradicts $\phi_2(\infty) = k_2$. □

Inspired by Theorem 1 in [2], we can obtain the existence of solutions for (1.5).

**Theorem 2.5.** *Assume that* $(J_1)-(J_2), (H_1)-(H_2)$ *hold. Then, for any* $c \in \left(c_0^*, c_1^*\right)$, (1.4) *admits a forced wave* $(\varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi))$ *connecting* $E_1$ *and* $E_2$. *Moreover,* $\varphi_1(\xi)$ *is nonincreasing,* $\varphi_i(\xi)(i = 2, 3)$ *are nondecreasing with respect to* $\xi \in \mathbb{R}$, *and* $\varphi_i(\xi) > 0(i = 1, 2, 3)$ *for all* $\xi \in \mathbb{R}$.

*Proof.* Let $\varphi_1 = \mathcal{L}_1 (r_1 - a_{12}\varphi_2 - a_{13}\varphi_3)$ denote the solution of the first equation of (1.5). By substituting $\varphi_2 = \varphi_3 = 0$ into the first equation of (1.5), it can be inferred from Theorem 2.1 that there exists a nonincreasing solution $\varphi_1 := \varphi_{1_0} = \mathcal{L}_1 (r_1)$ satisfying $\varphi_{1_0}(-\infty) = l_1, \varphi_{1_0}(\infty) = 0$. Moreover, $\mathcal{L}_1 (r_1)$ is nondecreasing with $r_1$. Similarly, let $\varphi_i = \mathcal{L}_i (r_i - a_{i1}\varphi_1)$ denote the solutions of the second and the third equations of (1.5), $i = 2, 3$. Then by substituting $\varphi_1 = \varphi_{1_0}$ into these two equations of (1.5), there exist positive solutions $\varphi_i := \varphi_{i_1} = \mathcal{L}_i (r_i - a_{i1}\varphi_{1_0})$ satisfying $\varphi_{i_1}(-\infty) = 0, \varphi_{i_1}(\infty) = k_i, i = 2, 3$. Thus, define the sequences $\{\varphi_{i_n}\}_{n\in\mathbb{N}}(i = 1, 2, 3)$ as follows:

$$
\begin{aligned}
&\varphi_{1_1} = \mathcal{L}_1 (r_1 - a_{12}\varphi_{2_1} - a_{13}\varphi_{3_1}), \\
&\varphi_{2_2} := \mathcal{L}_2 (r_2 - a_{21}\varphi_{1_1}), \varphi_{3_2} := \mathcal{L}_3 (r_3 - a_{31}\varphi_{1_1}), \\
&\cdots, \\
&\varphi_{1_n} := \mathcal{L}_1 (r_1 - a_{12}\varphi_{2_n} - a_{13}\varphi_{3_n}), \\
&\varphi_{2_{n+1}} := \mathcal{L}_2 (r_2 - a_{21}\varphi_{1_n}),\ \varphi_{3_{n+1}} := \mathcal{L}_3 (r_3 - a_{31}\varphi_{1_n}).
\end{aligned}
$$

Moreover, $\varphi_{1_n}$ is nonincreasing, and $\varphi_{i_n}(i = 2, 3)$ are nondecreasing with respect to $\xi$. For all $n \in \mathbb{N}$, by the monotonicity properties of $\mathcal{L}_i$, we can obtain $\varphi_{1_{n+1}} \le \varphi_{1_n}$, $\varphi_{i_{n+1}} \ge \varphi_{i_n}, i = 2, 3$. Then $\varphi_{i_n}$ are pointwise convergent since $\varphi_{i_n}$ are monotonous and bounded for $i = 1, 2, 3$,

$$
\varphi_{1_n} \to \varphi_1 := \inf_{n\in\mathbb{N}} \varphi_{1_n},\ \varphi_{i_n} \to \varphi_i := \sup_{n\in\mathbb{N}} \varphi_{i_n}, i = 2, 3.
$$

Noting that

$$
\begin{cases}
d_1 \left[\int_{\mathbb{R}} J_1(y)\varphi_{1_n}(\xi - y)\mathrm{d}y - \varphi_{1_n}(\xi)\right] + c\varphi'_{1_n} + b_1\varphi_{1_n} [r_1(\xi) - \varphi_{1_n} - a_{12}\varphi_{2_n} - a_{13}\varphi_{3_n}] = 0, \\
d_2 \left[\int_{\mathbb{R}} J_2(y)\varphi_{2_{n+1}}(\xi - y)\mathrm{d}y - \varphi_{2_{n+1}}(\xi)\right] + c\varphi'_{2_{n+1}} + b_2\varphi_{2_{n+1}} [r_2(\xi) - a_{21}\varphi_{1_n} - \varphi_{2_{n+1}} + a_{23}\varphi_{3_{n+1}}] = 0, \\
d_3 \left[\int_{\mathbb{R}} J_3(y)\varphi_{3_{n+1}}(\xi - y)\mathrm{d}y - \varphi_{3_{n+1}}(\xi)\right] + c\varphi'_{3_{n+1}} + b_3\varphi_{3_{n+1}} [r_3(\xi) - a_{31}\varphi_{1_n} + a_{32}\varphi_{2_{n+1}} - \varphi_{3_{n+1}}] = 0,
\end{cases}
$$

we state that $(\varphi_1, \varphi_2, \varphi_3)$ is a solution of (1.5) with $\varphi_i \in C^1(\mathbb{R}), i = 1, 2, 3$.

Because $\varphi_i(\xi)$ is monotone and bounded, the limit of $\varphi_i$ exists as $\xi \to \pm\infty, i = 1, 2, 3$. Let $\varphi_i^\pm := \varphi_i(\pm\infty), i = 1, 2, 3$. Notice that $\varphi_1$ is nonincreasing, and $\varphi_2, \varphi_3$ are nondecreasing with respect to $\xi \in \mathbb{R}$. Then, $\varphi_1^- > 0$ and $\varphi_i^+ > 0, i = 2, 3$. By letting $\xi \to \infty$, the first equation in (1.5) becomes

$$
b_1\varphi_1^+ [r_1(\infty) - \varphi_1^+ - a_{12}\varphi_2^+ - a_{13}\varphi_3^+] = 0.
$$

Because $r_1(\infty) \le 0$, there must be $\varphi_1^+ = 0$. Further, by the second and the third equations of (1.5), we have

$$
\begin{cases}
1 - \varphi_2^+ + a_{23}\varphi_3^+ = 0, \\
1 + a_{32}\varphi_2^+ - \varphi_3^+ = 0.
\end{cases}
$$

Clearly, $\varphi_i^+ = k_i,\ i = 2, 3$. By letting $\xi \to -\infty$, assume that $\varphi_2^- \ne 0, \varphi_3^- \ne 0$. Then we have

$$
\begin{cases}
r_2(-\infty) - a_{21}\varphi_1^- - \varphi_2^- + a_{23}\varphi_3^- = 0, \\
r_3(-\infty) - a_{31}\varphi_1^- + a_{32}\varphi_2^- - \varphi_3^- = 0.
\end{cases} \tag{2.13}
$$

It follows from (2.13) that $\varphi_2^-, \varphi_3^- < 0$, which contradicts $\varphi_i > 0, i = 2, 3$. If either $\varphi_2^- \neq 0, \varphi_3^- = 0$ or $\varphi_3^- \neq 0, \varphi_2^- = 0$, then it holds that $\varphi_3^- < 0$ or $\varphi_2^- < 0$. These are still contradictions. Therefore, we have $\varphi_i^- = 0, i = 2, 3$. Finally, from the first equation of (1.5), we obtain $\varphi_1^- = 1$ because $\varphi_1^- > 0$. □

**Lemma 2.6.** *Assume that $(J_1) - (J_2), (H_1) - (H_2)$ hold. For any $c \in \left(c_0^*, c_1^*\right)$, let $(\varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi))$ be a forced wave of* (1.4) *connecting $E_1$ and $E_2$. Then $0 < \varphi_i(\xi) < k_i (i = 1, 2, 3)$ for all $\xi \in \mathbb{R}$.*

*Proof.* We prove that $\varphi_1(\xi) > 0$ for all $\xi \in \mathbb{R}$ because the other cases can be handled similarly. Here, we will use the contradiction argument. Assume that there exists $\tilde{\xi}_1 \in \mathbb{R}$ such that $\varphi_1(\tilde{\xi}_1) = 0$. Obviously, $\varphi_1'(\tilde{\xi}_1) = 0$ because $0 \leq \varphi_1 \leq k_1$. Then, from the first equation of (1.5), $\varphi_1(\tilde{\xi}_1 - y) = 0$ for $y \in \{y \in \mathbb{R} | J_1(y) \neq 0\}$. By repeating this process, we obtain $\varphi_1(\xi) \equiv 0$ for all $\xi \in \mathbb{R}$, which is a contradiction. On the other hand, if there exists $\hat{\xi}_1 \in \mathbb{R}$ such that $\varphi_1(\hat{\xi}_1) = k_1$, then $\varphi_1'(\hat{\xi}_1) = 0$. It follows from the first equation of (1.5) that

$$0 = d_1 \left[ \int_{\mathbb{R}} J_1(y) \varphi_1(\hat{\xi}_1 - y) \mathrm{d}y - k_1 \right] + b_1 k_1 \left[ r_1(\hat{\xi}_1) - k_1 - a_{12} \varphi_2(\hat{\xi}_1) - a_{13} \varphi_3(\hat{\xi}_1) \right] < 0,$$

which is a contradiction. □

## 3. Gap formations

In this part, we need to make the following assumptions:

($J_1^*$) $J_i$ satisfies $(J_1)$ and $J_i$ is even, $i = 1, 2, 3$.

($H_1^*$) $r_1(\cdot) \in C(\mathbb{R})$ is nonincreasing with $-\infty < r_1(\infty) < 0 < r_1(-\infty) < \infty$ and for $i = 2, 3$, $r_i(\cdot) \in C(\mathbb{R})$ are nondecreasing with $-\infty < r_i(-\infty) < 0 < r_i(\infty) < \infty$. For convenience, denote $l_1 := r_1(-\infty)$, $l_i := r_i(\infty)$, $i = 2, 3$.

($H_3$) $r_2(-\infty) + a_{23} k_3 < 0$, $r_3(-\infty) + a_{32} k_2 < 0$.

To begin, consider the well-posedness of the following problem:

$$\begin{cases} u_t = d_1 (J_1 * u - u) + b_1 u [r_1(x - ct) - u - a_{12} w_1 - a_{13} w_2], \\ w_{1_t} = d_2 (J_2 * w_1 - w_1) + b_2 w_1 [r_2(x - ct) - a_{21} u - w_1 + a_{23} w_2], \\ w_{2_t} = d_3 (J_3 * w_2 - w_2) + b_3 w_2 [r_3(x - ct) - a_{31} u + a_{32} w_1 - w_2], \\ u(0, x) = u_0(x), w_1(0, x) = w_{10}(x), w_2(0, x) = w_{20}(x). \end{cases} \tag{3.1}$$

By the method of Theorem 2.1 in [17], we can get the following results.

**Lemma 3.1.** *For any $(u_0, w_{10}, w_{20}) \in \mathcal{X}_{\boldsymbol{k}}$,* (3.1) *admits a unique solution $(u(t, x), w_1(t, x), w_2(t, x))$ with $(u(0, x), w_1(0, x), w_2(0, x)) = (u_0, w_{10}, w_{20})$ and $\mathbf{0} \leq (u(t, x), w_1(t, x), w_2(t, x)) \leq \boldsymbol{k}$.*

**Theorem 3.2.** *Assume that $(J_1^*), (H_1^*) - (H_3)$ hold. Let $c > c_1^*$ and $(u(t, x), w_1(t, x), w_2(t, x))$ be a solution of* (3.1) *with $(u_0, w_{10}, w_{20}) \in \mathcal{X}_{\boldsymbol{k}}$. If there exists a constant $n_0 \in \mathbb{R}$ such that $u_0(x) = 0$ for all $x \geq n_0$, then for all constants $c_1, c_2, n_1, n_2 \in \mathbb{R}$ satisfying $c > c_2 > c_1 > c_1^*$, it holds that*

$$\sup_{x \geq c_1 t + n_1} u(t, x) \leq M_1 e^{-\gamma_1 t}, \ t \geq 0, \tag{3.2}$$

$$\sup_{x\le c_2t+n_2} w_1(t,x) \le M_2 e^{-\gamma_2 t},\ t\ge 0, \tag{3.3}$$

*and*

$$\sup_{x\le c_2t+n_2} w_2(t,x) \le M_3 e^{-\gamma_3 t},\ t\ge 0, \tag{3.4}$$

*where $M_i, \gamma_i$ are positive constants which can be defined later, $i = 1,2,3$.*

*Proof.* We prove (3.2) first. Because $J_1$ is even, by the first case in Remark 1.1, there exists $\lambda_1^* > 0$ such that

$$c_1^* = \min_{\lambda>0} f_1(\lambda) = f_1(\lambda_1^*).$$

Define

$$\bar{u}(t,x) = k_1 e^{-\lambda_1^*(x-c_1^*t-n_0)},$$

and we can check that

$$\bar{u}_t = d_1(J_1 * \bar{u} - \bar{u}) + b_1k_1\bar{u}.$$

Noting that $0 \le u_0(x) \le k_1$ for all $x\in\mathbb{R}$ and $u_0(x) = 0$ for all $x \ge n_0$, then $u_0(x) \le \bar{u}(0,x) = k_1 e^{-\lambda_1^*(x-n_0)}$. Because

$$u_t \le d_1(J_1 * u - u) + b_1k_1u,$$

by the comparison principle, we obtain

$$u(t,x) \le \bar{u}(t,x) = k_1 e^{-\lambda_1^*(x-c_1^*t-n_0)},\ t\ge 0,\ x\in\mathbb{R}.$$

By letting

$$\gamma_1 = \lambda_1^*(c_1 - c_1^*),\ M_1 = k_1 e^{-\lambda_1^*(n_1-n_0)},$$

we can get

$$\sup_{x\ge c_1t+n_1} u(t,x) \le k_1 e^{-\lambda_1^*(n_1-n_0)} \cdot e^{-\gamma_1 t} = M_1 e^{-\gamma_1 t},\ t\ge 0.$$

Second, we prove that (3.3) holds true. For any $\theta_1 \in (0, -(r_2(-\infty) + a_{23}k_3))$, there exists $\xi_2\in\mathbb{R}$ such that $r_2(\xi) \le r_2(-\infty) + \theta_1$ for all $\xi \le \xi_2$. By the second equation of (1.4), we have

$$w_{1_t} \le d_2(J_2 * w_1 - w_1) + b_2w_1[r_2(-\infty) + \theta_1 + a_{23}k_3],\ \text{for } x \le ct + \xi_2.$$

Now, we search for a supersolution of the following equation:

$$v_{1_t} = d_2(J_2 * v_1 - v_1) + b_2v_1[r_2(-\infty) + \theta_1 + a_{23}k_3],\ t>0,\ x\in\mathbb{R}. \tag{3.5}$$

Let

$$g_1(\lambda) = -c\lambda - d_2\left(\int_{\mathbb{R}} J_2(y)e^{-\lambda y}\mathrm{d}y - 1\right) - b_2[r_2(-\infty) + \theta_1 + a_{23}k_3].$$

Noting that $g_1(0) > 0,\ g_1''(\lambda) < 0$, there exists $\lambda_2^* > 0$ such that $g_1(\lambda_2^*) > 0$. Define

$$\bar{v}_1(t,x) = k_2\left[e^{\lambda_2^*(x-ct-\xi_2)} + e^{-\beta_2 t}\right],$$

where $\beta_2 \in (0, -b_2(r_2(-\infty) + \theta_1 + a_{23}k_3))$. Then we can verify that

$$\bar{v}_{1_t} - d_2(J_2 * \bar{v}_1 - \bar{v}_1) - b_2\bar{v}_1[r_2(-\infty) + \theta_1 + a_{23}k_3]$$

$$
\begin{aligned}
&= k_2 e^{\lambda_2^*(x-ct-\xi_2)} g_1(\lambda_2^*) + k_2 e^{-\beta_2 t}\left[-\beta_2 - b_2(r_2(-\infty) + \theta_1 + a_{23}k_3)\right] \\
&\geq 0,
\end{aligned}
$$

which means $\bar{v}_1$ is a supersolution of (3.5). Let $\tilde{w}_1(t,x) = \bar{v}_1(t,x) - w_1(t,x)$, it is clear that $\tilde{w}_1(t,x) \geq 0$ for all $x \geq ct + \xi_2$. From the above conclusions, we can conclude that

$$
\begin{cases}
\tilde{w}_{1_t} - d_2(J_2 * \tilde{w}_1 - \tilde{w}_1) - b_2\tilde{w}_1[r_2(-\infty) + \theta_1 + a_{23}k_3] \geq 0, \ t > 0, \ x \leq ct + \xi_2, \\
\tilde{w}_1(t,x) \geq 0, \ t > 0, \ x \geq ct + \xi_2, \\
\tilde{w}_1(0,x) \geq k_2\left[e^{\lambda_2^*(x-\xi_2)} + 1\right] - k_2 \geq 0, \ x \leq \xi_2.
\end{cases}
$$

Then by Lemma 4.7 in [21], we have $\tilde{w}_1(t,x) \geq 0$, which means that $\bar{v}_1(t,x) \geq w_1(t,x)$ for all $x \leq ct+\xi_2$ and $t \geq 0$. By choosing $t_0 \geq 0$ such that $c_2 t + n_2 \leq ct + \xi_2$ for all $t \geq t_0$ and letting

$$
\gamma_2 = \min\{\lambda_2^*(c - c_2), \beta_2\}, \ M_2 = k_2\left[e^{\lambda_2^*(n_2-\xi_2)} + 1\right],
$$

we can get

$$
\sup_{x \leq c_2 t + n_2} w_1(t,x) \leq \sup_{x \leq c_2 t + n_2} k_2\left[e^{\lambda_2(x-ct-\xi_2)} + e^{-\beta_2 t}\right] \leq M_2 e^{-\gamma_2 t}, \ t \geq t_0.
$$

Additionally, if necessary, we can increase the value of $M_2$ appropriately so that the above inequality holds for all $t \geq 0$.

Finally, for any $\theta_2 \in (0, -(r_3(-\infty) + a_{32}k_2))$, there exists $\xi_3 \in \mathbb{R}$ such that $r_3(\xi) \leq r_3(-\infty) + \theta_2$ for all $\xi \leq \xi_3$. Moreover, there exists $\lambda_3^* > 0$ such that

$$
g_2(\lambda_3^*) := -c\lambda_3^* - d_3\left(\int_{\mathbb{R}} J_3(y) e^{-\lambda_3^* y} \mathrm{d}y - 1\right) - b_3[r_3(-\infty) + \theta_2 + a_{32}k_2] > 0.
$$

By similar arguments, we can prove that (3.4) holds, where

$$
\gamma_3 = \min\{\lambda_3^*(c - c_2), \beta_3\}, \ M_3 = k_3\left[e^{\lambda_3^*(n_2-\xi_3)} + 1\right], \ \beta_3 \in (0, -b_3(r_3(-\infty) + a_{32}k_2 + \theta_2)).
$$

□

**Theorem 3.3.** *Assume that $(J_1^*), (H_1^*) - (H_3)$ hold. Let*

$$
\tilde{c}_i^* = \min_{\lambda > 0} \frac{d_i\left(\int_{\mathbb{R}} J_i(y) e^{\lambda y} \mathrm{d}y - 1\right) + b_i(l_i + a_{ij}k_j)}{\lambda}, \ i \neq j, \ i,j \in \{2,3\}.
$$

*Denote $\tilde{c}_0^* = \min\{-\tilde{c}_2^*, -\tilde{c}_3^*\}$ and $(u(t,x), w_1(t,x), w_2(t,x))$ as a solution of (3.1) with initial values $(u_0, w_{10}, w_{20}) \in \mathcal{X}_k$. If there exists a constant $m_i^0 \in \mathbb{R}$ such that $w_{i0}(x) = 0$ for all $x \leq m_i^0, i = 2,3$, then for $c < \tilde{c}_0^*$ and for all constants $\tilde{c}_1, \tilde{c}_2, m_1, m_2, m_3 \in \mathbb{R}$ satisfying $c < \tilde{c}_1 < \tilde{c}_2 < \tilde{c}_0^*$, it holds that*

$$
\sup_{x \geq \tilde{c}_1 t + m_1} u(t,x) \leq \tilde{M}_1 e^{-\nu_1 t}, \ t \geq 0, \tag{3.6}
$$

$$
\sup_{x \leq \tilde{c}_2 t + m_2} w_1(t,x) \leq \tilde{M}_2 e^{-\nu_2 t}, \ t \geq 0, \tag{3.7}
$$

*and*

$$
\sup_{x \leq \tilde{c}_2 t + m_3} w_1(t,x) \leq \tilde{M}_3 e^{-\nu_3 t}, \ t \geq 0, \tag{3.8}
$$

*where $\tilde{M}_j, \nu_j$ are positive constants which can be defined later, $j = 1,2,3$.*

*Proof.* For any $\eta_1 \in (0, -r_1(\infty))$, there exists $\xi_1 \in \mathbb{R}$ such that $r_1(\xi) \leq r_1(\infty) + \eta_1$ for all $\xi \geq \xi_1$. Then

$$u_t \leq d_1(J_1 * u - u) + b_1 u\left(r_1(\infty) + \eta_1\right), \ x \geq ct + \xi_1.$$

Let

$$h_1(\lambda) = c\lambda - d_1\left(\int_{\mathbb{R}} J_1(y)e^{\lambda y}\mathrm{d}y - 1\right) - b_1[r_1(\infty) + \eta_1].$$

Noting that $h_1(0) > 0,\ h_1''(\lambda) < 0$, there exists $\tilde{\lambda}_1^* > 0$ such that $h_1(\tilde{\lambda}_1^*) > 0$. Define

$$\hat{u}(t,x) = k_1\left[e^{-\tilde{\lambda}_1^*(x-ct-\xi_1)} + e^{-\beta_1 t}\right],$$

where $\beta_1 \in (0, -b_1(r_1(\infty) + \eta_1))$. Then we can verify that

$$\hat{u}_t - d_1(J_1 * \hat{u} - \hat{u}) - b_1\hat{u}\left(r_1(\infty) + \eta_1\right) = k_1 e^{-\tilde{\lambda}_1^*(x-ct-\xi_1)}h_1(\tilde{\lambda}_1^*) + k_1 e^{-\beta_1 t}\left[-\beta_1 - b_1(r_1(\infty) + \eta_1)\right] \geq 0,$$

which means $\hat{u}$ is a supersolution of

$$u_t = d_1(J_1 * u - u) + b_1 u\left(r_1(\infty) + \eta_1\right).$$

Similar to Theorem 3.2, we can get (3.6), where

$$\nu_1 = \min\left\{\tilde{\lambda}_1^*\left(\tilde{c}_1 - c\right), \beta_1\right\},\ \tilde{M}_1 = k_1\left[e^{-\tilde{\lambda}_1^*(m_1-\xi_1)} + 1\right].$$

Let

$$h_2(\lambda) = \frac{d_2\left(\int_{\mathbb{R}} J_2(y)e^{\lambda y}\mathrm{d}y - 1\right) + b_2(l_2 + a_{23}k_3)}{\lambda}.$$

There exists $\tilde{\lambda}_2^* > 0$ such that

$$\tilde{c}_2^* = \min_{\lambda>0} h_2(\lambda) = h_2(\tilde{\lambda}_2^*).$$

Define

$$\hat{w}_1(t,x) = k_2 e^{\tilde{\lambda}_2^*(x+\tilde{c}_2^* t - m_2^0)},$$

we can check that

$$\hat{w}_{1_t} = d_2(J_2 * \hat{w}_1 - \hat{w}_1) + b_2\hat{w}_1(l_2 + a_{23}k_3).$$

Noting that $0 \leq w_{10}(x) \leq k_2$ for all $x \in \mathbb{R}$ and $w_{10}(x) = 0$ for all $x \leq m_2^0$, then $w_{10}(x) \leq \hat{w}_1(0,x) = k_2 e^{\tilde{\lambda}_2^*(x-m_2^0)}$. Because

$$w_{1_t} \leq d_2(J_2 * w_1 - w_1) + b_2 w_1(l_2 + a_{23}k_3),$$

then by the comparison principle, we obtain

$$w_1(t,x) \leq \hat{w}_1(t,x) = k_2 e^{\tilde{\lambda}_2^*\left(x+\tilde{c}_2^* t - m_2^0\right)},\ t \geq 0,\ x \in \mathbb{R}.$$

By letting

$$\nu_2 = -\tilde{\lambda}_2^*(\tilde{c}_2 + \tilde{c}_2^*),\ \tilde{M}_2 = k_2 e^{\tilde{\lambda}_2^*\left(m_2 - m_2^0\right)},$$

we can get (3.7).

Similarly, by letting

$$\nu_3 = -\tilde{\lambda}_3^*(\tilde{c}_2 + \tilde{c}_3^*),\ \tilde{M}_3 = k_3 e^{\tilde{\lambda}_3^*\left(m_3 - m_0^3\right)},$$

we can prove (3.8), where

$$\tilde{c}_3^* = \min_{\lambda>0} \frac{d_3\left(\int_{\mathbb{R}} J_3(y)e^{\lambda y}\mathrm{d}y - 1\right) + b_3(l_3 + a_{32}k_2)}{\lambda}$$
$$= \frac{d_3\left(\int_{\mathbb{R}} J_3(y)e^{\tilde{\lambda}_3^* y}\mathrm{d}y - 1\right) + b_3(l_3 + a_{32}k_2)}{\tilde{\lambda}_3^*}.$$

□

**Remark 3.4.** *Theorems 3.2 and 3.3 show that when the shifting speed c exceeds the critical value* $c_1^*$ *(or is less than* $\tilde{c}_0^*$*), the system develops an expanding gap region whose width grows linearly in time. In this gap, the densities of all three species decay exponentially, effectively forming a local extinction zone.*

*Ecologically, this corresponds to a scenario where climate change drives the suitable habitat to move faster than the species can track via dispersal and reproduction. For the cooperative pair* $(w_1, w_2)$ *and the competitor u, if the habitat shift is too rapid, none of the species can maintain a viable population behind the moving front. Consequently, an ever-widening no-man's land emerges between the area already vacated by the retreating species and the area not yet reached by the advancing species, that is the gap.*

*This finding is consistent with the two-species competition model of Berestycki et al. [2], but our model additionally reveals that cooperation does not help species escape gap formation under rapid climate change; the cooperative species also go locally extinct when the shift is too fast. This provides a theoretical warning: Even with mutualistic interactions, if the rate of climate change exceeds the ecosystem's adaptive threshold, large-scale biodiversity loss and habitat fragmentation may still occur.*

## 4. Numeric simulations

In this section, we will verify the previous theoretical results through some numerical simulations. For the growth functions, we take

$$r_1(x-ct) = -\frac{4}{\pi}\arctan(x-ct),\ r_2(x-ct) = \frac{2}{\pi}\arctan(x-ct),\ r_3(x-ct) = \frac{3}{\pi}\arctan(x-ct).$$

Then we have $r_1(-\infty) = 2,\ r_2(\infty) = 1,\ r_3(\infty) = 1.5$. Let $d_1 = d_2 = d_3 = 1,\ b_1 = 0.5,\ b_2 = 1.1,\ b_3 = 0.8,\ a_{12} = 1.1,\ a_{13} = 1.2,\ a_{21} = 0.15,\ a_{23} = 0.2,\ a_{31} = 0.15,\ a_{32} = 0.3$. In the following, we will divide the verification of Theorem 2.5 and Theorems 3.2, 3.3 into two parts.

(1) We choose the following kernel function for $J_i, i = 1, 2, 3$ :

$$J_1(y) = J_2(y) = J_3(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{(y-1)^2}{2}},\ y \in \mathbb{R}.$$

By calculations, we obtain

$$c_1^* = \inf_{\lambda>0} \frac{\int_{\mathbb{R}} J_1(y)e^{\lambda y}\mathrm{d}y}{\lambda} \approx 3.63,$$

$$c_2^* = \inf_{\lambda>0} \frac{\int_0^1 J_2(y)e^{\lambda y}\mathrm{d}y + 0.1}{\lambda} \approx 0.621,$$
$$c_3^* = \inf_{\lambda>0} \frac{\int_0^1 J_3(y)e^{\lambda y}\mathrm{d}y + 0.2}{\lambda} \approx 0.677.$$

We choose $c = 2 \in (-0.621, 3.63)$ and $c = -0.5 \in (-0.621, 3.63)$, and the numeric results showed in Figures 1 and 2 verify Theorem 2.5; that is, there exists a forced wave connecting $(2, 0, 0)$ to $(0, 1.38, 1.91)$ for System (1.4).

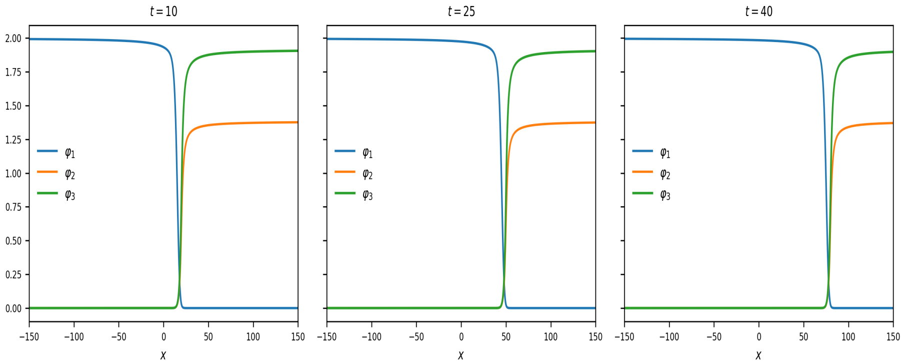


**Figure 1.** The forced wave travels with a speed $\mathrm{c} = 2$ at time $\mathrm{t} = 10,\ \mathrm{t} = 25,\ \mathrm{t} = 40,$ respectively.

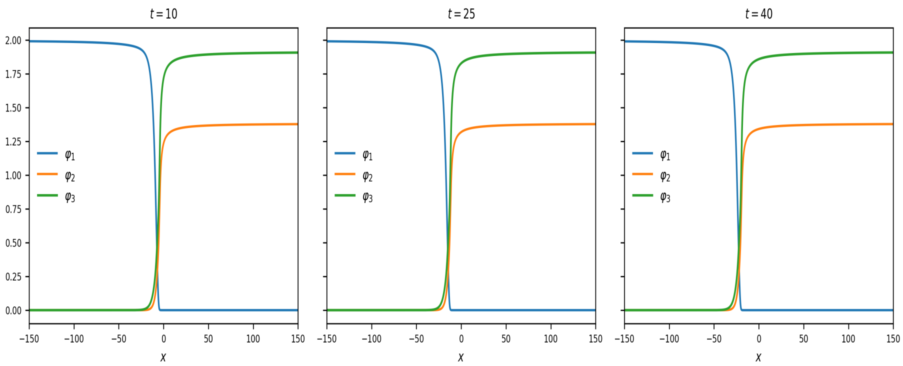


**Figure 2.** The forced wave travels with a speed $\mathrm{c} = -0.5$ at time $\mathrm{t} = 10,\ \mathrm{t} = 25,\ \mathrm{t} = 40,$ respectively.

(2) We choose the following kernel function for $J_i, i = 1, 2, 3$ :

$$J_1(y) = J_2(y) = J_3(y) = \begin{cases} \frac{1}{20(1-e^{-0.5})} e^{-\frac{|y|}{10}}, & y \in [-5, 5], \\ 0, & \text{elsewhere}, \end{cases}$$

and take the following initial data

$$u(x, 0) = \begin{cases} 2\sin(x - 10), & 10 \le x \le 10 + \pi, \\ 0, & \text{elsewhere}, \end{cases}$$

$$w_1(x,0) = \begin{cases} 1.38\sin(x-10), & 10 \le x \le 10+\pi, \\ 0, & \text{elsewhere}, \end{cases}$$

$$w_2(x,0) = \begin{cases} 1.91\sin(x-10), & 10 \le x \le 10+\pi, \\ 0, & \text{elsewhere}. \end{cases}$$

By calculations, we obtain

$$c_1^* = \inf_{\lambda>0} \frac{\int_{\mathbb{R}} J_1(y)e^{\lambda y}\mathrm{d}y}{\lambda} \approx 3.8,$$

$$\tilde{c}_2^* = \inf_{\lambda>0} \frac{\int_0^1 J_2(y)e^{\lambda y}\mathrm{d}y + 0.1}{\lambda} \approx 5.47,$$

$$\tilde{c}_3^* = \inf_{\lambda>0} \frac{\int_0^1 J_3(y)e^{\lambda y}\mathrm{d}y + 0.2}{\lambda} \approx 5.49.$$

By choosing $c = 4, c_1 = 3.8, c_2 = 3.9, n_1 = n_2 = 0$, Figure 3 verifies Theorem 3.2. By choosing $c = -5.6, \tilde{c}_1 = -5.55, \tilde{c}_2 = -5.5, m_1 = m_2 = m_3 = 0$, Figure 4 verifies Theorem 3.3.

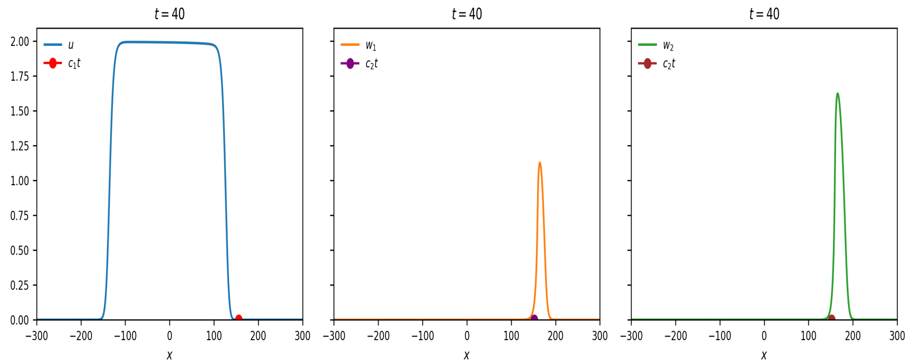


**Figure 3.** Gap formations are shown for the speed c = 4.

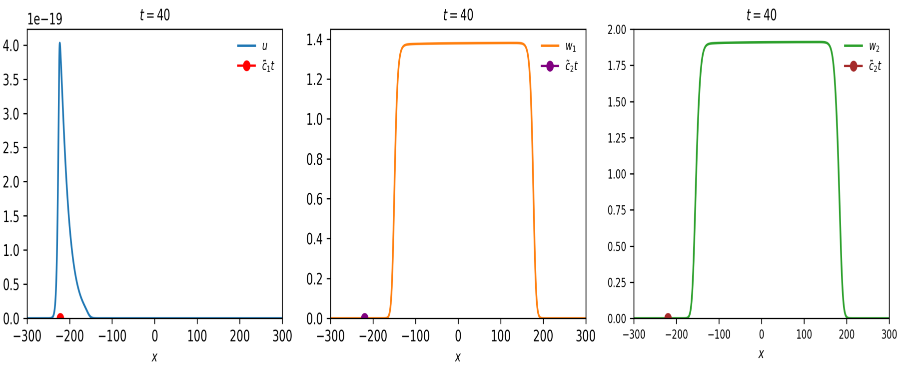


**Figure 4.** Gap formations are shown for the speed c = −5.6.

## 5. Discussion and conclusions

In this paper, we have investigated the existence of forced waves and gap formations for a three-species competitive-cooperative system with nonlocal dispersal and shifting habitat. By integrating spatial mobility and the dynamic interplay between competition and cooperation, the model better reflects real-world ecological scenarios where species or populations engage in both local interactions and spatial dispersal.

In establishing the existence of forced waves, the present study addresses the case where the growth functions change sign; see the hypothesis ($H_1$). Moreover, one may consider the alternative setting where growth functions $r_1(-\infty) > r_1(\infty) > 0$ and $r_i(\infty) > r_i(-\infty) > 0 (i = 2, 3)$ do not change sign, which will be left for future investigation. Moreover, the stability of the forced waves, especially their exponential stability, is an important open question that we plan to explore in subsequent work. Preliminary numerical simulations (Section 4) already suggest that the forced waves are stable under appropriate perturbations, and a rigorous proof using a squeezing technique or a comparison principle may be achievable.

In short, this research provides a systematic framework for analyzing forced waves in a class of three-species reaction-diffusion systems under climate change. It demonstrates how mobility, competition, and cooperation collectively influence spatial spread, providing insights that could guide management strategies in ecology and related disciplines. The interplay between nonlocal dispersal, asymmetric kernels, and sign-preserving growth functions remains a challenging but promising avenue for future research.

## Author contributions

Rui Yan: Conceptualization, Methodology, Writing–original draft, Writing–review and editing; Guirong Liu: Methodology, Validation, Formal analysis; Fangping Guo: Validation, Software; Xiaocui Li: Supervision, Writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare that they have no conflict of interest.

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