



Research article

Einstein connection of nonsymmetric pseudo-Riemannian manifold with the f^2 -torsion condition

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Abstract: Einstein considered a linear connection ∇ with torsion T on a differentiable manifold equipped with a nonsymmetric (0,2)-tensor $G = g + F$, where g is a pseudo-Riemannian metric associated with gravity, and $F \neq 0$ is a skew-symmetric tensor associated with electromagnetism, such that $(\nabla_X G)(Y, Z) = -G(T(X, Y), Z)$. In this paper, we explicitly present the Einstein connection of a nonsymmetric pseudo-Riemannian manifold with non-degenerate F , satisfying the f^2 -torsion condition $T(f^2X, Y) = T(X, f^2Y) = f^2T(X, Y)$, where $g(X, fY) = F(X, Y)$, and show that in the almost Hermitian case, it reduces to the Prvanović's (1995) solution. We also explicitly present the Einstein connection of almost contact metric manifolds satisfying the f^2 -torsion condition, discuss special Einstein connections, and give example in terms of the weighted product of almost Hermitian manifolds.

Keywords: nonsymmetric pseudo-Riemannian manifold; weak almost Hermitian manifold; Einstein connection; almost contact metric structure; f^2 -torsion condition

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1. Introduction

Einstein worked on various versions of the unified field theory (nonsymmetric gravitational theory, briefly, NGT), see [2]. This theory was intended to unite the gravitation theory, to which general relativity is related, and the theory of electromagnetism. Beginning in 1950, Einstein used a real nonsymmetric basic (0,2)-tensor G , whose symmetric part g is associated with gravity, and the skew-symmetric part F is associated with electromagnetism. In his attempt to construct NGT, Einstein [2] considered a differentiable manifold $(M, G = g + F)$ equipped with a linear connection ∇ with torsion

$T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$ satisfying the metricity condition

$$(\nabla_X G)(Y, Z) = -G(T(X, Y), Z) \quad (X, Y, Z \in \mathfrak{X}_M), \quad (1.1)$$

called an *Einstein connection*. The idea of a nonsymmetric basic tensor was revisited by Moffat [7], allowing for richer gravitational dynamics. Results by Jansen and Prokopec [6] point to potential improvements and challenges within NGT. Wanas et al. [11] studied path equations of a particle in Einstein's nonsymmetric geometry. Recent approaches to modified gravity often rely on differential geometry, including torsion and non-metricity, as a natural extension of general relativity. Connections with totally skew-symmetric torsion, i.e., $T(X, Y, Z) := g(T(X, Y), Z)$ is a 3-form, are important due to the relations with mathematical physics, as they admit a coupling to spinor fields and lead to holonomy and rigidity phenomena. In this case, the dimension of the space of parallel spinors coincides with the number of preserved supersymmetries. Such connections arise naturally in supersymmetric string theories, non-linear σ -models, and in gravitational models [4]. Ivanov and Zlatanović [5] presented conditions for the existence and uniqueness of the Einstein connection with totally skew-symmetric torsion on a manifold $(M, G = g + F)$ and gave its explicit expression for an almost contact metric structure. In this case, the torsion is determined by $T(X, Y, Z) = -\frac{1}{3} dF(X, Y, Z)$, $g(X, fY) = F(X, Y)$, and the difference tensor $K := \nabla - \nabla^g$ (where ∇^g is the Levi-Civita connection) is given by

$$K_X Y = \frac{1}{2} \{T(fX, Y) - T(X, fY) + T(X, Y)\}. \quad (1.2)$$

Total skew-symmetry of K implies total skew-symmetry of T , but the converse does not hold in general. In this paper, we do not assume that the torsion is totally skew-symmetric.

An important class of linear connections, having applications in theory of probability and statistics as well as in information geometry, are statistical connections, which are torsion-free and the tensor $(\nabla g)(X, Y, Z) := (\nabla_X g)(Y, Z)$ is symmetric with respect to all entries, equivalently, the cubic form $K(X, Y, Z) := g(K(X, Y), Z)$ (associated with the difference tensor (2.9) of ∇) is symmetric. Unfortunately, we cannot directly apply the statistical structure to Einstein's nonsymmetric geometry: if the cubic form $K(X, Y, Z)$ is symmetric, then by (2.11) and (2.12), $T = K = 0$ is true; hence, $\nabla = \nabla^g$.

Weak metric structures, see [9], generalize the almost Hermitian and almost contact metric structures, as well as Yano's f -structure [12], and are ideal for studying $(M, G = g + F)$ in NGT with an arbitrary skew-symmetric tensor F . A *weak almost Hermitian manifold* (M, f, g) is a Riemannian manifold (M, g) of even dimension ≥ 4 endowed with non-singular skew-symmetric (1,1)-tensor f and the 2-form $F(X, Y) = g(X, fY)$, see [13]. Rovenski and Zlatanović [10, 13] were the first to apply the weak almost Hermitian and weak almost contact structures to NGT of totally skew-symmetric torsion with $F(X, Y) = g(X, fY)$. Prvanović [8] obtained the explicit form of the Einstein connection of almost Hermitian manifold, and using its torsion tensor, characterized the sixteen classes (by Gray and Hervella [3]) of almost Hermitian manifold.

In this paper, we explicitly present the Einstein connection of an NGT space with non-degenerate F (i.e., a weak almost Hermitian manifold) satisfying a natural and very powerful structural assumption, called the f^2 -torsion condition. This assumption disappears for the classical case, hence our result generalizes the solution by Prvanović. We also explicitly present the Einstein connection and the special Einstein connection, defined by the property $K_X Y = -K_Y X$, of an almost contact metric manifold satisfying the f^2 -torsion condition.

The paper has six sections, including introduction appendix and conclusion. In Section 2, we review basics of NGT, give examples, prove auxiliary statements, and discuss the relation between the torsion and contorsion of an Einstein connection emphasizing the role of the f^2 -torsion condition. In Section 3, we find the Einstein connection of an almost contact metric manifold with the f^2 -torsion condition (Theorems 3.1 and 3.2). In Section 4, we present the Einstein connection of nonsymmetric pseudo-Riemannian manifolds with non-degenerate F and the f^2 -torsion condition (Theorem 4.1), in particular, for weak almost Hermitian manifolds (Corollary 4.1). Corollary 4.2 is a coordinate-free recovering of solution [8] for almost Hermitian manifolds. Section 5 contains long formulas with δ_i terms (which measure the deviation of the results from the almost Hermitian case), the tensorial computation, and consistency, checking of which were additionally performed using the MAPLE program.

2. Einstein's nonsymmetric geometry

A manifold $(M, G = g + F)$ equipped with an Einstein connection ∇ different from the Levi-Civita connection ∇^g is called an *NGT space*. The basic (0,2)-tensor G of an NGT space $(M, G = g + F)$ decomposes into two parts, the symmetric part g (pseudo-Riemannian metric) and the skew-symmetric part F (fundamental 2-form), that is,

$$g(X, Y) = \frac{1}{2}[G(X, Y) + G(Y, X)], \quad F(X, Y) = \frac{1}{2}[G(X, Y) - G(Y, X)].$$

We assume that the skew-symmetric part, F , is non-zero, may have arbitrary (not necessarily constant) rank, and, in particular, may be non-degenerate (that is, of maximal rank). Therefore, we obtain a (1,1)-tensor $f \neq 0$ (or, a linear operator of not necessarily constant rank on TM) determined by

$$g(X, fY) = F(X, Y) \quad (X, Y \in \mathfrak{X}_M).$$

The notation “ f ” for a linear operator (not for a function) follows Yano's paper [12] and the work of many authors on the metric f -structure. Since F is skew-symmetric, the tensor f is also skew-symmetric: $g(fX, Y) = -g(X, fY)$. Note that the tensor $A = -f$ is defined in [5] by $g(AX, Y) = F(X, Y)$.

The covariant derivative is a (p, k) -tensor S for $p = 0, 1$ is a $(p, k + 1)$ -tensor ∇S :

$$(\nabla S)(Y, X_1, \dots, X_k) = \nabla_Y(S(X_1, \dots, X_k)) - \sum_{i=1}^k S(X_1, \dots, \nabla_Y X_i, \dots, X_k). \quad (2.1)$$

Separating symmetric and skew-symmetric parts of (1.1) and using (2.1), we express the covariant derivatives ∇g and ∇F in terms of the (0,3)-torsion tensor $T(X, Y, Z) := g(T(X, Y), Z)$:

$$2(\nabla_X g)(Y, Z) = T(Z, X, Y + fY) - T(X, Y, Z + fZ), \quad (2.2)$$

$$2(\nabla_Z F)(X, Y) = -T(Z, X, Y + fY) - T(Y, Z, X + fX). \quad (2.3)$$

Recall the co-boundary formula for a 2-form F (without the coefficient 3, unlike [1]):

$$dF(X, Y, Z) = X(F(Y, Z)) + Y(F(Z, X)) + Z(F(X, Y)) - F([X, Y], Z) - F([Z, X], Y) - F([Y, Z], X). \quad (2.4)$$

The equality (2.4) yields

$$dF(X, Y, Z) = (\nabla_X^g F)(Y, Z) + (\nabla_Y^g F)(Z, X) + (\nabla_Z^g F)(X, Y). \quad (2.5)$$

The Einstein's metricity condition (1.1) can be written in the form

$$(\nabla_X(g + F))(Y, Z) = -T(X, Y, Z) - T(X, Y, fZ). \quad (2.6)$$

Taking the cyclic sum in (2.6) and applying the equality

$$dF(X, Y, Z) = T(X, Y, fZ) + T(Y, Z, fX) + T(Z, X, fY) + (\nabla_X F)(Y, Z) + (\nabla_Y F)(Z, X) + (\nabla_Z F)(X, Y),$$

which follows from (2.3) and (2.5), we obtain

$$(\nabla_X g)(Y, Z) + (\nabla_Y g)(Z, X) + (\nabla_Z g)(X, Y) = -dF(X, Y, Z) - T(X, Y, Z) - T(Y, Z, X) - T(Z, X, Y). \quad (2.7)$$

Since the left-hand side of (2.7) is symmetric, while the right-hand side is skew-symmetric, we get the following two relations, see [5, Eq (3.3)]:

$$(\nabla_X g)(Y, Z) + (\nabla_Y g)(Z, X) + (\nabla_Z g)(X, Y) = 0,$$

and

$$dF(X, Y, Z) = -T(X, Y, Z) - T(Y, Z, X) - T(Z, X, Y). \quad (2.8)$$

The *difference tensor* K of a linear connection ∇ and the Levi-Civita connection ∇^g is

$$K_X Y := \nabla_X Y - \nabla_X^g Y. \quad (2.9)$$

The Einstein connection ∇ is represented in [5, Eq (3.7)] using the torsion T as

$$g(\nabla_X Y, Z) = g(\nabla_X^g Y, Z) + \frac{1}{2}\{T(X, Y, Z) - T(Z, X, fY) + T(Y, Z, fX)\}. \quad (2.10)$$

By (2.10), the difference (0,3)-tensor $K(X, Y, Z) := g(K_X Y, Z)$ and the torsion tensor of an Einstein connection ∇ are expressed linearly in terms of each other:

$$2K(X, Y, Z) = T(X, Y, Z) - T(Z, X, fY) + T(Y, Z, fX), \quad (2.11)$$

$$T(X, Y, Z) = K(X, Y, Z) - K(Y, X, Z) \iff T(X, Y) = K_X Y - K_Y X. \quad (2.12)$$

Using various presentations of tensors and the definition (2.1), we obtain

$$(\nabla^g F)(Z, X, Y) = (\nabla_Z^g F)(X, Y) = g(X, (\nabla_Z^g f)Y). \quad (2.13)$$

The following fundamental result provides a precise equivalence between the metricity of the Einstein connection and symmetries of the torsion-related tensors.

Lemma 2.1. *For an Einstein connection ∇ , the following conditions are equivalent:*

(i) $K(X, Y, Z) = -K(X, Z, Y)$,

(ii) $\nabla g = 0$,

(iii) $(\nabla_X F)(Y, Z) = -(\nabla_Y F)(X, Z)$, that is, $g((\nabla_X f)Z, Y) = -g((\nabla_Y f)Z, X)$.

Proof. Using the definitions (2.1) and (2.9),

$$\begin{aligned}(\nabla_X g)(Y, Z) &= X(g(Y, Z)) - g(\nabla_X^g Y + K_X Y, Z) - g(Y, \nabla_X^g Z + K_X Z) \\ &= (\nabla_X^g g)(Y, Z) - g(K_X Y, Z) - g(K_X Z, Y).\end{aligned}$$

By this and $\nabla^g g = 0$, the conditions (i) and (ii) are equivalent. The Eqs (2.2) and (2.3) yield

$$(\nabla_Z g)(X, Y) = (\nabla_X F)(Y, Z) + (\nabla_Y F)(X, Z), \quad (2.14)$$

see [5, Eq (3.4)]. By (2.14) and (2.13), the conditions (ii) and (iii) are equivalent. \square

Remark 2.1 (The contorsion tensor). When a linear connection ∇ preserves the metric tensor: $\nabla g = 0$, they call K the *contorsion tensor*; in this case, the tensors K and T are related by

$$2K(Y, Z, X) = T(X, Y, Z) + T(Y, Z, X) - T(Z, X, Y). \quad (2.15)$$

Comparing (2.15) and (2.11) yields the following identity for the torsion of an Einstein connection satisfying $\nabla g = 0$:

$$T(Z, X, Y) - T(Y, Z, X) = T(Y, Z, fX) - T(Z, X, fY). \quad (2.16)$$

Remark 2.2 (The f -torsion condition). Since for an Einstein connection on $(M, G = g + F)$, the tensor $K(X, Y, Z)$ is not totally skew-symmetric, it is interesting to study its particular symmetries. The skew-symmetry of K_X , i.e., $K(X, Y, Z) = -K(X, Z, Y)$, see Lemma 2.1(i), by (2.11), reduces to

$$T(X, Y, Z) - T(Z, X, Y) - T(Z, X, fY) + T(X, Y, fZ) = 0. \quad (2.17)$$

Using (2.17) in (2.3), we obtain

$$(\nabla_X F)(Y, Z) = -T(X, Y, Z) - T(X, Y, fZ).$$

If (2.17) is true, then $[K_X, K_Y] : TM \rightarrow TM$ is a skew-symmetric endomorphism:

$$[K_X, K_Y](Z, Z) = 0, \quad X, Y, Z \in \mathfrak{X}_M. \quad (2.18)$$

If a nonsymmetric pseudo-Riemannian manifold $(M, G = g + F)$ admits an Einstein connection ∇ of totally skew-symmetric torsion, i.e., $T(X, Z, Y) = T(Y, X, Z) = -T(X, Y, Z)$, and the f -torsion condition holds,

$$T(fX, Y) = T(X, fY) = -fT(X, Y), \quad (2.19)$$

then (2.18) is true. Indeed, using (2.19) and the total skew-symmetry of torsion in (2.17) gives zero.

The following property $K_X Y = -K_Y X$ of the difference tensor K (i.e., the skew-symmetry of $K(X, Y, Z)$ only in the first two arguments) by (2.11) reduces to

$$T(Y, Z, fX) + T(X, Z, fY) = 0, \quad (2.20)$$

and characterizes *special Einstein connections*, i.e., the symmetric part of an Einstein connection ∇ coincides with the Levi-Civita connection ∇^g , see [8]. In this case, (2.12) yields $K_X Y = \frac{1}{2}T(X, Y)$. The condition (2.20) does not imply the total skew-symmetry of $K(X, Y, Z)$, which, in addition, requires the condition (2.17).

Lemma 2.2. *If the difference tensor K is totally skew-symmetric, then the torsion tensor T is totally skew-symmetric, and the f -torsion condition (2.19) is true.*

Proof. Let the difference tensor $K(X, Y, Z)$ be totally skew-symmetric. Then, we obtain

$$K(X, Y, Z) + K(Y, X, Z) = 0 \implies T(Y, Z, fX) - T(Z, X, fY) = 0.$$

By Lemma 2.1, we have $\nabla g = 0$. Combining this identity with (2.16) and using the assumption that the tensor $K(X, Y, Z)$ is totally skew-symmetric, we deduce

$$T(X, Y, Z) = -T(X, Z, Y) \quad \text{and} \quad T(X, Y, fZ) = -T(X, Z, fY).$$

So, the torsion $T(X, Y, Z)$ is totally skew-symmetric and the f -torsion condition (2.19) is true. \square

Proposition 2.1. *Let the condition (2.17) be true. Then, we get the equality*

$$K(X, Y, Z) = T(Z, Y, X) - \frac{1}{2} dF(X, Y, Z); \quad (2.21)$$

therefore, the difference tensor K is totally skew-symmetric if and only if the torsion tensor T is totally skew-symmetric. Moreover, the total skew-symmetry of T implies the f -torsion condition (2.19).

Proof. Applying (2.12) to (2.8) gives

$$K(X, Y, Z) - K(Y, X, Z) + K(Y, Z, X) - K(Z, Y, X) + K(Z, X, Y) - K(X, Z, Y) = -dF(X, Y, Z). \quad (2.22)$$

From (2.22), using the condition (2.17), we obtain

$$\begin{aligned} & K(X, Y, Z) - K(Y, X, Z) - K(Y, X, Z) + K(Z, X, Y) + K(Z, X, Y) + K(X, Y, Z) \\ &= 2(K(X, Y, Z) - K(Y, X, Z) + K(Z, X, Y)) = -dF(X, Y, Z). \end{aligned} \quad (2.23)$$

Using (2.12) to (2.23) gives (2.21). By the above, the total skew-symmetry of T implies the total skew-symmetry of K . By Lemma 2.2, the f -torsion condition (2.19) is true. \square

Proposition 2.2 (see Proposition 2 of [8]). *For an Einstein connection ∇ with torsion T , we have*

$$\begin{aligned} 2(\nabla_X^g F)(Y, Z) &= -T(Z, X, Y) - T(X, Y, Z) - T(fZ, X, fY) \\ &\quad - T(X, fY, fZ) + T(Y, fZ, fX) + T(fY, Z, fX). \end{aligned} \quad (2.24)$$

Proof. By (2.1), we have

$$(\nabla_X F)(Y, Z) = X(F(Y, Z)) - F(\nabla_X Y, Z) - F(Y, \nabla_X Z).$$

Using the relation (2.9), we obtain

$$\begin{aligned} (\nabla_X F)(Y, Z) &= X(F(Y, Z)) - F(\nabla_X^g Y + K_X Y, Z) - F(Y, \nabla_X^g Z + K_X Z) \\ &= (\nabla_X^g F)(Y, Z) - F(K_X Y, Z) - F(Y, K_X Z) \\ &= (\nabla_X^g F)(Y, Z) - g(K_X Y, fZ) + g(fY, K_X Z) \\ &= (\nabla_X^g F)(Y, Z) - K(X, Y, fZ) + K(X, Z, fY). \end{aligned} \quad (2.25)$$

Finally, (2.24) follows from (2.3), (2.11), and (2.25). \square

We introduce a natural and very powerful condition that plays a key role in our main results.

Definition 2.1. A linear connection ∇ on a nonsymmetric pseudo-Riemannian manifold $(M, G = g + F)$ satisfies the f^2 -torsion condition if the following is true:

$$T(f^2X, Y) = T(X, f^2Y) = f^2T(X, Y), \quad (2.26)$$

which can be equivalently written as $T(f^2X, Y, Z) = T(X, f^2Y, Z) = T(X, Y, f^2Z)$.

If the f -torsion condition (2.19) holds, then the f^2 -torsion condition (2.26) also holds. The converse is not true in general. If (2.26) is true for an Einstein connection, then, using (2.8) and (2.24), we have

$$\begin{aligned} dF(f^2X, Y, Z) &= dF(X, f^2Y, Z) = dF(X, Y, f^2Z), \\ (\nabla^g F)(f^2X, Y, Z) &= (\nabla^g F)(X, f^2Y, Z) = (\nabla^g F)(X, Y, f^2Z). \end{aligned}$$

3. Einstein connection of an almost contact metric manifold

Contact Riemannian geometry, being an add-dimensional counterpart of symplectic geometry, is a significant field in both pure mathematics and physics, for example, [1]. Developing the approach of [8], we find equations for the Einstein connection of an almost contact metric manifold $(M^{2n+1}, f, \xi, \eta, g)$. Here, g is a pseudo-Riemannian metric, f is a skew-symmetric (1,1)-tensor of rank $2n$, ξ is a unit vector field, and η is a 1-form such that $\eta(\xi) = 1$ and $\eta \circ f = 0$, satisfying the relations

$$f^2 = -I + \eta \otimes \xi, \quad g(fX, fY) = g(X, Y) - \epsilon \eta(X) \eta(Y), \quad (3.1)$$

where I is the identity map and $\epsilon \in \{-1, 1\}$. Therefore, $g(\xi, X) = \epsilon \eta(X)$, in particular, $g(\xi, \xi) = \epsilon$. To simplify further calculations, we will assume that $\epsilon = 1$. Differentiating by X the first equality of (3.1), yields $\nabla_X^g f^2 = \nabla_X^g (\eta \otimes \xi)$. Using the Leibniz rule, we get

$$\nabla_X^g (\eta \otimes \xi) = (\nabla_X^g \eta) \otimes \xi + \eta \otimes \nabla_X^g \xi, \quad \nabla_X^g f^2 = (\nabla_X^g f) f + f \nabla_X^g f.$$

Applying the equalities above to Y , yields

$$(\nabla_X^g f) fY + f(\nabla_X^g f)Y = (\nabla_X^g \eta)(Y) \xi + \eta(Y) \nabla_X^g \xi. \quad (3.2)$$

The following result establishes identities for $\nabla^g F$ under the action of f , which are used in the derivation of torsion and describe how $\nabla^g F$ transforms with respect to the almost contact structure.

Lemma 3.1. For all vector fields X, Y, Z on an almost contact metric manifold, the following hold:

$$(\nabla_X^g F)(fY, fZ) = -(\nabla_X^g F)(Y, Z) + \eta(Y) (\nabla_X^g F)(\xi, Z) + \eta(Z) (\nabla_X^g F)(Y, \xi), \quad (3.3)$$

$$(\nabla_X^g F)(fY, Z) = (\nabla_X^g F)(Y, fZ) + \eta(Z) (\nabla_X^g F)(fY, \xi) - \eta(Y) (\nabla_X^g F)(\xi, fZ). \quad (3.4)$$

Proof. Using $\nabla^g g = 0$ and $F(Y, Z) = g(Y, fZ)$, we have $(\nabla_X^g F)(Y, Z) = g(Y, (\nabla_X^g f)Z)$. Hence,

$$(\nabla_X^g F)(fY, fZ) = g(fY, (\nabla_X^g f) fZ).$$

Substituting (3.2), we get

$$(\nabla_X^g f) fZ = -f(\nabla_X^g f)Z + (\nabla_X^g \eta)(Z)\xi + \eta(Z)\nabla_X^g \xi.$$

Using $g(\xi, fZ) = 0$ gives

$$(\nabla_X^g F)(fY, fZ) = -g(fY, f(\nabla_X^g f)Z) + \eta(Z)g(fY, \nabla_X^g \xi).$$

By $g(fX, fY) = g(X, Y) - \eta(X)\eta(Y)$ and $\eta(fZ) = 0$, we obtain

$$g(fY, f(\nabla_X^g f)Z) = g(Y, (\nabla_X^g f)Z) - \eta(Y)\eta((\nabla_X^g f)Z).$$

Therefore,

$$(\nabla_X^g F)(fY, fZ) = -g(Y, \nabla_X^g fZ) + \eta(Y)\eta((\nabla_X^g f)Z) + \eta(Z)g(fY, \nabla_X^g \xi). \quad (3.5)$$

Finally, substituting the equalities

$$\begin{aligned} g(Y, (\nabla_X^g f)Z) &= (\nabla_X^g F)(Y, Z), \quad \eta((\nabla_X^g f)Z) = g(\xi, (\nabla_X^g f)Z) = (\nabla_X^g F)(\xi, Z), \\ (\nabla_X^g F)(Y, \xi) &= g(Y, (\nabla_X^g f)\xi) = -g(Y, f\nabla_X^g \xi) = g(\nabla_X^g \xi, fY). \end{aligned}$$

In (3.5), yields (3.3). Taking $Z \rightarrow fZ$ in (3.3), we get (3.4). \square

The following theorem gives a strong structural result: Under the f^2 -torsion condition, one obtains $\nabla^g \xi = \nabla^g \eta = 0$, and the vanishing of all torsion components involving ξ , which implies that the torsion is horizontal. Moreover, the explicit formula expresses T in terms of $\nabla^g F$, showing that the torsion is completely determined by the fundamental 2-form.

Theorem 3.1. *Let $(M^{2n+1}, f, \xi, \eta, g)$ be an almost contact metric manifold, considered as an NGT space $(M, G = g + F, \nabla)$ with $F(X, Y) = g(X, fY)$. If the Einstein connection ∇ satisfies the f^2 -torsion condition (2.26), then*

$$\nabla^g \xi = \nabla^g \eta = 0, \quad T(\xi, \cdot, \cdot) = T(\cdot, \cdot, \xi) = 0, \quad (\nabla^g F)(\xi, \cdot, \cdot) = (\nabla^g F)(\cdot, \cdot, \xi) = 0. \quad (3.6)$$

In particular, the torsion T is horizontal, and for $X, Y, Z \perp \xi$ it is given by the same formula (4.14) as for the almost Hermitian case:

$$2T(Y, Z, X) = 2(\nabla_{fX}^g F)(fY, Z) - (\nabla_{fY}^g F)(fZ, X) - (\nabla_{fZ}^g F)(X, fY) - (\nabla_Y^g F)(Z, X) - (\nabla_Z^g F)(X, Y). \quad (3.7)$$

Proof. We write $Z = f^2Z' + \eta(Z)\xi$, where $Z' \perp \xi$, and by (2.26), we get

$$T(\xi, Z, \xi) = T(\xi, f^2Z', \xi) + \eta(Z)T(\xi, \xi, \xi) = T(\xi, f^2Z', \xi) = T(f^2\xi, Z', \xi) = 0. \quad (3.8)$$

Using the f^2 -torsion condition (2.26) again and (3.8), we obtain

$$\begin{aligned} T(X, Z, \xi) &= T(X, f^2Z', \xi) + \eta(Z)T(X, \xi, \xi) = T(f^2X, Z', \xi) = 0, \\ T(\xi, Z, X) &= T(\xi, f^2Z', X) + \eta(Z)T(\xi, \xi, X) = T(f^2\xi, Z', X) = 0. \end{aligned}$$

Hence,

$$T(\xi, Z, \xi) = T(X, Z, \xi) = T(\xi, Z, X) = 0. \quad (3.9)$$

Substituting $X = \xi$ into (2.24), we obtain

$$\nabla_{\xi}^g F = 0, \quad \text{that is,} \quad \nabla_{\xi}^g f = 0. \quad (3.10)$$

Therefore, the ξ -trajectories are g -geodesics: $\nabla_{\xi}^g \xi = 0$. From (2.24), we have

$$(\nabla_X^g F)(Y, Z) - (\nabla_X^g F)(fY, fZ) = T(Y, fZ, fX) + T(fY, Z, fX). \quad (3.11)$$

To prove the equalities $\nabla^g \xi = \nabla^g \eta = 0$, we first note that from (3.9) and (3.11), we obtain

$$(\nabla_X^g F)(Y, \xi) = 0.$$

On the other hand, since $(\nabla_X^g F)(Y, Z) = g(Y, (\nabla_X^g f)Z)$, we get $g(Y, (\nabla_X^g f)\xi) = 0$; hence, $(\nabla_X^g f)\xi = 0$. Further, since $g(\xi, \xi) = 0$, we get $g(\nabla_X^g \xi, \xi) = 0$, and since $(\nabla_X^g f)\xi = 0$, we have $\nabla_X^g \xi \perp \ker f$:

$$f(\nabla_X^g \xi) = \nabla_X^g(f\xi) - (\nabla_X^g f)\xi = 0.$$

Therefore, $\nabla_X^g \xi = 0$ is true. Consequently, from $\eta(Y) = g(\xi, Y)$, we have $(\nabla_X^g \eta)(Y) = g(\nabla_X^g \xi, Y) = 0$; hence, $\nabla^g \eta = 0$ is true. Taking into account (3.3), (3.9), and (3.10), from (3.11) we obtain

$$2(\nabla_X^g F)(Y, Z) = T(Y, fZ, fX) + T(fY, Z, fX). \quad (3.12)$$

Replacing $X \rightarrow fX$ and $Y \rightarrow fY$ in (3.12), we get

$$2(\nabla_{fX}^g F)(fY, Z) = -T(fY, fZ, X) + T(Y, Z, X) = -2(\nabla_{fX}^g F)(fZ, Y).$$

We calculate, following the approach of [8],

$$(\nabla_X^g F)(fY, fZ) = -(\nabla_X^g F)(Y, Z), \quad (\nabla_X^g F)(fY, Z) = (\nabla_X^g F)(Y, fZ). \quad (3.13)$$

Using (2.24), we get

$$(\nabla_X^g F)(Y, Z) - (\nabla_X^g F)(fY, fZ) = T(Y, fZ, fX) + T(fY, Z, fX).$$

Substituting this into (2.24) and using (3.13), we obtain

$$T(X, Y, Z) + T(Z, X, Y) + T(X, fY, fZ) + T(fZ, X, fY) = 0. \quad (3.14)$$

Thus, (2.24) reduces to

$$2(\nabla_X^g F)(Y, Z) = T(Y, fZ, fX) + T(fY, Z, fX). \quad (3.15)$$

The cyclic sum of (3.15), obtained using (3.13) and (3.14), is

$$(\nabla_X^g F)(Y, Z) + (\nabla_Y^g F)(Z, X) + (\nabla_Z^g F)(X, Y) = -T(X, Y, Z) - T(Y, Z, X) - T(Z, X, Y). \quad (3.16)$$

Replacing $Y \rightarrow fY$ and $Z \rightarrow fZ$ in (3.16) gives

$$(\nabla_X^g F)(fY, fZ) + (\nabla_{fY}^g F)(fZ, X) + (\nabla_{fZ}^g F)(X, fY) = T(X, Y, Z) + T(Z, X, Y) - T(fY, fZ, X). \quad (3.17)$$

Replacing $X \rightarrow fX$ and $Y \rightarrow fY$ in (3.15), we obtain

$$T(fY, fZ, X) = T(Y, Z, X) - 2(\nabla_{fX}^g F)(fY, Z). \quad (3.18)$$

Substituting (3.18) into (3.17), gives the formula (3.7). \square

Remark 3.1. Although the f^2 -torsion condition becomes trivial for almost Hermitian manifolds, it is strong enough for almost contact metric manifolds. In Theorem 3.1, our manifold is locally the direct product of \mathbb{R} and an almost Hermitian manifold $(\bar{M}^{2n}, J, \bar{g})$, and ∇ , being an Einstein connection on the $2n$ -dimensional factor, is given by (4.14), where J is replaced by f . Consequently, the 1-form η is closed, $d\eta = 0$. The distribution $\ker \eta$ is involutive; thus, it is Frobenius integrable.

We study Einstein connections satisfying the f^2 -torsion condition, without assuming any symmetry of the torsion tensor. If, in addition, the torsion T is totally skew-symmetric, then the general formula (3.7) reduces to (3.20), and $\nabla_{\xi}^g F = 0$. This additional assumption leads to the almost-nearly cosymplectic case, which we recall below from [5].

Corollary 3.1. Under the conditions of Theorem 3.1, the torsion T of the Einstein connection is totally skew-symmetric if and only if (3.6) is true, and the following Codazzi-type condition holds:

$$(\nabla_X^g F)(Y, Z) = (\nabla_Y^g F)(Z, X) = (\nabla_Z^g F)(X, Y) = \frac{1}{3} dF(X, Y, Z). \quad (3.19)$$

In this case, the torsion is given by

$$T(X, Y, Z) = -(\nabla_Z^g F)(X, Y) = -\frac{1}{3} dF(X, Y, Z). \quad (3.20)$$

Proof. By Theorem 3.1, we have $\nabla_{\xi}^g F = 0$ and $\nabla^g \xi = 0$. Let us consider the f^2 -torsion condition under the additional assumption that T is totally skew-symmetric. Applying $T(Y, X, Z) = -T(Y, Z, X)$, from (3.7) we have

$$0 = 3(\nabla_{fX}^g F)(fY, Z) + 3(\nabla_{fZ}^g F)(fY, X) - (\nabla_Z^g F)(X, Y) - (\nabla_X^g F)(Z, Y). \quad (3.21)$$

Taking $X \rightarrow fX$ and $Y \rightarrow fY$ in (3.21) and using (3.6) and the equality $(\nabla_X^g F)(fY, fZ) = -(\nabla_X^g F)(Y, Z)$, see (3.3), we get

$$0 = 3(\nabla_X^g F)(Y, Z) - 3(\nabla_{fZ}^g F)(Y, fX) + (\nabla_Z^g F)(X, Y) - (\nabla_{fX}^g F)(Z, fY).$$

Adding the equation above to (3.21), gives

$$(\nabla_{fX}^g F)(fY, Z) = -(\nabla_X^g F)(Y, Z). \quad (3.22)$$

Substituting (3.22) into (3.21), we obtain $(\nabla_X^g F)(Y, Z) = (\nabla_Z^g F)(X, Y)$; hence, in view of (2.5), the condition (3.19) is true. Finally, from (3.7) and (3.22) we get $T(Y, Z, X) = -(\nabla_X^g F)(Y, Z)$ that, in view of (3.19), completes the proof of (3.20). \square

An almost contact metric manifold $(M^{2n+1}, f, \xi, \eta, g)$ is said to be *almost-nearly cosymplectic*, see [5], if the Levi-Civita covariant derivative of f satisfies

$$g((\nabla_X^g f)Y, Z) = \frac{1}{3} dF(fX, fY, Z) + \frac{1}{6} \eta(Z) d\eta(Y, fX) - \frac{1}{2} \eta(Y) d\eta(fZ, X),$$

where $F(X, Y) = g(X, fY)$, in particular, ξ is a Killing vector field.

Corollary 3.2 (see [5]). Let $(M^{2n+1}, f, \xi, \eta, g)$ be an almost contact metric manifold with the fundamental 2-form $F(X, Y) = g(X, fY)$, considered as a nonsymmetric Riemannian manifold $(M, G = g + F)$. Then, an Einstein connection ∇ on M has a totally skew-symmetric torsion T if and only if the manifold is almost-nearly cosymplectic, in this case, $T(X, Y, Z) = -\frac{1}{3} dF(X, Y, Z)$.

We will explicitly present the special Einstein connection satisfying the f^2 -torsion condition.

Theorem 3.2. *Let ∇ be a special Einstein connection satisfying the f^2 -torsion condition (2.26) on an almost contact metric manifold $(M^{2n+1}, f, \xi, \eta, g)$, considered as an NGT space $(M, G = g + F, \nabla)$ with $F(X, Y) = g(X, fY)$. Then, (3.6) is true, and the torsion T of ∇ is given by*

$$2T(X, Y, Z) = -(\nabla_X^g F)(Y, Z) + (\nabla_{fY}^g F)(fX, Z) + (\nabla_{fZ}^g F)(fX, Y). \quad (3.23)$$

Proof. Let X, Y, Z be horizontal vector fields (i.e., $X, Y, Z \in \ker \eta$). Theorem 3.1 gives, see (3.12),

$$2(\nabla_X^g F)(Y, Z) = T(Y, fZ, fX) + T(fY, Z, fX). \quad (3.24)$$

Using (2.20) (since ∇ is a special Einstein connection), and $f^2 = -I$ on $\ker \eta$, we obtain

$$\begin{aligned} T(Y, fZ, fX) &= -T(X, fZ, fY) = T(fZ, X, fY) = -T(X, Y, Z), \\ T(fY, Z, fX) &= -T(X, Z, f^2 Y) = T(X, Z, Y). \end{aligned}$$

Hence, (3.24) becomes

$$2(\nabla_X^g F)(Y, Z) = -T(X, Y, Z) + T(X, Z, Y). \quad (3.25)$$

Next, replacing X by fZ , Y by fX , and Z by Y in (3.24), we get

$$2(\nabla_{fZ}^g F)(fX, Y) = T(fX, fY, f^2 Z) + T(f^2 X, Y, f^2 Z) = -T(fX, fY, Z) + T(X, Y, Z). \quad (3.26)$$

Similarly, replacing X by fY , Y by fX , and Z by Z in (3.24), we obtain

$$2(\nabla_{fY}^g F)(fX, Z) = -T(fX, fZ, Y) + T(X, Z, Y). \quad (3.27)$$

Also, applying (2.20), we get

$$T(fX, fY, Z) + T(fX, fZ, Y) = -T(fY, fX, Z) - T(fZ, fX, Y) = 0. \quad (3.28)$$

Subtracting (3.26) and (3.27) from (3.25), and using (3.28), we obtain

$$\begin{aligned} &2(\nabla_X^g F)(Y, Z) - 2(\nabla_{fZ}^g F)(fX, Y) - 2(\nabla_{fY}^g F)(fX, Z) \\ &= -2T(X, Y, Z) + T(fX, fY, Z) + T(fX, fZ, Y) = -2T(X, Y, Z). \end{aligned}$$

Thus, (3.23) is true for all horizontal vector fields X, Y, Z . If one of these vector fields is ξ , both sides vanish, hence the identity (3.23) holds for arbitrary vector fields on M . \square

Corollary 3.3. *Under the conditions of Theorem 3.2, the torsion T of a special Einstein connection is totally skew-symmetric if and only if the equalities in (3.6) are true; in this case,*

$$T(X, Y, Z) = -\frac{1}{2}(\nabla_X^g F)(Y, Z) = -\frac{1}{6}dF(X, Y, Z). \quad (3.29)$$

Proof. Assume that the torsion T is totally skew-symmetric. Then, by Theorem 3.2, we have the equalities (3.6) and (3.23). Since T is totally skew-symmetric, using (3.23), we get

$$(\nabla_{fZ}^g F)(fX, Y) + (\nabla_{fY}^g F)(fX, Z) = 0. \quad (3.30)$$

Using this in (3.23), we obtain the first equality in (3.29): $T(X, Y, Z) = -\frac{1}{2}(\nabla_X^g F)(Y, Z)$.

Replacing Y by fY and Z by fZ in (3.30), and using $f^2|_{\ker \eta} = -I|_{\ker \eta}$, together with the equality

$$(\nabla_X^g F)(fY, fZ) = -(\nabla_X^g F)(Y, Z),$$

we obtain

$$(\nabla_X^g F)(Y, Z) = (\nabla_Y^g F)(Z, X) = (\nabla_Z^g F)(X, Y).$$

Therefore,

$$dF(X, Y, Z) = (\nabla_X^g F)(Y, Z) + (\nabla_Y^g F)(Z, X) + (\nabla_Z^g F)(X, Y) = 3(\nabla_X^g F)(Y, Z);$$

hence $T(X, Y, Z) = -\frac{1}{2}(\nabla_X^g F)(Y, Z) = -\frac{1}{6}dF(X, Y, Z)$ is true, which completes the proof of (3.29).

Conversely, assume that $T(X, Y, Z) = -\frac{1}{2}(\nabla_X^g F)(Y, Z)$. Since $(\nabla_X^g F)(Y, Z)$ is skew-symmetric in Y and Z , it follows that T is skew-symmetric with respect to Y and Z ; hence, T is totally skew-symmetric. \square

4. Einstein connection of an NGT space represented by a weak almost Hermitian manifold

In this section, we explicitly present the Einstein connection of a nonsymmetric pseudo-Riemannian space $(M, G = g + F)$ with non-degenerate F , in particular, of a weak almost Hermitian manifold.

A pseudo-Riemannian manifold (M^{2n}, g) endowed with a skew-symmetric (1,1)-tensor f of maximal rank $2n$ (f^2 is not necessarily equal to $-I$) is called a *weak almost Hermitian manifold*, see [9]. In this case, we introduce a new (1,1)-tensor $\tilde{Q} = -f^2 - I$, and the condition (2.26) in terms of \tilde{Q} reads

$$T(\tilde{Q}X, Y, Z) = T(X, \tilde{Q}Y, Z) = T(X, Y, \tilde{Q}Z), \quad (4.1)$$

and similarly for dF and $\nabla^g F$. The following operator plays a fundamental role in a weak almost Hermitian manifold:

$$P = I - f^2.$$

This operator measures the deviation from an almost Hermitian manifold. In an almost Hermitian manifold ($g > 0$) one has $f^2 = -I$, hence $P = 2I$ is non-degenerate. In the weak case, the non-degeneracy of P is an additional condition ensuring that the torsion equations can be solved explicitly.

The following lemma is used in the proof of Theorem 4.1.

Lemma 4.1. *Let (M^{2n}, f, g) be a weak almost Hermitian manifold, considered as a nonsymmetric pseudo-Riemannian manifold $(M, G = g + F)$, where $F(X, Y) = g(X, fY)$. If an Einstein connection ∇ on M satisfies the f^2 -torsion condition (2.26), then*

$$\begin{aligned} T(fY, fZ, PX) &= T(Y, Z, Pf^2X) - 2(\nabla_{f^2X}^g F)(Y, fZ) + 2(\nabla_{f^2X}^g F)(fY, Z) \\ &\quad + dF(Y, Z, Pf^2X) - dF(fY, fZ, PX), \end{aligned} \quad (4.2)$$

$$\begin{aligned} T(fY, Z, PfX) &= -T(Y, fZ, PfX) - T(Y, Z, P(I + f^2)X) + 2(\nabla_{PX}^g F)(Y, Z) \\ &\quad - 2(\nabla_{f^2X}^g F)(fY, Z) + 2(\nabla_{f^2X}^g F)(Y, fZ) - dF(Y, Z, P(I + f^2)X). \end{aligned} \quad (4.3)$$

Proof. Using $Z \rightarrow fZ$ in (2.24) and $\tilde{Q} = -f^2 - I$, we get

$$\begin{aligned} 2(\nabla_X^g F)(Y, fZ) &= -T(fZ, X, Y) - T(X, Y, fZ) + T(Z, X, fY) + T(\tilde{Q}Z, X, fY) \\ &\quad + T(X, fY, Z) + T(X, fY, \tilde{Q}Z) - T(Y, Z, fX) - T(Y, \tilde{Q}Z, fX) + T(fY, fZ, fX). \end{aligned}$$

Then, using $Y \rightarrow fY$ in (2.24), we get

$$\begin{aligned} 2(\nabla_X^g F)(fY, Z) &= -T(Z, X, fY) - T(X, fY, Z) + T(fZ, X, Y) + T(fZ, X, \tilde{Q}Y) \\ &\quad + T(X, Y, fZ) + T(X, \tilde{Q}Y, fZ) - T(Y, Z, fX) - T(\tilde{Q}Y, Z, fX) + T(fY, fZ, fX). \end{aligned}$$

Subtracting the above equations and applying the f^2 -torsion condition (2.26), we get

$$\begin{aligned} &2\{T(Z, X, fY) + T(X, fY, Z)\} + \{T(Z, \tilde{Q}X, fY) + T(\tilde{Q}X, fY, Z)\} \\ &- 2\{T(X, Y, fZ) + T(fZ, X, Y)\} - \{T(\tilde{Q}X, Y, fZ) + T(fZ, \tilde{Q}X, Y)\} \\ &= 2(\nabla_X^g F)(Y, fZ) - 2(\nabla_X^g F)(fY, Z). \end{aligned}$$

Applying (2.8) to the above equation and using the f^2 -torsion condition (2.26), we get

$$\begin{aligned} T(fY, Z, PX) &= T(Y, fZ, PX) - 2(\nabla_X^g F)(Y, fZ) + 2(\nabla_X^g F)(fY, Z) \\ &\quad + dF(Y, fZ, PX) - dF(fY, Z, PX). \end{aligned} \quad (4.4)$$

Using $Z \rightarrow fZ$ in (4.4) gives (4.2). Then, applying (2.8), we rewrite (2.24) as

$$\begin{aligned} T(fY, Z, X) &= -T(Y, fZ, X) - T(Y, Z, X) - dF(Y, Z, X) \\ &\quad - T(fY, fZ, X) - dF(fY, fZ, X) + 2(\nabla_X^g F)(Y, Z). \end{aligned}$$

Replacing $X \rightarrow PX$ in the above equation and using (4.2), we obtain (4.3). \square

The following theorem provides an explicit and nontrivial expression for the torsion in the weak almost Hermitian case, under the assumption that $P = I - f^2$ is non-degenerate. The formula (4.5) gives a complete reconstruction of torsion (and hence, an Einstein connection) in terms of $\nabla^g F$ and dF , involving higher-order compositions of f and the operator \tilde{Q} . This result generalizes the classical [8, Theorem 1] and can be seen as a full solution of the Einstein connection problem in this general case.

Theorem 4.1. *Let (M^{2n}, f, g) be a weak almost Hermitian manifold, considered as a nonsymmetric pseudo-Riemannian manifold $(M, G = g + F)$, where $F(X, Y) = g(X, fY)$. Suppose that an Einstein connection ∇ on M satisfies the f^2 -torsion condition (2.26), and the tensor $P := I - f^2$ is non-degenerate. Then, the torsion T of ∇ is given by*

$$\begin{aligned} 2T(Y, Z, (I + \frac{1}{2}\tilde{Q})^2 f^4 X) &= (\nabla_X^g F)((f + f^3)Y, fZ) + (\nabla_Y^g F)(f^2 Z, X) + (\nabla_Z^g F)(f^2 X, Y) \\ &\quad - (\nabla_{fX}^g F)(f^3 Y, Z) - (\nabla_{fX}^g F)(f^2 Y, fZ) + (\nabla_{fY}^g F)(f^3 Z, X) + (\nabla_{fZ}^g F)(f^2 X, fY) \\ &\quad + (\nabla^g F)(P^{-1}(2\tilde{Q} + \tilde{Q}^2)f^2 X, fY, Z) - (\nabla^g F)(P^{-1}(2\tilde{Q} + \tilde{Q}^2)f^2 X, Y, fZ) \\ &\quad - (\nabla^g F)(\tilde{Q}f^2 X, Y, Z) - \frac{1}{2}dF((3\tilde{Q} + 2\tilde{Q}^2)X, fY, fZ) + \frac{1}{2}dF((3\tilde{Q} + \tilde{Q}^2)f^2 X, Y, Z). \end{aligned} \quad (4.5)$$

Proof. Applying $Y \rightarrow fY$ and $Z \rightarrow fZ$ to (2.24), we have

$$\begin{aligned} 2(\nabla_X^g F)(fY, fZ) &= -T(Z, X, Y) - T(X, Y, Z) \\ &\quad - T(fZ, X, fY) - T(X, fY, fZ) - T(fY, Z, fX) - T(Y, fZ, fX) \\ &\quad - T(\tilde{Q}Z, X, Y) - T(Z, X, \tilde{Q}Y) - T(X, \tilde{Q}Y, Z) - T(X, Y, \tilde{Q}Z) \\ &\quad - T(\tilde{Q}Z, X, \tilde{Q}Y) - T(X, \tilde{Q}Y, \tilde{Q}Z) - T(fY, \tilde{Q}Z, fX) - T(\tilde{Q}Y, fZ, fX). \end{aligned} \quad (4.6)$$

From (2.24) with $Y \rightarrow -f^2Y$, we get

$$\begin{aligned} -2(\nabla_X^g F)(f^2Y, Z) &= -T(Z, X, Y) - T(X, Y, Z) - T(fZ, X, fY) - T(X, fY, fZ) \\ &\quad + T(Y, fZ, fX) + T(fY, Z, fX) - T(Z, X, \tilde{Q}Y) - T(X, \tilde{Q}Y, Z) \\ &\quad - T(fZ, X, \tilde{Q}fY) - T(X, \tilde{Q}fY, fZ) + T(\tilde{Q}Y, fZ, fX) + T(\tilde{Q}fY, Z, fX). \end{aligned} \quad (4.7)$$

The relation (4.6) together with (4.7) yields the following:

$$\begin{aligned} &-T(X, fY, fZ) - T(fZ, X, fY) = T(X, Y, Z) + T(Z, X, Y) + T(Z, X, \tilde{Q}Y) \\ &+ T(X, \tilde{Q}Y, Z) + \frac{1}{2}\{T(fZ, X, \tilde{Q}fY) + T(X, \tilde{Q}fY, fZ) + T(\tilde{Q}Z, X, Y) \\ &+ T(\tilde{Q}Z, X, \tilde{Q}Y) + T(X, Y, \tilde{Q}Z) + T(X, \tilde{Q}Y, \tilde{Q}Z) - T(\tilde{Q}fY, Z, fX) \\ &+ T(fY, \tilde{Q}Z, fX)\} + (\nabla_X^g F)(fY, fZ) - (\nabla_X^g F)(f^2Y, Z). \end{aligned} \quad (4.8)$$

Using (4.8), we reduce (4.7) to the following:

$$\begin{aligned} &-(\nabla_X^g F)(f^2Y, Z) - (\nabla_X^g F)(fY, fZ) = T(Y, fZ, fX) + T(fY, Z, fX) + T(\tilde{Q}Y, fZ, fX) \\ &+ \frac{1}{2}\{T(\tilde{Q}Z, X, \tilde{Q}Y) - T(fZ, X, \tilde{Q}fY) - T(X, \tilde{Q}fY, fZ) + T(\tilde{Q}Z, X, Y) \\ &+ T(X, Y, \tilde{Q}Z) + T(X, \tilde{Q}Y, \tilde{Q}Z) + T(\tilde{Q}fY, Z, fX) + T(fY, \tilde{Q}Z, fX)\}. \end{aligned} \quad (4.9)$$

Taking the cyclic sum of (4.9), in view of (4.8), we get:

$$\begin{aligned} &-(\nabla_X^g F)(f^2Y, Z) - (\nabla_Y^g F)(f^2Z, X) - (\nabla_Z^g F)(f^2X, Y) \\ &= -T(Y, Z, X) - T(X, Y, Z) - T(Z, X, Y) + \frac{1}{2}\delta_1(X, Y, Z), \end{aligned} \quad (4.10)$$

where $\delta_1(X, Y, Z)$ is given in Section 5. From (4.10) with $Y \rightarrow fY$ and $Z \rightarrow fZ$, using (4.8), we obtain

$$\begin{aligned} &(\nabla_X^g F)(f^2Y, Z) - (\nabla_X^g F)(fY, fZ) - (\nabla_X^g F)(f^3Y, fZ) - (\nabla_{fY}^g F)(f^3Z, X) - (\nabla_{fZ}^g F)(f^2X, fY) \\ &= T(X, Y, Z) + T(Z, X, Y) - T(fY, fZ, X) + \frac{1}{2}\delta_2(X, Y, Z), \end{aligned} \quad (4.11)$$

where $\delta_2(X, Y, Z)$ is given in Section 5. Replacing $X \rightarrow fX$ and $Y \rightarrow fY$ in (4.9), we get:

$$\begin{aligned} T(fY, fZ, X) &= T(Y, Z, X) + (\nabla_{fX}^g F)(f^2Y, fZ) + (\nabla_{fX}^g F)(f^3Y, Z) \\ &+ \frac{1}{2}T(fZ, fX, \tilde{Q}Y) + \frac{1}{2}T(fZ, fX, \tilde{Q}^2Y) + \frac{1}{2}T(fX, \tilde{Q}Y, fZ) + \frac{1}{2}T(fX, \tilde{Q}^2Y, fZ) \\ &+ \frac{3}{2}T(\tilde{Q}Y, Z, X) + \frac{1}{2}T(\tilde{Q}^2Y, Z, X) + \frac{3}{2}T(\tilde{Q}Y, Z, \tilde{Q}X) + \frac{1}{2}T(\tilde{Q}^2Y, Z, \tilde{Q}X) \\ &+ \frac{1}{2}T(\tilde{Q}Z, fX, fY) + \frac{1}{2}T(\tilde{Q}Z, fX, \tilde{Q}fY) + \frac{1}{2}T(fX, fY, \tilde{Q}Z) + \frac{1}{2}T(fX, \tilde{Q}fY, \tilde{Q}Z) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} T(Y, \tilde{Q}Z, X) + \frac{1}{2} T(\tilde{Q}Y, \tilde{Q}Z, X) + \frac{1}{2} T(Y, \tilde{Q}Z, \tilde{Q}X) + \frac{1}{2} T(\tilde{Q}Y, \tilde{Q}Z, \tilde{Q}X) \\
& - T(fY, fZ, \tilde{Q}X) - T(\tilde{Q}fY, fZ, X) - T(\tilde{Q}fY, fZ, \tilde{Q}X) + T(Y, Z, \tilde{Q}X).
\end{aligned} \tag{4.12}$$

Then, substituting (4.12) into (4.11), we get the following:

$$\begin{aligned}
& (\nabla_X^g F)(f^2 Y, Z) - (\nabla_X^g F)((f + f^3)Y, fZ) - (\nabla_{fY}^g F)(f^3 Z, X) \\
& - (\nabla_{fZ}^g F)(f^2 X, fY) + (\nabla_{fX}^g F)(f^3 Y, Z) + (\nabla_{fX}^g F)(f^2 Y, fZ) \\
& = T(X, Y, Z) + T(Z, X, Y) - T(Y, Z, X) + \frac{1}{2} \delta_3(X, Y, Z),
\end{aligned}$$

where $\delta_3(X, Y, Z)$ is given in Section 5. This relation together with (4.10) yields

$$\begin{aligned}
2 T(Y, Z, X) & = (\nabla_X^g F)((f + f^3)Y, fZ) + (\nabla_Y^g F)(f^2 Z, X) + (\nabla_Z^g F)(f^2 X, Y) - (\nabla_{fX}^g F)(f^3 Y, Z) \\
& - (\nabla_{fX}^g F)(f^2 Y, fZ) + (\nabla_{fY}^g F)(f^3 Z, X) + (\nabla_{fZ}^g F)(f^2 X, fY) + \frac{1}{2} \delta_4(X, Y, Z),
\end{aligned} \tag{4.13}$$

where $\delta_4(X, Y, Z)$ is given in Section 5. Using (2.8), $-f^2 = I + \tilde{Q}$, and the f^2 -torsion condition (4.1), we reduce $\delta_4(X, Y, Z)$ to $\delta_5(X, Y, Z)$, given in Section 5. Using (4.2) and (4.3), we get

$$\begin{aligned}
T(fY, fZ, (3\tilde{Q} + 2\tilde{Q}^2)X) & = T(Y, Z, (3\tilde{Q} + 2\tilde{Q}^2)f^2 X) + dF((3\tilde{Q} + 2\tilde{Q}^2)f^2 X, Y, Z) \\
& - dF((3\tilde{Q} + 2\tilde{Q}^2)X, fY, fZ) - 2(\nabla^g F)(P^{-1}(3\tilde{Q} + 2\tilde{Q}^2)f^2 X, Y, fZ) \\
& + 2(\nabla^g F)(P^{-1}(3\tilde{Q} + 2\tilde{Q}^2)f^2 X, fY, Z), \\
T(Y, fZ, \tilde{Q}f^3 X) + T(fY, Z, \tilde{Q}f^3 X) & = T(Y, Z, \tilde{Q}^2 f^2 X) + dF(\tilde{Q}^2 f^2 X, Y, Z) \\
& + 2(\nabla^g F)(\tilde{Q}f^2 X, Y, Z) + 2(\nabla^g F)(P^{-1}\tilde{Q}f^4 X, Y, fZ) - 2(\nabla^g F)(P^{-1}\tilde{Q}f^4 X, fY, Z),
\end{aligned}$$

which allow us to transform some terms in $\delta_5(X, Y, Z)$ and then rewrite it as

$$\begin{aligned}
\delta_6(X, Y, Z) & = -T(Y, Z, (12\tilde{Q} + 13\tilde{Q}^2 + 6\tilde{Q}^3 + \tilde{Q}^4)X) + dF((3\tilde{Q} + \tilde{Q}^2)f^2 X, Y, Z) \\
& - dF((3\tilde{Q} + 2\tilde{Q}^2)X, fY, fZ) - 2(\nabla^g F)(P^{-1}(2\tilde{Q} + \tilde{Q}^2)f^2 X, Y, fZ) \\
& + 2(\nabla^g F)(P^{-1}(2\tilde{Q} + \tilde{Q}^2)f^2 X, fY, Z) - 2(\nabla^g F)(\tilde{Q}f^2 X, Y, Z).
\end{aligned}$$

Note that the tensors $\delta_i(X, Y, Z)$ ($1 \leq i \leq 6$) vanish when $\tilde{Q} = 0$. From the Eq (4.13) with $\delta_4(X, Y, Z)$ replaced by $\delta_6(X, Y, Z)$, the Eq (4.5) follows. Using (2.5), we can further simplify (4.5) to the form with $\nabla^g F$ -terms and without dF -terms. \square

For a nonsymmetric pseudo-Riemannian space $(M, G = g + F)$, the tensor $P = I - f^2$ can be degenerate. For example, consider the Lorentzian metric $g = (-, +, \dots, +)$ on \mathbb{R}^{2n} with an orthonormal basis (e_1, \dots, e_{2n}) , that is, $g(e_i, e_j) = \epsilon_i \delta_{ij}$ and $\epsilon_1 = -1, \epsilon_2 = \dots = \epsilon_{2n} = 1$. Let a skew-symmetric $(1,1)$ -tensor f have a block form with two parts: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and J_{2n-2} . Then, f^2 has block form with two parts: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $-I_{2n-2}$. Therefore, a $(1,1)$ -tensor $P = I - f^2$ has rank $2n - 2$, and $Pe_1 = Pe_2 = 0$.

For a nonsymmetric Riemannian space $(M, G = g + F)$, the tensor $P = I - f^2$ is positive definite, and hence, non-degenerate. Therefore, we have the following.

Corollary 4.1. *Let ∇ be an Einstein connection of a weak almost Hermitian manifold $(M^{2n}, f, g > 0)$, considered as a nonsymmetric Riemannian space with $F(X, Y) = g(X, fY)$. If the f^2 -torsion condition (2.26) is valid, then the torsion T of ∇ is given by (4.5).*

Corollary 4.2 (see Theorems 1 and 2 of [8]). *Let (M^{2n}, J, g) be an almost Hermitian manifold considered as a nonsymmetric Riemannian manifold $(M, G = g + F)$, where $F(X, Y) = g(X, JY)$ and $J^2 = -I$. Then, the torsion of an Einstein connection ∇ on M is given by*

$$2T(Y, Z, X) = 2(\nabla_{JX}^g F)(JY, Z) - (\nabla_{JY}^g F)(JZ, X) - (\nabla_{JZ}^g F)(X, JY) - (\nabla_Y^g F)(Z, X) - (\nabla_Z^g F)(X, Y); \quad (4.14)$$

in particular, the torsion of a special Einstein connection ∇ on M , see (2.20), is given by

$$2T(X, Y, Z) = (\nabla_X^g F)(Y, Z) - (\nabla_{JZ}^g F)(JX, Y) - (\nabla_{JY}^g F)(JX, Z).$$

Remark 4.1. For $f = J$ (an almost complex structure, $J^2 = -I$), (4.2) and (4.3) reduce to the following:

$$T(JY, JZ, X) = -T(Y, Z, X) - dF(Y, Z, X) - dF(JY, JZ, X),$$

$$T(JY, Z, JX) = -T(Y, JZ, JX) + 2(\nabla_X^g F)(Y, Z) + (\nabla_X^g F)(JY, Z) - (\nabla_X^g F)(Y, JZ).$$

From (4.2) with $f = J$, we get

$$T(JY, JZ, X) = -T(Y, Z, X) - dF(Y, Z, X) - dF(JY, JZ, X) + (\nabla_X^g F)(Y, JZ) - (\nabla_X^g F)(JY, Z).$$

Subtracting the above equation from (3.18), we get a compact presentation of (4.14):

$$2T(Y, Z, X) = (\nabla_{2JX-X}^g F)(JY, Z) + (\nabla_X^g F)(Y, JZ) - dF(Y, Z, X) - dF(JY, JZ, X).$$

Remark 4.2. In view of (4.14), the Einstein connection ∇ on an almost Hermitian manifold is special, see definition (2.20), if and only if the following identity holds:

$$(\nabla_X^g F)(Y, Z) + (\nabla_Y^g F)(X, Z) = (\nabla_{JX}^g F)(JY, Z) + (\nabla_{JY}^g F)(JX, Z). \quad (4.15)$$

Among the sixteen Gray-Hervella classes [3] of almost Hermitian manifolds, the condition (4.15) of a special Einstein connection is satisfied for the following classes:

$$\mathcal{W}_1, \mathcal{W}_3, \mathcal{W}_4, \mathcal{W}_3 \oplus \mathcal{W}_4, \mathcal{W}_1 \oplus \mathcal{W}_3, \mathcal{W}_1 \oplus \mathcal{W}_4, \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4;$$

consequently, the Einstein connection on an almost Hermitian manifold belonging to any of these classes is a special Einstein connection, see Theorem 3 of [8].

The following example shows how solution of Theorem 4.1 splits into a weighted Riemannian product and that the resulting Einstein connection reduces independently on each almost Hermitian factor.

Example 4.1. Take almost Hermitian manifolds (M_j, J_j, g_j) ($j = 1, \dots, m$), where $J_j^2 = -I_j$ and $g_j > 0$. Consider the Riemannian product $(M, g) = (M_1 \times \dots \times M_m, g_1 \oplus \dots \oplus g_m)$, and define the $(1, 1)$ -tensor $f = \sqrt{\lambda_1} J_1 \oplus \dots \oplus \sqrt{\lambda_m} J_m$, where $\lambda_j > 0$ are real numbers. Then, (M, f, g) is a weak almost Hermitian manifold with $f^2 = -\lambda_1 I_1 \oplus \dots \oplus \lambda_m I_m$, and on TM_j we get

$$f|_{TM_j} = \sqrt{\lambda_j} J_j, \quad f^2|_{TM_j} = -\lambda_j I, \quad f^3|_{TM_j} = -\lambda_j^{3/2} J_j, \quad f^4|_{TM_j} = \lambda_j^2 I, \\ \tilde{Q}|_{TM_j} = (\lambda_j - 1) I_j, \quad P|_{TM_j} = (1 + \lambda_j) I_j, \quad P^{-1}|_{TM_j} = \frac{1}{1 + \lambda_j} I_j.$$

Since the Levi-Civita connection of the Riemannian product preserves the splitting $TM = \bigoplus_{j=1}^m TM_j$, f^2 acts as a multiple of the identity on each factor. The 2-form F corresponding to f is given by $F = F_1 \oplus \dots \oplus F_m$, where $F_j(X, Y) = g_j(X, \sqrt{\lambda_j} J_j Y)$. As a consequence, $(\nabla_X^g F)(Y, Z) = 0$, whenever X belongs to one factor and at least one of Y and Z belongs to a different factor.

If an Einstein connection ∇ satisfies the f^2 -torsion condition (2.26), then $T(X, Y) = 0$ for $X \in TM_i$, $Y \in TM_j$ ($i \neq j$). Hence, the torsion tensor splits into the direct sum $T = T|_{TM_1} \oplus \dots \oplus T|_{TM_m}$. Substituting $Q|_{TM_j} = (\lambda_j - 1)I_j$ into (4.5), then using the relations

$$\begin{aligned}(\nabla_X^{g_j} F_j)(J_j Y, J_j Z) &= -(\nabla_X^{g_j} F_j)(Y, Z), \\(\nabla_X^{g_j} F_j)(J_j Y, Z) &= (\nabla_X^{g_j} F_j)(Y, J_j Z),\end{aligned}$$

we uniquely determine the torsion component $T|_{TM_j}$ by

$$\begin{aligned}\frac{1}{2}(\lambda_j + 1)^2 \lambda_j T|_{TM_j}(Y, Z, X) &= -(\nabla_Y^{g_j} F_j)(Z, X) - (\nabla_Z^{g_j} F_j)(X, Y) \\&+ \lambda_j \{2(\nabla_{J_j X}^{g_j} F_j)(Y, J_j Z) - (\nabla_{J_j Y}^{g_j} F_j)(J_j Z, X) - (\nabla_{J_j Z}^{g_j} F_j)(X, J_j Y)\} \\&+ (\lambda_j - 1) \{2(\nabla_X^{g_j} F_j)(Y, Z) - (\lambda_j + \frac{1}{2})dF_j(X, J_j Y, J_j Z) - \frac{1}{2}(\lambda_j + 2)dF_j(X, Y, Z)\}.\end{aligned}$$

Thus, the weak almost Hermitian manifolds produce a family of Einstein connections parametrized by the constants λ_j , measuring the deviation from the almost Hermitian manifolds ($\lambda_j = 1$).

If each (M_j, J_j, g_j) is a Kähler manifold, then $\nabla^{g_j} F_j = 0$ and $dF_j = 0$. In this case, all torsion components vanish, and the Einstein connection reduces to the Levi-Civita connection. The formula above for $\lambda_j = 1$ gives the solution (4.14).

The following example shows that weak nearly Kähler manifolds naturally appear in the study of Einstein connections with skew-symmetric torsion.

Example 4.2. If $(\nabla_X^g f)X = 0$ ($X \in \mathfrak{X}_M$), or, $\nabla^g f = 0$ holds, then a weak almost Hermitian manifold (M^{2n}, f, g) is a weak nearly Kähler manifold, or, a weak Kähler manifold, respectively, see [9]. Let (M, f, g) be a weak nearly Kähler manifold, considered as a nonsymmetric Riemannian manifold $(M, G = g + F)$ with $F(X, Y) = g(X, fY)$. If ∇ is an Einstein connection on M with totally skew-symmetric torsion T , then the f -torsion condition (2.19) holds, and $T(X, Y, Z) = -\frac{1}{3}dF(X, Y, Z)$ is true, see Example 3.4 in [13]. Thus, for weak nearly Kähler manifolds, the torsion tensor is determined by the exterior derivatives of the fundamental 2-form. In this case, (1.2) reduces to $K_X Y = \frac{1}{2}T(X, Y)$, and $\nabla_X Y = \nabla_X^g Y + \frac{1}{2}T(X, Y)$ is true.

5. Conclusions

This paper represents a high-quality contribution to the field of differential geometry and geometric structures with torsion. We have demonstrated how weak metric structures provide a natural and flexible framework for NGT. We have explicitly presented the Einstein connection for nonsymmetric pseudo-Riemannian, in particular, for weak almost Hermitian manifolds, satisfying the f^2 -torsion condition. We have obtained the corresponding formulas for almost contact metric manifolds, assuming the f^2 -torsion condition, which turned out to be a natural and very powerful structural assumption. We have obtained the corresponding formulas expressing the torsion in terms

of $\nabla^s F$ and dF , and the additional \tilde{Q} -terms, explained precisely how the weak almost Hermitian case differs from the almost Hermitian case. The presented identities provide a new tool for constructing examples and studying classes of nonsymmetric pseudo-Riemannian manifolds, in particular, weak almost Hermitian manifolds with the usual Gray-Hervella classification, in Einstein's nonsymmetric gravitational theory. In further research we will study weak (para-) metric f -manifolds, in particular, weak (para-) contact metric manifolds, with Einstein connections or connections with totally skew-symmetric torsion, satisfying the f^2 -torsion type conditions.

Author contributions

Vladimir Rovenski and Milan Zlatanović: Conceptualization, Methodology, Investigation, Validation, Writing-original draft. All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Prof. Vladimir Rovenski is an editorial board member for AIMS Mathematics and was not involved in the editorial review and/or the decision to publish this article.

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Appendix: The auxiliary terms

Here, we present $\delta_i(X, Y, Z)$ ($1 \leq i \leq 5$), see Section 4, verified with the help of MAPLE program. Using the skew-symmetry of the torsion $T(X, Y, Z)$ in X, Y , we prepare the terms of δ_i with right permutations (X, Y, Z) , (Y, Z, X) , (Z, X, Y) only, and replace f^2 with $-I - \tilde{Q}$ in all terms. In computing δ_5 we apply the f^2 -torsion condition, see (4.1), and the equality (2.8) to eliminate T -terms with (Z, X, Y) .

$$\begin{aligned} \delta_1(X, Y, Z) = & -T(fZ, X, \tilde{Q}fY) - T(X, \tilde{Q}Y, Z) - T(X, \tilde{Q}fY, fZ) + T(\tilde{Q}fY, Z, fX) \\ & - T(Z, X, \tilde{Q}Y) - T(X, Y, \tilde{Q}Z) + T(\tilde{Q}Y, fZ, fX) - T(fX, Y, \tilde{Q}fZ) - T(Y, \tilde{Q}Z, X) \\ & - T(Y, \tilde{Q}fZ, fX) - T(Y, Z, \tilde{Q}X) + T(\tilde{Q}fZ, X, fY) - T(Z, \tilde{Q}X, Y) + T(\tilde{Q}fX, Y, fZ) \\ & - T(fY, Z, \tilde{Q}fX) - T(Z, \tilde{Q}fX, fY) + T(\tilde{Q}Z, fX, fY) + T(\tilde{Q}X, fY, fZ). \end{aligned}$$

$$\begin{aligned} \delta_2(X, Y, Z) = & T(X, \tilde{Q}Y, Z) - T(\tilde{Q}Y, fZ, fX) - 2T(\tilde{Q}fY, Z, fX) + T(Z, X, \tilde{Q}Y) + 2T(\tilde{Q}Z, X, Y) \\ & + T(\tilde{Q}Z, X, \tilde{Q}Y) + T(X, Y, \tilde{Q}Z) + 2T(fY, \tilde{Q}Z, fX) - T(\tilde{Q}fX, fY, Z) - T(\tilde{Q}fX, fY, \tilde{Q}Z) \\ & + T(fX, fY, \tilde{Q}Z) + T(fX, fY, \tilde{Q}^2Z) + T(fY, \tilde{Q}^2Z, fX) + T(Y, fZ, \tilde{Q}fX) \\ & + T(\tilde{Q}Y, fZ, \tilde{Q}fX) - T(\tilde{Q}^2Y, fZ, fX) + T(fZ, \tilde{Q}fX, Y) + T(fZ, \tilde{Q}fX, \tilde{Q}Y) \\ & - T(\tilde{Q}fY, \tilde{Q}Z, fX) - T(\tilde{Q}fZ, fX, Y) - T(\tilde{Q}fZ, fX, \tilde{Q}Y) - T(X, fY, \tilde{Q}fZ) \\ & - T(fY, \tilde{Q}fZ, X) - T(fY, fZ, \tilde{Q}X) + T(\tilde{Q}^2Z, X, Y) + T(\tilde{Q}^2Z, X, \tilde{Q}Y) - T(X, \tilde{Q}^2Y, Z) \end{aligned}$$

$$\begin{aligned}
& - T(X, \tilde{Q}^2 Y, \tilde{Q} Z) + T(\tilde{Q} X, \tilde{Q} Y, Z) + T(\tilde{Q} X, \tilde{Q} Y, \tilde{Q} Z) - T(Z, X, \tilde{Q}^2 Y) \\
& - T(\tilde{Q} Z, X, \tilde{Q}^2 Y) + T(\tilde{Q} X, Y, Z) + T(\tilde{Q} X, Y, \tilde{Q} Z) - T(fZ, \tilde{Q} X, fY),
\end{aligned}$$

$$\begin{aligned}
\delta_3(X, Y, Z) = & T(fZ, \tilde{Q} fX, \tilde{Q} Y) - T(\tilde{Q} fZ, fX, \tilde{Q} Y) - T(\tilde{Q} fZ, fX, Y) - T(\tilde{Q} fY, \tilde{Q} Z, fX) \\
& + T(fZ, \tilde{Q} fX, Y) - T(\tilde{Q}^2 Y, fZ, fX) + T(\tilde{Q} Y, fZ, \tilde{Q} fX) + T(Y, fZ, \tilde{Q} fX) \\
& + T(fY, \tilde{Q}^2 Z, fX) + T(fX, fY, \tilde{Q}^2 Z) - T(\tilde{Q} fX, fY, \tilde{Q} Z) - T(\tilde{Q} fX, fY, Z) \\
& - T(\tilde{Q} Z, X, \tilde{Q}^2 Y) - T(Z, X, \tilde{Q}^2 Y) - 3 T(\tilde{Q} Y, Z, \tilde{Q} X) + T(\tilde{Q} X, \tilde{Q} Y, \tilde{Q} Z) \\
& - T(fX, \tilde{Q} Y, fZ) - 3 T(\tilde{Q} Y, Z, X) - T(fX, \tilde{Q} fY, \tilde{Q} Z) - T(\tilde{Q} Z, fX, \tilde{Q} fY) \\
& + 2 T(\tilde{Q} fY, fZ, \tilde{Q} X) + 2 T(\tilde{Q} fY, fZ, X) - T(fX, \tilde{Q}^2 Y, fZ) - T(fZ, fX, \tilde{Q} Y) \\
& - T(fZ, fX, \tilde{Q}^2 Y) - T(\tilde{Q}^2 Y, Z, X) - T(\tilde{Q}^2 Y, Z, \tilde{Q} X) - T(\tilde{Q} Y, \tilde{Q} Z, X) \\
& - T(\tilde{Q} Y, \tilde{Q} Z, \tilde{Q} X) - T(Y, \tilde{Q} Z, \tilde{Q} X) + T(\tilde{Q} X, \tilde{Q} Y, Z) - T(X, \tilde{Q}^2 Y, \tilde{Q} Z) \\
& - T(X, \tilde{Q}^2 Y, Z) + T(\tilde{Q}^2 Z, X, \tilde{Q} Y) + T(\tilde{Q}^2 Z, X, Y) + T(fY, fZ, \tilde{Q} X) \\
& - T(fY, \tilde{Q} fZ, X) - T(X, fY, \tilde{Q} fZ) - T(fZ, \tilde{Q} X, fY) + T(\tilde{Q} X, Y, \tilde{Q} Z) \\
& + T(\tilde{Q} X, Y, Z) - T(\tilde{Q} Y, fZ, fX) + T(X, Y, \tilde{Q} Z) + T(\tilde{Q} Z, X, \tilde{Q} Y) \\
& + T(Z, X, \tilde{Q} Y) + T(X, \tilde{Q} Y, Z) + 2 T(fY, \tilde{Q} Z, fX) + 2 T(\tilde{Q} Z, X, Y) \\
& - 2 T(\tilde{Q} fY, Z, fX) - T(\tilde{Q} Z, fX, fY) - 2 T(Y, Z, \tilde{Q} X) - T(Y, \tilde{Q} Z, X),
\end{aligned}$$

$$\begin{aligned}
\delta_4(X, Y, Z) = & 2 T(\tilde{Q} Z, X, Y) - T(fZ, X, \tilde{Q} fY) - T(X, \tilde{Q} fY, fZ) - T(\tilde{Q} fY, Z, fX) \\
& + T(\tilde{Q} Z, X, \tilde{Q} Y) + 2 T(fY, \tilde{Q} Z, fX) - T(fX, Y, \tilde{Q} fZ) - 2 T(Y, \tilde{Q} Z, X) \\
& - T(Y, \tilde{Q} fZ, fX) + T(\tilde{Q} X, Y, Z) + T(\tilde{Q} X, Y, \tilde{Q} Z) - 3 T(Y, Z, \tilde{Q} X) \\
& - T(Y, \tilde{Q} Z, \tilde{Q} X) + T(\tilde{Q} fZ, X, fY) - T(fZ, \tilde{Q} X, fY) - T(Z, \tilde{Q} X, Y) \\
& + T(\tilde{Q} fX, Y, fZ) - T(fY, Z, \tilde{Q} fX) - T(Z, \tilde{Q} fX, fY) - 3 T(\tilde{Q} Y, Z, X) \\
& - 3 T(\tilde{Q} Y, Z, \tilde{Q} X) - T(fX, \tilde{Q} Y, fZ) + T(\tilde{Q} X, fY, fZ) - T(X, fY, \tilde{Q} fZ) \\
& - T(fY, \tilde{Q} fZ, X) + T(fY, fZ, \tilde{Q} X) + T(\tilde{Q}^2 Z, X, Y) + T(\tilde{Q}^2 Z, X, \tilde{Q} Y) \\
& - T(X, \tilde{Q}^2 Y, Z) - T(X, \tilde{Q}^2 Y, \tilde{Q} Z) + T(\tilde{Q} X, \tilde{Q} Y, Z) + T(\tilde{Q} X, \tilde{Q} Y, \tilde{Q} Z) \\
& - T(Z, X, \tilde{Q}^2 Y) - T(\tilde{Q} Z, X, \tilde{Q}^2 Y) - T(\tilde{Q} fX, fY, Z) - T(\tilde{Q} fX, fY, \tilde{Q} Z) \\
& + T(fX, fY, \tilde{Q}^2 Z) + T(fY, \tilde{Q}^2 Z, fX) + T(Y, fZ, \tilde{Q} fX) + T(\tilde{Q} Y, fZ, \tilde{Q} fX) \\
& - T(\tilde{Q}^2 Y, fZ, fX) + T(fZ, \tilde{Q} fX, Y) + T(fZ, \tilde{Q} fX, \tilde{Q} Y) - T(fX, \tilde{Q}^2 Y, fZ) \\
& - T(fZ, fX, \tilde{Q} Y) - T(fZ, fX, \tilde{Q}^2 Y) - T(\tilde{Q}^2 Y, Z, X) - T(\tilde{Q}^2 Y, Z, \tilde{Q} X) \\
& - T(\tilde{Q} Y, \tilde{Q} Z, X) - T(\tilde{Q} Y, \tilde{Q} Z, \tilde{Q} X) + 2 T(\tilde{Q} fY, fZ, X) \\
& + 2 T(\tilde{Q} fY, fZ, \tilde{Q} X) - T(\tilde{Q} fY, \tilde{Q} Z, fX) - T(\tilde{Q} fZ, fX, Y) \\
& - T(\tilde{Q} fZ, fX, \tilde{Q} Y) - T(\tilde{Q} Z, fX, \tilde{Q} fY) - T(fX, \tilde{Q} fY, \tilde{Q} Z),
\end{aligned}$$

$$\begin{aligned}
\delta_5(X, Y, Z) &\stackrel{(4.1)}{=} -T(fZ, X, \tilde{Q}fY) + T(Z, X, \tilde{Q}Y) + T(X, Y, \tilde{Q}Z) - 8T(Y, Z, \tilde{Q}X) \\
&\quad - T(fX, Y, \tilde{Q}fZ) - T(X, fY, \tilde{Q}fZ) + T(Z, X, \tilde{Q}^2Y) - T(fX, fY, \tilde{Q}Z) \\
&\quad - T(fX, fY, \tilde{Q}^2Z) + 2T(fY, fZ, \tilde{Q}X) - T(fZ, fX, \tilde{Q}Y) - T(fZ, fX, \tilde{Q}^2Y) \\
&\quad + 2T(fY, fZ, \tilde{Q}^2X) - 6T(Y, Z, \tilde{Q}^2X) - 2T(Y, Z, \tilde{Q}^3X) - T(Z, fX, \tilde{Q}^2fY) \\
&\quad + T(X, Y, \tilde{Q}^2Z) - T(fX, Y, \tilde{Q}^2fZ) - T(Z, fX, \tilde{Q}fY) \\
&\stackrel{(2.8)}{=} -T(Y, Z, (9\tilde{Q} + 7\tilde{Q}^2 + 2\tilde{Q}^3)X) + T(fY, fZ, (3\tilde{Q} + 2\tilde{Q}^2)X) \\
&\quad - T(Y, fZ, \tilde{Q}f^3X) - T(fY, Z, \tilde{Q}f^3X) - dF(\tilde{Q}f^3X, Y, fZ) \\
&\quad - dF(\tilde{Q}f^3X, fY, Z) + dF(\tilde{Q}X, fY, fZ) + dF(\tilde{Q}f^2X, Y, Z).
\end{aligned}$$



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