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Research article

Global dynamics in a generalized slow-fast predator-prey model

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Abstract: This paper studies a generalized slow-fast predator-prey model extended from the classic Gause model, where the predator reproduction rate is taken as a small singular perturbation parameter. First, by applying Dulac's criterion, we derive the sufficient conditions under which the local asymptotic stability of the system's unique positive equilibrium implies its global asymptotic stability in the first quadrant. Second, based on the geometric singular perturbation theory, we prove the existence of a unique relaxation oscillation surrounding the positive equilibrium, and show that this relaxation oscillation converges to the transcritical slow-fast cycle in the Hausdorff distance as the perturbation parameter approaches zero. Finally, with relaxation oscillation properties and Zhang Zhifen's limit cycle uniqueness theorem, we further derive sufficient conditions for the existence and uniqueness of stable limit cycles. The results enrich dynamical researches on slow-fast predator-prey systems and are applicable to related ecological dynamic analyses.

Keywords: predator-prey model; Slow-fast system; global asymptotic stability; relaxation oscillation; limit cycle; geometric singular perturbation theory
Mathematics Subject Classification: 34C07, 34C23, 34D15

1. Introduction

Predator-prey models are fundamental in theoretical ecology, providing insights into the dynamics of biological systems that involve interacting species. The study of global stability and the existence of limit cycles in these models is crucial to understand the long-term behavior of ecological systems. In the present study, we consider the following generalized slow-fast predator-prey system:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = xg(x) - y\Phi(x), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \varepsilon y\left(p(x) - q(x)\right), \end{cases} \tag{1.1}$$

where x and y represent the densities of the prey and predator populations, respectively. $0 < \varepsilon \ll 1$ means the reproductive rate of the predator is much smaller than that of the prey. $g(x)$ denotes the prey's intrinsic growth rate, $\Phi(x)$ stands for the functional response, $p(x)$ is predator's growth rate, and $q(x)$ is the predator's death rate. The functions $g(x)$, $\Phi(x)$, $p(x)$, and $q(x)$ are smooth and satisfy the following assumptions:

(i) $g(0) = \alpha > 0$, $g'(x) < 0$ for all $x \geq 0$, and there exists $K > 0$ such that $g(K) = 0$. This means the prey follows logistic growth with a positive intrinsic growth rate and a finite environmental carrying capacity K.

(ii) $\Phi(0) = 0$, $\Phi'(x) > 0$ for all $x \geq 0$, and $\Phi(x) = x\hat{\Phi}(x)$ for some smooth function $\hat{\Phi}(x)$. This implies that predation only occurs when the prey exists, and the predation efficiency rises with increasing prey density.

(iii) $p(0) = 0$ and $p'(x) \geq 0$ for all $x \geq 0$. This shows that the predator cannot grow without the prey, and higher prey density improves the predator's growth.

(iv) $q(0) > 0$, $q'(x) \leq 0$ for all $x \geq 0$, and $\lim\limits_{x\to\infty} q(x) = q_\infty > 0$. This indicates that the predator has a natural mortality even without the prey, and its mortality decreases with more prey but remains positive in the limit.

Note that the assumptions in System (1.1) are biologically meaningful. Setting $p(x) = \Phi(x)$ and $q(x)$ as a constant in System (1.1) yields the classical Gause predator-prey system.

The global stability and the existence, uniqueness, and non-uniqueness of limit cycles in predator-prey models without slow-fast time scales have long been extensively investigated. Liu, Zegeling, and Huang recently explored the application of Liénard transformations to predator-prey systems in [1], with a primary focus on the limit cycle uniqueness of the classical Gause and Leslie-Gower systems with different functional responses. Zhang and Yan further derived the necessary and sufficient conditions for the nonexistence of limit cycles in Leslie-Gower predator-prey models [2]. Other notable advances include Dai, Zhao, and Sang's proof of the existence of four limit cycles in a Leslie-type predator-prey system with a generalized Holling type III functional response [3], and Xiao and Zhang's systematic study on the uniqueness and nonexistence of limit cycles for general predator-prey systems [4], with numerous other relevant contributions available in [5–10]. Inspired by these fruitful studies on non-slow-fast predator-prey models, we extend such research to the slow-fast framework and investigate the global stability as well as the existence and uniqueness of limit cycles for a generalized slow-fast predator-prey model given by (1.1).

For predator-prey models with slow-fast time scales, relaxation oscillations represent a special type of limit cycle, and the existence of such limit cycles has also attracted considerable research attention in recent years. The geometric singular perturbation theory has become a key tool to investigate relaxation oscillation phenomena in slow-fast predator-prey models, spawning many insightful studies.

For example, Zhao and Shen [11] analyzed the slow-fast dynamics of classical predator-prey models with the Allee effect, including the existence of relaxation oscillations and canards, and Zhu and Liu further investigated canard cycles and relaxation oscillations in a slow-fast Leslie-Gower predator-prey model with the Allee effect [12]. Su and Zhang [13] explored the global stability, relaxation oscillations, and canard explosions of a predator-prey model with a sigmoid functional response, while Chen and Zhang also studied the slow-fast dynamics including relaxation oscillations of predator-prey models with the sigmoid functional response [14]. Chowdhury, Banerjee, and Petrovskii examined canards, relaxation oscillations, and pattern formation in several slow-fast predator-prey models [15, 16], and Shen, Hsu, and Yang investigated the existence of relaxation oscillations for slow-fast intraguild predation models with evolutionary effects [17]. Beyond ecological models, Hsu and Ruan also probed into relaxation oscillations and the entry-exit function in multidimensional slow-fast systems [18], thus providing a general theoretical reference for relevant research. In addition, Li et al. studied relaxation oscillations for a Leslie-type predator-prey model with a Holling type I functional response [19], and Li, Li, and Wu analyzed relaxation oscillations and canard explosions in a Leslie-Gower predator-prey system with an additive Allee effect [20].

Beyond the above research on two-dimensional slow-fast predator-prey systems, abundant supplementary studies concerning multi-scale biochemical oscillators, core geometric singular perturbation theories, classical pest interaction models, and spatially extended piecewise smooth ecological systems have been documented. Research on circadian clock and genetic regulatory slow-fast oscillators that exhibit relaxation bursting phenomena can be referred to in the work of Chen, Duan and Li on the reduction of a three-dimensional circadian oscillator model [21], their subsequent study on the Tyson-Hong-Thron-Novak circadian oscillator [22], and the analysis of relaxation oscillations and canards in a regulated two-gene model by De Maesschalck, Kiss and Kovács [23]. A series of foundational papers constructing general entry-exit mechanisms and oscillation counting theories for singularly perturbed planar systems are summarized in the entry-exit theorem and relaxation oscillation criteria established by Ai and Sadhu [24], the criterion for relaxation oscillations in predator-prey and epidemic models developed by Hsu and Wolkowicz [25], the number and stability analysis of relaxation oscillations by Hsu [26], the classical relaxation oscillation study in predator-prey systems by Liu, Xiao and Yi [27], and the canard cycle analysis for Holling-type predator-prey systems by Li and Zhu [28]. Several typical non-diffusive slow-fast population models, such as forest pest interaction systems and Leslie-type predator-prey variants, are analyzed in the spruce-budworm interaction model with Holling type II response by Tai and Zhang [29], the relaxation oscillations of a slow-fast predator-prey model with a piecewise smooth functional response by Li, Wang and Wu [30], the travelling waves in the Holling–Tanner model with weak diffusion by Ghazaryan, Manukian and Schecter [31], the relaxation oscillations in a slow-fast modified Leslie-Gower model by Wang and Zhang [32], and the degenerate transcritical bifurcation analysis in a slow-fast Leslie-Gower model by Zhong and Shen [33]. Recent advances covering diffusive predator-prey models, piecewise smooth functional response systems and singular perturbation epidemic models are presented in the spatio-temporal slow-fast predator-prey system with reproductive Allee effect studied by Fu and Liu [34], the traveling front in a diffusive model with Beddington-DeAngelis response by Dong and Liu [35], the canard cycle and nonsmooth bifurcation analysis in a piecewise-smooth continuous predator-prey model by Zhu and Liu [36], the existence of traveling wave solutions in a singularly perturbed predator-prey equation with diffusion by Zhu and Liu [37], the canard explosion in a singularly perturbed SIS epidemic model by Wu, Zhang and Xie [38], the slow-fast dynamics of a piecewise-smooth Leslie-Gower model with Holling type I response and weak Allee effect by Wu and Xie [39], the relaxation oscillations of a piecewise-smooth slow-fast Bazykin model by Wu, Lu and Xie [40], and the relaxation oscillations in predator-prey systems with piecewise smooth functional responses by Huang, Huzak and Yao [41]. Further valuable discussions on canard orbits, traveling waves and singular bifurcations are contained in the bibliographies of the above literatures.

Against this research backdrop, this work aims to further explore the dynamics of slow-fast predator-prey systems by investigating a generalized slow-fast extension of the classical Gause predator-prey model, where the predator's reproductive rate is treated as a small singular perturbation parameter $0 < \varepsilon \ll 1$.

First, this paper establishes global stability conditions using the Lyapunov method and the Dulac criterion, and analyzes how the monotonicity of prey-isocline affects the equilibrium stability, so as to enrich relevant stability research results. Based on the geometric singular perturbation theory, we prove the existence of relaxation oscillations and verify that these oscillations converge to slow-fast cycles when the small perturbation parameter approaches zero. Combined with the characteristics of relaxation oscillations and Zhang Zhifen's theorem, we analyze the existence and uniqueness of limit cycles, and establish explicit sufficient conditions for stable limit cycles. The research methods and conclusions in this paper are universal, and can be used for the dynamic analysis of various predator-prey ecological models.

Previous studies have extensively explored the existence and uniqueness of limit cycles in conventional population models, as well as relaxation oscillations within slow-fast ecological systems. Compared with these works, this paper makes distinct improvements. Most existing slow-fast predator-prey models adopt fixed functional response forms and only analyze local dynamics. Herein, we establish a more generalized model with variable growth and mortality functions to enhance its universality. Beyond merely verifying the existence of relaxation oscillations, we further establish global stability criteria via the Lyapunov method and the Dulac criterion, and investigate the relationship between the sign of the derivative of the prey isocline at x_* and the stability of the equilibrium. Furthermore, combining the geometric singular perturbation theory with Zhang Zhifen's theorem, we not only confirm the limit cycle existence but also strictly prove their uniqueness, and obtain easy-to-use sufficient conditions. The established results are applicable to a wide range of ecological models, which further enriches the dynamical research theories for slow-fast predator-prey systems.

The paper is organized as follows: Section 2 introduces the slow-fast structure of the predator-prey model; Section 3 establishes the existence of positive equilibria and key properties of the associated critical manifold; Section 4 is devoted to the analysis of global stability; Section 5 proves the existence of relaxation oscillations surrounding the positive equilibrium; and Section 6 addresses the existence and uniqueness of limit cycles.

2. Slow-fast structure

By introducing the slow time scale $\tau = \varepsilon t$ into System (1.1), we obtain the following transformed system:

$$\begin{cases} \varepsilon \dfrac{\mathrm{d}x}{\mathrm{d}\tau} = xg(x) - y\Phi(x), \\ \dfrac{\mathrm{d}y}{\mathrm{d}\tau} = y\left(p(x) - q(x)\right). \end{cases} \tag{2.1}$$

Here, the original variable t is called the fast time, while τ is referred to as the slow time. Accordingly, System (1.1) and (2.1) are termed the fast system and the slow system, respectively, and they are equivalent for any $\varepsilon > 0$.

By formally setting $\varepsilon = 0$ in the fast system (1.1), we derive the fast subsystem as follows:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = xg(x) - y\Phi(x), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = 0. \end{cases}$$

Similarly, substituting $\varepsilon = 0$ into the slow system (2.1) yields the slow subsystem as follows:

$$\begin{cases} 0 = xg(x) - y\Phi(x), \\ \dfrac{\mathrm{d}y}{\mathrm{d}\tau} = y\left(p(x) - q(x)\right). \end{cases} \tag{2.2}$$

Now, we introduce the following function:

$$F(x) = \frac{xg(x)}{\Phi(x)}.$$

Then, the critical manifold (or critical set) of System (1.1) is then defined as follows:

$$C_0 := \{(x, y) \mid y = F(x)\} \cup C_y := \{(x, y) \mid x = 0\}.$$

3. Existence of positive equilibria and properties of the critical manifold

System (1.1) always has two boundary equilibria: $E_0(0, 0)$ and $E_K(K, 0)$. The conditions for the existence and positions of positive equilibria are concluded as follows.

Theorem 3.1. *System* (1.1) *has a unique positive equilibrium* $E_*(x_*, y_*)$ *iff* $p(0) < q(0)$ *and* $p(K) > q(0)$.

Proof. Denote $G(x) = p(x) - q(x)$. Evidently, $G'(x) > 0$ from assumptions. Based on the given conditions,

$$G(0) = p(0) - q(0) < 0,$$

$$G(K) = p(K) - q(K) > p(K) - q(0) > 0;$$

hence, there exists a unique $x_* \in (0, K)$ such that $G(x_*) = p(x_*) - q(x_*) = 0$, (i.e., System (1.1) has a unique positive equilibrium). This completes the proof of the theorem. □

Theorem 3.2. *Assume that*

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{xg(x)}{\Phi(x)}\right) < 0.$$

Then, the critical manifold C_0 admits a unique nonnormally hyperbolic point $D(x_M, F(x_M))$ with $x_M \in (0, K)$, as illustrated in Figure 1. Moreover, it satisfies the following:

$$F'(x) > 0, \quad x \in (0, x_M) \quad \text{and} \quad F'(x) < 0, \quad x \in (x_M, K).$$

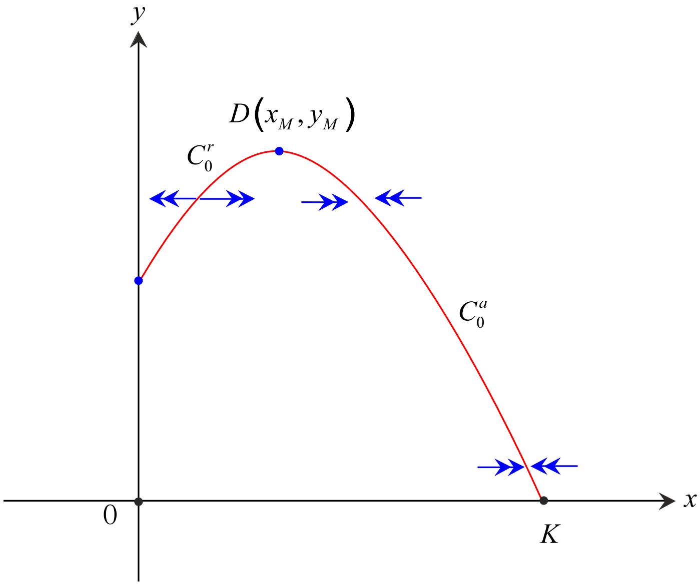

Figure 1. Illustration of the critical manifold geometry. The critical manifold C_0 (red curve) is comprised of two normally hyperbolic submanifolds: C_0^r (repelling) and C_0^a (attracting). Additionally, the two nonnormally hyperbolic points $D(x_M, y_M)$ and $(0, F(0))$ (blue dots) are shown. Double arrows denote fast flow directions.

The nonnormally hyperbolic point D divides the critical manifold C_0 into two normally hyperbolic components:

$$C_0^a := \{(x, y) : x_M < x < K,\ y = F(x)\}, \quad C_0^r := \{(x, y) : 0 < x < x_M,\ y = F(x)\},$$

where C_0^a is attracting and C_0^r is repelling.

Proof. Since $g(0) = \alpha > 0$, $g(K) = 0$, $\Phi(0) = 0$, and $\Phi'(x) > 0$ for $x \geq 0$, we obtain

$$F(K) = \frac{Kg(K)}{\Phi(K)} = 0$$

and

$$F(0) = \lim_{x\to 0} F(x) = \lim_{x\to 0} \frac{xg(x)}{\Phi(x)} = \lim_{x\to 0} \frac{g(x) + xg'(x)}{\Phi'(x)} = \frac{\alpha}{\Phi'(0)} > 0.$$

From the condition

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{xg(x)}{\Phi(x)}\right) < 0,$$

it follows that $F(x)$ attains a unique maximum at $x_M \in (0, K)$ with $F'(x_M) = 0$, which is the unique nonnormally hyperbolic point. Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{xg(x)}{\Phi(x)}\right) > 0 \quad \text{for } 0 < x < x_M$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{xg(x)}{\Phi(x)}\right) < 0 \quad \text{for } x_M < x < K.$$

Accordingly, C_0^a is attracting and C_0^r is repelling. □

4. Global stability

First, we investigate the local stability of E_* by calculating the Jacobian matrix as follows:

$$J(E_*) = \begin{pmatrix} g(x_*) + x_* g'(x_*) - y_* \Phi'(x_*) & -\Phi(x_*) \\ \varepsilon y_* \left(p'(x_*) - q'(x_*)\right) & 0 \end{pmatrix}.$$

Then, we have $DetJ(E_*) = \varepsilon \Phi(x_*) y_* \left(p'(x_*) - q'(x_*)\right)$ and $trJ(E_*) = g(x_*) + x_* g'(x_*) - y_* \Phi'(x_*)$. Clearly, $DetJ(E_*) > 0$ by the assumptions. Then, the local stability of E_* can be obtained by studying $trJ(E_*)$.

Theorem 4.1. *If the derivative of the prey-isocline at x_* is negative (positive), then the equilibrium E_* is locally asymptotically stable (unstable).*

Proof. The prey-isocline of System (1.1) is given by $y = \dfrac{xg(x)}{\Phi(x)}$. Direct computation yields the following:

$$\begin{aligned} \operatorname{tr} J(E_*) &= g(x_*) + x_* g'(x_*) - y_* \Phi'(x_*) \\ &= g(x_*) + x_* g'(x_*) - \frac{x_* g(x_*) \Phi'(x_*)}{\Phi(x_*)} \\ &= x_* g(x_*) \left(\frac{\mathrm{d}}{\mathrm{d}x} \ln \frac{xg(x)}{\Phi(x)}\right)\Bigg|_{x=x_*}. \end{aligned}$$

Thus, $\operatorname{tr} J(E_*) < 0$ provided that

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln \frac{xg(x)}{\Phi(x)}\bigg|_{x=x_*} < 0, \quad \text{i.e.,} \quad \frac{\mathrm{d}}{\mathrm{d}x} \frac{xg(x)}{\Phi(x)}\bigg|_{x=x_*} < 0,$$

and $\operatorname{tr} J(E_*) > 0$ whenever

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln \frac{xg(x)}{\Phi(x)}\bigg|_{x=x_*} > 0, \quad \text{i.e.,} \quad \frac{\mathrm{d}}{\mathrm{d}x} \frac{xg(x)}{\Phi(x)}\bigg|_{x=x_*} > 0.$$

The proof is complete. □

Theorem 4.2. *$E_*(x_*, y_*)$ is globally stable in the first quadrant when*

$$\left(\frac{xg(x)}{\Phi(x)} - y_*\right)(x - x_*) \le 0.$$

Proof. Construct the following Lyapunov function:

$$V(x, y) = \int_{x_*}^{x} \frac{p(s) - q(s)}{\Phi(s)} \mathrm{d}s + \frac{1}{\varepsilon}\left(y - y_* - y_* \ln \frac{y}{y_*}\right).$$

Global stability is established by verifying the following two properties:

(i) $V(x, y)$ is positive definite with respect to E_*; and

(ii) $\dot{V}(x, y) \le 0$ for all $x > 0$, $y > 0$.

Property (i). Let $\varphi(z) = z - 1 - \ln z$ for $z > 0$. Then, $\varphi'(z) = 1 - 1/z$, which implies $\varphi(z) \ge 0$ for all $z > 0$, with equality if and only if $z = 1$. Thus,

$$\frac{1}{\varepsilon}\left(y - y_* - y_* \ln \frac{y}{y_*}\right) = \frac{y_*}{\varepsilon}\varphi\left(\frac{y}{y_*}\right) \ge 0,$$

with equality if and only if $y = y_*$.

Since $p'(x) \ge 0$ and $q'(x) \le 0$, the function $G(x) = p(x) - q(x)$ is strictly increasing. From $G(x_*) = 0$ it follows that $(x - x_*)G(x) \ge 0$. Together with $\Phi(x) > 0$ for $x > 0$, one obtains

$$\int_{x_*}^{x} \frac{p(s) - q(s)}{\Phi(s)} \mathrm{d}s \ge 0,$$

with equality if and only if $x = x_*$.

Hence, $V(x, y) \ge 0$ for all $x > 0, y > 0$, and $V(x, y) = 0$ if and only if $(x, y) = (x_*, y_*)$.

Property (ii). The time derivative of $V(x, y)$ along solutions of System (1.1) satisfies the following:

$$\begin{aligned}
\dot{V}(x, y) &= \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial V}{\partial y} \cdot \frac{dy}{dt} \\
&= \frac{p(x) - q(x)}{\Phi(x)}\big(xg(x) - y\Phi(x)\big) + \frac{1}{\varepsilon}\left(1 - \frac{y_*}{y}\right)\varepsilon y\big(p(x) - q(x)\big) \\
&= \big(p(x) - q(x)\big)\left(\frac{xg(x)}{\Phi(x)} - y\right) + \big(p(x) - q(x)\big)(y - y_*) \\
&= \big(p(x) - q(x)\big)\left(\frac{xg(x)}{\Phi(x)} - y_*\right).
\end{aligned}$$

The monotonicity of $G(x) = p(x) - q(x)$ implies the following:

$$(x - x_*)(p(x) - q(x)) \ge 0.$$

Under the assumption

$$\left(\frac{xg(x)}{\Phi(x)} - y_*\right)(x - x_*) \le 0,$$

one has

$$\big(p(x)-q(x)\big)\left(\frac{xg(x)}{\Phi(x)}-y_*\right)\le 0,$$

which yields $\dot V(x,y)\le 0$ for all $x>0, y>0$.

By the Lyapunov stability theorem for autonomous systems, the equilibrium E_* is globally asymptotically stable in the first quadrant. □

Theorem 4.3. *Assume that the equilibrium E_* is locally asymptotically stable, and*

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{xg(x)}{\Phi(x)}<0,\quad \frac{\mathrm{d}}{\mathrm{d}x}\frac{p(x)-q(x)}{\Phi(x)}>0$$

hold for $0\le x\le K$. Then, E_* is globally asymptotically stable in the positive quadrant.

Proof. Denote the right-hand side of System (1.1) by the following:

$$F_1(x,y)=xg(x)-y\Phi(x),\quad F_2(x,y)=\varepsilon y\big(p(x)-q(x)\big).$$

Construct the Dulac function of the form

$$\mu(x,y)=\big(\Phi(x)\big)^{\alpha}y^{\delta},$$

where α,δ are undetermined real constants. The divergence of the vector field $(\mu F_1,\mu F_2)$ reads as follows:

$$\begin{aligned}D:=&\frac{\partial\big(\mu F_1(x,y)\big)}{\partial x}+\frac{\partial\big(\mu F_2(x,y)\big)}{\partial y}\\ =&-y^{\delta+1}\big(\Phi(x)\big)^{\alpha}\Phi'(x)(1+\alpha)+\big(\Phi(x)\big)^{\alpha-1}y^{\delta}\Big(\alpha\Phi'(x)xg(x)+xg'(x)\Phi(x)+g(x)\Phi(x)\\ &+\varepsilon(\delta+1)\Phi(x)\big(p(x)-q(x)\big)\Big).\end{aligned}$$

Set $\alpha=-1$ and $\beta=\delta+1$. After simplification, the divergence can be rewritten as

$$D=\big(\Phi(x)\big)^{-2}y^{\delta}L(x),$$

with the auxiliary function

$$\begin{aligned}L(x)&=\Phi(x)\big(xg'(x)+g(x)\big)-\Phi'(x)xg(x)+\varepsilon\beta\Phi(x)\big(p(x)-q(x)\big)\\ &=\Phi^2(x)\left(\frac{xg(x)}{\Phi(x)}\right)'+\varepsilon\beta\Phi(x)\big(p(x)-q(x)\big).\end{aligned}$$

Differentiating $L(x)$ yields the following:

$$\begin{aligned}L'(x)=&2\Phi(x)\Phi'(x)\left(\frac{xg(x)}{\Phi(x)}\right)'+\Phi^2(x)\left(\frac{xg(x)}{\Phi(x)}\right)''\\ &+\varepsilon\beta\Big(\Phi'(x)\big(p(x)-q(x)\big)+\Phi(x)\big(p'(x)-q'(x)\big)\Big).\end{aligned}$$

The choice $\alpha = -1$ enables variable separation and expresses the divergence D as the product of the positive term $\big(\Phi(x)\big)^{-2} y^{\delta}$ and the auxiliary function $L(x)$. Accordingly, the sign of D coincides with that of $L(x)$, that is, $D < 0$ if and only if $L(x) < 0$. The formula $\beta = \delta + 1$ is merely defined to simplify the expression.

It is shown below that there exists a suitable constant β that ensures the Dulac function $\mu(x, y) = (\Phi(x))^{\alpha} y^{\delta}$ with $\alpha = -1$ and $\beta = \delta + 1$ satisfies the inequality $L(x) < 0$.

The remainder of the proof is divided into four steps:

(i) There exists $\delta_1 > 0$ such that $L(x) < 0$ for all $x \in (0, \delta_1)$;

(ii) $L(x_*) < 0$;

(iii) if
$$\beta < -\frac{Kg'(K)}{\varepsilon\big(p(K) - q(K)\big)},$$
then $L(K) < 0$; and

(iv) if
$$\beta < -\frac{\max\limits_{\delta_1 \le x \le K} \left(\dfrac{xg(x)}{\Phi(x)}\right)''}{\varepsilon \max\limits_{\delta_1 \le x \le K} \left(\dfrac{p(x) - q(x)}{\Phi(x)}\right)'},$$
then $L(x)$ has no zero in $[\delta_1, K]$.

It follows from $L(0) = 0$ and $L'(0) = \varepsilon\beta\Phi'(0)\big(p(0) - q(0)\big) < 0$ that there exists $\delta_1 > 0$ such that $L(x) < 0$ for all $x \in (0, \delta_1)$.

Since $p(x_*) = q(x_*)$ at the positive equilibrium x_*, the term $\varepsilon\beta\Phi(x_*)(p(x_*) - q(x_*))$ vanishes, and by Theorem 4.1, the local asymptotic stability of E_* ensures that $\left(\frac{xg(x)}{\Phi(x)}\right)'\Big|_{x=x_*} < 0$, which, together with $\Phi^2(x_*) > 0$, implies
$$L(x_*) = \Phi^2(x_*) \left(\frac{xg(x)}{\Phi(x)}\right)'\Big|_{x=x_*} < 0.$$

It can be verified that $p(K) > q(K)$. By the assumption for the unique positive equilibrium in Theorem 3.1, we have $p(K) > q(0)$. Since $q'(x) \le 0$ for all $x \ge 0$, the function $q(x)$ is non-increasing, which implies $q(K) \le q(0)$. Combining these two inequalities yields $p(K) > q(0) \ge q(K)$, so $p(K) > q(K)$ holds. At the right endpoint $x = K$, direct calculation gives
$$L(K) = K\Phi(K)g'(K) + \varepsilon\beta\Phi(K)\big(p(K) - q(K)\big) < 0$$
provided
$$\beta < -\frac{Kg'(K)}{\varepsilon\big(p(K) - q(K)\big)}.$$

Now, we prove $L(x) < 0$ for all $x \in [\delta_1, K]$ by contradiction. For contradiction, suppose that there exists $x_1 \in [\delta_1, K]$ such that $L(x_1) = 0$. We choose x_1 to be the first zero of $L(x)$ in $[\delta_1, K]$, so that $L(x) < 0$ for all $x \in [\delta_1, x_1)$. At such a first zero, we must have the following:
$$L'(x_1) \ge 0.$$

From the assumption $L(x_1) = 0$, we have the following:

$$\Phi(x_1)\left(\frac{xg(x)}{\Phi(x)}\right)'\bigg|_{x=x_1} = -\varepsilon\beta\big(p(x_1) - q(x_1)\big).$$

Substituting this equality into $L'(x)\big|_{x=x_1}$ yields the following:

$$\begin{aligned} L'(x)\big|_{x=x_1} &= \varepsilon\beta\Big(\Phi(x_1)\big(p'(x_1) - q'(x_1)\big) - \Phi'(x_1)\big(p(x_1) - q(x_1)\big)\Big) + \Phi^2(x_1)\left(\frac{xg(x)}{\Phi(x)}\right)''\bigg|_{x=x_1} \\ &\le \Phi^2(x_1)\left[\varepsilon\beta \max_{\delta_1\le x\le K}\left(\frac{p(x) - q(x)}{\Phi(x)}\right)' + \max_{\delta_1\le x\le K}\left(\frac{xg(x)}{\Phi(x)}\right)''\right]. \end{aligned}$$

Thus, $L'(x)\big|_{x=x_1} < 0$ holds whenever

$$\beta < -\frac{\max\limits_{\delta_1\le x\le K}\left(\dfrac{xg(x)}{\Phi(x)}\right)''}{\varepsilon\max\limits_{\delta_1\le x\le K}\left(\dfrac{p(x) - q(x)}{\Phi(x)}\right)'}.$$

This contradicts $L'(x_1) \ge 0$. Therefore, no such zero x_1 exists, and $L(x) < 0$ for all $x \in [\delta_1, K]$.

Consequently, there exists a Dulac function $\mu(x, y) = (\Phi(x))^\alpha y^\delta$ with $\alpha = -1$ and $\beta = \delta + 1$ satisfying

$$0 < \beta < \min\left\{-\frac{Kg'(K)}{\varepsilon\big(p(K) - q(K)\big)},\ -\frac{\max\limits_{\delta_1\le x\le K}\left(\dfrac{xg(x)}{\Phi(x)}\right)''}{\varepsilon\max\limits_{\delta_1\le x\le K}\left(\dfrac{p(x) - q(x)}{\Phi(x)}\right)'}\right\},$$

such that the divergence $D < 0$ holds for all $x > 0, y > 0$.

By the Dulac criterion, System (1.1) has no closed orbits in the first quadrant. By combining with the local asymptotic stability of E_*, the equilibrium E_* is globally asymptotically stable. □

5. Relaxation oscillations

In this section, we are concerned with the existence of relaxation oscillations in the framework of the geometric singular perturbation theory. To this end, we first recall and introduce the concept of the entry-exit function, which plays a key role in detecting the occurrence of relaxation oscillations.

To be precise, consider the following general slow-fast system:

$$\begin{cases} \dfrac{\mathrm{d}s}{\mathrm{d}t} = sF(s, z, b, k), \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = kG(s, z, b, k), \end{cases}$$

where $s \ge 0$, $z \ge 0$ are state variables, $b \in \mathbb{R}^k$ is a vector parameter, and $0 < k \ll 1$ is the small singular perturbation parameter that separates the fast and slow time scales.

We impose the following fundamental assumptions on the system:

(i) The functions F and G are sufficiently smooth with respect to all their arguments. This regularity condition guarantees the local existence and uniqueness of solutions for the initial value problems associated with the system.

(ii) For every $z > 0$, we have $G(0, z, b, 0) < 0$. This condition ensures that, on the singular manifold $s = 0$, the slow flow evolves in the direction of decreasing z as the time t increases.

(iii) There exists a constant $Z_0 > 0$ such that

$$F(0, z, b, 0) < 0 \quad \text{for } z > Z_0, \qquad F(0, z, b, 0) > 0 \quad \text{for } 0 < z < Z_0.$$

Consequently, the singular manifold $s = 0$ is normally hyperbolic attracting for $z > Z_0$ and normally hyperbolic repelling for $0 < z < Z_0$, while the point $(0, Z_0)$ is the unique nonnormally hyperbolic point that separates these two regions.

Now, let (s_0, z_0) be an initial point near the z-axis such that $z_0 > Z_0$ and $s_0 > 0$ is sufficiently small. Consider the orbit of the system with $k > 0$ starting from (s_0, z_0). By the classical Fenichel theory, for $z > Z_0$, the z-axis is normally hyperbolic attracting. Therefore, the trajectory first approaches the z-axis exponentially fast without crossing it, and then moves slowly downward along the z-axis within an $O(k)$-neighborhood.

When the orbit crosses the threshold $z = Z_0$, the local dynamics abruptly change: for $z < Z_0$, the z-axis becomes normally hyperbolic repelling. As a result, the orbit will depart from a neighborhood of the z-axis at some point $(0, r_k(z_0))$ and eventually intersect the vertical line $s = s_0$ again. Here, $r_k(z_0)$ satisfies the limit

$$\lim_{k \to 0} r_k(z_0) = r_0(z_0),$$

where $r_0(z_0)$ is the corresponding exit point in the singular limit $k = 0$.

The limiting exit point $r_0(z_0)$ is uniquely determined by the following integral equation:

$$\int_{z_0}^{r_0(z_0)} \frac{F(0, z, b, 0)}{G(0, z, b, 0)} \, \mathrm{d}z = 0.$$

The implicit function $r_0(\cdot)$ defined by the above integral is known as the entry-exit function. This function establishes a precise relation between the entry point and the exit point of orbits passing near the singular manifold, and it serves as a powerful tool for to establish the existence of relaxation oscillations. The geometric mechanism is schematically illustrated in Figure 2.

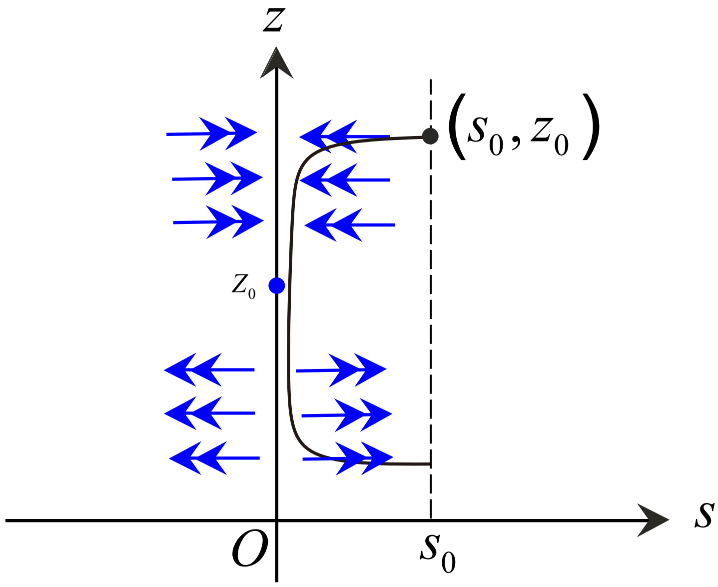

Figure 2. Schematic illustration of the entry-exit mechanism. The orbit starts at (s_0, z_0) and first approaches the z-axis exponentially fast. Then, it evolves slowly downward along the z-axis within an $O(k)$-neighborhood. After crossing the nonnormally hyperbolic point $(0, Z_0)$ (blue dot), the orbit departs from the z-axis at $(0, r_k(z_0))$ and eventually intersects the line $s = s_0$ again. The entry-exit function $r_0(\cdot)$ describes the relation between the entry point $(0, z_0)$ and the exit point $(0, r_0(z_0))$ in the singular limit $k = 0$.

Proposition 5.1. *Consider System* (1.1) *under the following conditions:*

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{xg(x)}{\Phi(x)}\right) < 0, \quad p(0) < q(0), \quad \textit{and} \quad p(K) > q(0).$$

For each $y_0 > \dfrac{\alpha}{\Phi'(0)}$*, there exists a unique* $y_z \in \left(0, \dfrac{\alpha}{\Phi'(0)}\right)$ *such that*

$$I(y_z) = \int_{y_0}^{y_z} \frac{F_0(y)}{G_0(y)}\,\mathrm{d}y = 0, \tag{5.1}$$

where

$$F_0(y) = \alpha - y\hat{\Phi}(0), \quad G_0(y) = y\big(p(0) - q(0)\big).$$

Proof. For each $y_0 > \dfrac{\alpha}{\Phi'(0)}$, the integral $I(y_z)$ can be expressed as follows

$$I(y_z) = \int_{y_0}^{y_z} \frac{F_0(y)}{G_0(y)}\,\mathrm{d}y = \int_{y_0}^{y_z} \frac{\alpha - y\hat{\Phi}(0)}{y\big(p(0) - q(0)\big)}\,\mathrm{d}y.$$

Since $p(0) < q(0)$, we immediately obtain the following:

$$p(0) - q(0) < 0.$$

First, we consider the limit as $y_z \to 0^+$. Direct computation yields the following:

$$\lim_{y_z\to 0^+} I(y_z) = \int_{y_0}^{0^+} \frac{\alpha - y\hat{\Phi}(0)}{y\big(p(0)-q(0)\big)}\,\mathrm{d}y$$
$$= \frac{1}{p(0)-q(0)}\big(\alpha \ln y - \hat{\Phi}(0)\,y\big)\Big|_{y_0}^{0^+}.$$

Since $\alpha > 0$, it follows that $\alpha \ln y \to -\infty$ as $y \to 0^+$, and $\hat{\Phi}(0)\,y \to 0$. Combined with $p(0)-q(0) < 0$, we conclude that

$$\lim_{y_z\to 0^+} I(y_z) = +\infty.$$

Next, we investigate the limit as $y_z \to \left(\frac{\alpha}{\Phi'(0)}\right)^-$. For any $y \in \left(\frac{\alpha}{\Phi'(0)}, y_0\right)$, we have the following:

$$\alpha - y\hat{\Phi}(0) < 0.$$

Thus, the integrand satisfies

$$\frac{\alpha - y\hat{\Phi}(0)}{y\big(p(0)-q(0)\big)} > 0,$$

which implies

$$\lim_{y_z\to\left(\frac{\alpha}{\Phi'(0)}\right)^-} I(y_z) = \int_{y_0}^{\frac{\alpha}{\Phi'(0)}} \frac{\alpha - y\hat{\Phi}(0)}{y\big(p(0)-q(0)\big)}\,\mathrm{d}y < 0.$$

Furthermore, for all $y_z \in \left(0, \frac{\alpha}{\Phi'(0)}\right)$, direct differentiation gives

$$\frac{\mathrm{d}I(y_z)}{\mathrm{d}y_z} = \frac{\alpha - y_z\hat{\Phi}(0)}{y_z\big(p(0)-q(0)\big)} < 0,$$

which shows that $I(y_z)$ is strictly decreasing on $\left(0, \frac{\alpha}{\Phi'(0)}\right)$.

Accordingly, by the Intermediate Value Theorem, there exists a unique $y_z \in \left(0, \frac{\alpha}{\Phi'(0)}\right)$ such that $I(y_z) = 0$. This completes the proof. □

Let x_R be the x-coordinate of the intersection point between the horizontal line $y = y_R = P_0(y_M)$ and the attracting critical manifold C_0^a, where y_M and y_R are related by the integral equation (5.1), that is,

$$\int_{y_M}^{P_0(y_M)} \frac{F_0(y)}{G_0(y)}\,\mathrm{d}y = 0.$$

Now, we define a transcritical slow-fast cycle Γ_0 as follows:

$$\Gamma_0 := \{(x, y_R) \mid x \in (0, x_R)\} \cup \{(0, y) \mid y \in (y_R, y_M)\}$$
$$\cup \{(x, y_M) \mid x \in (0, x_M)\} \cup \{(x, y) \mid x \in (x_M, x_R),\ y = F(x)\}.$$

Let U_0 be a sufficiently small tubular neighborhood of Γ_0. On the basis of the above analysis, the existence and uniqueness of relaxation oscillations for System (1.1) are established in the following theorem. The corresponding geometric structure is illustrated in Figure 3.

Theorem 5.2. *Consider System* (1.1) *under the following conditions:*

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{xg(x)}{\Phi(x)}\right) < 0, \quad p(0) < q(0), \quad \textit{and} \quad p(K) > q(0).$$

Assume that $\frac{\mathrm{d}}{\mathrm{d}x}\frac{xg(x)}{\Phi(x)}\Big|_{x=x_*} > 0$. *Then, there exists a unique relaxation oscillation* $\Gamma_\varepsilon \subset U_0$ *surrounding the unique positive equilibrium* $E_*(x_*, y_*)$*, and as* $\varepsilon \to 0$*,* Γ_ε *converges to the transcritical slow-fast cycle* Γ_0 *in the Hausdorff distance.*

Proof. By Theorem 3.1, System (1.1) admits a unique positive equilibrium $E_*(x_*, y_*)$ if and only if $p(0) < q(0)$ and $p(K) > q(0)$. By Theorem 3.2, if

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{xg(x)}{\Phi(x)}\right) < 0,$$

then the critical manifold C_0 has a unique nonnormally hyperbolic point $D(x_M, F(x_M))$ with $x_M \in (0, K)$, as shown in Figure 3. The nonnormally hyperbolic point D divides the critical manifold C_0 into two normally hyperbolic components:

$$C_0^a := \{(x,y) : x_M < x < K,\ y = F(x)\}, \quad C_0^r := \{(x,y) : 0 < x < x_M,\ y = F(x)\},$$

where C_0^a is attracting, and C_0^r is repelling. By Theorem 4.1, if

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{xg(x)}{\Phi(x)}\Big|_{x=x_*} > 0,$$

then the unique positive equilibrium $E_*(x_*, y_*)$ is locally asymptotically unstable and lies on the repelling branch C_0^r of the critical manifold.

Let $\delta_1, \delta_2, x_s > 0$ be sufficiently small constants. Two cross sections Σ_{in} and Σ_{out} at the fixed abscissa x_s are introduced to construct the Poincaré map for the analysis of periodic orbits of System (1.1). Specifically,

$$\Sigma_{in} = \{(x_s, y) \mid y \in (y_M - \delta_1, y_M + \delta_1)\},$$
$$\Sigma_{out} = \{(x_s, y) \mid y \in (P_0(y_M) - \delta_2, P_0(y_M) + \delta_2)\},$$

where y_M and $P_0(y_M)$ satisfy the entry-exit relation (5.1) (see Proposition 5.1), that is,

$$\int_{y_M}^{P_0(y_M)} \frac{F_0(y)}{G_0(y)}\,\mathrm{d}y = 0.$$

A Poincaré map $\Pi : \Sigma_{in} \to \Sigma_{in}$ is induced by the flow of System (1.1), expressed as the composition $\Pi = \Pi_2 \circ \Pi_1$ with

$$\Pi_1 : \Sigma_{in} \to \Sigma_{out}, \quad \Pi_2 : \Sigma_{out} \to \Sigma_{in},$$

where Π_1 and Π_2 denote the mappings along trajectories between the corresponding cross sections. The cross sections and the Poincaré map are schematically illustrated in Figure 3.

First, we analyze the mapping Π_1 in detail. By the entry-exit function and Proposition 5.1, there exists a positive constant δ_2 (depending on δ_1) such that for every $y \in (y_M - \delta_1, y_M + \delta_1)$ (i.e., for every point $(x_s, y) \in \Sigma_{in}$), there exists a unique value $P_\varepsilon(y) \in (P_0(y_M) - \delta_2, P_0(y_M) + \delta_2)$ satisfying the convergence condition $\lim_{\varepsilon \to 0} P_\varepsilon(y) = P_0(y)$ in the C^1-topology. In other words, the trajectory of System (1.1) starting from any point $(x_s, y) \in \Sigma_{in}$ will necessarily intersect the cross section Σ_{out} at the unique point $(x_s, P_\varepsilon(y))$, due to the regularity of the system's flow and the choice of sufficiently small δ_1 and δ_2. Therefore, the mapping $\Pi_1 : \Sigma_{in} \to \Sigma_{out}$ can be explicitly defined as follows:

$$\Pi_1(x_s, y) = (x_s, P_\varepsilon(y)).$$

Next, we turn to the analysis of the mapping $\Pi_2 : \Sigma_{out} \to \Sigma_{in}$. To investigate the behavior of Π_2, we consider two arbitrary orbits Γ_ε^1 and Γ_ε^2 of System (1.1) that start from distinct points in the cross section Σ_{out}. According to the Fenichel theory, which characterizes the persistence and smoothness of slow manifolds in slow-fast systems, combined with Theorem 2.1 of [42] (which establishes results regarding the attraction of trajectories to slow manifolds), both orbits Γ_ε^1 and Γ_ε^2 will be attracted to the slow manifold C_ε^a, that is, a perturbation of C_0^a at an exponential rate of $O(e^{-\frac{1}{\varepsilon}})$. This exponential attraction implies that the orbits rapidly approach C_ε^a as they evolve, and once they reach the slow manifold, they move along C_ε^a until they pass through the jump point (x_M, y_M), where the slow flow of System (2.2) undergoes a rapid transition (jump). At the jump point (x_M, y_M), the two orbits Γ_ε^1 and Γ_ε^2 exponentially contract toward each other, which is a key property leading to the contraction of the Poincaré map Π. After this exponential contraction, the orbits continue to evolve and rapidly converge back to the incoming cross section Σ_{in}, thus completing the mapping Π_2.

By combining the properties of Π_1 and Π_2, we can now establish the contraction property of the composite Poincaré map $\Pi = \Pi_2 \circ \Pi_1$. Specifically, the exponential contraction of orbits at the jump point (x_M, y_M), together with the smoothness and regularity of Π_1, implies that Π is a contraction mapping on Σ_{in} with an exponential contraction rate of $O(e^{-\frac{1}{\varepsilon}})$. By the Contraction Mapping Theorem, every contraction mapping on a non-empty, closed, bounded, and convex metric space has a unique fixed point. Since Σ_{in} is a closed and convex interval (hence a suitable metric space) and Π is a contraction on Σ_{in}, it follows that Π has a unique fixed point in Σ_{in}. This fixed point corresponds to a unique trajectory of System (1.1) that intersects Σ_{in} at the fixed point and is periodic, (i.e., the unique relaxation oscillation $\Gamma_\varepsilon \subset U_0$, where U_0 is the tubular neighborhood of the transcritical slow-fast cycle Γ_0 defined earlier). This periodic orbit Γ_ε intersects Σ_{in} at the fixed point for all sufficiently small ε (i.e., $0 < \varepsilon \ll 1$). Furthermore, by applying Fenichel's theorem again and Theorem 2.1 of [42], we can conclude that as $\varepsilon \to 0$, the relaxation oscillation Γ_ε converges to the transcritical slow-fast cycle Γ_0 in the Hausdorff distance, which characterizes the convergence of compact sets and is consistent with the conclusion of Theorem 5.2.

The proof is complete. □

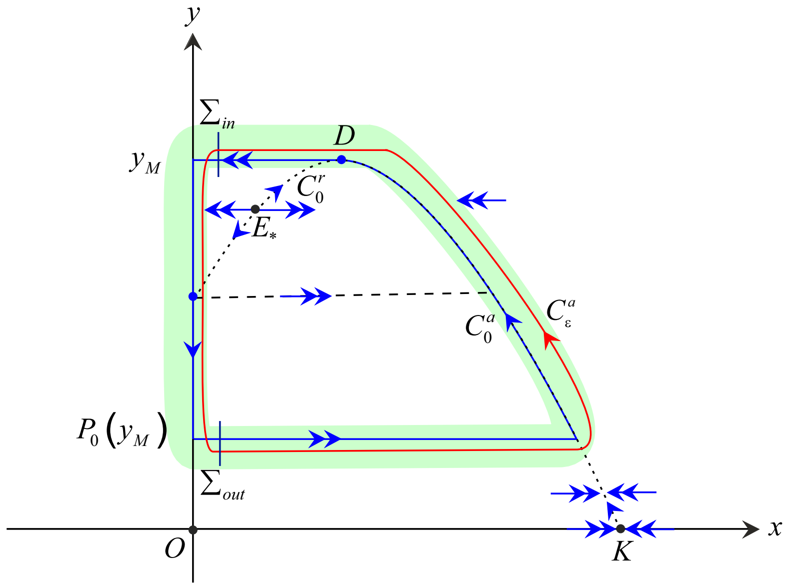

Figure 3. Relaxation oscillation Γ_ε (red) converges to the transcritical slow-fast cycle Γ_0 (blue dashed) as $\varepsilon \to 0$. The blue double arrows indicate the direction of the fast subsystem flow, the blue single arrows represent the direction of the slow subsystem flow, and the red arrows denote the direction of the full system flow. E_* is the unique positive equilibrium, Σ^{in} and Σ^{out} are the Poincaré cross sections, and $P_0(y_M)$ is the corresponding entry-exit function.

6. Uniqueness of limit cycles

Next, we discuss the existence and uniqueness of limit cycles of System (1.1). Rewrite system (1.1) in the following form:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \Phi(x)\left(F(x) - y\right), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y\chi(x), \end{cases} \tag{6.1}$$

where $F(x) = \frac{xg(x)}{\Phi(x)}$, $\chi(x) = \varepsilon\left(p(x) - q(x)\right)$. Define $H(x) := -\frac{\Phi(x)F'(x)}{\chi(x)}$.

Theorem 6.1. *Consider System* (6.1)*. Assume that*

(i) $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\dfrac{xg(x)}{\Phi(x)} < 0$ *and* $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{p(x)-q(x)}{\Phi(x)} > 0$ *for* $0 \le x \le K$*;*

(ii) $F'(x_*) > 0$ *and* $H(x)$ *is a non-decreasing function on* $(0, x_*) \cup (x_*, +\infty)$.

Then, System (6.1) *has a unique stable limit cycle surrounding* $E_*(x_*, y_*)$.

Proof. The existence of the limit cycles follows from Theorem 5.2

Next, we prove the uniqueness of the limit cycles. Moving positive equilibrium $E_*(x_*, y_*)$ to the origin by letting $x_1 = x - x_*$, $y_1 = y - y_*$ gives the following:

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \Phi(x_1 + x_*)\left(F(x_1 + x_*) - y_1 - y_*\right), \\ \dfrac{\mathrm{d}y_1}{\mathrm{d}t} = (y_1 + y_*)\chi(x_1 + x_*). \end{cases} \tag{6.2}$$

Then, letting $x_1 = \xi(u)$, $y_1 = \eta(v)$ in System (6.2) gives the following:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = \dfrac{\Phi(\xi(u) + x_*)}{\xi'(u)} \left(F(\xi(u) + x_*) - \eta(v) - y_*\right), \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{\eta(v) + y_*}{\eta'(v)} \chi(\xi(u) + x_*). \end{cases} \tag{6.3}$$

Set

$$\begin{cases} \dfrac{\Phi(\xi(u) + x_*)}{\xi'(u)} = 1, \xi(0) = 0, \\ \dfrac{\eta(v) + y_*}{\eta'(v)} = 1, \eta(0) = 0; \end{cases} \tag{6.4}$$

then, $\eta(v) = y_*(e^v - 1)$. Since $\xi(u)$ satisfies the conditions of the existence and uniqueness theorem of the solution, $\xi(u)$ exists and is unique. Substituting (6.4) into (6.3) yields the following Liénard system:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = -\eta(v) - (y_* - F(\xi(u) + x_*)) \equiv -\eta(v) - M(u), \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = \chi(\xi(u) + x_*) \equiv \psi(u). \end{cases} \tag{6.5}$$

Recall Zhang Zhifen's uniqueness theorem for limit cycles [43]. Consider the following Liénard system:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = -\eta(v) - M(u), \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = \psi(u), \end{cases}$$

where the following conditions are satisfied:

(i) $M'(u), \psi(u) \in C^0(\mathbb{R})$, and $\psi(u)$ satisfies the Lipschitz condition;

(ii) $u\psi(u) > 0$ for $u \neq 0$, and $\Psi(u) = \int_0^u \psi(s)\mathrm{d}s \to +\infty$ as $|u| \to +\infty$;

(iii) $v\eta(v) > 0$ for all $v \neq 0$, $\eta(\pm\infty) = \infty$, $\eta(v) \in C^0(\mathbb{R})$. $\eta(v)$ is monotonic on $\mathbb{R}$, satisfies the Lipschitz condition, and has left and right derivatives at $v = 0$ with $\eta'_+(0)\eta'_-(0) \neq 0$;

(iv) the ratio $\frac{M'(u)}{\psi(u)}$ is non-decreasing on $\mathbb{R} \setminus \{0\}$, and $\frac{M'(u)}{\psi(u)}$ is not a constant when $0 < |u| \ll 1$.

Under these assumptions, the system admits at most one limit cycle around the origin, which is unique and stable if it exists.

All hypotheses of Zhang Zhifen's uniqueness theorem are explicitly verified for the transformed Liénard system (6.5):

(i) Since $\xi'(u) = \Phi(\xi(u) + x_*) > 0$ and $\chi(\cdot)$ is smooth, $M'(u)$ and $\psi(u)$ are continuous on $\mathbb{R}$. Moreover, $\psi(u)$ is continuously differentiable and thus satisfies the Lipschitz condition.

(ii) From $p'(x) \geq 0$ and $q'(x) \leq 0$, it follows that $\chi'(x) \geq 0$. Since $\chi(x_*) = 0$, one has $(x - x_*)\chi(x) > 0$, which implies $\xi(u)\chi(\xi(u) + x_*) > 0$. Note that $\xi(0) = 0$ and $\xi'(u) = \Phi(\xi(u) + x_*) > 0$, so u and $\xi(u)$ share the same sign. Thus, $u\psi(u) = u\chi(\xi(u) + x_*) > 0$ for $u \neq 0$. Let $\Psi(u) = \int_0^u \psi(s)\mathrm{d}s$. From $\psi(0) = 0$ and $\psi'(u) = \chi'(x)\xi'(u) = \chi'(x)\Phi(x) > 0$, it is deduced that $\Psi(u) \to +\infty$ as $|u| \to +\infty$.

(iii) The function $\eta(v) = y_*(e^v - 1)$ is continuous on $\mathbb{R}$ with $\eta(0) = 0$. One has $v\eta(v) = y_* v(e^v - 1) > 0$ for all $v \neq 0$. Moreover, $\eta(+\infty) = +\infty$ and $\eta(-\infty) = -y_*$. Since $\eta'(v) = y_* e^v > 0$, $\eta(v)$ is strictly increasing and satisfies the Lipschitz condition. The function $\eta(v)$ is differentiable at $v = 0$ with $\eta'(0) = y_* > 0$, so $\eta'_+(0)\eta'_-(0) = y_*^2 > 0$. Since the analysis is restricted to the first quadrant, the condition $\eta(-\infty) = -y_*$ can be used instead of $\eta(-\infty) \to +\infty$.

(iv) Clearly, $M(0) = 0$. Let $m(u) = M'(u) = -F'(x)\xi'(u) = -F'(x)\Phi(x)$. Then,

$$\frac{m(u)}{\psi(u)} = -\frac{\Phi(x)F'(x)}{\chi(x)} = H(x).$$

Since $x = \xi(u) + x_*$ is strictly increasing and $H(x)$ is non-decreasing, the ratio $m(u)/\psi(u)$ is non-decreasing with respect to u. Furthermore, $H(x)$ is not a constant for a sufficiently small $|x|$, so $\frac{M'(u)}{\psi(u)}$ is not a constant when $0 < |u| \ll 1$.

Therefore, the uniqueness of the limit cycle is established by Zhang Zhifen's uniqueness theorem [43]. □

7. Conclusions

This paper studies the global dynamics of a generalized slow-fast predator-prey model extended from the classical Gause system, where the predator reproductive rate serves as a small singular perturbation parameter $0 < \varepsilon \ll 1$. Combining Lyapunov theory, Dulac's criterion, geometric singular perturbation theory, Fenichel theory, and Zhang Zhifen's limit cycle uniqueness theorem, we systematically analyze the stability of equilibrium, relaxation oscillations and limit cycles, and draw the main conclusions as follows:

First, we derive the existence and uniqueness condition for the positive coexistence equilibrium $E_*(x_*, y_*)$. The system has exactly one positive equilibrium if and only if $p(0) < q(0)$ and $p(K) > q(0)$. When $\frac{d^2}{dx^2}\left(\frac{xg(x)}{\Phi(x)}\right) < 0$, the critical manifold C_0 possesses a unique non-normally hyperbolic fold point $D(x_M, y_M)$, which divides C_0 into an attracting branch C_0^a and a repelling branch C_0^r. The sign of $F'(x_*)$ determines local stability of E_*: $F'(x_*) < 0$ yields local asymptotic stability, while $F'(x_*) > 0$ leads to local instability.

Second, two groups of sufficient conditions for global asymptotic stability of E_* are established. By constructing a dedicated Lyapunov function, we prove global stability under $\left(\frac{xg(x)}{\Phi(x)} - y_*\right)(x - x_*) \leq 0$. Using a power-type Dulac function, we further show that if E_* is locally stable, $\left(\frac{xg(x)}{\Phi(x)}\right)'' < 0$ and $\left(\frac{p(x)-q(x)}{\Phi(x)}\right)' > 0$ on $[0, K]$, the system contains no closed orbits in the first quadrant, so E_* is globally asymptotically stable.

Third, we prove the existence and singular limit behavior of relaxation oscillations. When E_* is locally unstable with concave critical manifold and the equilibrium existence assumptions satisfied, there exists a unique relaxation oscillation Γ_ε surrounding E_* for sufficiently small ε. Based on the entry-exit function and contraction mapping theorem for Poincaré maps, we confirm that Γ_ε converges to the transcritical slow-fast cycle Γ_0 in Hausdorff distance as $\varepsilon \to 0$.

Fourth, we obtain sufficient conditions for the existence and uniqueness of a stable limit cycle. Under the concave critical manifold condition, monotonicity of $\frac{p(x)-q(x)}{\Phi(x)}$, local instability of E_* and monotonicity of $H(x) = -\frac{\Phi(x)F'(x)}{\chi(x)}$ on $(0, x_*) \cup (x_*, +\infty)$, we convert the original model into a Liénard system. All assumptions of Zhang Zhifen's uniqueness theorem are verified, which together with the relaxation oscillation result guarantee a stable limit cycle enclosing E_*.

Compared with existing slow-fast predator-prey literature relying on fixed functional responses, our generalized model with arbitrary smooth monotonic vital functions enjoys wider ecological applicability. This work integrates global stability, geometric singular perturbation analysis and classical limit cycle theory into a unified framework, linking nullcline geometry to long-term population behaviors. The theoretical results can be applied to various classic ecological models to analyze species coexistence, steady states and periodic outbreaks.

Potential extensions include incorporating spatial diffusion, Allee effects, piecewise smooth response functions or multi-species interactions. Further research can also investigate canard orbits, canard explosions and singular bifurcations near the fold point $D(x_M, y_M)$.

Author contributions

Cheng Wang: Funding acquisition, Conceptualization, Writing–original draft, Writing–review & editing; Qianqian Zhao: Funding acquisition, Conceptualization, Software, Writing–review & editing; Yanru Xie: Writing–original draft, Writing–review & editing. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this research.

Conflict of interest

The authors declare that they have no conflicts of interest to disclose.

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References

1. Y. Liu, A. Zegeling, W. Huang, The application of Liénard transformations to predator-prey systems, *Qual. Theory Dyn. Syst.*, **23** (2024), Paper No. 91, 33. https://doi.org/10.1007/s12346-023-00947-0

2. D. Zhang, P. Yan, Necessary and sufficient conditions for the nonexistence of limit cycles of Leslie-Gower predator-prey models, *Appl. Math. Lett.*, **71** (2017), 1–5. https://doi.org/10.1016/j.aml.2017.03.008

3. Y. Dai, Y. Zhao, B. Sang, Four limit cycles in a predator-prey system of Leslie type with generalized Holling type III functional response, *Nonlinear Anal. Real World Appl.*, **50** (2019), 218–239. https://doi.org/10.1016/j.nonrwa.2019.04.003

4. D. Xiao, Z. Zhang, On the uniqueness and nonexistence of limit cycles for predator-prey systems, *Nonlinearity*, **16** (2003), 1185–1201. https://doi.org/10.1088/0951-7715/16/3/321

5. J. Huang, S. Ruan, J. Song, Bifurcations in a predator-prey system of Leslie type with generalized Holling type III functional response, *J. Differential Equations*, **257** (2014), 1721–1752. https://doi.org/10.1016/j.jde.2014.04.024

6. R. E. Kooij, A. Zegeling, Co-existence of a period annulus and a limit cycle in a class of predator-prey models with group defense, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **31** (2021), Paper No. 2150154, 11. https://doi.org/10.1142/S0218127421501546

7. E. González-Olivares, H. Meneses-Alcay, B. González-Yañez, J. Mena-Lorca, A. Rojas-Palma, R. Ramos-Jiliberto, Multiple stability and uniqueness of the limit cycle in a Gause-type predator-prey model considering the Allee effect on prey, *Nonlinear Anal. Real World Appl.*, **12** (2011), 2931–2942. https://doi.org/10.1016/j.nonrwa.2011.04.003

8. K. Hasík, On a predator-prey system of Gause type, *J. Math. Biol.*, **60** (2010), 59–74. https://doi.org/10.1007/s00285-009-0257-8

9. T. W. Hwang, Uniqueness of the limit cycle for Gause-type predator-prey systems, *J. Math. Anal. Appl.*, **238** (1999), 179–195. https://doi.org/10.1006/jmaa.1999.6520

10. S. Ruan, D. Xiao, Global analysis in a predator-prey system with nonmonotonic functional response, *SIAM J. Appl. Math.*, **61** (2000/01), 1445–1472. https://doi.org/10.1137/S0036139999361896

11. L. Zhao, J. Shen, Canards and homoclinic orbits in a slow-fast modified May-Holling-Tanner predator-prey model with weak multiple Allee effect, *Discrete Contin. Dyn. Syst. Ser. B*, **27** (2022), 6745–6769. https://doi.org/10.3934/dcdsb.2022018

12. Z. Zhu, X. Liu, Canard cycles and relaxation oscillations in a singularly perturbed Leslie-Gower predator-prey model with Allee effect, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **32** (2022), Paper No. 2250071, 23. https://doi.org/10.1142/S0218127422500717

13. W. Su, X. Zhang, Global stability and canard explosions of the predator-prey model with the sigmoid functional response, *SIAM J. Appl. Math.*, **82** (2022), 976–1000. https://doi.org/10.1137/21M1437755

14. X. Chen, X. Zhang, Dynamics of the predator-prey model with the Sigmoid functional response, *Stud. Appl. Math.*, **147** (2021), 300–318. https://doi.org/10.1111/sapm.12382

15. P. R. Chowdhury, S. Petrovskii, M. Banerjee, Oscillations and pattern formation in a slow-fast prey-predator system, *Bull. Math. Biol.*, **83** (2021), Paper No. 110, 41. https://doi.org/10.1007/s11538-021-00941-0

16. P. R. Chowdhury, M. Banerjee, S. Petrovskii, Canards, relaxation oscillations, and pattern formation in a slow-fast ratio-dependent predator-prey system, *Appl. Math. Model.*, **109** (2022), 519–535. https://doi.org/10.1016/j.apm.2022.04.022

17. J. Shen, C. H. Hsu, T. H. Yang, Fast-slow dynamics for intraguild predation models with evolutionary effects, *J. Dynam. Differential Equations*, **32** (2020), 895–920. https://doi.org/10.1007/s10884-019-09744-3

18. T. H. Hsu, S. Ruan, Relaxation oscillations and the entry-exit function in multidimensional slow-fast systems, *SIAM J. Math. Anal.*, **53** (2021), 3717–3758. https://doi.org/10.1137/19M1295507

19. S. Li, X. Wang, X. Li, K. Wu, Relaxation oscillations for Leslie-type predator-prey model with Holling type I response functional function, *Appl. Math. Lett.*, **120** (2021), Paper No. 107328, 6. https://doi.org/10.1016/j.aml.2021.107328

20. K. Li, S. Li, K. Wu, Relaxation Oscillations and Canard Explosion of a Leslie-Gower Predator-Prey System with Addictive Allee Effect, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **32** (2022), Paper No. 2250168, 9. https://doi.org/10.1142/S0218127422501681

21. S. Chen, J. Duan, J. Li, Effective reduction of a three-dimensional circadian oscillator model, *Discrete Contin. Dyn. Syst. Ser. B*, **26** (2021), 5407–5419. https://doi.org/10.3934/dcdsb.2020349

22. S. Chen, J. Duan, J. Li, Dynamics of the Tyson-Hong-Thron-Novak circadian oscillator model, *Phys. D*, **420** (2021), Paper No. 132869, 16. https://doi.org/10.1016/j.physd.2021.132869

23. P. De Maesschalck, G. Kiss, A. Kovacs, Relaxation oscillations and canards of a regulated two-gene model, *J. Math. Anal. Appl.*, **502** (2021), Paper No. 125144, 18. https://doi.org/10.1016/j.jmaa.2021.125144

24. S. Ai, S. Sadhu, The entry-exit theorem and relaxation oscillations in slow-fast planar systems, *J. Differential Equations*, **268** (2020), 7220–7249. https://doi.org/10.1016/j.jde.2019.11.067

25. T. H. Hsu, G. S. K. Wolkowicz, A criterion for the existence of relaxation oscillations with applications to predator-prey systems and an epidemic model, *Discrete Contin. Dyn. Syst. Ser. B*, **25** (2020), 1257–1277. https://doi.org/10.3934/dcdsb.2019219

26. T. H. Hsu, Number and stability of relaxation oscillations for predator-prey systems with small death rates, *SIAM J. Appl. Dyn. Syst.*, **18** (2019), 33–67. https://doi.org/10.1137/18M1166705

27. W. Liu, D. Xiao, Y. Yi, Relaxation oscillations in a class of predator–prey systems, *J. Differential Equations*, **188** (2003), 306–331. https://doi.org/10.1016/S0022-0396(02)00076-1

28. C. Li, H. Zhu, Canard cycles for predator–prey systems with Holling types of functional response, *J. Differential Equations*, **254** (2013), 879–910. http://dx.doi.org/10.1016/j.jde.2012.10.003

29. Y. M. Tai, Z. Zhang, Relaxation oscillations in a spruce-budworm interaction model with Holling's type II functional response, *Discrete Contin. Dyn. Syst. Ser. B*, **26** (2021), 2173–2199. https://doi.org/10.3934/dcdsb.2021027

30. S. Li, C. Wang, K. Wu, Relaxation oscillations of a slow-fast predator-prey model with a piecewise smooth functional response, *Appl. Math. Lett.*, **113** (2021), Paper No. 106852, 6. https://doi.org/10.1016/j.aml.2020.106852

31. A. Ghazaryan, V. Manukian, S. Schecter, Travelling waves in the Holling-Tanner model with weak diffusion, *Proc. A.*, **471** (2015), 20150045, 16. https://doi.org/10.1098/rspa.2015.0045

32. C. Wang, X. Zhang, Relaxation oscillations in a slow-fast modified Leslie-Gower model, *Appl. Math. Lett.*, **87** (2019), 147–153. https://doi.org/10.1016/j.aml.2018.07.029
33. L. Zhong, J. Shen, Degenerate transcritical bifurcation point can be an attractor: a case study in a slow-fast modified Leslie-Gower model, *Qual. Theory Dyn. Syst.*, **21** (2022), Paper No. 76, 11. https://doi.org/10.1007/s12346-022-00608-8
34. Y. Fu, X. Liu, Dynamics of a spatio-temporal slow-fast predator-prey system with reproductive Allee effect and a generalist predator, *Qual. Theory Dyn. Syst.*, **24** (2025), Paper No. 223, 49. https://doi.org/10.1007/s12346-025-01386-9
35. H. Dong, X. Liu, Traveling front in a diffusive predator-prey model with Beddington-DeAngelis functional response, *Math. Methods Appl. Sci.*, **48** (2025), 3682–3711. https://doi.org/10.1002/mma.10508
36. Z. Zhu, X. Liu, Canard cycle and nonsmooth bifurcation in a piecewise-smooth continuous predator-prey model, *Math. Comput. Simulation*, **227** (2025), 477–499. https://doi.org/10.1016/j.matcom.2024.08.017
37. Z. Zhu, X. Liu, Existence of traveling wave solutions in a singularly perturbed predator-prey equation with spatial diffusion, *Discrete Contin. Dyn. Syst.*, **44** (2024), 78–115. https://doi.org/10.3934/dcds.2023097
38. X. Wu, P. Zhang, F. Xie, Canard explosion of a singularly perturbed SIS epidemic model with a non-monotone incidence rate and a logistic-type population birth rate, *Commun. Nonlinear Sci. Numer. Simul.*, **154** (2026), Paper No. 109547, 21. https://doi.org/10.1016/j.cnsns.2025.109547
39. X. Wu, F. Xie, Slow-fast dynamics of a piecewise-smooth Leslie-Gower model with Holling type-I functional response and weak Allee effect, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **34** (2024), Paper No. 2450086, 28. https://doi.org/10.1142/S021812742450086X
40. X. Wu, S. Lu, F. Xie, Relaxation oscillations of a piecewise-smooth slow-fast Bazykin's model with Holling type I functional response, *Math. Biosci. Eng.*, **20** (2023), 17608–17624. https://doi.org/10.3934/mbe.2023782
41. J. Huang, R. Huzak, J. Yao, Relaxation oscillations in predator-prey systems with piecewise smooth functional responses, *J. Differential Equations*, **453** (2026), Paper No. 113907, 33. https://doi.org/10.1016/j.jde.2025.113907
42. M. Krupa, P. Szmolyan, Extending geometric singular perturbation theory to nonhyperbolic points–fold and canard points in two dimensions, *SIAM J. Math. Anal.*, **33** (2001), 286–314. http://dx.doi.org/10.1137/S0036141099360919
43. Z. F. Zhang, T. R. Ding, W. Z. Huang, Z. X. Dong, *Qualitative theory of differential equations*, vol. 101 of Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, 1992. Translated from the Chinese by Anthony Wing Kwok Leung. https://doi.org/10.1090/mmono/101

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