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Research article

Consecutive-k-out-of-n: F systems with imperfect protection measures

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Abstract: Consecutive-k-out-of-n: F systems are widely used in reliability modeling to describe systems whose failure is triggered by the occurrence of k consecutive component failures. In this paper, we introduce a consecutive-k-out-of-n: F structure equipped with imperfect protection blocks. The system is partitioned into consecutive blocks of size k, each of which is operative with a predetermined probability. Under the proposed design, whenever a protection block is operative, the failure rate of the components it contains is reduced; otherwise, the components retain a higher baseline failure rate. This modeling framework captures both local dependence and stochastic protection effects. We first develop a recursive scheme for computing the system's reliability, while a Markov embedding, which leads to an efficient matrix-based representation, is also established. The asymptotic behavior of the system's reliability is also under investigation. In addition, we consider a cost-constrained optimization problem, where the protection parameters are selected to maximize reliability under limited resources. Several numerical results and graphical representations illustrate the impact of protection effectiveness and system size, while the performance improvement of the optimized protected structure compared to its unprotected counterpart is also confirmed under different design scenarios.

Keywords: consecutive-k-out-of-n: F systems; reliability analysis; imperfect protection blocks; stochastic effectiveness; asymptotic behavior; optimization problem
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1. Introduction

Over the past decades, consecutive-type systems have received considerable attention in reliability theory due to their ability to model systems in which local dependence among neighboring components plays a crucial role. Such models arise naturally in a wide range of applications, including communication networks, fault-tolerant systems, biological sequence analysis, and signal processing. In these contexts, system behavior is often determined by the occurrence of patterns of consecutive failures, rather than by isolated component malfunctions, making consecutive-type structures particularly relevant for reliability assessment (see, e.g., Derman et al. [1] or Chiang and Niu [2]).

Consecutive-type systems generalize the classical consecutive-k-out-of-n: F models, which consist of n components arranged in a linear or circular configuration. In such a design, the system fails if and only if at least k consecutive components fail. These models have been extensively investigated in the literature (see Li et al. [3] or Triantafyllou and Koutras [4]), along with several extensions, such as the so-called r-within-consecutive-k-out-of-n systems and other configurations that capture more complex failure mechanisms (see, e.g., Kamalja and Amrutkar [5] or Eryilmaz [6]). Some recent results on the aforementioned topic have been provided by Ye et al. [7], Yi et al. [8], and Dembińska and Nikolov [9].

Despite the rich literature on consecutive-k-out-of-n: F systems, relatively limited attention has been given to models incorporating imperfect protection mechanisms, where protection does not eliminate failures but instead reduces their likelihood in a probabilistic manner. In many practical settings, such as cybersecurity systems, distributed computing environments, and biological processes, protective mechanisms (e.g., redundancy schemes, filtering procedures, or error-correction protocols) operate under uncertainty and provide only partial mitigation of failure risks. Consequently, it is more realistic to model protection as a stochastic improvement rather than as a deterministic guarantee.

Motivated by these considerations, we introduce a consecutive-k-out-of-n: F system equipped with imperfect protection blocks. Although the imperfect protection-block mechanism introduced in the present paper appears to be new in the context of consecutive-k-out-of-n: F systems, the broader idea that reliability-enhancement mechanisms may operate imperfectly has been widely studied in reliability engineering. For instance, imperfect fault coverage models consider situations in which fault-tolerant or recovery mechanisms fail to completely prevent system failure (see, e.g., Myers [10]). From a different point of view, imperfect maintenance and imperfect repair models account for maintenance actions that only partially restore system reliability rather than returning the system to an as-good-as-new condition (see, e.g., Doyen and Gaudoin [11]).

Within the proposed framework, the system is partitioned into consecutive blocks of size k, each of which is operative with a pre-determined probability. When a block is operative, the failure rate of the components it contains is reduced, whereas in the absence of protection, the components retain a higher baseline failure rate. This modeling framework captures both the local dependence inherent in consecutive failure patterns and the probabilistic nature of protection mechanisms.

First, we develop a recursive scheme for the computation of the system reliability, based on a block-level representation of the failure process. In addition, we establish a Markov embedding of the system and derive a matrix formulation that allows for efficient evaluation of reliability.

On the other hand, we analyze the asymptotic behavior of the system reliability as the system size increases, showing that it exhibits exponential decay determined by the spectral radius of an associated transition matrix. Furthermore, we investigate a cost-constrained optimization problem, in which the parameters of the protection mechanism are specified to maximize system reliability under limited resources. Numerical results are presented to illustrate the effect of protection effectiveness and system parameters, as well as to compare the optimized protected system with its unprotected counterpart.

The remainder of the paper is organized as follows: Section 2 introduces the proposed reliability model, while necessary notations and definitions are also provided. Section 3 offers the main results of the paper, such as the recursive scheme for the system reliability and its asymptotic behavior; we also shed light on an optimization problem in terms of its overall reliability. Section 4 provides several numerical results, and Section 5 summarizes the main contribution of the present work, while some directions for future research are also mentioned.

2. The proposed consecutive-type reliability model

Let us next consider the case of a consecutive-*k*-out-of-*n*: *F* structure consisting of *n* components. To improve the reliability of such a model, protective measures may be applied to strengthen the system's overall robustness. This might include adding redundancy, utilizing more reliable materials, implementing fault-tolerant designs, or applying protective coatings and shielding to reduce the probability of component failures.

Motivated by this, we next study the reliability of consecutive-*k*-out-of-*n*: *F* structures equipped with imperfect protection measures. The so-called protection blocks, which have been recently studied by Fang et al. [12] and Eryilmaz [13] for a different reliability design known as consecutive-*k*-out-*n*: *G* system, help to reduce the failure probability of the components inside the block. Although protection-block mechanisms have recently been studied in the context of consecutive-type structures, the present work focuses on the fundamentally different consecutive-*k*-out-of-*n*: *F* setting. In the latter case, system failure is driven by runs of consecutive component failures rather than runs of successful components. Consequently, the underlying probabilistic structure and reliability representation differ substantially from those arising in protection-based consecutive-*k*-out-of-*n*: *G* systems studied by Fang et al. [12] and Eryilmaz [13].

Within the proposed framework, each block can protect exactly k consecutive components. The assumption that protection blocks have fixed size k is primarily motivated by analytical tractability and by the desire to align the protection mechanism with the critical run length defining system failure.

The system consists of n components arranged in a linear sequence and is modeled as a consecutive-k-out-of-n:F structure, meaning that the system fails if and only if at least k consecutive components fail. To incorporate protection, the sequence of components is partitioned into consecutive, non-overlapping blocks of size k. More precisely, the first block covers components $1, \dots, k$, the second block covers components $k+1, \dots, 2k$, and so on. If n is not a multiple of k, the final block may contain fewer than k components. Thus, the entire system is structurally covered by a collection of disjoint protection blocks.

Each block is associated with a binary random variable Z_b, indicating whether the protection mechanism in that block is operative. The index b is used to enumerate the protection blocks in the system. Specifically, $b = 1,2, \dots, B$, where $B = \lceil n/k \rceil$, and each value of b identifies a distinct block consisting of consecutive components. In other words, the components of the structure are partitioned into successive groups of size k, where $B_b = \{(b-1)k+1, \dots, bk\}, b = 1,2, \dots$ indicate the positions of the components within the $b-th$ block.

We next assume that $Z_b = 1$ if the block is active and $Z_b = 0$ otherwise. The random variables Z_b's are independent and identically distributed with $P(Z_b = 1) = \alpha$. This assumption implies that protection is applied across the system in a stochastic manner; namely, all blocks are present, but only a random subset of them is effectively protected at any given realization.

The protection mechanism operates locally within each block. Conditioning on the state of the $b-th$ block, all components within that block share the same failure probability. If the block is operative ($Z_b = 1$), the components experience a reduced failure probability q_1, reflecting the effect of protection. If the block has failed ($Z_b = 0$), the components fail with a higher baseline probability q_0, where $0 < q_1 < q_0 < 1$. Components belonging to different blocks are independent conditional on the block states, while components within the same block are identically distributed but still conditionally independent given Z_b. Based on the abovementioned argumentation, we may define the failure probability of the components within the $b-th$ block as follows:

$$q(Z_b) = \begin{cases} q_1, & Z_b = 1, \\ q_0, & Z_b = 0, \end{cases} \quad (2.1)$$

where $0 < q_1 < q_0 < 1$.

It is clear that components belonging to the same protection block are conditionally independent and identically distributed given the block protection state Z_b. Unconditionally, dependence is induced through the common latent block variable.

Although the model is developed in a general probabilistic framework, it is motivated by practical situations in which local protection mechanisms may operate imperfectly. The following example illustrates how the proposed structure naturally arises in a real-world reliability setting. Consider a linear wireless sensor network deployed along a pipeline. Communication is considered operational provided that no run of k consecutive sensor failures occurs. Protection blocks correspond to local backup-power modules covering groups of neighboring sensors. When a protection module is operative, the failure probability of the associated sensors is reduced from q_0 to q_1. However, protection modules may themselves be unavailable with probability $1-\alpha$, leading naturally to the proposed imperfect-protection model.

To sum up, the system can be viewed as a sequence of alternating protected and unprotected segments, determined by the realization of the block states. Although protection blocks cover the entire system structurally, their activation is probabilistic. Therefore, not all components are protected simultaneously. This structure [$C_{k,n}(a, q_1, q_0, F)$, hereafter] creates a heterogeneous environment in which runs of consecutive failures depend not only on the individual failure probabilities but also on the spatial arrangement of active and inactive protection blocks. Consequently, the reliability of the system reflects the interplay between the block-level randomness and the local failure dynamics within each block.

3. Main results

In the present section, we shall provide the main contribution of the paper. More precisely, the reliability of the proposed consecutive-*k*-out-of-*n*: *F* structure $C_{k,n}(a, q_1, q_0, F)$ is studied in some detail, while its asymptotic behavior is also under investigation. In addition, a cost-constrained optimization problem is also considered.

For a given component failure probability q, we define $Q(q)$ as the transition matrix describing the evolution of the current run length of consecutive failures. The states of the system are indexed by $i = 0,1, \ldots, k-1$, where state i represents that the system currently ends with exactly i consecutive failed components, provided that no run of k consecutive failures has yet occurred.

The matrix $Q(q)$ specifies the probabilities of transitioning between these states when a new component is observed. In particular, with probability $1-q$, the next component functions and the system returns to state 0, while with probability q, the next component fails and the run length increases by one, unless this would produce k consecutive failures, in which case the process exits the transient state space. The matrix $Q(q)$ has a quite simple structure reflecting the evolution of the failure run length. More precisely, it takes the following form:

$$Q(q) = \begin{pmatrix} 1-q & q & 0 & \cdots & 0 \\ 1-q & 0 & q & \cdots & 0 \\ 1-q & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & q \\ 1-q & 0 & 0 & \cdots & 0 \end{pmatrix}. \quad (3.1)$$

Kindly note that $Q(q)$ is the transient substochastic matrix associated with the non-failed states $\{0, \dots, k-1\}$. The absorbing failure state is not included in $Q(q)$. Therefore, the last row sums to $1-q$, since an additional failure from state $k-1$ leads to absorption.

Lemma 1 characterizes the Markovian evolution of the system at the block level through a linear recursion. Actually, it provides the fundamental state recursion, which forms the basis for the computation of the system reliability later on.

Lemma 1. Let us denote by u_m the row vector of transient state probabilities of a $C_{k,n}(a, q_1, q_0, F)$ structure after m complete protection blocks, where the states correspond to the current run length of consecutive failures. Then, the state probabilities satisfy the recurrence relationship

$$u_{m+1} = u_m M, m \geq 0, \quad (3.2)$$

with initial condition

$$u_0 = e_0 = (1,0, \dots,0), \quad (3.3)$$

where

$$M = \alpha Q(q_1)^k + (1-\alpha) Q(q_0)^k. \quad (3.4)$$

Proof. By conditioning on the state of a given protection block, the evolution of the run length of consecutive failures over the k components of the block is provided by $Q(q_1)^k$ if the block is operative and by $Q(q_0)^k$ otherwise. Since the block states are independent and identically distributed with $P(Z_b = 1) = \alpha$, the unconditional transition across a block can be expressed with the aid of the following mixture:

$$M = E\left[Q(q(Z_b))^k\right] = \alpha Q(q_1)^k + (1-\alpha) Q(q_0)^k.$$

Since all blocks are assumed to be independent from each other, we may conclude that the same transition applies from one block to the next, and the desired recursive scheme in (3.2) is readily deduced.

□

The following proposition provides an explicit representation of the reliability of the $C_{k,n}(a, q_1, q_0, F)$ system with the aid of the state recursion established in the previous lemma.

Proposition 1. Let $n = mk + r$, where $m = \lfloor n/k \rfloor$ and $0 \leq r < k$. Then the reliability of the $C_{k,n}(a, q_1, q_0, F)$ system is given by

$$R_{n,k}(\alpha, q_1, q_0) = e_0 M^m N_r \mathbf{1}, \quad (3.5)$$

where

$$N_r = \alpha Q(q_1)^r + (1-\alpha) Q(q_0)^r,$$

while

$$e_0 = (1,0, \dots,0)$$

and $\mathbf{1}$ is the k-dimensional column vector of units.

Proof. After m complete protection blocks, the transient state distribution can be viewed as (see Lemma 1)

$$u_m = e_0 M^m.$$

For the remaining r components, given the state of the corresponding protection block, the evolution is determined by $Q(q_1)^r$ or $Q(q_0)^r$. Averaging over the block state, the transition matrix N_r for these remaining r components can be expressed via the following formula:

$$N_r = \alpha Q(q_1)^r + (1-\alpha) Q(q_0)^r.$$

Thereof, the transient state distribution after $n = mk + r$ components is

$$e_0 M^m N_r.$$

The system operates when no run of k consecutive failures is observed, namely when the process stays within the transient state space. Based on this argument, the desired result in (3.5) is straightforward.

□

The result provided by Proposition 1 constitutes a natural analogue of the recursion in the traditional consecutive-k-out-of-n: F model, apart from the fact that the recurrence is now developed in terms of the protection blocks instead of the corresponding components, since the protection mechanism acts on blocks of exactly k consecutive components. Indeed, the classical consecutive-k-out-of-n:F ecurrence (see Chiang and Niu [2]) can be viewed as a special case of Proposition 1 of the present work. Setting $\alpha = 1$ and $q_1 = q_0 = q$, the block transition matrix reduces to $M = Q(q)^k$, and the model coincides with the standard component-wise recursion.

While Lemma 1 provides the recursive evolution of the state probabilities, it does not explicitly identify the underlying stochastic structure. The following proposition clarifies that this recursion admits a Markov chain interpretation at the block level.

Proposition 2. Let us denote by Y_m the length of the terminal run of consecutive failures at the end of the m-th protection block of a $C_{k,n}(a, q_1, q_0, F)$ structure, provided that no run of k consecutive failures has occurred. Then the process $(Y_m)_{m\geq 0}$ is a time-homogeneous Markov chain on the state space $\{0,1,\dots,k-1\}$, with transition matrix given in (3.4).

Proof. Since Y_m records the current run length of consecutive failures at the boundary of the $m-th$ block, it is readily observed that the transition from Y_m to Y_{m+1} depends on the outcomes of the components within the next block and on its protection state. In addition, by conditioning on $Y_m = i$, the evolution over the next block is determined by the transition matrices $Q(q_1)^k$ or $Q(q_0)^k$, depending on whether the block operates or not.

Since the block states are independent and identically distributed and do not depend on the history of the process, it follows that the distribution of Y_{m+1} conditional on the entire past depends only on Y_m. Therefore, the process (Y_m) can be viewed as a Markov chain with

$$P(Y_{m+1} = j \mid Y_m, Y_{m-1}, \dots) = P(Y_{m+1} = j \mid Y_m). \tag{3.6}$$

Finally, the transition probabilities coincide with those derived in Lemma 1 [see (3.4) therein].

□

The results proved previously provide a complete recursive and probabilistic description of the system at the block level. In particular, the Markov embedding established in Proposition 2 allows us to shed light on the long-term performance of the system. We next focus on the asymptotic behavior of the reliability of a $C_{k,n}(a, q_1, q_0, F)$ structure by establishing some corresponding exponential bounds.

We first establish a basic structural property of the block transition matrix, which shall be proved useful in the sequel.

Lemma 2. The matrices $Q(q_0)^k$, $Q(q_1)^k$ and the block transition matrix M of a $C_{k,n}(a, q_1, q_0, F)$ structure have strictly positive entries.

Proof. For any failure probability $q \in (0,1)$, both failure and success occur with positive probability. Therefore, starting from any initial state, namely any current run length of consecutive failures, it is plausible to reach any other transient state within at most k steps with positive probability. This implies that all entries of the matrix $Q(q)^k$, which describes the evolution over k consecutive components, are strictly positive.

Additionally, given that $Q(q_1)^k$ and $Q(q_0)^k$ are strictly positive matrices, M, as a convex combination of these matrices with positive weights, shall also have all of its entries strictly positive (see, e.g., Berman and Plemmons [14]).

□

The positivity of the transition matrix, along with the Markov structure established previously, allows us to characterize the asymptotic behavior of the reliability of the $C_{k,n}(a, q_1, q_0, F)$ system.

Proposition 3. Let $n = mk + r$, where $0 \leq r < k$. Then there exist positive constants c_r^- and c_r^+ such that the reliability of the $C_{k,n}(a, q_1, q_0, F)$ structure satisfies the following inequality:

$$c_r^- \, \rho(M)^m \leq R_{n,k}^{PB} \leq c_r^+ \, \rho(M)^m, m \geq 0, \tag{3.7}$$

where $\rho(M) \in (0,1)$ denotes the spectral radius of matrix M.

Proof. We have already proved that the evolution of the $C_{k,n}(a, q_1, q_0, F)$ system at block boundaries is described by a Markov chain with transition matrix M with strictly positive entries (see Proposition 2 and Lemma 2). Recalling the Perron–Frobenius theorem for primitive nonnegative matrices (see, e.g., Berman and Plemmons [14]), we may deduce that its spectral radius $\rho(M)$ is a simple positive eigenvalue such that

$$Mv = \rho(M)v, \tag{3.8}$$

where v denotes a strictly positive right eigenvector.

Since the absorption within the substochastic matrix M occurs with positive probability, its row sums are strictly less than or equal to one, and at least one row sum is strictly less than one (see, e.g., Norris [15]). Therefore,

$$0 < \rho(M) < 1.$$

For a fixed $r \in \{0,1, \dots, k-1\}$, we define the following vector

$$y_r = N_r \mathbf{1},$$

which contains strictly positive entries and represents the reliabilities of the remaining r components. Since both y_r and v have strictly positive entries, their component-wise ratios are bounded, and hence, there exist constants $a_r, b_r > 0$ such that (see, e.g., Horn and Johnson [16])

$$a_r v \leq y_r \leq b_r v. \tag{3.9}$$

Multiplying by M^m, we obtain that

$$a_r M^m v \leq M^m y_r \leq b_r M^m v.$$

We next recall that $M^m v = \rho(M)^m v$ [see (3.8)], and the following inequality holds true

$$a_r \rho(M)^m v \leq M^m y_r \leq b_r \rho(M)^m v$$

or equivalently

$$a_r(e_0 v)\rho(M)^m \le e_0 M^m y_r \le b_r(e_0 v)\rho(M)^m,$$

where $e_0 = (1,0,\dots,0)$.
Applying (3.5), the result we are chasing is readily obtained by setting

$$c_r^- = a_r(e_0 v), c_r^+ = b_r(e_0 v). \tag{3.10}$$

□

The result offered by Proposition 3 shows that the reliability decreases exponentially with the number of protection blocks, at a rate determined by $\rho(M)$. To illustrate the implementation of the exponential bounds, consider the case

$$k = 2, \alpha = 0.6, q_0 = 0.30, q_1 = 0.10.$$

It is clear that for the special case $k = 2$, the following holds true:

$$Q(q) = \begin{pmatrix} 1-q & q \\ 1-q & 0 \end{pmatrix}.$$

Therefore, we readily observe that

$$Q(q_0) = \begin{pmatrix} 0.7 & 0.3 \\ 0.7 & 0 \end{pmatrix}, Q(q_1) = \begin{pmatrix} 0.9 & 0.1 \\ 0.9 & 0 \end{pmatrix}.$$

The block transition matrix can be expressed via the next formula:

$$M = \alpha Q(q_1)^2 + (1-\alpha)Q(q_0)^2,$$

which can be simplified as

$$M = \begin{pmatrix} 0.820 & 0.138 \\ 0.682 & 0.138 \end{pmatrix}.$$

So, the spectral radius $\rho(M)$ is equal to 0.937691. For even $n = 2m$, we have $r = 0$, and the constants obtained from Proposition 3 are given below:

$$c_0^- = 1.000000, c_0^+ = 1.172567.$$

Therefore,

$$1.000000\,\rho(M)^m \le R_{2m,2}^{PB} \le 1.172567\,\rho(M)^m.$$

Table 1 provides some numerical comparisons between the exact reliability and the corresponding exponential bounds.

Table 1. Comparisons between the exact value and corresponding bounds.

n	m	Exact value	Lower bound	Upper bound
10	5	0.740982	0.724933	0.850032
20	10	0.537162	0.525527	0.616216
30	15	0.389406	0.380972	0.446715
40	20	0.282293	0.276179	0.323838
50	25	0.204644	0.200211	0.234761
60	30	0.148353	0.145140	0.170186

It is easily observed that the exact reliability lies within the proposed bounds for all cases considered.

To shed more light on the effect of the protection mechanism, it is instructive to compare the proposed $C_{k,n}(a, q_1, q_0, F)$ model with the traditional consecutive-k-out-of-n: F systems. This comparison could provide insight into how block protection enhances the reliability behavior.

Proposition 4. If $R_{n,k}(q)$ corresponds to the reliability of the traditional consecutive-k-out-of-n:F system consisting of components with common unreliability q, then the reliability of the $C_{k,n}(a, q_1, q_0, F)$ structure satisfies the following inequality:

$$R_{n,k}(q_0) \leq R_{n,k}(\alpha, q_1, q_0) \leq R_{n,k}(q_1). \tag{3.11}$$

Proof. For every protection block, the effective component failure probability lies between q_1 and q_0. Since the reliability of a consecutive-k-out-of-n: F system is nonincreasing with respect to the component failure probabilities, the protected system is stochastically bounded between the two classical systems with common failure probabilities q_0 and q_1. Therefore, the desired result is effortlessly concluded.

□

Proposition 4 shows that, in terms of stochastic ordering, the protected system lies between two classical consecutive systems: the fully unprotected system with unreliability q_0 and the fully protected system with unreliability q_1.

We next turn our attention to a simple optimization problem, examining how the level of protection should be chosen under a cost constraint. More precisely, we formulate a general cost-constrained design problem in which the protection parameters are determined to maximize the reliability of the overall structure under a pre-specified budget.

Proposition 5. Let $C(\alpha, q_1)$ be a cost function and $C_{\max} > 0$ express the available budget for constructing a $C_{k,n}(a, q_1, q_0, F)$ system. We define the set

$$D = \{(\alpha, q_1): 0 \leq a \leq 1, 0 < q_1 < q_0, C(\alpha, q_1) \leq C_{\max}\}, \tag{3.12}$$

which includes all feasible values of the design parameters α, q_1. We also assume that both the reliability of the $C_{k,n}(a, q_1, q_0, F)$ system and the cost function $C(\alpha, q_1)$ increase with respect to α and decrease with respect to q_1. If $(\alpha^*, q_1^*) \in D$ is an optimal solution of

$$\max_{(\alpha, q_1) \in D} R_{n,k}(\alpha, q_1, q_0), \tag{3.13}$$

then the budget constraint is active, namely $C(\alpha^*, q_1^*) = C_{\max}$, under the assumption that it does not lie on the natural parameter bounds ($\alpha = 1$ or $q_1 = 0$).

Proof. Let us consider the case where the pair (α^*, q_1^*) is optimal but does not utilize the full budget; namely, let us assume that

$$C(\alpha^*, q_1^*) < C_{\max}.$$

Given that (α^*, q_1^*) is not on the parameter bounds $\alpha = 1$ or $q_1 = 0$, it is possible to improve slightly the protection design. In simple words, one can either increase α or decrease q_1 by a sufficiently small amount while remaining within the feasible region.

Recalling the monotonicity of the reliability mentioned before, increasing α or decreasing q_1 leads to the improvement of the system's reliability. Moreover, because the original budget constraint is not fully utilized, such a small improvement can be made without violating the restriction

$$C(\alpha, q_1) \leq C_{\max}.$$

Thereof, there exists another feasible design with reliability strictly larger than the one achieved by the solution (α^*, q_1^*), contradicting its optimality. Therefore, an interior optimal solution cannot leave unused budget, and hence, the following equality should hold true:

$$C(\alpha^*, q_1^*) = C_{\max}.$$

This completes the proof.

□

It is worth noting that the aforementioned result does not require a specific functional form for the cost. It only relies on the natural assumption that improving the protection mechanism, either by increasing the probability of operative blocks or by reducing the protected failure probability, requires additional resources. Therefore, the result provided by Proposition 5 applies to linear, nonlinear, or alternative cost functions, given that the stated monotonicity conditions are satisfied.

The optimization problem reflects a trade-off between protection availability (α) and protection effectiveness (q_1), which may lead to multiple optimal designs. In practice, different combinations of protection probability and effectiveness may yield similar performance, reflecting an inherent trade-off between these two design parameters.

4. Numerical results

In this section, we illustrate several numerical results for the proposed $C_{k,n}(a, q_1, q_0, F)$ system. First, we carry out a numerical experimentation for the exact reliability values obtained from the block-recursive representation and compare them with the corresponding two-sided exponential bounds.

Table 2 depicts the exact reliability values along with the corresponding lower and upper bound for several values of n, k and different protection scenarios.

The results in Table 2 confirm that the exact reliability lies within the proposed theoretical bounds in all cases examined. As expected, reliability decreases as n increases, since systems of larger size have a higher probability of containing a run of k consecutive failures. On the other hand, for fixed n, increasing the value of parameter k generally improves the overall reliability of the resulting structure, because a longer run of failures is now required for the failure of the overall system.

Additionally, Table 2 highlights the impact of the protection parameters. As it is easily observed, larger values of α lead to a larger probability that blocks operate under the reduced unreliability q_1, while smaller values of q_1 correspond to more effective protection. Consequently, scenarios with higher α and lower q_1 exhibit better performance. Needless to say, the comparison among alternative scenarios provides a direct numerical illustration of the protective effect described in the previous section.

On the other hand, it is of some practical interest to investigate the cost-constrained optimization problem for the protection mechanism. For illustration purposes, we assume that the cost function is defined as

$$C(\alpha, q_1) = c_\alpha \alpha + c_q (q_0 - q_1),$$

where c_α measures the cost of improving the reliability of the protection blocks, while c_q corresponds to the cost of reducing the component failure probability under an operative protection block.

Table 2. Exact value and bounds for the reliability of the $C_{k,n}(a, q_1, q_0, F)$ structure.

Design parameters $a, q_1, q_0)$	k	n	Exact value	Lower bound	Upper bound
$(0.6, 0.05, 0.25)$	3	50	0.837842	0.833248	0.838470
		70	0.777844	0.771508	0.792436
		90	0.722057	0.718686	0.821007
	4	50	0.962981	0.961023	0.967156
		70	0.947782	0.945855	0.951892
		90	0.932824	0.930927	0.936869
	5	50	0.991460	0.990946	1
		70	0.987860	0.987348	1
		90	0.984273	0.983763	1
$(0.7, 0.05, 0.30)$	3	50	0.815416	0.809563	0.816168
		70	0.748169	0.740729	0.762646
		90	0.686334	0.683262	0.774560
	4	50	0.950765	0.947926	0.955815
		70	0.930780	0.928001	0.935724
		90	0.911215	0.908494	0.916055
	5	50	0.986569	0.985887	1
		70	0.980976	0.980297	1
		90	0.975414	0.974739	1
$(0.8, 0.08, 0.30)$	3	50	0.855949	0.851599	0.856517
		70	0.801758	0.795890	0.814640
		90	0.750857	0.747962	0.849026
	4	50	0.965483	0.963511	0.969149
		70	0.951301	0.949359	0.954914
		90	0.937328	0.935414	0.940887
	5	50	0.991120	0.990676	1
		70	0.987414	0.986971	1
		90	0.983721	0.983280	1
$(0.9, 0.08, 0.35)$	3	50	0.888950	0.885179	0.889371
		70	0.845868	0.840971	0.856343
		90	0.804725	0.802658	0.894651
	4	50	0.972044	0.970288	0.974927
		70	0.960523	0.958787	0.963371
		90	0.949138	0.947423	0.951953
	5	50	0.991962	0.991654	1
		70	0.988642	0.988335	1
		90	0.985333	0.985028	1

To investigate the sensitivity of the optimal design to the underlying cost structure, three cost scenarios were considered. In Scenario 1, equal weights are assigned to improving the protection availability parameter and reducing the protected failure probability, corresponding to a balanced cost structure $c_\alpha = 1, c_q = 1$. Scenario 2 assumes that increasing the probability of an operative protection block is relatively less expensive than reducing the protected failure probability, and therefore, $c_\alpha =$

$0.5, c_q = 1$. Conversely, Scenario 3 corresponds to the case where improving the effectiveness of an operative protection block is relatively less expensive, leading to $c_\alpha = 1, c_q = 0.5$.

These cost scenarios allow us to examine how the optimal protection strategy changes when the relative costs of improving protection availability and protection effectiveness vary.

For each pair (n, k), $n = 60,80,100$ and $k = 3,4,5$, we numerically solve the constrained maximization problem

$$\max_{\alpha, q_1} R_{n,k}(\alpha, q_1, q_0)$$

under the following restrictions

$$C(\alpha, q_1) \leq C_{\max}, \quad 0 \leq \alpha \leq 1, \quad 0 < q_1 < q_0,$$

while $q_0 = 0.3, C_{\max} = 0.9$.

The reliability values of the resulting optimal protected structures are compared with the corresponding ones of the unprotected consecutive-k-out-of-n:F system with failure probability q_0.

Table 3 presents the optimal design parameters α^* and q_1^*, the corresponding cost $C(\alpha^*, q_1^*)$, the optimized reliability of the proposed $C_{k,n}(a, q_1, q_0, F)$ system, and the corresponding one for the traditional consecutive-k-out-of-n:F system with common unreliability q_0.

The last column of Table 3 depicts the improvement ratio, defined as the ratio between the reliability of the optimized $C_{k,n}(a, q_1, q_0, F)$ structure and the reliability of the corresponding unprotected one. More precisely, for each pair (n, k), each entry of the last column of Table 2 is computed with the aid of the following expression:

$$\text{Improvement ratio} = \frac{R_{n,k}(\alpha^*, q_1^*, q_0)}{R_{n,k}(q_0)}, \tag{3.14}$$

where $R_{n,k}(\alpha^*, q_1^*, q_0)$ is the reliability of the optimally protected system, and $R_{n,k}(q_0)$ is the reliability of the unprotected consecutive-k-out-of-n:F system consisting of components with common unreliability q_0. It goes without saying that values larger than one indicate an improvement due to the protection mechanism. For instance, an improvement ratio equal to 2 practically means that the protected system is twice as reliable as the unprotected one under the same values of the design parameters n, k, and q_0.

Based on the numerical results presented in Table 3, it is clear that the optimized $C_{k,n}(a, q_1, q_0, F)$ system consistently outperforms the unprotected system. This was actually expected, since the theoretical comparison result (see Proposition 4) reveals that the reliability of the $C_{k,n}(a, q_1, q_0, F)$ structure lies above the reliability of the unprotected system with failure probability q_0. The improvement is particularly visible for larger values of n, where the occurrence of runs of failed components with large size becomes more likely in the absence of protection.

Under the balanced-cost Scenario 1 $(c_\alpha, c_q) = (1,1)$, the optimal solutions allocate the available budget between improving the protection availability α and reducing the protected failure probability q_1. The resulting optimal values lie within the range $0.65 \leq \alpha^* \leq 0.71$ and $0.05 \leq q_1^* \leq 0.11$, indicating a compromise between the two protection mechanisms.

When improving the protection availability becomes relatively inexpensive (Scenario 2), the optimizer strongly favors increasing α. In fact, the optimal solution is located at the boundary $\alpha^* = 1$ for all examined (n, k) combinations, while q_1^* is reduced to its minimum feasible value. Consequently, the resulting reliability values are extremely close to one, indicating that the available budget is most effectively spent on ensuring that protection is almost always operative.

By contrast, when reducing the protected failure probability becomes relatively less expensive (Scenario 3), the optimal solutions place greater emphasis on decreasing q_1. Compared with Scenario 1, the values of α^* increase only moderately, whereas q_1^* decreases substantially. The corresponding reliability values are consistently higher than those obtained under Scenario 1 but generally lower than those achieved under Scenario 2.

Table 3. Cost-constrained optimization problem for the $C_{k,n}(a, q_1, q_0, F)$ structure.

Cost scenario	n	k	Optimal choice (α^*, q_1^*)	$R_{n,k}(\alpha, q_1, q_0)$	$R_{n,k}(q_0)$	Improvement ratio
Scenario 1	60	3	$(0.66, 0.06)$	0.742059	0.306331	2.422413
		4	$(0.69, 0.09)$	0.929079	0.716106	1.297404
		5	$(0.71, 0.11)$	0.981266	0.907690	1.081058
	80	3	$(0.65, 0.05)$	0.672786	0.204149	3.295558
		4	$(0.68, 0.08)$	0.905834	0.637414	1.421107
		5	$(0.71, 0.11)$	0.974794	0.877047	1.111451
	100	3	$(0.65, 0.05)$	0.607011	0.136052	4.461608
		4	$(0.68, 0.08)$	0.883171	0.567369	1.556608
		5	$(0.71, 0.11)$	0.968366	0.847438	1.142698
Scenario 2	60	3	$(0.85, 0.08)$	0.863959	0.306331	2.820348
		4	$(0.86, 0.09)$	0.968864	0.716106	1.352961
		5	$(0.91, 0.13)$	0.993075	0.907690	1.094068
	80	3	$(0.85, 0.08)$	0.823302	0.204149	4.032844
		4	$(0.86, 0.09)$	0.958388	0.637414	1.503556
		5	$(0.91, 0.13)$	0.990659	0.877047	1.129540
	100	3	$(0.85, 0.08)$	0.782817	0.136052	5.753804
		4	$(0.86, 0.09)$	0.948025	0.567369	1.670914
		5	$(0.91, 0.13)$	0.988249	0.847438	1.166162
Scenario 3	60	3	$(0.76, 0.02)$	0.840673	0.306331	2.744331
		4	$(0.76, 0.02)$	0.957785	0.716106	1.337490
		5	$(0.77, 0.04)$	0.988788	0.907690	1.089345
	80	3	$(0.76, 0.02)$	0.795247	0.204149	3.895421
		4	$(0.76, 0.02)$	0.943800	0.637414	1.480670
		5	$(0.77, 0.04)$	0.984942	0.877047	1.123021
	100	3	$(0.76, 0.02)$	0.748859	0.136052	5.504207
		4	$(0.76, 0.02)$	0.930019	0.567369	1.639177
		5	$(0.77, 0.04)$	0.981111	0.847438	1.157738

*Note: All designs have been constructed under the choice $q_0 = 0.3, C_{\max} = 0.9$.

Based on Table 3, it is easily observed that for all cost scenarios, reliability increases with the threshold parameter k, while the effect of the system size n is opposite. An interesting observation is that the optimal protection parameters are remarkably stable across different system sizes. For a given cost scenario, nearly identical values of (α^*, q_1^*) are obtained for $n = 60,80,100$. This suggests that the optimal allocation strategy is driven primarily by the relative cost structure rather than by the system size itself.

Figure 1 compares the reliability of the optimized $C_{k,n}(a, q_1, q_0, F)$ system with the corresponding one of the traditional consecutive-k-out-of-n:F system for several values of n and k.

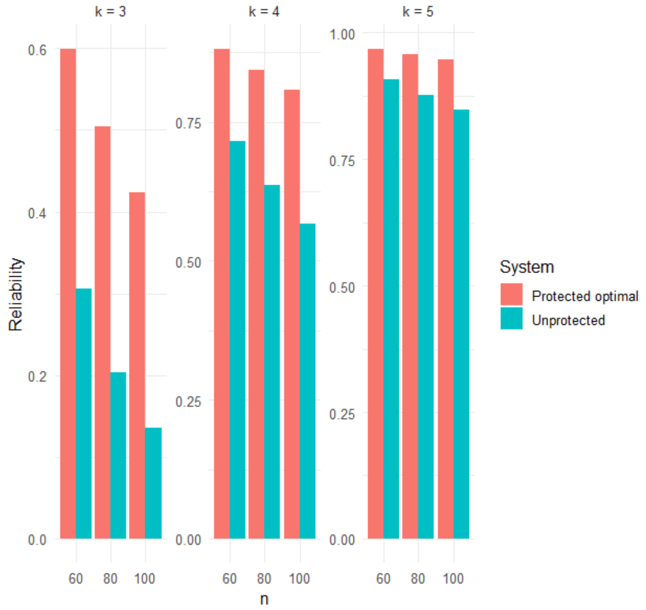

Figure 1. Comparison of the optimized $C_{k,n}(a, q_1, q_0, F)$ structure versus the traditional consecutive-k-out-of-n:F system.

As expected, the protected system consistently outperforms the unprotected one, confirming the effectiveness of the proposed protection mechanism. A more detailed inspection of Figure 1 reveals several important observations. First, the relative improvement seems to be more pronounced as the system size n increases. This happens because such systems are more susceptible to the occurrence of long runs of consecutive failures and therefore benefit more from even moderate levels of protection.

On the other hand, the magnitude of the improvement depends on the value of k. For smaller values of k, where system failure is more likely, the implementation of protection measures leads to a substantial enhancement of its performance. In contrast, for larger values of k, protection seems to be less beneficial.

In addition, Figure 2 measures the relative benefit of optimal protection compared to the unprotected system. More specifically, the improvement ratio defined in (3.14) is depicted therein, and we focus on larger values of this ratio, which indicate stronger reliability gains due to the protection mechanism.

A closer inspection of Figure 2 reveals that the improvement ratio, for all three cost scenarios, typically increases with the system size n. This reflects the fact that larger systems are more vulnerable to consecutive failures and therefore benefit more strongly from the implementation of protection measures. In contrast, for small systems, the baseline reliability is already relatively high.

Moreover, the ratio seems to be highly influenced by the value of parameter k. For smaller values of k, where failure is easier to trigger, the relative improvement is more apparent. However, when k is large, the system becomes more robust, and the relative advantage of protection tends to be less beneficial.

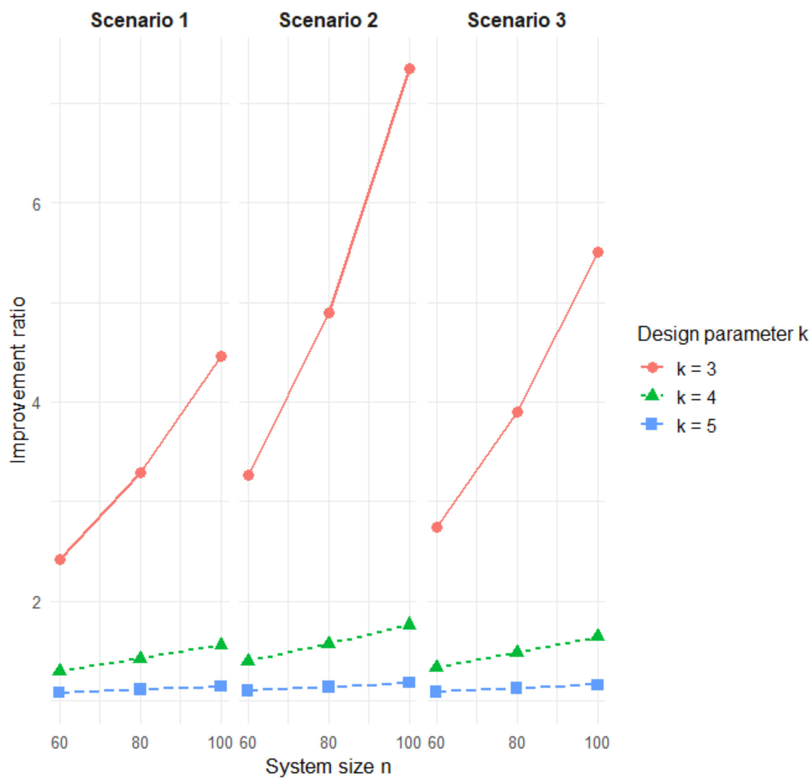

Figure 2. Reliability improvement ratio under optimized protection.

Additionally, the relative ranking of the three cost scenarios is consistent across all examined values of design parameters n, k. The optimal protection strategy is strongly influenced by the assumed cost structure. In particular, making improvements in protection availability relatively less expensive leads to the greatest overall reliability gains. Finally, the separation between the curves corresponding to different values of k becomes more pronounced as n increases. This suggests that the choice of the design parameter k becomes increasingly important in large systems, where the cumulative effect of consecutive-failure mechanisms is stronger.

5. Conclusions

In the present work, the reliability of the consecutive-k-out-of-n: F system equipped with imperfect protection blocks is investigated. By exploiting a Markov embedding at the block level, a tractable recursive representation is derived, while the asymptotic exponential behavior of system reliability is studied in detail. In addition, a cost-constrained optimization framework is considered, while some numerical evidence illustrating the impact of protection parameters on system performance is provided.

A limitation of the present model is the assumption that protection-block states are independent and that component failures are conditionally independent given the block state. In practice, shared resources, environmental factors, or common-cause effects may induce dependence both between protection blocks and between components. Extending the proposed framework to accommodate such dependence structures remains an interesting topic for future work.

On the other hand, the proposed reliability model relies on an additional simplifying assumption, namely the fixed block size equal to k. While this choice leads to a convenient block-recursive representation, alternative protection structures involving block sizes different from k, variable block lengths, or overlapping protection regions constitute interesting directions for future research. To be more precise, a natural direction for future research is to relax the independence assumption and allow for dependence among the block states. Another plausible extension is to consider protection blocks of random size. Under such a framework, structural heterogeneity could be adopted in the modified design. These extensions constitute interesting directions for future work and may lead to a unified framework for the analysis and optimal design of protected consecutive-type systems under realistic operational circumstances.

Another limitation of the present work is the assumption of homogeneous components. In heterogeneous systems, component failure probabilities may vary across locations, leading to nonstationary transition structures and more complex reliability calculations. Extending the proposed imperfect-protection framework to heterogeneous consecutive systems remains an interesting topic for future investigation. Another promising direction is the investigation of multiple interacting protection modules and the extension of the proposed imperfect protection mechanism to continuous-k-out-of-n reliability structures.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Ioannis S. Triantafyllou is the Guest Editor of special issue "Statistical Reliability Modeling with Applications" for AIMS Mathematics. Ioannis S. Triantafyllou was not involved in the editorial review and the decision to publish this article.

References

1. C. Derman, G. J. Lieberman, S. M. Ross, On the consecutive-k-out-of-n: F system, *IEEE T. Reliab.*, **31** (1982), 57–63. http://doi.org/10.1109/TR.1982.5221229
2. D. T. Chiang, S. C. Niu, Reliability of Consecutive-k-out-of-n: F System, *IEEE T. Reliab.*, **30** (1981), 87–89. http://doi.org/10.1109/TR.1981.5220981
3. M. Li, L. Hu, R. Peng, Z. Bai, Reliability modeling for repairable circular consecutive-k-out-of-n: F systems with retrial feature, *Reliab. Eng. Syst. Safe.*, **216** (2021), 107957. http://doi.org/10.1016/j.ress.2021.107957
4. I. S. Triantafyllou, M. V. Koutras, On the signature of coherent systems and applications, *Probab. Eng. Inform. Sc.*, **22** (2008), 19–35. http://doi.org/10.1017/S0269964808000028
5. K. K. Kamalja, K. P. Amrutkar, Reliability and reliability importance of weighted-within-consecutive-out-of-system, *IEEE T. Reliab.*, **67** (2018), 951–969. http://doi.org/10.1109/TR.2018.2826065

6. S. Eryilmaz, Component importance for linear consecutive-k-Out-of-n and m-Consecutive-k -Out-of-n systems with exchangeable components, *Nav. Res. Log.*, **60** (2013), 313–320. https://doi.org/10.1002/nav.21535
7. T. Ye, L. Liu, Y. Zhou, H. Gao, Effective Noisy-matrix of Bayesian network for scalable m-consecutive-k-out-of-n: F models with overlapping from -1 to k-1, *Reliab. Eng. Syst. Safe.*, **265** (2026), 111578, https://doi.org/10.1016/j.ress.2025.111578
8. H. Yi, N. Balakrishnan, X. A. Li, General type of linear consecutive-k systems, *Methodol. Comput. Appl.*, **28** (2026). https://doi.org/10.1007/s11009-026-10246-1
9. A. Dembińska, N. I. Nikolov, Reliability properties of k-out-of-n systems with several cold standby units, *J. Comput. Appl. Math.*, **477** (2026), 117203, https://doi.org/10.1016/j.cam.2025.117203
10. A. Myers, *Complex system reliability*, London: Springer-Verlag, 2010. https://doi.org/10.1007/978-1-84996-414-2
11. L. Doyen, O. Gaudoin, Classes of imperfect repair models based on reduction of failure intensity or virtual age, *Reliab. Eng. Syst. Safe.*, **84** (2004), 45–56, https://doi.org/10.1016/S0951-8320(03)00173-X
12. S. Fang, R. Yu, D. Wu, R. Peng, Reliability of a consecutive k-out-of-n: G system with protection blocks, *Int. J. Gen. Syst.*, **54** (2025), 421–439. https://doi.org/10.1080/03081079.2024.2402308
13. S. Eryilmaz, On reliability of consecutive k-out-of-n: G system equipped with protection blocks, *Int. J. Gen. Syst.*, **55** (2026), 129–144. https://doi.org/10.1080/03081079.2025.2463945
14. A. Berman, R. J. Plemmons, *Nonnegative matrices in the mathematical sciences*, 2 Eds., Philadelphia: SIAM, 1994. https://doi.org/10.1137/1.9781611971262
15. J. R. Norris, *Markov chains*, New York: Cambridge University Press, 1997. https://doi.org/10.1017/CBO9780511810633
16. R. J. Horn, C. R. Johnson, *Matrix analysis*, 2 Eds., New York: Cambridge University Press, 2012. https://doi.org/10.1017/CBO9780511810817