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Research article

Balmer spectrum of tensor n-angulated categories

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Abstract: In this paper, we first introduce the notions of thick and prime tensor ideals in tensor n-angulated categories and then obtain the Balmer spectrum of such categories. We consider the topological properties of this spectrum. Moreover, we introduce the notion of radical tensor ideals in tensor n-angulated categories and show that they can be parametrized by supports of subcategories. Furthermore, we provide Balmer's classification theorem in terms of Thomason subsets of the spectrum space.

Keywords: tensor n-angulated category; Balmer spectrum; thick; radical; Thomason subset
Mathematics Subject Classification: 18F30,18G25,18G80

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1. Introduction

The classification problem on subcategories is a hot topic of great interest. For instance, the classification problem of Serre subcategories has been studied by many scholars. A typical result was given by Gabriel [1], who established, over a commutative Noetherian ring R, a bijective correspondence between the set of Serre subcategories of the category mod R of finitely generated modules and that of specialization-closed subsets of the prime spectrum Spec(R). Thomason [2] classified dense triangulated subcategories of triangulated categories using Grothendieck groups. Kanda [3] classified Serre subcategories of Noetherian abelian categories using open subsets of the atom spectrum. Matsui [4] classified dense resolving and coresolving subcategories of exact categories via Grothendieck groups. For more work on the classification of subcategories, we refer to [5–7] and the references therein.

The study of tensor triangular geometry began with the work of Balmer [8], who introduced a new geometric space, namely the Balmer spectrum, in the context of braided tensor triangulated categories. The Balmer spectrum is endowed with a Zariski topology, where the supports of objects form a closed basis of the spectrum. One of Balmer's major contributions is the classification theorem for subcategories: There is a one-to-one correspondence between radical ideals of a tensor triangulated category and Thomason subsets of its Balmer spectrum. For more applications of tensor triangulated categories, see [9, 10].

A higher-dimensional analog of triangulated categories, called n-angulated categories, was recently introduced by Geiss, Keller, and Oppermann [11]. For $n = 3$, one recovers the usual triangulated categories, meanwhile for larger n, such structures arise naturally from cluster tilting subcategories of triangulated categories. In [12], Bergh and Thaule introduced the notion of a tensor n-angulated category; that is, an n-angulated category equipped with a symmetric tensor product compatible with the n-angulated structure. For such categories, the Grothendieck group naturally becomes a commutative ring, which one calls the Grothendieck ring. They proved that the ideals of this ring correspond bijectively to the complete and dense n-angulated tensor ideals and that prime ideals correspond to n-angulated tensor prime ideals.

In this paper, we aim to study the Balmer spectrum of a tensor n-angulated category and the subcategory classification problem.

In Section 2, we recall the notions of tensor n-angulated categories. Then, we introduce the notion of thick tensor ideals and prime tensor ideals and obtain the Balmer spectrum. We will show that this spectrum space is not empty and is a T_0 space. We also characterize the irreducible closed subsets of this spectrum space.

In Section 3, we introduce the notion of radical tensor ideals. Then, we give the classification theorem of radical ideals (see Theorems 3.4 and 3.8). More precisely, let $\mathcal{T}$ be a nonzero tensor n-angulated category. We prove the existence of prime $\otimes$-ideals; study the topological properties of $\operatorname{Spc}(\mathcal{T})$, including the T_0-property and irreducible closed subsets; and establish a classification theorem for radical $\otimes$-ideals via Thomason subsets.

2. Tensor n-angulated categories

Let $n \geq 3$ be an integer. We first recall the notion of tensor n-angulated categories as follows.

Definition 2.1. ([12, Section 5]) A *tensor n-angulated category* is a triple $(\mathcal{T}, \Sigma, \otimes, \mathbb{1})$ such that

(1) $(\mathcal{T}, \Sigma)$ is an n-angulated category ([11, Definition 2.1]).
(2) $(\mathcal{T}, \otimes, \mathbb{1})$ is a symmetric monoidal category with the unit $\mathbb{1}$ and the natural isomorphisms $\alpha : (- \otimes (- \otimes -)) \to ((- \otimes -) \otimes -)$, $\lambda : (\mathbb{1} \otimes -) \to (-)$, and $\rho : (- \otimes \mathbb{1}) \to (-)$.
(3) There are natural isomorphisms $l : (- \otimes (\Sigma -)) \to \Sigma(- \otimes -)$ and $r : ((\Sigma -) \otimes -) \to \Sigma(- \otimes -)$.
(4) For every object $a \in \mathcal{T}$, the endofunctors $(a \otimes -)$ and $(- \otimes a)$ on $\mathcal{T}$ are n-angulated functors, together with the natural isomorphisms l and r, respectively.
(5) For any $a \in \mathcal{T}$, the diagrams

$$\begin{tikzcd}
& \Sigma(\mathbb{1} \otimes a) \arrow[dr, "\Sigma\lambda"] & & & \Sigma(a \otimes \mathbb{1}) \arrow[dr, "\Sigma\rho"] & \\
\mathbb{1} \otimes \Sigma a \arrow[ur, "l"] \arrow[rr, "\lambda"] & & \Sigma a & \Sigma a \otimes \mathbb{1} \arrow[ur, "r"] \arrow[rr, "\rho"] & & \Sigma a
\end{tikzcd}$$

commute.

(6) For any $a, b \in \mathcal{T}$, the diagram

$$\begin{tikzcd}
\Sigma a \otimes \Sigma b \arrow[r, "r"] \arrow[d, "l"] & \Sigma(a \otimes \Sigma b) \arrow[d, "\Sigma l"] \\
\Sigma(\Sigma a \otimes b) \arrow[r, "\Sigma r"] & \Sigma^2(a \otimes b)
\end{tikzcd}$$

anticommutes.

It is well known that the notion of ideals plays a crucial role in ring theory. We also need the categorical version of ideals.

In the following, we always assume that $\mathcal{T} = (\mathcal{T}, \Sigma, \otimes, \mathbb{1})$ is a skeletally small tensor n-angulated category, and all subcategories are additive and full subcategories which are closed under isomorphisms.

Definition 2.2. A *thick tensor ideal* (or *thick $\otimes$-ideal*) $\mathcal{I}$ of $\mathcal{T}$ is a subcategory such that the following conditions are satisfied:

(a) $\mathcal{I}$ satisfies the $(n-1)/n$-property (which is called a *complete* subcategory in [12, Definition 4.1]): Given any n-angle $a_1 \to a_2 \to \cdots \to a_n \to \Sigma a_1$, if any $n-1$ terms of $a_1, a_2, \cdots, a_n$ belong to $\mathcal{I}$, then so does the last one.
(b) $\mathcal{I}$ is closed under taking direct summands.
(c) $\mathcal{I}$ is a $\otimes$-ideal; that is, for any $a \in \mathcal{I}$ and $b \in \mathcal{T}$, $a \otimes b \in \mathcal{I}$.

Remark 2.3. The intersection of any collection of thick $\otimes$-ideals is again a thick $\otimes$-ideal.

Given a collection $\mathcal{X}$ of objects of $\mathcal{T}$, we denote by $\langle \mathcal{X} \rangle$ the smallest thick $\otimes$-ideal of $\mathcal{T}$ which contains $\mathcal{X}$. By Remark 2.3, $\langle \mathcal{X} \rangle$ is exactly the intersection of all thick $\otimes$-ideals which contain $\mathcal{X}$.

Now, we give the notion of prime $\otimes$-ideals in a tensor n-angulated category.

Definition 2.4. Let $\mathcal{P}$ be a proper thick $\otimes$-ideal of a tensor n-angulated category $\mathcal{T}$. Then $\mathcal{P}$ is called *prime* provided that $a \otimes b \in \mathcal{P}$ implies $a \in \mathcal{P}$ or $b \in \mathcal{P}$.

- We denote by $\mathrm{Spc}(\mathcal{T})$ the set of all prime $\otimes$-ideals of $\mathcal{T}$ and call it the *Balmer spectrum* of $\mathcal{T}$.
- Any collection $\mathcal{S}$ of objects of $\mathcal{T}$ we denote by

$$V(\mathcal{S}) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \mid \mathcal{S} \cap \mathcal{P} = \emptyset\}.$$

Remark 2.5. (1) $\bigcap_{i \in I} V(\mathcal{S}_i) = V(\bigcup_{i \in I} \mathcal{S}_i)$.
(2) $V(\mathcal{S}_1) \cup V(\mathcal{S}_2) = V(\mathcal{S}_1 \oplus \mathcal{S}_2)$, where $\mathcal{S}_1 \oplus \mathcal{S}_2 = \{a_1 \oplus a_2 \mid a_1 \in \mathcal{S}_1, a_2 \in \mathcal{S}_2\}$.
(3) $V(\mathcal{T}) = \emptyset$.
(4) $V(\emptyset) = \mathrm{Spc}(\mathcal{T})$.

Thus, the collection $\{V(\mathcal{S}) \mid \forall \mathcal{S} \subseteq \mathcal{T}\}$ defines the closed subsets of a topology on $\mathrm{Spc}(\mathcal{T})$, which is called the *Zariski topology*.

We denote by $U(\mathcal{S}) := \mathrm{Spc}(\mathcal{T}) \backslash V(\mathcal{S}) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \mid \mathcal{S} \cap \mathcal{P} \neq \emptyset\}$ the open complement of $V(\mathcal{S})$ for any $\mathcal{S} \subseteq \mathcal{T}$.

For any object $a \in \mathcal{T}$, we denote by

$$\mathrm{supp}(a) := V(\{a\}) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \mid a \notin \mathcal{P}\}$$

the closed subset of the topological space $\mathrm{Spc}(\mathcal{T})$ and call it the *support* of the object a.

In the following, we show that prime $\otimes$-ideals must exist in any nonzero tensor n-angulated category.

We say that a collection $\mathcal{S}$ of objects of $\mathcal{T}$ is $\otimes$-*multiplicative* provided that $\mathbb{1} \in \mathcal{S}$ and $\mathcal{S}$ is $\otimes$-closed; that is, if $a_1, a_2 \in \mathcal{S}$, then $a_1 \otimes a_2 \in \mathcal{S}$.

Lemma 2.6. *Let $\mathcal{T}$ be a nonzero tensor n-angulated category and $\mathcal{I}$ a proper thick $\otimes$-ideal of $\mathcal{T}$. Assume that $\mathcal{S}$ is a $\otimes$-multiplicative collection such that $\mathcal{I} \cap \mathcal{S} = \emptyset$. Then, we can find a prime $\otimes$-ideal $\mathcal{P}$ of $\mathcal{T}$ which satisfies $\mathcal{I} \subseteq \mathcal{P}$ and $\mathcal{P} \cap \mathcal{S} = \emptyset$.*

Proof. Let $\mathcal{F}$ be the collection consisting of those proper thick $\otimes$-ideals $\mathcal{J}$ of $\mathcal{T}$ such that

1. $\mathcal{J} \cap \mathcal{S} = \emptyset$.
2. $\mathcal{I} \subseteq \mathcal{J}$.
3. for any $c \in \mathcal{S}$ and $a \in \mathcal{T}$, if $a \otimes c \in \mathcal{J}$, then $a \in \mathcal{J}$.

Let $\mathcal{J}_0 := \{a \in \mathcal{T} : \text{ there exists } c \in \mathcal{S} \text{ such that } a \otimes c \in \mathcal{I}\}$. First, we check that $\mathcal{J}_0$ is a proper thick $\otimes$-ideal of $\mathcal{T}$. Indeed,

- $\mathcal{J}_0$ satisfies the $(n-1)/n$-property: Given an n-angle $a_1 \longrightarrow a_2 \longrightarrow \cdots \longrightarrow a_n \longrightarrow \Sigma a_1$, without loss of generality, we can assume $a_1, \cdots, a_{n-1} \in \mathcal{J}_0$. Then, there exist $b_1, \cdots, b_{n-1} \in \mathcal{S}$ with $b_1 \otimes a_1, \cdots, b_{n-1} \otimes a_{n-1} \in \mathcal{I}$. Because $\otimes$ is n-angulated, we have an n-angle
$$b_1 \otimes \cdots \otimes b_{n-1} \otimes a_1 \longrightarrow \cdots \longrightarrow b_1 \otimes \cdots \otimes b_{n-1} \otimes a_{n-1} \longrightarrow b_1 \otimes \cdots \otimes b_{n-1} \otimes a_n \longrightarrow \Sigma(b_1 \otimes \cdots \otimes b_{n-1} \otimes a_1).$$
Moreover, because $\mathcal{I}$ is an ideal, $b_1 \otimes \cdots \otimes b_{n-1} \otimes a_1, \cdots, b_1 \otimes \cdots \otimes b_{n-1} \otimes a_{n-1} \in \mathcal{I}$. By the thickness of $\mathcal{I}$, we also have $b_1 \otimes \cdots \otimes b_{n-1} \otimes a_n \in \mathcal{I}$, and because $\mathcal{S}$ is $\otimes$-multiplicative, and $b_1 \otimes \cdots \otimes b_{n-1} \in \mathcal{S}$, it follows that $a_n \in \mathcal{J}_0$.
- $\mathcal{J}_0$ is closed under direct summands: Suppose $a \oplus b \in \mathcal{J}_0$. Then, there exists $c \in \mathcal{S}$ with $c \otimes (a \oplus b) \in \mathcal{I}$. However $c \otimes (a \oplus b) = (c \otimes a) \oplus (c \otimes b)$ and $\mathcal{I}$ is thick, so we get $c \otimes a \in \mathcal{I}$, $c \otimes b \in \mathcal{I}$, and hence, $a, b \in \mathcal{J}_0$.
- $\mathcal{J}_0$ is a $\otimes$-ideal: Let $a \in \mathcal{J}_0$; that is, there exists $c \in \mathcal{S}$ with $a \otimes c \in \mathcal{I}$. For any $b \in \mathcal{T}$, because $\mathcal{I}$ is a $\otimes$-ideal, $b \otimes a \otimes c \in \mathcal{I}$, and hence, $b \otimes a \in \mathcal{J}_0$.
- $\mathcal{J}_0$ is proper: In fact, $\mathbb{1} \notin \mathcal{J}_0$. Otherwise, $\mathbb{1} \in \mathcal{J}_0$ implies that there exists $c \in \mathcal{S}$ such that $c \cong \mathbb{1} \otimes c \in \mathcal{I}$, which is a contradiction with the assumption $\mathcal{S} \cap \mathcal{I} = \emptyset$.

Second, we show that $\mathcal{J}_0$ satisfies the above conditions (1)–(3).

1. Suppose $\mathcal{J}_0 \cap \mathcal{S} \neq \emptyset$, and let $a \in \mathcal{J}_0 \cap \mathcal{S}$. Then, there exists $c \in \mathcal{S}$ with $a \otimes c \in \mathcal{I}$. Moreover, because $a \in \mathcal{S}$, $a \otimes c \in \mathcal{S}$, and hence, $a \otimes c \in \mathcal{S} \cap \mathcal{I}$, which is a contradiction with $\mathcal{S} \cap \mathcal{I} = \emptyset$. Thus, $\mathcal{J}_0 \cap \mathcal{S} = \emptyset$.
2. $\forall a \in \mathcal{I}$, because $\mathbb{1} \in \mathcal{S}$, and $\mathbb{1} \otimes a \cong a \in \mathcal{I}$; then, $a \in \mathcal{J}_0$. Thus, $\mathcal{I} \subseteq \mathcal{J}_0$.
3. Let $a \in \mathcal{T}$ and $c \in \mathcal{S}$ such that $a \otimes c \in \mathcal{J}_0$. Then, there exists $b \in \mathcal{S}$ such that $a \otimes c \otimes b \in \mathcal{I}$. Because $\mathcal{S}$ is $\otimes$-multiplicative, $c \otimes b \in \mathcal{S}$, and hence, $a \in \mathcal{J}_0$.

Following this, we know that the collection $\mathcal{F}$ is nonempty. By Zorn's lemma, there exists an element $\mathcal{P} \in \mathcal{F}$ maximal for inclusion. We claim that $\mathcal{P}$ is a prime $\otimes$-ideal. Indeed, assume that $a \otimes b \in \mathcal{P}$, and $b \notin \mathcal{P}$. Let $\mathcal{J}_1 = \{d \in \mathcal{T} \mid a \otimes d \in \mathcal{P}\}$. As a similar argument to $\mathcal{J}_0$, we can check that $\mathcal{J}_1$ is a thick $\otimes$-ideal, which contains $\mathcal{P}$ properly because $b \in \mathcal{J}_1 \backslash \mathcal{P}$. By the maximality of $\mathcal{P}$ in $\mathcal{F}$, we know that $\mathcal{J}_1$ does not belong to $\mathcal{F}$. However $\mathcal{J}_1$ satisfies the conditions (2) and (3), so $\mathcal{J}_1$ can not satisfy the condition (1); that is, $\mathcal{J}_1 \cap \mathcal{S} \neq \emptyset$. Thus, there is $d \in \mathcal{S}$ and $d \in \mathcal{J}_1$; that is, $a \otimes d \in \mathcal{P}$. By the condition (3) for $\mathcal{P}$, we have $a \in \mathcal{P}$, as desired. □

Proposition 2.7. *Let $\mathcal{T}$ be a nonzero tensor n-angulated category.*

(a) Let $\mathcal{S}$ be a $\otimes$-multiplicative collection, and $0 \notin \mathcal{S}$. Then, we can find a prime $\otimes$-ideal $\mathcal{P}$ of $\mathcal{T}$ such that $\mathcal{P} \cap \mathcal{S} = \emptyset$.

(b) Let $\mathcal{I}$ be a thick proper $\otimes$-ideal of $\mathcal{T}$. Then, we can find a maximal thick proper $\otimes$-ideal $\mathcal{J}$ of $\mathcal{T}$ which contains $\mathcal{I}$.

(c) Each maximal thick proper $\otimes$-ideal of $\mathcal{T}$ is prime.

(d) The spectrum $\mathrm{Spc}(\mathcal{T})$ *is nonempty.*

Proof. (a) Let $\mathcal{I} = \{0\}$; it is a proper thick $\otimes$-ideal. Clearly, $\mathcal{I} \cap \mathcal{S} = \emptyset$, so there exists a prime $\otimes$-ideal $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$ such that $\mathcal{P} \cap \mathcal{S} = \emptyset$ by Lemma 2.6.

(b) Let $\mathcal{S} := \{\mathbb{1}\}$. Then, the prime $\otimes$-ideal $\mathcal{P}$ in the proof of Lemma 2.6 is a maximal proper thick $\otimes$-ideal which contains $\mathcal{I}$.

(c) Let $\mathcal{I}$ be a maximal proper thick $\otimes$-ideal, and $\mathcal{S} = \{\mathbb{1}\}$. Then, $\mathcal{I} \cap \mathcal{S} = \emptyset$. By Lemma 2.6, there exists a prime $\otimes$-ideal $\mathcal{P}$ such that $\mathcal{I} \subseteq \mathcal{P}$. By the maximality, $\mathcal{I} = \mathcal{P}$ is prime.

(d) Follows from (a). □

Corollary 2.8. *Let $\mathcal{T}$ be a tensor n-angulated category, and let $a \in \mathcal{T}$. The following are equivalent:*

(1) $a \in \bigcap_{\mathcal{P} \in \mathrm{Spc}(\mathcal{T})} \mathcal{P}$.

(2) $U(a) = \mathrm{Spc}(\mathcal{T})$.

(3) $\mathrm{supp}(a) = \emptyset$.

(4) a is a $\otimes$-nilpotent object, which means that there exists a positive integer n such that $a^{\otimes n} = 0$.

Proof. (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) are clear.

(1) $\Longrightarrow$ (4). Assume that a is not $\otimes$-nilpotent; that is, for any integer $n \geq 0$, $a^{\otimes n} \neq 0$. Set $\mathcal{S} := \{a^{\otimes n} \mid \forall\ n \geq 0\}$. Then, $\mathcal{S}$ is a $\otimes$-multiplicative collection, and $0 \notin \mathcal{S}$. By Proposition 2.7 (a), there exists a prime $\otimes$-ideal $\mathcal{P}$ such that $\mathcal{P} \cap \mathcal{S} = \emptyset$. In particular, $a \notin \mathcal{P}$, which is a contradiction. Thus, a is $\otimes$-nilpotent.

(4) $\Longrightarrow$ (1). Suppose $a^{\otimes n} = 0$. Because every prime $\otimes$-ideal $\mathcal{P}$ contains 0, then $a^{\otimes n} \in \mathcal{P}$, and thus, $a \in \mathcal{P}$ by the definition of prime $\otimes$-ideals. Therefore, $a \in \bigcap_{\mathcal{P} \in \mathrm{Spc}(\mathcal{T})} \mathcal{P}$. □

Corollary 2.9. *Let $\mathcal{T}$ be a tensor n-angulated category, and $a \in \mathcal{T}$. The following are equivalent:*

(1) $a \notin \bigcup_{\mathcal{P} \in \mathrm{Spc}(\mathcal{T})} \mathcal{P}$.

(2) $U(a) = \emptyset$.

(3) $\mathrm{supp}(a) = \mathrm{Spc}(\mathcal{T})$.

(4) $\langle a \rangle = \mathcal{T}$; that is, the object a generates $\mathcal{T}$ as a thick $\otimes$-ideal.

Proof. (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) are clear.

(4) $\Longrightarrow$ (1). If $\langle a \rangle = \mathcal{T}$, then a does not belong to any proper thick $\otimes$-ideal of $\mathcal{T}$. Of course, a does not belong to any prime $\otimes$-ideal.

(1) $\Longrightarrow$ (4). Suppose $\langle a \rangle \subsetneq \mathcal{T}$; that is, $\langle a \rangle$ is a proper thick $\otimes$-ideal of $\mathcal{T}$. Then, by Proposition 2.7 (b) and (c), we can find a prime $\otimes$-ideal $\mathcal{P}$ such that $\langle a \rangle \subseteq \mathcal{P}$. Thus, $a \in \mathcal{P}$, which is a contradiction. □

For a topological space X, we denote by $\mathcal{X}_{\mathrm{cl}}(X)$ the collection of all closed subsets of X and by $\mathcal{X}_{\mathrm{op}}(X)$ the collection of all open subsets of X.

Let $\mathcal{T}$ be a tensor n-angulated category and $a \in \mathcal{T}$. We remark that $U(a) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \mid a \in \mathcal{P}\} = \mathrm{Spc}(\mathcal{T}) \backslash \mathrm{supp}(a)$.

Lemma 2.10. *Let $\mathcal{T}$ be a tensor n-angulated category. The assignment $U(-) : \mathcal{T} \to \mathcal{X}_{\mathrm{op}}(\mathrm{Spc}(\mathcal{T}))$ given by $a \mapsto U(a)$ satisfies the following properties:*

(1) $U(0) = \mathrm{Spc}(\mathcal{T})$, $U(\mathbb{1}) = \emptyset$.
(2) $U(a \oplus b) = U(a) \cap U(b)$.
(3) *Given any n-angle,* $a_1 \to a_2 \to \cdots \to a_n \to \Sigma a_1$. *For any* $i \in \{1, 2, \cdots, n\}$, $U(a_i) \supseteq U(a_1) \cap \cdots \cap U(a_{i-1}) \cap U(a_{i+1}) \cap \cdots \cap U(a_n)$.
(4) $U(a \otimes b) = U(a) \cup U(b)$.

Proof. The assertions follow directly from the definitions and Remark 2.5. □

Lemma 2.11. *Let $\mathcal{T}$ be a tensor n-angulated category. The assignment* $\mathrm{supp}(-) : \mathcal{T} \to \mathcal{X}_{\mathrm{cl}}(\mathrm{Spc}(\mathcal{T}))$ *given by* $a \mapsto \mathrm{supp}(a)$ *satisfies the following properties:*

(1) $\mathrm{supp}(0) = \emptyset$, $\mathrm{supp}(\mathbb{1}) = \mathrm{Spc}(\mathcal{T})$.
(2) $\mathrm{supp}(a \oplus b) = \mathrm{supp}(a) \cup \mathrm{supp}(b)$.
(3) *Given any n-angle,* $a_1 \to a_2 \to \cdots \to a_n \to \Sigma a_1$. *For any* $i \in \{1, 2, \cdots, n\}$, $\mathrm{supp}(a_i) \subseteq \mathrm{supp}(a_1) \cup \cdots \cup \mathrm{supp}(a_{i-1}) \cup \mathrm{supp}(a_{i+1}) \cup \cdots \cup \mathrm{supp}(a_n)$.
(4) $\mathrm{supp}(a \otimes b) = \mathrm{supp}(a) \cap \mathrm{supp}(b)$.

Proof. The assertions follow directly from the definitions and Remark 2.5. □

It is easy to check that $U(\mathcal{S}) = \bigcup_{a \in \mathcal{S}} U(a)$ for any $\mathcal{S} \subseteq \mathcal{T}$; thus, $\{U(a) \mid a \in \mathcal{T}\}$ is a basis of the topology on $\mathrm{Spc}(\mathcal{T})$. Dually, $\{\mathrm{supp}(a) \mid a \in \mathcal{T}\}$ is a basis of closed subsets on $\mathrm{Spc}(\mathcal{T})$.

Proposition 2.12. *Let $\mathcal{T}$ be a tensor n-angulated category and $\omega \subseteq \mathrm{Spc}(\mathcal{T})$ a subset of the spectrum. Its closure is*

$$\overline{\omega} = \bigcap_{\substack{a \in \mathcal{T} \text{ s.t.} \\ \omega \subseteq \mathrm{supp}(a)}} \mathrm{supp}(a).$$

Proof. It follows directly from that $\{\mathrm{supp}(a)\}_{a \in \mathcal{T}}$ is a basis of closed subsets. □

Now, we show that $\mathrm{Spc}(\mathcal{T})$ is a T_0 space.

Proposition 2.13. *Let $\mathcal{T}$ be a tensor n-angulated category. For any point $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$, its closure is $\overline{\{\mathcal{P}\}} = \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{T}) \mid \mathcal{Q} \subseteq \mathcal{P}\}$. In particular, if $\overline{\{\mathcal{P}_1\}} = \overline{\{\mathcal{P}_2\}}$, then $\mathcal{P}_1 = \mathcal{P}_2$.*

Proof. Let $\mathcal{S}_0 = \mathcal{T} \setminus \mathcal{P}$, $V(\mathcal{S}_0) := \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{T}) \mid \mathcal{S}_0 \cap \mathcal{Q} = \emptyset\}$. Clearly, $\mathcal{P} \in V(\mathcal{S}_0)$, and if $\mathcal{P} \in V(\mathcal{S})$ for some subset $\mathcal{S} \subseteq \mathcal{T}$, then $\mathcal{S} \subseteq \mathcal{S}_0$, and hence, $V(\mathcal{S}_0) \subseteq V(\mathcal{S})$. This shows that $V(\mathcal{S}_0)$ is the smallest closed subset which contains the point $\mathcal{P}$; that is, $\overline{\{\mathcal{P}_1\}} = V(\mathcal{S}_0) = \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{T}) \mid \mathcal{Q} \subseteq \mathcal{P}\}$.

Moreover, $\mathcal{P}_1 \in \overline{\{\mathcal{P}_1\}} \subseteq \overline{\{\mathcal{P}_2\}}$ implies $\mathcal{P}_1 \subseteq \mathcal{P}_2$, and $\mathcal{P}_2 \in \overline{\{\mathcal{P}_2\}} \subseteq \overline{\{\mathcal{P}_1\}}$ implies $\mathcal{P}_2 \subseteq \mathcal{P}_1$. Thus, $\overline{\{\mathcal{P}_1\}} = \overline{\{\mathcal{P}_2\}}$ implies that $\mathcal{P}_1 = \mathcal{P}_2$. □

Proposition 2.14. *Let $\mathcal{T}$ be a nonzero tensor n-angulated category. Then, there exists minimal prime $\otimes$-ideals in $\mathcal{T}$.*

Proof. We argue using Zorn's lemma. Consider any nonempty chain $C \subseteq \mathrm{Spc}(\mathcal{T})$, and set $\mathcal{I}' := \bigcap_{\mathcal{I}\in C} \mathcal{I}$. We claim that $\mathcal{I}'$ is a prime $\otimes$-ideal. To see this, assume $a_1 \notin \mathcal{I}'$ and $a_2 \notin \mathcal{I}'$. Then, for each $i = 1, 2$ we can find $\mathcal{I}_i \in C$ with $a_i \notin \mathcal{I}_i$. Because C is totally ordered by inclusion, let $\mathcal{I}_0$ denote the smaller of $\mathcal{I}_1$ and $\mathcal{I}_2$ (so $\mathcal{I}_0 = \min\{\mathcal{I}_1, \mathcal{I}_2\}$). Then, $a_1, a_2 \notin \mathcal{I}_0$, and therefore, $a_1 \otimes a_2 \notin \mathcal{I}_0$. Because $\mathcal{I}' \subseteq \mathcal{I}_0$, we conclude $a_1 \otimes a_2 \notin \mathcal{I}'$. Hence, $\mathcal{I}'$ is a prime $\otimes$-ideal, as desired. □

Corollary 2.15. *Let $\mathcal{T}$ be a nonzero tensor n-angulated category. Then, the topological space* $\mathrm{Spc}(\mathcal{T})$ *have a closed point. More generally, for any nonempty closed subset Z of* $\mathrm{Spc}(\mathcal{T})$*, Z contains at least one closed point.*

Proof. Let $Z \subseteq \mathrm{Spc}(\mathcal{T})$ be a nonempty closed subset, and pick $\mathcal{P} \in Z$. By Proposition 2.14, there exists a minimal prime $\otimes$-ideal $\mathcal{P}'$ contained in $\mathcal{P}$. Proposition 2.13 tells us that $\overline{\{\mathcal{P}\}} = \{\mathcal{I} \in \mathrm{Spc}(\mathcal{T}) \mid \mathcal{I} \subseteq \mathcal{P}\}$, so $\mathcal{P}'$ belongs to $\overline{\{\mathcal{P}\}} \subseteq Z$. The minimality of $\mathcal{P}'$ then yields

$$\overline{\{\mathcal{P}'\}} = \{\mathcal{I} \in \mathrm{Spc}(\mathcal{T}) \mid \mathcal{I} \subseteq \mathcal{P}'\} = \{\mathcal{P}'\},$$

which means $\mathcal{P}'$ is a closed point. □

Definition 2.16. Let $\mathcal{T}$ be a nonzero tensor n-angulated category and Z a nonempty closed subset of $\mathrm{Spc}(\mathcal{T})$. Then, Z is called *irreducible* provided that for any open subsets U_1 and U_2 in $\mathrm{Spc}(\mathcal{T})$, the condition $Z \cap U_1 \cap U_2 = \emptyset$ implies $Z \cap U_1 = \emptyset$ or $Z \cap U_2 = \emptyset$.

Proposition 2.17. *Let $\mathcal{T}$ be a nonzero tensor n-angulated category and Z a nonempty closed subset of* $\mathrm{Spc}(\mathcal{T})$*. The following are equivalent:*

- *(i) Z is irreducible.*
- *(ii) For all $a, b \in \mathcal{T}$, if $U(a \oplus b) \cap Z = \emptyset$, then $U(a) \cap Z = \emptyset$ or $U(b) \cap Z = \emptyset$.*
- *(iii) $P_Z := \{a \in \mathcal{T} \mid U(a) \cap Z \neq \emptyset\}$ is a prime $\otimes$-ideal.*

If one of the above conditions holds, then $Z = \overline{\{P_Z\}}$; that is, any nonempty irreducible closed subset of $\mathrm{Spc}(\mathcal{T})$ *has a unique generic point.*

Proof. $(i) \Rightarrow (ii)$ is clear because $U(a \oplus b) = U(a) \cap U(b)$.

$(ii) \Rightarrow (iii)$ Let $a, b \in P_Z$; that is, $U(a) \cap Z \neq \emptyset$, and $U(b) \cap Z \neq \emptyset$. Then, $U(a \oplus b) \cap Z \neq \emptyset$ by the contrapositive of condition (ii), and hence, $a \oplus b \in P_Z$.

Assume $a \oplus b \in P_Z$. Then, $U(a \oplus b) \cap Z \neq \emptyset$. However, $U(a \oplus b) = U(a) \cap U(b)$; thus, $(U(a) \cap Z) \cap (U(b) \cap Z) \neq \emptyset$, and so $U(a) \cap Z \neq \emptyset$, and $U(b) \cap Z \neq \emptyset$; that is, $a, b \in P_Z$.

Given an n-angle, $a_1 \longrightarrow a_2 \longrightarrow \cdots \longrightarrow a_n \longrightarrow \Sigma a_1$. For any $i \in \{1, 2, \cdots, n\}$, assume that $a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n \in P_Z$; that is, $U(a_k) \cap Z \neq \emptyset$ for any $k \in \{1, \cdots, i-1, i+1, \cdots, n\}$. Clearly, $a_i \in \langle a_1 \oplus \cdots \oplus a_{i-1} \oplus a_{i+1} \oplus \cdots \oplus a_n \rangle$ by the $(n-1)/n$-property, and so $U(a_1 \oplus \cdots \oplus a_{i-1} \oplus a_{i+1} \oplus \cdots \oplus a_n) \subseteq U(a_i)$. By the first statement, $a_1 \oplus \cdots \oplus a_{i-1} \oplus a_{i+1} \oplus \cdots \oplus a_n \in P_Z$; that is, $U(a_1 \oplus \cdots \oplus a_{i-1} \oplus a_{i+1} \oplus \cdots \oplus a_n) \cap Z \neq \emptyset$. Then, $U(a_i) \cap Z \neq \emptyset$, and hence, $a_i \in P_Z$.

Finally, let $a \in P_Z$, $b \in \mathcal{T}$. Then, $U(a) \cap Z \neq \emptyset$. However, $U(a \otimes b) = U(a) \cup U(b)$, so $U(a \otimes b) \cap Z = (U(a) \cup U(b)) \cap Z \supseteq U(a) \cap Z \neq \emptyset$. Thus, $a \otimes b \in P_Z$.

Therefore, P_Z is a prime $\otimes$-ideal.

$(iii) \Rightarrow (i)$ It suffices to prove that $Z = \overline{\{P_Z\}} := \{I \in \mathrm{Spc}(\mathcal{T}) \mid I \subseteq P_Z\}$.

Let $I \in Z$. For any $a \in I$, $I \in U(a)$, and hence, $I \in U(a) \cap Z$. Thus, $U(a) \cap Z \neq \emptyset$, which means that $a \in P_Z$. Thus, $I \subseteq P_Z$, and then $I \in \overline{\{P_Z\}}$. Hence, $Z \subseteq \overline{\{P_Z\}}$.

Conversely, because Z is closed, $Z = \overline{Z} = \bigcap_{\substack{a \in \mathcal{T} \\ Z \subseteq \mathrm{supp}(a)}} \mathrm{supp}(a)$. To show $\overline{\{P\}} \subseteq Z$, it suffices to show $\overline{\{P_Z\}} \subseteq \overline{Z}$, or equivalently, to show $P_Z \in \overline{Z}$. To do it, let $a \in \mathcal{T}$ satisfying $Z \subseteq \mathrm{supp}(a)$. Then, every prime $\otimes$-ideal in Z does not contain a, and hence, $U(a) \cap Z = \emptyset$, which shows that $a \notin P_Z$. Thus, $P_Z \in \mathrm{supp}(\mathcal{S})$. Therefore, $P_Z \in \bigcap_{\substack{a \in \mathcal{T} \\ Z \subseteq \mathrm{supp}(a)}} \mathrm{supp}(a) = \overline{Z}$. □

Corollary 2.18. *Let $\mathcal{T}$ be a nonzero tensor n-angulated category. The following are equivalent:*

(1) $\mathrm{Spc}(\mathcal{T})$ *is irreducible.*
(2) For any $a, b \in \mathcal{T}$, if $\langle a \oplus b\rangle = \mathcal{T}$, then $\langle a\rangle = \mathcal{T}$ or $\langle b\rangle = \mathcal{T}$.

Proof. By Corollary 2.9, for any $a \in \mathcal{T}$, $U(a) = \emptyset$ if and only if $\langle a\rangle = \mathcal{T}$. Then, by Proposition 2.17, $\mathrm{Spc}(\mathcal{T})$ is irreducible if and only if for any $a, b \in \mathcal{T}$, if $U(a \oplus b) = \emptyset$, then $U(a) = \emptyset$ or $U(b) = \emptyset$ if and only if for any $a, b \in \mathcal{T}$, if $\langle a \oplus b\rangle = \mathcal{T}$, then $\langle a\rangle = \mathcal{T}$ or $\langle b\rangle = \mathcal{T}$. □

3. Radical ⊗-ideals and classifications

Now, we give the notion of radical ⊗-ideals.

Definition 3.1. Let $\mathcal{T}$ be a tensor n-angulated category and $\mathcal{I}$ a thick $\otimes$-ideal of $\mathcal{T}$. The *radical* $\sqrt{\mathcal{I}}$ is defined as follows:

$$\sqrt{\mathcal{I}} := \bigcap_{\substack{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \\ \mathcal{P} \supseteq \mathcal{I}}} \mathcal{P}.$$

A thick $\otimes$-ideal $\mathcal{I}$ is called *radical* if $\sqrt{\mathcal{I}} = \mathcal{I}$.

We denote by $\mathbf{Rad}(\mathcal{T})$ the set of all radical $\otimes$-ideals of $\mathcal{T}$.

Let W be a subset of $\mathrm{Spc}(\mathcal{T})$. We define $\mathrm{supp}^{-1}(W) := \{a \in \mathcal{T} \mid \mathrm{supp}(a) \subseteq W\}$. On the other hand, let $\mathcal{S}$ be a subcategory of $\mathcal{T}$. We define $\mathrm{supp}(\mathcal{S}) = \bigcup_{a \in \mathcal{S}} \mathrm{supp}(a)$.

Lemma 3.2. *Let $\mathcal{T}$ be a tensor n-angulated category and $\mathcal{S}$ a subcategory of $\mathcal{T}$. Then,*

(1) $\mathcal{P} \in \mathrm{supp}(\mathcal{S})$ *if and only if* $\mathcal{S} \nsubseteq \mathcal{P}$.
(2) $\mathrm{supp}(\sqrt{<\mathcal{S}>}) = \mathrm{supp}(<\mathcal{S}>) = \mathrm{supp}(\mathcal{S})$.

Proof. (1) $\mathcal{P} \in \mathrm{supp}(\mathcal{S})$ if and only if there exists $a \in \mathcal{S}$ such that $a \notin \mathcal{P}$ if and only if $\mathcal{S} \nsubseteq \mathcal{P}$.

(2) It follows from (1) and the fact that $\sqrt{<\mathcal{S}>} \nsubseteq \mathcal{P} \Leftrightarrow <\mathcal{S}> \nsubseteq \mathcal{P} \Leftrightarrow \mathcal{S} \nsubseteq \mathcal{P}$ for a prime $\otimes$-ideal $\mathcal{P}$. □

Lemma 3.3. *Let $\mathcal{T}$ be a tensor n-angulated category and W a subset of* $\mathrm{Spc}(\mathcal{T})$.

(1) $\mathrm{supp}^{-1}(W)$ *is a thick ⊗-ideal of $\mathcal{T}$.*
(2) Let $\mathcal{S}$ be a subcategory of $\mathcal{T}$. Then, $\mathrm{supp}^{-1}(\mathrm{supp}(\mathcal{S}))$ *is a radical ⊗-ideal. In fact,* $\mathrm{supp}^{-1}(\mathrm{supp}(\mathcal{S})) = \sqrt{<\mathcal{S}>}$.

Proof. Because $\mathrm{supp}(0) = \emptyset \subseteq W$, we have $0 \in \mathrm{supp}^{-1}(W)$. Thus, $\mathrm{supp}^{-1}(W)$ is nonempty.

Given an n-angle, $a_1 \to a_2 \to \cdots \to a_n \to \Sigma a_1$ in $\mathcal{T}$. If for any $i \in \{1, 2, \cdots, n\}$, $a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n$ belong to $\mathrm{supp}^{-1}(W)$, and then

$$\mathrm{supp}(a_1), \cdots, \mathrm{supp}(a_{i-1}), \mathrm{supp}(a_{i+1}), \cdots, \mathrm{supp}(a_n) \subseteq W.$$

On the other hand, $\mathrm{supp}(a_i) \subseteq \mathrm{supp}(a_1) \cup \cdots \cup \mathrm{supp}(a_{i-1}) \cup \mathrm{supp}(a_{i+1}) \cup \cdots \cup \mathrm{supp}(a_n)$. Then, $\mathrm{supp}(a_i) \subseteq W$, which means that $a_i \in \mathrm{supp}^{-1}(W)$.

Suppose that $a \oplus b \in \mathrm{supp}^{-1}(W)$. Then, $\mathrm{supp}(a \oplus b) \subseteq W$. However, $\mathrm{supp}(a \oplus b) = \mathrm{supp}(a) \cup \mathrm{supp}(b)$, so $\mathrm{supp}(a), \mathrm{supp}(b) \subseteq W$. Thus, $a, b \in \mathrm{supp}^{-1}(W)$.

Let $a \in \mathrm{supp}^{-1}(W)$, and $b \in \mathcal{T}$. Then, $\mathrm{supp}(a) \subseteq W$. On the other hand, $\mathrm{supp}(a \otimes b) = \mathrm{supp}(a) \cap \mathrm{supp}(b)$, so we have $\mathrm{supp}(a \otimes b) \subseteq W$. Then, $a \otimes b \in \mathrm{supp}^{-1}(W)$.

Therefore, $\mathrm{supp}^{-1}(W)$ is a thick $\otimes$-ideal of $\mathcal{T}$.

(2) First, $\sqrt{< \mathcal{S} >} \subseteq \mathrm{supp}^{-1}(\mathrm{supp}(\mathcal{S}))$. Indeed, let $a \in \sqrt{< \mathcal{S} >}$, and $\mathcal{P}' \in \mathrm{supp}(a)$; that is, $a \notin \mathcal{P}'$. Then, $\mathcal{S} \nsubseteq \mathcal{P}'$; otherwise, $\mathcal{P}' \supseteq < \mathcal{S} >$, and then $a \in \mathcal{P}'$. This implies that $\mathcal{P}' \in \mathrm{supp}(\mathcal{S})$ by Lemma 3.2, and then $a \in \mathrm{supp}^{-1}(\mathrm{supp}(\mathcal{S}))$.

Now, we prove that $\mathrm{supp}^{-1}(\mathrm{supp}(\mathcal{S})) \subseteq \sqrt{< \mathcal{S} >}$. Let $a \in \mathrm{supp}^{-1}(\mathrm{supp}(\mathcal{S}))$. Assume $a \notin \sqrt{< \mathcal{S} >}$; then, there exists $\mathcal{P} \supseteq \mathcal{S}$ such that $a \notin \mathcal{P}$, which means $\mathcal{P} \in \mathrm{supp}(a)$. Because $\mathrm{supp}(a) \subseteq \mathrm{supp}(\mathcal{S})$, we have $\mathcal{P} \in \mathrm{supp}(\mathcal{S})$. This shows that $\mathcal{S} \nsubseteq \mathcal{P}$, which is a contradiction. Thus, $a \in \sqrt{< \mathcal{S} >}$. □

Now, we define the parameter set of supports of subcategories of $\mathcal{T}$ as follows:

$$\mathbf{Para}(\mathcal{T}) := \{\mathrm{supp}(\mathcal{S}) \mid \mathcal{S} \text{ is a subcategory of } \mathcal{T}\}.$$

The following theorem shows that the set $\mathbf{Para}(\mathcal{T})$ can parameterize all radical $\otimes$-ideals of $\mathcal{T}$.

Theorem 3.4. *There is an order-preserving bijection*

$$\begin{tikzcd} \mathbf{Para}(\mathcal{T}) \arrow[r, shift left, "\mathrm{supp}^{-1}(-)"] & \mathbf{Rad}(\mathcal{T}) \arrow[l, shift left, "\mathrm{supp}(-)"] \end{tikzcd}.$$

Proof. First, for a subcategory $\mathcal{S}$ of $\mathcal{T}$, $\mathrm{supp}(\mathrm{supp}^{-1}(\mathrm{supp}(\mathcal{S}))) = \mathrm{supp}(\mathcal{S})$: $\mathrm{supp}^{-1}(\mathrm{supp}(\mathcal{S})) = \sqrt{< \mathcal{S} >}$ by Lemma 3.3, and then $\mathrm{supp}(\mathrm{supp}^{-1}(\mathrm{supp}(\mathcal{S}))) = \mathrm{supp}(\sqrt{< \mathcal{S} >}) = \mathrm{supp}(< \mathcal{S} >) = \mathrm{supp}(\mathcal{S})$ by Lemma 3.2.

Second, for a radical $\otimes$-ideal $\mathcal{I}$ of $\mathcal{T}$, $\mathrm{supp}^{-1}(\mathrm{supp}(\mathcal{I})) = \mathcal{I}$: Because $\mathcal{I}$ is a radical $\otimes$-ideal, $\sqrt{< \mathcal{I} >} = \mathcal{I}$. By Lemma 3.3, $\mathrm{supp}^{-1}(\mathrm{supp}(\mathcal{I})) = \sqrt{< \mathcal{I} >} = \mathcal{I}$. □

Lemma 3.5. *Let $\mathcal{T}$ be a tensor n-angulated category, $a \in \mathcal{T}$ an object, and $\mathcal{S}$ a collection of objects of $\mathcal{T}$. The following are equivalent:*

(1) $U(a) \subseteq U(\mathcal{S})$.
(2) $V(\mathcal{S}) \subseteq \mathrm{supp}(a)$.
(3) There exists $b_1, \cdots, b_n \in \mathcal{S}$ such that $b_1 \otimes \cdots \otimes b_n \in \langle a \rangle$.

Proof. (1) $\Longleftrightarrow$ (2) is clear.

Let

$$\mathcal{S}' = \{b_1 \otimes b_2 \otimes \cdots \otimes b_n \mid \forall b_1, b_2, \cdots, b_n \in \mathcal{S}, \forall n \in \mathbb{N}\} \cup \{\mathbb{1}\}.$$

Then, the condition (3) is equivalent to the condition $\mathcal{S}' \cap \langle a \rangle \neq \emptyset$.

On the other hand, because $\mathcal{S} \subseteq \mathcal{S}'$,

$$U(\mathcal{S}) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \mid \mathcal{P} \cap \mathcal{S} \neq \emptyset\} \subseteq \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \mid \mathcal{P} \cap \mathcal{S}' \neq \emptyset\} := U(\mathcal{S}').$$

Now, assume $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$ with $\mathcal{S} \cap \mathcal{P} = \emptyset$. Then, for any $b = b_1 \otimes \cdots \otimes b_n \in \mathcal{S}'$ with $b_1, \cdots, b_n \in \mathcal{S}$, we have $b_1, \cdots, b_n \notin \mathcal{P}$, and hence, $b = b_1 \otimes \cdots \otimes b_n \notin \mathcal{P}$ because $\mathcal{P}$ is prime. It follows that $\mathcal{P} \cap \mathcal{S}' = \emptyset$, and thus, we have $U(\mathcal{S}') \subseteq U(\mathcal{S})$. Therefore, $U(\mathcal{S}) = U(\mathcal{S}')$.

To prove (1) $\Longleftrightarrow$ (3), it suffices to show that

$$U(a) \subseteq U(\mathcal{S}') \Longleftrightarrow \mathcal{S}' \cap \langle a \rangle \neq \emptyset.$$

Indeed, assume $\mathcal{S}' \cap \langle a \rangle \neq \emptyset$. For any $\mathcal{P} \in U(a)$, because $a \in \mathcal{P}$, we have $\langle a \rangle \subseteq \mathcal{P}$, and hence, $\mathcal{P} \cap \mathcal{S}' \neq \emptyset$; that is, $\mathcal{P} \in U(\mathcal{S}')$. Thus, $U(a) \subseteq U(\mathcal{S}')$. Conversely, assume $\mathcal{S}' \cap \langle a \rangle = \emptyset$. By Lemma 2.6 (with $\mathcal{I} := \langle a \rangle$ and $\mathcal{S} := \mathcal{S}'$), we can find a prime $\otimes$-ideal $\mathcal{P}$ such that $\langle a \rangle \subseteq \mathcal{P}$, and $\mathcal{P} \cap \mathcal{S}' = \emptyset$; that is, $\mathcal{P} \in U(a)$, but $\mathcal{P} \notin U(\mathcal{S}')$. Thus, $U(a) \nsubseteq U(\mathcal{S}')$. □

Proposition 3.6. *Let $\mathcal{T}$ be a tensor n-angulated category.*

(a) For any object $a \in \mathcal{T}$, the open subset $U(a)$ is quasi-compact. In particular, $\mathrm{Spc}(\mathcal{T})$ *is quasi-compact.*

(b) Any quasi-compact open subset of $\mathrm{Spc}(\mathcal{T})$ *is of the form $U(a)$ for some object $a \in \mathcal{T}$.*

Proof. (a) Take an open cover $\{U(\mathcal{S}_i)\}_{i \in I}$ of $U(a)$. Write $\mathcal{S} := \bigcup_{i \in I} \mathcal{S}_i$. Then, $U(a) \subseteq \bigcup_{i \in I} U(\mathcal{S}_i) = U(\mathcal{S})$. By Lemma 3.5, one can find $b_1, \cdots, b_n \in \mathcal{S}$ such that $b_1 \otimes \cdots \otimes b_n \in \langle a \rangle$. Clearly, these $b_1, \cdots, b_n$ lie in $\bigcup_{i \in I_0} \mathcal{S}_i$ for some finite subset $I_0 \subseteq I$. Now, for any $\mathcal{P} \in U(a)$, we have $b_1 \otimes \cdots \otimes b_n \in \langle a \rangle \subseteq \mathcal{P}$. Because $\mathcal{P}$ is prime, at least one b_i belongs to $\mathcal{P}$, and hence, $\mathcal{P} \cap (\bigcup_{i \in I_0} \mathcal{S}_i) \neq \emptyset$; that is, $\mathcal{P} \in U(\bigcup_{i \in I_0} \mathcal{S}_i) = \bigcup_{i \in I_0}(U(\mathcal{S}_i))$. Thus, $U(a) \subseteq \bigcup_{i \in I_0}(U(\mathcal{S}_i))$.

(b) Suppose $\mathbf{U} = U(\mathcal{S})$ is a quasi-compact open subset for some $\mathcal{S} \subseteq \mathcal{T}$. Then, $\mathbf{U} = \bigcup_{a \in \mathcal{S}} U(a)$, so $\{U(a)\}_{a \in \mathcal{S}}$ forms an open cover of $\mathbf{U}$. By quasi-compactness, there exist $a_1, \cdots, a_n \in \mathcal{S}$ such that

$$\mathbf{U} = U(a_1) \cup \cdots \cup U(a_n) = U(a_1 \otimes \cdots \otimes a_n).$$

□

Corollary 3.7. *Suppose that $U(\mathcal{S}) = \mathrm{Spc}(\mathcal{T})$ for a class $\mathcal{S}$ of objects of $\mathcal{T}$. Then, there exist $b_1, \cdots, b_n \in \mathcal{S}$ such that $b_1 \otimes \cdots \otimes b_n = 0$.*

Proof. Observe that $U(0) = \mathrm{Spc}(\mathcal{T}) = U(\mathcal{S})$. Applying Lemma 3.5 yields $b_1, \cdots, b_n \in \mathcal{S}$ with $b_1 \otimes \cdots \otimes b_n = 0$. □

Let X be a topological space and Y a subset of X. Then, Y is called a *Thomason* subset of X if $Y = \bigcup_{i \in \Lambda} Y_i$, where each complementary set Y_i^c of Y_i is quasi-compact and open.

Let $\mathcal{S}$ be a subcategory of $\mathcal{T}$. By Proposition 3.6(1), $\mathrm{supp}(\mathcal{S}) = \bigcup_{a \in \mathcal{S}} \mathrm{supp}(a)$ is a Thomason subset of $\mathrm{Spc}(\mathcal{T})$. On the other hand, let $Y = \bigcup_{i \in \Lambda} Y_i$ be a Thomason subset of $\mathrm{Spc}(\mathcal{T})$. By Proposition 3.6(2), $Y_i^c = U(a_i)$ for some $a_i \in \mathcal{T}$, and then $Y_i = \mathrm{Spc}(\mathcal{T}) \setminus U(a_i) = \mathrm{supp}(a_i)$. Thus, $Y = \bigcup_{i \in \Lambda} \mathrm{supp}(a_i)$. Following this, we have

Theorem 3.8. *Let $\mathcal{T}$ be a tensor n-angulated category.*

(1) $\mathbf{Para}(\mathcal{T}) = \{$*Thomason subsets of* $\mathrm{Spc}(\mathcal{T})\}$.

(2) There is an order-preserving bijection

$$\begin{tikzcd} \{\textit{Thomason subsets of}\ \mathrm{Spc}(\mathcal{T})\} \arrow[r, shift left, "\mathrm{supp}^{-1}(-)"] & \mathbf{Rad}(\mathcal{T}) \arrow[l, shift left, "\mathrm{supp}(-)"] \end{tikzcd}.$$

Definition 3.9. A topological space is called *noetherian* if any open subset is quasi-compact.

Corollary 3.10. *The topological space* $\mathrm{Spc}(\mathcal{T})$ *is noetherian if and only if any closed subset of* $\mathrm{Spc}(\mathcal{T})$ *is of the form* $\mathrm{supp}(a)$ *for some object* $a \in \mathcal{T}$.

Proof. By Proposition 3.6, an open subset of $\mathrm{Spc}(\mathcal{T})$ is quasi-compact if and only if it is of the form $U(a)$ for some $a \in \mathcal{T}$. Note that $\mathrm{Spc}(\mathcal{T}) \setminus U(a) = \mathrm{supp}(a)$.

Thus, $\mathrm{Spc}(\mathcal{T})$ is noetherian $\Longleftrightarrow$ any open subset is of the form $U(a)$ for some $a \in \mathcal{T}$.
$\Longleftrightarrow$ any closed subset is of the form $\mathrm{supp}(a)$ for some $a \in \mathcal{T}$. □

Recall that a subset Y of a topological space X is called *specialization-closed* if it is a union of closed subsets.

Corollary 3.11. *Assume that* $\mathrm{Spc}(\mathcal{T})$ *is a noetherian topological space. Then, there is an order-preserving bijection*

$$\{\textit{specialization-closed subsets of } \mathrm{Spc}(\mathcal{T})\} \underset{\mathrm{supp}(-)}{\overset{\mathrm{supp}^{-1}(-)}{\rightleftarrows}} \mathbf{Rad}(\mathcal{T}) \,.$$

Remark 3.12. A subset Y of a topological space X is specialization-closed if and only if for any $y \in X$, $y \in Y$ implies $\overline{\{y\}} \subseteq Y$.

Proof. Assume that $Y = \bigcup_i Y_i$, where each Y_i is a closed subset. If $y \in Y$, then $y \in Y_i$ for some i, and hence, $\overline{\{y\}} \subseteq Y_i \subseteq Y$.

Conversely, assume $\overline{\{y\}} \subseteq Y$ for any $y \in Y$. Then, $Y = \bigcup_{y \in Y} \overline{\{y\}}$, and hence, Y is specialization-closed. □

To conclude the paper, we provide an equivalent characterization of a thick ⊗-ideal being a radical ⊗-ideal.

Lemma 3.13. *Let* $\mathcal{I}$ *be a proper thick* ⊗*-ideal of* $\mathcal{T}$. *The radical* $\sqrt{\mathcal{I}}$ *is a proper thick* ⊗*-ideal of* $\mathcal{T}$, *and*

$$\sqrt{\mathcal{I}} = \bigcap_{\substack{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \\ \text{s.t. } \mathcal{P} \supseteq \mathcal{I}}} \mathcal{P} = \{a \in \mathcal{T} \mid \textit{there exists an integer } n \geq 1 \textit{ such that } a^{\otimes n} \in \mathcal{I}\}.$$

In particular, if $\mathcal{P}$ *is prime, then* $\sqrt{\mathcal{P}} = \bigcap_{\substack{\mathcal{P}' \in \mathrm{Spc}(\mathcal{T}) \\ \text{s.t. } \mathcal{P}' \supseteq \mathcal{P}}} \mathcal{P}' = \mathcal{P}$. *Thus, prime* ⊗*-ideals are radical.*

Proof. Observe that any prime ⊗-ideal $\mathcal{P}$ containing $\mathcal{I}$ automatically contains $\sqrt{\mathcal{I}}$. Consequently, the collection $\{a \in \mathcal{T} \mid \text{there exists an integer } n \geq 1 \text{ such that } a^{\otimes n} \in \mathcal{I}\} \subseteq \bigcap_{\substack{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \\ \text{s.t. } \mathcal{P} \supseteq \mathcal{I}}} \mathcal{P}$.

For the reverse inclusion, pick $a \in \bigcap_{\substack{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \\ \text{s.t. } \mathcal{P} \supseteq \mathcal{I}}} \mathcal{P}$. Define the ⊗-multiplicative collection $\mathcal{S} := \{a^{\otimes n} \mid n \geq 1\} \cup \{\mathbb{1}\}$.

Suppose $\mathcal{S} \cap \mathcal{I} = \emptyset$. Then, Lemma 2.6 provides a prime ⊗-ideal $\mathcal{P}$ such that $\mathcal{I} \subseteq \mathcal{P}$, and $\mathcal{P} \cap \mathcal{S} = \emptyset$. The condition $\mathcal{P} \cap \mathcal{S} = \emptyset$ forces $a \notin \mathcal{P}$, contradicting our choice of a. Hence, $\mathcal{S} \cap \mathcal{I} \neq \emptyset$. Because $\mathbb{1} \notin \mathcal{I}$, we deduce that there exists $n \geq 1$ such that $a^{\otimes n} \in \mathcal{I}$. This establishes that $\{a \in \mathcal{T} \mid \text{there exists an integer } n \geq 1 \text{ such that } a^{\otimes n} \in \mathcal{I}\} = \bigcap_{\substack{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \\ \text{s.t. } \mathcal{P} \supseteq \mathcal{I}}} \mathcal{P}$.

Furthermore, the intersection of proper thick ⊗-ideals remains a thick ⊗-ideal, so $\sqrt{\mathcal{I}}$ is a proper thick ⊗-ideal. □

Proposition 3.14. *The following are equivalent:*

(i) Any thick $\otimes$*-ideal of* $\mathcal{T}$ *is radical.*
(ii) $\forall\, a \in \mathcal{T},\ a \in \langle a{\otimes}a\rangle$.

Proof. $(i) \Rightarrow (ii)$ By hypothesis, $\langle a{\otimes}a\rangle$ is radical, meaning $\langle a{\otimes}a\rangle = \sqrt{\langle a{\otimes}a\rangle} = \bigcap_{\substack{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \\ \mathcal{P} \supseteq \langle a\otimes a\rangle}} \mathcal{P}$. For any $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$ with $\langle a{\otimes}a\rangle \subseteq \mathcal{P}$, primeness forces $a \in \mathcal{P}$, and therefore, $a \in \langle a{\otimes}a\rangle$.

$(ii) \Rightarrow (i)$ Let $\mathcal{I}$ be an arbitrary thick $\otimes$-ideal of $\mathcal{T}$. The inclusion $\mathcal{I} \subseteq \sqrt{\mathcal{I}}$ is obvious. We now show $\sqrt{\mathcal{I}} \subseteq \mathcal{I}$. By Lemma 3.13, it suffices to prove that $a^{\otimes n} \in \mathcal{I}$ implies $a \in \mathcal{I}$. Induction on n reduces us to the case $n = 2$. Assume $a{\otimes}a \in \mathcal{I}$. Then, $\langle a{\otimes}a\rangle \subseteq \mathcal{I}$. Condition (ii) gives $a \in \langle a{\otimes}a\rangle$, whence $a \in \mathcal{I}$, as required. □

4. Conclusions

In this paper, we have systematically developed the theory of the Balmer spectrum for tensor n-angulated categories. We introduced the notions of thick tensor ideals and prime tensor ideals in this higher-categorical setting, and established the existence of prime $\otimes$-ideals in any nonzero tensor n-angulated category. The associated Balmer spectrum, endowed with the Zariski topology, was shown to be a nonempty T_0 space, and we provided a complete characterization of its irreducible closed subsets via generic points.

Furthermore, we introduced the concept of radical $\otimes$-ideals and proved that they can be parametrized by supports of subcategories. The main classification results establish order-preserving bijections between the set of radical $\otimes$-ideals and the set of Thomason subsets of the spectrum. In the noetherian case, this correspondence simplifies to specialization-closed subsets.

These results extend Balmer's foundational work on tensor triangulated categories to the higher n-angulated framework, and provide a geometric approach to the classification problem of subcategories in tensor n-angulated categories. It would be interesting in future work to compute concrete examples of such spectra for specific n-angulated categories arising from cluster tilting theory.

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

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