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Research article

Three-way decision with regret theory and T-spherical fuzzy interactional Aczél-Alsina aggregation operators for multi-attribute decision-making

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Abstract: The three-way decision (TWD) has attracted extensive attention in multi-attribute decision-making (MADM) for its ability to incorporate decision-makers' (DMs') preferences in uncertain environments. However, models often struggle to effectively integrate objective evaluation values with DMs' psychological behaviors when handling T-spherical fuzzy (T-SF) information, leading to inadequate performance in comparing and classifying alternatives. To address this issue, we proposed a novel regret-theory-based TWD approach in the T-SF environment. First, a relative utility function (RUF) was constructed by integrating regret theory (RT) to reasonably reflect DMs' psychological expectations and relative utility differences in different scenarios. Second, the T-spherical fuzzy interactional Aczél-Alsina (T-SFIAA) aggregation operator enabled robust aggregation information, and the TOPSIS method was employed to compute probabilities, enabling the construction of an action-set-based decision model. Third, thresholds for three-way partitions were adaptively derived, and finally, extensive parameter analysis and comparative experiments with existing methods demonstrated the superior stability and effectiveness of the novel approach. The results indicated that the proposed method significantly enhances the scientific validity and feasibility of MADM in complex and uncertain environments, offering a novel solution for decision-making in the T-SF context.

Keywords: T-spherical fuzzy set (T-SFS); regret theory (RT); three-way decision (TWD); Aczél-Alsina (AA) aggregation operator; relative utility function (RUF); multi-attribute decision-making (MADM)
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1. Introduction

The core of the decision-making process lies in selecting the optimal option from the available options, which requires decision-makers (DMs) to complete a series of steps, including information collection, data verification, and final decision-making. In complex decision-making scenarios, the quality of alternatives is often determined by multiple attributes. Therefore, multi-attribute decision-making (MADM) methods enable DMs to make reasonable decisions while considering multiple criteria (see, e.g., [1,2]).

The three-way decision (TWD) theory was first proposed by Yao [3] within the framework of rough sets, interpreting the positive region, boundary region, and negative region as three action categories: accepting, delaying, and rejecting. This partition not only extends traditional rough set methods but also aligns better with cognitive logic in real-world decision-making. Subsequently, Yao [4] systematically organized the TWD theory, promoting it to become a research hotspot in the decision-making field. In the application and extension of TWD, Du et al. [5] constructed a novel multi-attribute three-way decision method integrating grey relational analysis and the technique for order preference by similarity to ideal solution (TOPSIS). Wang et al. [6] extended the classical TOPSIS by adding a third middle reference point, proposed a novel Best-Mean-Worst reference point-based TOPSIS (BMW-TOPSIS) model, and led to a three-way TOPSIS framework. Although the aforementioned methods extend traditional TOPSIS, they rely on rational decision-making assumptions and fail to incorporate DMs' psychological preferences, thus imposing limitations on their applicability. To address this gap, Yan et al. [7] proposed a novel TWD approach based on regret theory (RT) and TOPSIS, which establishes a maximized expected revenue model for benefit attributes and a minimized expected loss model for cost attributes, respectively, and achieves more accurate classification and ranking through integrated decision domains.

Regret theory-based TWD models are primarily constructed within the frameworks of intuitionistic fuzzy sets (IFS) or Pythagorean fuzzy sets (PyFS). However, these frameworks limit the ability to handle fuzzy information in complex decision-making scenarios, often making it difficult for DMs to arrive at accurate judgments. Therefore, to overcome this expressive limitation, we extend the multi-attribute TWD by combining it with RT in a T-spherical fuzzy (T-SF) environment, which possesses strong generalization capabilities and extensive applications in advanced decision-making. Subsequently, we briefly introduce T-spherical fuzzy sets (T-SFSs), TWD, and RT, outline the motivation and contributions, and present the paper's structure.

Our motivations for this paper are as follows:

- Although MADM theory has made significant progress in handling fuzzy information, traditional fuzzy sets are limited by their linear constraints, making it difficult to effectively process complex fuzzy information. Furthermore, the traditional accept-reject two-way decision framework lacks the flexibility of deferred decision-making, which hinders its ability to meet decision-making demands in high-risk scenarios. Therefore, by integrating T-SFSs with three-way decision-making to construct a classification framework, we aim to provide new theoretical and methodological support for MADM in complex uncertain environments.
- Although Hussain et al. [8] introduced the AA operator into the T-SF environment, the constructed aggregation operators follow the traditional approach of independently aggregating membership, neutrality, and non-membership degrees, without fully considering their inherent

interrelationships. Furthermore, the constructed AA operators on T-spherical fuzzy numbers (T-SFNs) may not be closed. Therefore, we construct a novel T-spherical fuzzy interactional Aczél–Alsina (T-SFIAA) aggregation operator to overcome the shortcomings of existing AA operators, providing a theoretical foundation for more accurate and practical T-SF information aggregation.
- RT constitutes a key component of behavioral decision theory. Depending on whether outcomes meet their expectations, DMs experience rejoicing or regret, which guides their choices. T-SFSs, as the latest extension of fuzzy sets, offer stronger generalization capabilities and greater flexibility than PFSs. The combination of these two into a T-SF regret function can be applied to complex decision-making scenarios in fuzzy environments. However, few researchers have used this function for TWD to create more cognitively realistic loss functions.

The major contributions of this paper can be summarized as follows:

- In practical complex decision-making, traditional methods struggle with fuzzy information and multi-level classification. They also fail to account for DMs' psychological behaviors adequately. Here, we construct a relative utility function (RUF) in a T-SF environment. This innovation not only enables the decision model to adapt to complex, uncertain scenarios but also enhances its explanatory power regarding decision behavior.
- To address the shortcomings of AA operators in the T-SF environment, we propose a T-spherical Fuzzy interactional Aczél-Alsina (T-SFIAA) aggregation operator. We rigorously prove that the proposed operator satisfies properties such as idempotency, monotonicity, boundedness, and closure under T-SFNs. In this context, the AA operator addresses the information loss common in traditional aggregation. Furthermore, its adjustable parameters enhance the model's flexibility and practicality.
- We integrate the improved RUF with the TWD model to propose a new TWMADM framework. By introducing a parameterized state set, the framework significantly enhances the flexibility of partitioning the action set. It enables the model to dynamically adjust decision rules according to DMs' risk preferences and specific decision contexts.

The structure of this paper is as follows: In Section 2, we present the research status of T-SFSs, TWD, and RT. Then, the fundamental concepts and properties of them are systematically reviewed in Section 3, which provides a theoretical foundation for subsequent model construction. In Section 4, we propose the T-SFIAA aggregation operator based on an analysis of the limitations of AA operators, along with rigorous proofs of their operational properties. In Section 5, we construct a novel TWD model by integrating the T-SF regret function and the T-SFIAAWG-TOPSIS method for classification and ranking under T-SF information. In Section 6, we validate the feasibility, stability, and superiority of the proposed method through a case study, parameter sensitivity analysis, and comparative and ablation experiments. In the last section, we summarize the research findings and major contributions of the paper, point out the limitations of the model, and provide an outlook on future research directions.

2. Literature review

2.1. T-SFSs

Zadeh [9] first introduced the concept of fuzzy sets in 1965, providing a novel mathematical tool for handling uncertainty. This theory has developed rapidly. Atanassov [10] extended this work by proposing the IFS model. This model characterizes the fuzziness of decision information using membership degree μ and non-membership degree ν, subject to the constraint $0 \leq \mu + \nu \leq 1$. However, IFSs are not suitable for situations where the sum of membership and non-membership degrees exceeds 1. To address this, Yager et al. (see [11, 12]) introduced PyFSs and q-rung orthopair fuzzy sets (q-ROFSs). PyFSs require $0 \leq \mu^2 + \nu^2 \leq 1$, and q-ROFSs require $0 \leq \mu^q + \nu^q \leq 1$ $(q \geq 1)$. These extensions enable membership and non-membership degrees to span a larger range. Considering DMs' hesitant attitudes during evaluation, Cuong [13] proposed picture fuzzy sets (PFSs). Based on IFSs, PFSs use membership degree μ, non-membership degree ν, and neutral degree η, with $0 \leq \mu+\eta+\nu \leq 1$. Mahmood et al. [14] further generalized this by proposing spherical fuzzy sets (SFSs) and T-spherical fuzzy sets (T-SFSs). SFSs satisfy $0 \leq \mu^2+\eta^2+\nu^2 \leq 1$, and T-SFSs satisfy $0 \leq \mu^t+\eta^t+\nu^t \leq 1$ $(t \geq 1)$. T-SFSs generalize the earlier types of fuzzy sets. By setting appropriate values for μ, η, ν, and t, T-SFSs reduce to IFSs, PyFSs, q-ROFSs, PFSs, SFSs, and others. Due to their broad applicability, T-SFSs have developed rapidly. Hussain et al. [8] developed these operators on T-SF information using Aczél-Alsina (AA) T-norms and T-conorms. To capture relationships among input data, Wang et al. [15] introduced T-spherical fuzzy Aczél-Alsina Hamy mean (T-SFAAHM) operators by combining the Hamy mean with AA operators for effective aggregation. Similarly, Li et al. [16] employed AA T-norm and T-conorm to define spherical cubic fuzzy aggregation operators and integrated them with the Renyi entropy weight method and WASPAS to evaluate service quality of crowdsourcing logistics platforms. Petchimuthu et al. [17] employed the Yager T-norm to construct aggregation operators in the q-ROFS framework, integrating power prioritized weights to handle extreme values, while the constructed operators lack interactions between dimensions. Anjum et al. [18] used the Dombi T-norm in the linear Diophantine fuzzy Z-number environment, which is sensitive to input differences but computationally heavy and lacks dimension interaction.

Consequently, Zeng et al. [19] introduced new operational laws to propose T-spherical fuzzy Einstein geometric interaction operators and averaging interactive aggregation operators. Ju et al. [20] proposed the T-spherical fuzzy weighted averaging interaction (T-SFWAI) operator and the T-spherical fuzzy weighted geometric interaction (T-SFWGI) operator, and discussed their application in group decision-making. Subsequently, Wang et al. [21] combined interaction operational laws with power average and Heronian mean operators, proposing the T-spherical fuzzy interaction power Heronian mean (T-SFIPHM) operator. Following this line of research, Akram et al. [22] integrated interaction operational laws with the power Bonferroni mean, introducing the T-spherical fuzzy interaction power Bonferroni mean (TSFI-PBM) operator and the T-spherical fuzzy weighted interaction power Bonferroni mean (TSFWIPBM) operator. However, such operators are not closed in the T-SFNs, which alone undermines their reliability for decision-making in the T-SF environment.

2.2. TWD

Since Yao [23] proposed the TWD framework based on decision-theoretic rough sets (DTRS) in 2009, this method has gradually developed into an important theoretical tool for dealing with decision-making problems under uncertain information. It has been widely applied to MADM problems such as conflict analysis, investment evaluation, and other fields. The basic idea of TWD is to divide objects into three regions: Acceptance, rejection, and delay, which fully reflects the tolerance and gradualness toward uncertainty and risk in practical decision-making processes. In early studies, different objects were usually assigned the same loss function, which ignored the impact of individual attribute differences on risk assessment. To overcome this problem, Jia and Liu [24] proposed a relative loss function based on object evaluation values, constructed loss functions and threshold calculation methods through attribute evaluation values, and verified the model's effectiveness through an example of project investment. Subsequently, Liu et al. [25] introduced intuitionistic fuzzy numbers (IFNs) into TWD, constructing a three-way multi-attribute decision-making (TWMADM) model based on relative loss functions, effectively improving the model's expressive power and applicability in fuzzy environments. Zhan et al. [26] focused on incomplete information environments and constructed a TWD model in incomplete fuzzy decision information systems based on utility theory, effectively addressing the common problem of missing attributes in reality. Furthermore, Mondal et al. [27] introduced RT and prospect theory into TWD, establishing a three-way multi-attribute model that integrates behavioral decision preferences, reflecting human emotional fluctuations and psychological expectations when facing risks. Tang and Zhang [28] proposed a three-stage ranking model by introducing the maximum k-means clustering algorithm into the TWD framework. They also developed a dynamic loss function and conditional probability calculation procedures suitable for multi-attribute ranking problems. Tang and Qiao [29] presented a novel TWD method based on triangular norms for attribute weighting and distance fusion in incomplete hybrid information systems, and validated, through medical diagnosis examples, that its classification and stability surpass existing models. Research on TWD primarily focuses on addressing issues in incomplete information systems. However, studies on handling high-dimensional complex decision scenarios remain relatively scarce, making it challenging to meet the requirements for classifying high-dimensional decision information in big data environments.

2.3. RT

RT was proposed by Bell [30] in 1982 to explain individuals' behavioral deviations from traditional expected utility theory under uncertain conditions, emphasizing that people in decision-making experience regret or rejoicing emotional reactions due to comparisons of outcomes with other alternatives. Unlike traditional MADM methods centered on rational maximization, RT incorporates emotional factors into the decision-making process, making it more aligned with actual human behavior. In recent years, TWD models have widely integrated RT ideas, forming RT-based TWD models. Deng et al. [31] proposed a RT-based TWD method, using interval fuzzy numbers to characterize incomplete multi-scale information systems and constructing three-partition rules driven by maximum bi-partition and preference indices. Subsequently, Zhan et al. [32] introduced triangular fuzzy numbers and dependency measures, constructing a generalized TWD framework that further enhanced the model's behavioral explanatory power and applicability. Zhang et al. [33] proposed a TWD method based on prospect-regret theory in Pythagorean fuzzy environments to improve the accuracy and practicality of uncertain information processing in MADMs. Wang et al. [34] addressed dynamic MADM problems by integrating RT and TWD concepts, constructing a time-weighted dynamic TWD model that enables aggregation processing of interval-valued fuzzy information across time dimensions. However, research combining RT and TWD remains scarce in high-dimensional decision environments. Most researchers have not effectively extended to high-dimensional scenarios, limiting their ability to process complex information in big data contexts.

In summary, the application of RT, from traditional fuzzy decision-making to integration with TWD models, has significantly expanded its role in capturing psychological behaviors under uncertainty. The integration of RT with the T-SFIAAWG operator and TWD frameworks represents a promising method for addressing the challenges posed by complex high-dimensional decision environments.

3. Preliminaries

3.1. T-SFSs

Definition 1. *[14] Let X be a universe of discourse. A T-SFS A on X is defined as*

$$A = \{\langle x, \mu_A(x), \eta_A(x), \nu_A(x)\rangle \mid x \in X\},$$

where $\mu_A(x)$, $\eta_A(x)$, and $\nu_A(x)$ represent the membership degree, neutrality degree, and non-membership degree of element x with respect to the set A, respectively. For all $x \in X$, these degrees satisfy $\mu_A(x), \eta_A(x), \nu_A(x) \in [0, 1]$, and $0 \le \mu_A^t(x) + \eta_A^t(x) + \nu_A^t(x) \le 1$, where $t \in \mathbb{Z}^+$. The refusal degree of element x with respect to A is given by $\pi_A(x) = \sqrt{1 - \mu_A^t(x) - \eta_A^t(x) - \nu_A^t(x)}$.

For convenience, we call $F = \langle \mu, \eta, \nu\rangle$ a T-spherical fuzzy number (T-SFN) on X, where μ, η, ν are the membership degree, neutrality degree, and non-membership degree, respectively. They satisfy $0 \le \mu, \eta, \nu, \mu^t + \eta^t + \nu^t \le 1$. The set of all T-SFNs on X is denoted by Φ. Denote $0_\Phi = \langle 0, 0, 1\rangle$ and $1_\Phi = \langle 1, 0, 0\rangle$.

Remark 1. *If $t = 1$, then the T-SFS A reduces to a PFS.*
If $t = 2$, then A represents a spherical fuzzy set.

Definition 2. *[14] Let $F_1 = \langle \mu_1, \eta_1, \nu_1\rangle$, $F_2 = \langle \mu_2, \eta_2, \nu_2\rangle \in \Phi$. The basic operations for them are defined as follows:*

(1) $F_1 \subseteq F_2 \Leftrightarrow \mu_1 \le \mu_2,\ \eta_1 \le \eta_2,\ \nu_1 \ge \nu_2$;
(2) $F_1 = F_2 \Leftrightarrow F_1 \subseteq F_2,\ F_2 \subseteq F_1$;
(3) $F_1 \vee F_2 = \langle \max\{\mu_1, \mu_2\},\ \min\{\eta_1, \eta_2\},\ \min\{\nu_1, \nu_2\}\rangle$;
(4) $F_1 \wedge F_2 = \langle \min\{\mu_1, \mu_2\},\ \min\{\eta_1, \eta_2\},\ \max\{\nu_1, \nu_2\}\rangle$;
(5) $F_1^c = \langle \nu_1,\ \eta_1,\ \mu_1\rangle$.

Definition 3. *[35] Let $F_1 = \langle \mu_1, \eta_1, \nu_1\rangle$, $F_2 = \langle \mu_2, \eta_2, \nu_2\rangle \in \Phi$, and $r \ge 0$. The Minkowski distance $d_r(F_1, F_2)$ between F_1 and F_2 is defined as*

$$d_r(F_1, F_2) = \sqrt[r]{|\mu_1^t - \mu_2^t|^r + |\eta_1^t - \eta_2^t|^r + |\nu_1^t - \nu_2^t|^r + |\pi_1^t - \pi_2^t|^r}. \tag{3.1}$$

When $r = 1$, the distance is the Manhattan distance; when $r = 2$, it is the Euclidean distance; when $r \to +\infty$, it is the Chebyshev distance.

To simplify calculations, we use the Manhattan distance in subsequent examples.

Definition 4. *[14] For any $F = \langle \mu, \eta, \nu\rangle \in \Phi$, the score function $S_M(F)$ of F is defined as*

$$S_M(F) = \mu^t - \nu^t. \tag{3.2}$$

To obtain a more precise ranking, the definitions of the accuracy function and the missing function for T-SFNs, along with the novel score function, accuracy functions and ranking rules, are given below.

To constrain the score range to $[0, 1]$, the score function is adjusted as the following new score function:

$$S(F) = \frac{\mu^t - \nu^t + 1}{2}. \tag{3.3}$$

Furthermore, the first accuracy function $A_c(F)$ and the second accuracy function $A_d(F)$ of F are, respectively, defined as

$$A_c(F) = \mu^t + \nu^t, \tag{3.4}$$

$$A_d(F) = \mu^t + \eta^t + \nu^t. \tag{3.5}$$

Definition 5. *Let $F_1 = \langle\mu_1, \eta_1, \nu_1\rangle$, $F_2 = \langle\mu_2, \eta_2, \nu_2\rangle \in \Phi$. Then*

(1) If $S(F_1) < S(F_2)$, then F_1 is inferior to F_2 in the new rules, denoted by $F_1 \prec_S F_2$;

(2) If $S(F_1) = S(F_2)$, and

(2.1) if $A_c(F_1) < A_c(F_2)$, then $F_1 \prec_S F_2$;

(2.2) if $A_c(F_1) = A_c(F_2)$, and

(2.2.1) if $A_d(F_1) < A_d(F_2)$, then $F_1 \prec_S F_2$;

(2.2.2) if $A_d(F_1) = A_d(F_2)$, then F_1 is similar to F_2 in the new rules, denoted by $F_1 =_S F_2$.

If $F_1 \prec_S F_2$ or $F_1 =_S F_2$, we denote it as $F_1 \preccurlyeq_S F_2$.

Theorem 1. $(\Phi, \preccurlyeq_S)$ *is a complete lattice.*

Proof. According to Theorem 3 in Wu [35], the conclusion can be proved similarly. We omit it. □

3.2. RT

Bell [30] and Loomes and Sugden [36] proposed RT from a psychological perspective. This suggests that DMs compare their situation with other scenarios. If they realize that they could have achieved better outcomes by choosing different alternatives, they may experience regret; conversely, if they find their choices led to relatively good results, they may experience rejoice. The theory consists of two major components: The current utility function and the regret function.

Definition 6. *[37] Assume a DM has a set of alternatives $Z = \{z_1, z_2, \ldots, z_m\}$ to choose from. Let d_i represent the decision attribute value or comprehensive evaluation value of the alternative z_i (i=1, 2, ..., m), and $\gamma \in (0, 1)$ denote the risk aversion coefficient. The current utility function of d_i is defined as*

$$v(d_i) = \frac{1 - \mathrm{e}^{-\gamma d_i}}{\gamma}. \tag{3.6}$$

The parameter γ serves as a scaling factor that adjusts the utility values to a specific range to meet the computational requirements of the model. It influences the curvature of the utility function, thereby reflecting the DM's overall attitude toward risk.

Next, the regret function is expressed as follows:

$$h(\Delta v_i) = 1 - \mathrm{e}^{-\delta \Delta v_i}, \tag{3.7}$$

where $\delta \in [0, +\infty)$ represents the regret aversion coefficient, and $\Delta v_i = v(d_i) - v(d^*)$, with $d^* = \max\limits_{1 \le i \le m}\{d_i\}$. If $\Delta v_i < 0$, then $h(\Delta v_i) < 0$, indicating regret as a psychological penalty. If $\Delta v_i = 0$, it means the current choice is the optimal one, and the DM experiences rejoicing (or zero regret), in which case $h(\Delta v_i) = 0$.

Finally, the perceived utility function is calculated as follows:

$$I(d_i) = v(d_i) + h(\Delta v_i). \tag{3.8}$$

This function's main contribution is its integration of objective outcomes and the subjective psychological biases inherent in human decision-making. Thereby, it enables DMs to make choices with the laws of human cognition.

3.3. TWD

The fundamental idea of TWD is to partition the universal set into three categories and adopt corresponding strategies for each category. This model is referred to as the "Trisecting-And-Acting (T&A) model" [38]. Based on Bayesian decision-making process, Yao [3] introduced the traditional TWD theory. Let $O = \{H, \neg H\}$ be a set of two states, representing whether an object is in H or not. Let $I = \{I_P, I_B, I_N\}$ be a set of three actions. For any object z, $I_P, I_B,$, and I_N represent $z \in POS(H)$ (acceptance), $z \in BND(H)$ (deferment), $z \in NEG(H)$ (rejection), respectively. When $z \in H$, λ^{PP}, λ^{BP}, and λ^{NP} represent the loss functions under three actions, I_P, I_B, and I_N, respectively. Similarly, λ^{PN}, λ^{BN}, and λ^{NN} represent the loss functions under three actions, I_P, I_B, and I_N , respectively, when $z \in \neg H$. Finally, we calculate the expected loss $R(I_\star|[z])$ ($\star = P, B, N$) for each alternative z using the following formula:

$$\begin{aligned} R(I_P|[z]) &= \lambda^{PP} Pr(H|[z]) + \lambda^{PN} Pr(\neg H|[z]), \\ R(I_B|[z]) &= \lambda^{BP} Pr(H|[z]) + \lambda^{BN} Pr(\neg H|[z]), \\ R(I_N|[z]) &= \lambda^{NP} Pr(H|[z]) + \lambda^{NN} Pr(\neg H|[z]), \end{aligned} \tag{3.9}$$

where $Pr(H|[z]) = \dfrac{|H \cap [z]|}{|[z]|}$ and $Pr(\neg H|[z]) = 1 - Pr(H|[z])$ denote the conditional probability that the equivalence class of alternative z belongs to the desirable state H and $\neg H$, respectively.

Using Bayesian minimum decision theory, the action with the smallest loss is selected. Moreover, the loss of the deferral strategy generally lies between that of the active and passive strategies. Then the classification rules are

(1) If $R(I_P|[z]) \leq R(I_B|[z])$ and $R(I_P|[z]) \leq R(a_N|[z])$, then $z \in POS(H)$;

(2) If $R(I_B|[z]) \leq R(I_P|[z])$ and $R(I_B|[z]) \leq R(I_N|[z])$, then $z \in BND(H)$;

(3) If $R(I_N|[z]) \leq R(I_P|[z])$ and $R(I_N|[z]) \leq R(I_B|[z])$, then $z \in NEG(H)$.

Consider a loss function with $\lambda^{PP} \leq \lambda^{BP} \leq \lambda^{NP}$ and $\lambda^{NN} \leq \lambda^{BN} \leq \lambda^{PN}$. Additionally, $Pr(H|[z]) + Pr(\neg H|[z])$=1. Finally, the classification rules are

(1) If $Pr(H|[z]) \geq \alpha$, then $z \in POS(H)$;

(2) If $\beta < Pr(H|[z]) < \alpha$, then $z \in BND(H)$;

(3) If $Pr(H|[z]) \leq \beta$, then $z \in NEG(H)$.

Where

$$\alpha = \frac{(\lambda^{PN} - \lambda^{BN})}{(\lambda^{PN} - \lambda^{BN}) + (\lambda^{BP} - \lambda^{PP})},$$
$$\beta = \frac{(\lambda^{BN} - \lambda^{NN})}{(\lambda^{BN} - \lambda^{NN}) + (\lambda^{NP} - \lambda^{BP})}.$$

Furthermore, let $\widehat{\lambda}^{PP} = 0$, $\widehat{\lambda}^{NN} = 0$, $\widehat{\lambda}^{BP} = \lambda^{BP} - \lambda^{PP}$, $\widehat{\lambda}^{NP} = \lambda^{NP} - \lambda^{PP}$, $\widehat{\lambda}^{PN} = \lambda^{PN} - \lambda^{NN}$, and $\widehat{\lambda}^{BN} = \lambda^{BN} - \lambda^{NN}$. The loss functions are replaced by the relative loss functions. Then the thresholds α and β also change accordingly.

4. T-SFIAA aggregation operators

4.1. T-SF operational laws and their limitations

In the field of fuzzy set theory and MADM, AA operators are constructed based on an important parameterized family of T-norms and T-conorms proposed by Aczél and Alsina [39]. This family of operators enables smooth transition between strict algebraic operators and extreme limit operators through parameter M, thereby providing greater flexibility in information aggregation.

Definition 7. *[39] The Aczél-Alsina T-norm is defined as*

$$T(\alpha,\beta) = \begin{cases} T_c(\alpha,\beta), & \text{if } M = 0, \\ \min(\alpha,\beta), & \text{if } M \to \infty, \\ e^{-\left((-\ln\alpha)^M + (-\ln\beta)^M\right)^{\frac{1}{M}}}, & \text{otherwise}. \end{cases}$$

Additionally, the Aczél-Alsina T-conorm is defined as

$$S(\alpha,\beta) = \begin{cases} S_c(\alpha,\beta), & \text{if } M = 0, \\ \max(\alpha,\beta), & \text{if } M \to \infty, \\ 1 - e^{-\left((-\ln(1-\alpha))^M + (-\ln(1-\beta))^M\right)^{\frac{1}{M}}}, & \text{otherwise}. \end{cases}$$

Note that $T_c(\alpha,\beta) = \alpha \cdot \beta$ and $S_c(\alpha,\beta) = \alpha + \beta - \alpha \cdot \beta$ are the algebraic T-norm and T-conorm, which correspond to the special case $M = 0$ of the Aczél-Alsina norms. Applying AA operators to the T-SF environment reveals significant limitations in research, which constitutes one of the innovative motivations of this paper. Hussain et al. [8] initially constructed the theoretical framework of AA operators in the T-SF environment.

Definition 8. *[8] Let $F_1 = \langle \mu_1, \eta_1, \nu_1 \rangle, F_2 = \langle \mu_2, \eta_2, \nu_2 \rangle \in \Phi$, and $M \geq 1$. The T-spherical fuzzy Aczél-Alsina sum $F_1 \oplus_{\mathrm{AA}} F_2$ and Aczél-Alsina product $F_1 \otimes_{\mathrm{AA}} F_2$ are defined as*

$$F_1 \oplus_{\mathrm{AA}} F_2 = \left\langle \left(S(\mu_1^t, \mu_2^t)\right)^{\frac{1}{t}}, \left(T(\eta_1^t, \eta_2^t)\right)^{\frac{1}{t}}, \left(T(\nu_1^t, \nu_2^t)\right)^{\frac{1}{t}} \right\rangle, \tag{4.1}$$

$$F_1 \otimes_{\mathrm{AA}} F_2 = \left\langle \left(T(\mu_1^t, \mu_2^t)\right)^{\frac{1}{t}}, \left(S(\eta_1^t, \eta_2^t)\right)^{\frac{1}{t}}, \left(S(\nu_1^t, \nu_2^t)\right)^{\frac{1}{t}} \right\rangle. \tag{4.2}$$

Here, T and S are the Aczél-Alsina T-norms and T-conorms, respectively.

Specifically, they define the following operations:

Definition 9. *[8, 15] Let $F = \langle \mu, \eta, \nu \rangle$, $F_1 = \langle \mu_1, \eta_1, \nu_1 \rangle, F_2 = \langle \mu_2, \eta_2, \nu_2 \rangle \in \Phi$, $\rho \in \mathbb{R}$ and $M \geq 1$. The basic AA operations of T-SFNs are defined as*

(1) $F_1 \oplus_{\mathrm{AA}} F_2 = \left\langle \left(1 - e^{-\left((-\ln(1-\mu_1^t))^M + (-\ln(1-\mu_2^t))^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(e^{-\left((-\ln(\eta_1^t))^M + (-\ln(\eta_2^t))^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(e^{-\left((-\ln(\nu_1^t))^M + (-\ln(\nu_2^t))^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}} \right\rangle,$

(2) $F_1 \otimes_{\mathrm{AA}} F_2 = \left\langle \left(\mathrm{e}^{-\left((-\ln(\mu_1^t))^M + (-\ln(\mu_2^t))^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(1 - \mathrm{e}^{-\left((-\ln(1-\eta_1^t))^M + (-\ln(1-\eta_2^t))^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(1 - \mathrm{e}^{-\left((-\ln(1-\nu_1^t))^M + (-\ln(1-\nu_2^t))^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}} \right\rangle$,

(3) $(\rho F)_{\mathrm{AA}} = \left\langle \left(1 - (1 - \mu^t)^{\rho^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left((\eta^t)^{\rho^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left((\nu^t)^{\rho^{\frac{1}{M}}}\right)^{\frac{1}{t}} \right\rangle$,

(4) $(F^{\rho})_{\mathrm{AA}} = \left\langle \left((\mu^t)^{\rho^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(1 - (1 - \eta^t)^{\rho^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(1 - (1 - \nu^t)^{\rho^{\frac{1}{M}}}\right)^{\frac{1}{t}} \right\rangle$.

Although Hussain et al. [8] extended the AA operator to the T-SF environment, their operations do not achieve interactions among dimensions. To capture the intrinsic correlations among membership, neutrality, and non-membership degrees, Akram et al. [22] introduced interactional operational laws for T-SFNs, leading to the following definition.

Definition 10. *[22] Let $F = \langle \mu, \eta, \nu \rangle$, $F_1 = \langle \mu_1, \eta_1, \nu_1 \rangle, F_2 = \langle \mu_2, \eta_2, \nu_2 \rangle \in \Phi$, $\rho \in \mathbb{R}$ and $M \geq 1$. Then the interaction operators are defined as*

(1) $F_1 \oplus_{\mathrm{PBM}} F_2 = \Big\langle \left(1 - (1 - \mu_1^t)(1 - \mu_2^t)\right)^{\frac{1}{t}},$
$\left((1 - \mu_1^t)(1 - \mu_2^t) - (1 - \mu_1^t - \eta_1^t - \nu_1^t)(1 - \mu_2^t - \eta_2^t - \nu_2^t) - \eta_1^t \eta_2^t\right)^{\frac{1}{t}},$
$\left((1 - \mu_1^t)(1 - \mu_2^t) - (1 - \mu_1^t - \eta_1^t - \nu_1^t)(1 - \mu_2^t - \eta_2^t - \nu_2^t) - \eta_1^t \eta_2^t\right)^{\frac{1}{t}} \Big\rangle,$

(2) $F_1 \otimes_{\mathrm{PBM}} F_2 = \Big\langle \left((1 - \nu_1^t)(1 - \nu_2^t) - (1 - \mu_1^t - \eta_1^t - \nu_1^t)(1 - \mu_2^t - \eta_2^t - \nu_2^t) - \eta_1^t \eta_2^t\right)^{\frac{1}{t}},$
$\left(1 - (1 - \eta_1^t)(1 - \eta_2^t)\right)^{\frac{1}{t}}, \left(1 - (1 - \nu_1^t)(1 - \nu_2^t)\right)^{\frac{1}{t}} \Big\rangle,$

(3) $(\rho F)_{\mathrm{PBM}} = \Big\langle \left(1 - (1 - \mu_1^t)^{\rho}\right)^{\frac{1}{t}}, \left((1 - \mu_1^t)^{\rho} - (1 - \mu_1^t - \eta_1^t - \nu_1^t)^{\rho} - \eta_1^{t\rho}\right)^{\frac{1}{t}},$
$\left((1 - \mu_1^t)^{\rho} - (1 - \mu_1^t - \eta_1^t - \nu_1^t)^{\rho} - \eta_1^{t\rho}\right)^{\frac{1}{t}} \Big\rangle,$

(4) $(F^{\rho})_{\mathrm{PBM}} = \Big\langle \left((1 - \nu_1^t)^{\rho} - (1 - \mu_1^t - \eta_1^t - \nu_1^t)^{\rho} - \eta_1^{t\rho}\right)^{\frac{1}{t}}, \left(1 - (1 - \eta_1^t)^{\rho}\right)^{\frac{1}{t}}, \left(1 - (1 - \nu_1^t)^{\rho}\right)^{\frac{1}{t}} \Big\rangle.$

Akram et al. [22] proposed new interaction operational laws that achieve interaction among three dimensions. However, the computational results of their operators may cause the sum of the t-th powers of the three degrees to exceed 1. Thus, such operators are not closed on Φ. The special counterexamples are shown in the next subsection.

4.2. Novel T-SFIAA aggregation operators

To overcome the limitations of the aforementioned non-interactional aggregation, Wu et al. [35] introduced an improved aggregation model; the picture fuzzy interactional Aczél-Alsina aggregation operator in picture fuzzy environments. Inspired by the interactional operations proposed by others, Wu et al. introdeced the interactional operational laws for picture fuzzy numbers, which can fundamentally overcome the limitation of non-closeness. We extend them into T-SF environments.

Definition 11. *Let $F_1 = \langle \mu_1, \eta_1, \nu_1 \rangle, F_2 = \langle \mu_2, \eta_2, \nu_2 \rangle \in \Phi$, and $M \geq 1$. The T-SF interactional Aczél-Alsina sum $F_1 \oplus_{AA} F_2$ and Aczél-Alsina product $F_1 \otimes_{AA} F_2$ are defined as*

$$F_1 \oplus_{\mathrm{AA}} F_2 = \left\langle (S(\mu_1^t, \mu_2^t))^{\frac{1}{t}}, (T(\eta_1^t + \nu_1^t, \eta_2^t + \nu_2^t) - T(\nu_1^t, \nu_2^t))^{\frac{1}{t}}, (T(\nu_1^t, \nu_2^t))^{\frac{1}{t}} \right\rangle, \tag{4.3}$$

$$F_1 \otimes_{\mathrm{AA}} F_2 = \left\langle\ (T(\mu_1^t, \mu_2^t))^{\frac{1}{t}}, (T(\mu_1^t + \eta_1^t, \mu_2^t + \eta_2^t) - T(\mu_1^t, \mu_2^t))^{\frac{1}{t}}, (S(\nu_1^t, \nu_2^t))^{\frac{1}{t}}\ \right\rangle. \tag{4.4}$$

Here, T and S are the Aczél-Alsina T-norms and T-conorms, respectively.

Then we have the novel T-SF interactional AA operators as follows:

Definition 12. *Let $F = \langle \mu, \eta, \nu \rangle$, $F_1 = \langle \mu_1, \eta_1, \nu_1 \rangle$, $F_2 = \langle \mu_2, \eta_2, \nu_2 \rangle \in \Phi$, $M \geq 1$, and $\rho \in \mathbb{R}$. The novel T-spherical fuzzy interactional Aczél-Alsina (T-SFIAA) operators are defined as follows:*

(1) $$F_1 \oplus_{\mathrm{AA}} F_2 = \left\langle\ \left(1 - e^{-\left((-\ln(1-\mu_1^t))^M + (-\ln(1-\mu_2^t))^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \right.$$
$$\left(e^{-\left((-\ln(\eta_1^t+\nu_1^t))^M + (-\ln(\eta_2^t+\nu_2^t))^M\right)^{\frac{1}{M}}} - e^{-\left((-\ln(\nu_1^t))^M + (-\ln(\nu_2^t))^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}},$$
$$\left. \left(e^{-\left((-\ln(\nu_1^t))^M + (-\ln(\nu_2^t))^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}\ \right\rangle,$$

(2) $$F_1 \otimes_{\mathrm{AA}} F_2 = \left\langle\ \left(e^{-\left((-\ln(\mu_1^t))^M + (-\ln(\mu_2^t))^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \right.$$
$$\left(e^{-\left((-\ln(\mu_1^t+\eta_1^t))^M + (-\ln(\mu_2^t+\eta_2^t))^M\right)^{\frac{1}{M}}} - e^{-\left((-\ln(\mu_1^t))^M + (-\ln(\mu_2^t))^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}},$$
$$\left. \left(1 - e^{-\left((-\ln(1-\nu_1^t))^M + (-\ln(1-\nu_2^t))^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}\ \right\rangle,$$

(3) $$(\rho F)_{\mathrm{AA}} = \left\langle\ \left(1 - (1-\mu^t)^{\rho^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left((\eta^t + \nu^t)^{\rho^{\frac{1}{M}}} - (\nu^t)^{\rho^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left((\nu^t)^{\rho^{\frac{1}{M}}}\right)^{\frac{1}{t}}\ \right\rangle,$$

(4) $$(F^\rho)_{\mathrm{AA}} = \left\langle\ \left((\mu^t)^{\rho^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left((\mu^t + \eta^t)^{\rho^{\frac{1}{M}}} - (\mu^t)^{\rho^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(1 - (1-\nu^t)^{\rho^{\frac{1}{M}}}\right)^{\frac{1}{t}}\ \right\rangle.$$

Here, $\eta^t + \nu^t$ denotes the integrated non-support intensity, reflecting the intrinsic interaction between neutral and negative attitudes. Conversely, $\mu^t + \eta^t$ denotes the integrated support intensity, capturing the interaction between positive and neutral attitudes. This symmetric treatment follows the cognitive logic of merging similar attitude information into unified dimensions for aggregation, while preserving the closure property of the proposed operators. To illustrate the advantages of the proposed operators over existing ones, we provide the following examples.

Example 1. *Let $F_1 = \langle 0.0090, 0.0565, 0.8885 \rangle$, $F_2 = \langle 0.3069, 0.6054, 0.6846 \rangle \in \Phi$, here $t = 2$, and $M = 2$. According to Definition 9 compute $F_1 \otimes_{\mathrm{AA}} F_2 = \langle 0.0078, 0.6054, 0.9022 \rangle$. However,* $0.0078^2 + 0.6054^2 + 0.9022^2 \approx 1.1805 > 1$. *Thus, Hussain's operator $F_1 \otimes_{\mathrm{AA}} F_2$ is not closed in Φ. In contrast, for the operator proposed in Definition 12, $F_1 \otimes_{\mathrm{AA}} F_2 = \langle 0.0078, 0.0134, 0.9022 \rangle$, and* $0.0078^2 + 0.0134^2 + 0.9022^2 \approx 0.8141 < 1$.

Example 2. *Let $F = \langle 0.3065, 0.7083, 0.5038 \rangle \in \Phi$, here $t = 2$, $\rho = 2$, and $M = 2$. According to Definition 9 compute $(F^\rho)_{\mathrm{AA}} = \langle 0.1878, 0.7916, 0.5823 \rangle$. However,* $0.1878^2 + 0.7916^2 + 0.5823^2 \approx 1.0009 > 1$. *Thus, Hussain's operator $(F^\rho)_{\mathrm{AA}}$ is not closed in Φ. By contrast, for the operator proposed in Definition 12, $(F^\rho)_{\mathrm{AA}} = \langle 0.1878, 0.6673, 0.5823 \rangle$, and* $0.1878^2 + 0.6673^2 + 0.5823^2 \approx 0.8196 < 1$.

Example 3. *Let $F_1 = \langle 0.90, 0.05, 0.05 \rangle$, $F_2 = \langle 0.1, 0.8, 0.1 \rangle \in \Phi$, here $t = 3$ and $M = 2$. According to Definition 10 compute $F_1 \otimes_{\mathrm{PBM}} F_2 = \langle 0.9536, 0.8000, 0.1040 \rangle$. However,* $0.9536^3 + 0.8000^3 + 0.1040^3 \approx 1.3804 > 1$. *Thus, Akram's operator $F_1 \otimes_{\mathrm{PBM}} F_2$ is not closed in Φ. In contrast, for the operator* *proposed in Definition 12,* $F_1 \otimes_{\mathrm{AA}} F_2 = \langle 0.9000, 0.0614, 0.0229 \rangle$*, and* $0.9000^3 + 0.0614^3 + 0.0229^3 \approx 0.7292 < 1$.

The above examples illustrate that neither the operators in Definition 9 nor those in Definition 10 preserve closure in Φ. By contrast, the operators introduced in Definition 12 are closed and thus suitable for constructing T-SF aggregation operators. Furthermore, we prove the closure of the novel interaction operators.

The detailed proof of the following theorems can be found in the appendix.

Theorem 2. *The four basic operators in Definition 12 are all T-SFNs.*

Theorem 3. *Let* $F_1 = \langle \mu_1, \eta_1, \nu_1 \rangle, F_2 = \langle \mu_2, \eta_2, \nu_2 \rangle \in \Phi$*, and* $\rho \in \mathbb{R}$*. Then the following hold:*
(1) $F_1 \oplus_{\mathrm{AA}} F_2 = F_2 \oplus_{\mathrm{AA}} F_1$,
(2) $F_1 \otimes_{\mathrm{AA}} F_2 = F_2 \otimes_{\mathrm{AA}} F_1$,
(3) $(\rho(F_1 \oplus_{\mathrm{AA}} F_2))_{\mathrm{AA}} = (\rho F_1)_{\mathrm{AA}} \oplus_{\mathrm{AA}} (\rho F_2)_{\mathrm{AA}}$,
(4) $((F_1 \otimes_{\mathrm{AA}} F_2)^{\rho})_{\mathrm{AA}} = (F_1^{\rho})_{\mathrm{AA}} \otimes_{\mathrm{AA}} (F_2^{\rho})_{\mathrm{AA}}$.

Based on Definition 12, we can aggregate the T-SF information.

Theorem 4. *Let* $F_j = \langle \mu_j, \eta_j, \nu_j \rangle \in \Phi$ $(j = 1, 2, \ldots, n)$*,* $M \geq 1$*, and* $\Omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ *be the weight vector with* $\omega_j \in (0, 1)$ *and* $\sum_{j=1}^{n} \omega_j = 1$*. Then the T-spherical fuzzy interactional Aczél-Alsina weighted average (T-SFIAAWA) aggregation operator is a mapping T-SFIAAWA:* $\Phi^n \to \Phi$*, and*

$$\begin{aligned}
&\text{T-SFIAAWA}(F_1, F_2, \ldots, F_n) = \bigoplus_{j=1}^{n}{}_{\mathrm{AA}} \left(\omega_j F_j\right)_{\mathrm{AA}} \\
&= \left\langle \left(1 - e^{-\left(\sum_{j=1}^{n} \omega_j \left(-\ln\left(1-\mu_j^t\right)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(e^{-\left(\sum_{j=1}^{n} \omega_j \left(-\ln\left(\eta_j^t + \nu_j^t\right)\right)^M\right)^{\frac{1}{M}}} - e^{-\left(\sum_{j=1}^{n} \omega_j \left(-\ln(\nu_j^t)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(e^{-\left(\sum_{j=1}^{n} \omega_j \left(-\ln(\nu_j^t)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}} \right\rangle.
\end{aligned} \tag{4.5}$$

Theorem 5. *(Idempotency) If* $F_j = \langle \mu, \eta, \nu \rangle \in \Phi$ $(j = 1, 2, \ldots, n)$*, with* $\omega_j \in (0, 1)$ *and* $\sum_{j=1}^{n} \omega_j = 1$*, then,*

$$\text{T-SFIAAWA}(F_1, F_2, \ldots, F_n) = \langle \mu, \eta, \nu \rangle.$$

Theorem 6. *(Monotonicity) Let* $G_j = \langle \mu_{G_j}, \eta_{G_j}, \nu_{G_j} \rangle$ *and* $L_j = \langle \mu_{L_j}, \eta_{L_j}, \nu_{L_j} \rangle$ *be two collections of T-SFNs* $(j = 1, 2, \ldots, n)$*, with weights* $\omega_j \in (0, 1)$ *and* $\sum_{j=1}^{n} \omega_j = 1$*. If for all j,* $G_j \subseteq L_j$*, i.e.,* $\mu_{G_j} \leq \mu_{L_j}$*,* $\eta_{G_j} \leq \eta_{L_j}$*, and* $\nu_{G_j} \geq \nu_{L_j}$*, then,*

$$\text{T-SFIAAWA}(G_1, G_2, \ldots, G_n) \preccurlyeq_S \text{T-SFIAAWA}(L_1, L_2, \ldots, L_n).$$

Theorem 7. *(Boundedness) Let* $F_j = \langle \mu_j, \eta_j, \nu_j \rangle$ *be a collection of T-SFNs* $(j = 1, 2, \ldots, n)$*, with weights* $\omega_j \in (0, 1)$ *and* $\sum_{j=1}^{n} \omega_j = 1$*. Then,*

$$\begin{aligned}
F_{Min} = \left\langle \min_{1 \leq j \leq n}\{\mu_j\}, \min_{1 \leq j \leq n}\{\eta_j\}, \max_{1 \leq j \leq n}\{\nu_j\} \right\rangle &\preccurlyeq_S \textit{T-SFIAAWA}(F_1, F_2, ..., F_n) \\
&\preccurlyeq_S F_{Max} = \left\langle \max_{1 \leq j \leq n}\{\mu_j\}, \left(1 - \max_{1 \leq j \leq n}\{\mu_j^t\} - \min_{1 \leq j \leq n}\{\nu_j^t\}\right)^{\frac{1}{t}}, \min_{1 \leq j \leq n}\{\nu_j\} \right\rangle.
\end{aligned}$$

Similarly, we have the following T-spherical fuzzy interactional Aczél-Alsina weighted geometric operators.

Theorem 8. *Let $F_j = \langle \mu_j, \eta_j, \nu_j \rangle$ be a collection of T-SFNs ($j = 1, 2, \ldots, n$), $\omega_j \in (0, 1)$ and $\sum_{j=1}^{n} \omega_j = 1$. Then the T-spherical fuzzy interactional Aczél-Alsina weighted geometric (T-SFIAAWG) aggregation operator is also a mapping T-SFIAAWG: $\Phi^n \to \Phi$, and*

$$\text{T-SFIAAWG}(F_1, F_2, \ldots, F_n) = \underset{j=1_{\text{AA}}}{\overset{n}{\otimes}} \left(F_j^{\omega_j}\right)_{\text{AA}}$$

$$= \left\langle \left(\mathrm{e}^{-\left(\sum_{j=1}^{n} \omega_j\left(-\ln(\mu_j^t)\right)^M\right)^{\frac{1}{M}}} \right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\sum_{j=1}^{n} \omega_j\left(-\ln(\mu_j^t+\eta_j^t)\right)^M\right)^{\frac{1}{M}}} - \mathrm{e}^{-\left(\sum_{j=1}^{n} \omega_j\left(-\ln(\mu_j^t)\right)^M\right)^{\frac{1}{M}}} \right)^{\frac{1}{t}}, \left(1 - \mathrm{e}^{-\left(\sum_{j=1}^{n} \omega_j\left(-\ln(1-\nu_j^t)\right)^M\right)^{\frac{1}{M}}} \right)^{\frac{1}{t}} \right\rangle. \quad (4.6)$$

Theorem 9. *(Idempotency) If $F_j = \langle \mu, \eta, \nu \rangle \in \Phi$ for all $j = 1, 2, \ldots, n$, with $\omega_j \in (0, 1)$ and $\sum_{j=1}^{n} \omega_j = 1$, then,*

$$\textit{T-SFIAAWG}(F_1, F_2, \ldots, F_n) = \langle \mu, \eta, \nu \rangle.$$

Theorem 10. *(Monotonicity) Let $G_j = \langle \mu_{G_j}, \eta_{G_j}, \nu_{G_j} \rangle$ and $L_j = \langle \mu_{L_j}, \eta_{L_j}, \nu_{L_j} \rangle$ be two collections of T-SFNs ($j = 1, 2, \ldots, n$), with weights $\omega_j \in (0, 1)$ and $\sum_{j=1}^{n} \omega_j = 1$. If for all j, $G_j \subseteq L_j$, i.e., $\mu_{G_j} \leq \mu_{L_j}$, $\eta_{G_j} \leq \eta_{L_j}$, and $\nu_{G_j} \geq \nu_{L_j}$, then,*

$$\textit{T-SFIAAWG}(G_1, G_2, \ldots, G_n) \preccurlyeq_S \textit{T-SFIAAWG}(L_1, L_2, \ldots, L_n).$$

Theorem 11. *(Boundedness) Let $F_j = \langle \mu_j, \eta_j, \nu_j \rangle \in \Phi$ ($j = 1, 2, \ldots, n$), with weights $\omega_j \in (0, 1)$ and $\sum_{j=1}^{n} \omega_j = 1$. Then,*

$$F_{Min} = \left\langle \min_{1 \leq j \leq n}\{\mu_j\}, \min_{1 \leq j \leq n}\{\eta_j\}, \max_{1 \leq j \leq n}\{\nu_j\} \right\rangle \preccurlyeq_S \textit{T-SFIAAWG}(F_1, F_2, ..., F_n)$$

$$\preccurlyeq_S F_{Max} = \left\langle \max_{1 \leq j \leq n}\{\mu_j\}, \left(1 - \left(\max_{1 \leq j \leq n}\{\mu_j^t\} + \min_{1 \leq j \leq n}\{\nu_j^t\}\right)\right)^{\frac{1}{t}}, \min_{1 \leq j \leq n}\{\nu_j\} \right\rangle.$$

5. TWMADM method in T-SF environments

Let $Z = \{z_1, z_2, \ldots, z_m\}$ denote the set of alternatives, where z_i represents the i-th alternative, $\varrho = \{\varrho_1, \varrho_2, \ldots, \varrho_n\}$ denote the set of attributes, where ϱ_j represents the j-th attribute, and $F_{ij} = \langle \mu_{ij}, \eta_{ij}, \nu_{ij} \rangle \in \Phi$ be the T-SF attribute value of the alternative z_i respect to the attribute ϱ_j, which satisfy the condition $0 \leq \mu_{ij}, \eta_{ij}, \nu_{ij}, \mu_{ij}^t + \eta_{ij}^t + \nu_{ij}^t \leq 1$ ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$). We say $(Z, \varrho, (F_{ij})_{m \times n})$ is a T-spherical fuzzy information system (T-SFIS).

For the sake of discussion, it is assumed below that ϱ_j is a benefit attribute (unless otherwise stated).

5.1. Attribute weights based on the entropy weight method

Suppose $S_{ij} = S(F_{ij})$ is the score function of F_{ij} based on Eq (3.3). Generally, each attribute has different importance or influence in practical decision-making problems. The weights of all attributes can be represented by a vector $\Omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$, where ω_j denotes the weight of ϱ_j, $\omega_j \in (0, 1)$, and $\sum_{j=1}^{n} \omega_j = 1$. Since the attribute weights are unknown, this paper employs the entropy weight method [40] to determine the attribute weights. The calculation process is as follows.

For $\varrho_j \in \varrho$, normalize each column of attribute values:

$$\varphi(F_{ij}) = \frac{S_{ij}}{\sum_{k=1}^{m} S_{kj}}, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n. \tag{5.1}$$

Then calculate the information entropy of each attribute:

$$E_j = -\frac{1}{\ln m} \sum_{i=1}^{m} \varphi(F_{ij}) \ln \varphi(F_{ij}), \quad j = 1, 2, \ldots, n. \tag{5.2}$$

Here, $\varphi(F_{ij})$ represents the proportion of the score of the i-th alternative relative to the j-th attribute in the total alternatives. Finally, the objective weight of the j-th attribute is obtained as

$$\omega_j = \frac{1 - E_j}{\sum_{l=1}^{n} (1 - E_l)}. \tag{5.3}$$

5.2. Conditional probability based on the TOPSIS method with T-SFIAAWG aggregation operators

In the context of TWMADM, the traditional TOPSIS model struggles to effectively handle the three-dimensional fuzzy information inherent to T-SFNs. This often results in information loss during the aggregation of multi-attribute data, leading to inaccuracies in identifying the ideal solutions and calculating the closeness degrees, which compromises the reliability of the decision outcome. To address this issue, the introduction of the T-SFIAAWG aggregation operator is essential. This operator was designed for the T-SF environment, with its core AA operator constructed based on T-norms and T-conorms, enabling it to capture the correlations among decision attributes and thereby overcoming the limitation of traditional weighted average operators that assume attribute independence.

Regarding the choice of operators, we adopt the T-SFIAAWG aggregation operator over its arithmetic counterpart for two reasons. First, it is insensitive to extreme values, aligning with the conservative principle of ideal solution construction in TOPSIS. Second, its multiplicative structure preserves the proportional relationships among membership, neutrality, and non-membership degrees, ensuring consistency with subsequent distance calculations. As shown in Table 1, both operators yield highly consistent rankings, confirming the robustness of this choice.

Table 1. Comparison of ranking results under T-SFIAAWA and T-SFIAAWG operators.

Operator	Ranking results
T-SFIAAWG	$z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$
T-SFIAAWA	$z_7 > z_2 > z_1 > z_8 > z_4 > z_5 > z_6 > z_3$

For each alternative z_i, aggregate all the attribute values to $F_i^* = \text{T-SFIAAWG}(F_{i1}, F_{i2}, \ldots, F_{in}) \triangleq \langle \mu_i^*, \eta_i^*, \nu_i^* \rangle$ based on Eq (4.6), here the weight ω_j is obtained by Eq (5.3). The three components are calculated as follows:

$$\begin{aligned}
\mu_i^* &= \left(\mathrm{e}^{-\left(\sum\limits_{j=1}^{n} \omega_j (-\ln(\mu_{ij}^t))^M \right)^{\frac{1}{M}}} \right)^{\frac{1}{t}}, \\
\eta_i^* &= \left(\mathrm{e}^{-\left(\sum\limits_{j=1}^{n} \omega_j (-\ln(\eta_{ij}^t + \mu_{ij}^t))^M \right)^{\frac{1}{M}}} - \mathrm{e}^{-\left(\sum\limits_{j=1}^{n} \omega_j (-\ln(\mu_{ij}^t))^M \right)^{\frac{1}{M}}} \right)^{\frac{1}{t}}, \\
\nu_i^* &= \left(1 - \mathrm{e}^{-\left(\sum\limits_{j=1}^{n} \omega_j (-\ln(1-\nu_{ij}^t))^M \right)^{\frac{1}{M}}} \right)^{\frac{1}{t}}.
\end{aligned} \tag{5.4}$$

Then determine the positive and negative ideal solutions for each attribute ϱ_j $(j = 1, 2 \ldots, n)$ as follows:

$$\begin{aligned}
G_j^+ &= \left\langle \max_{1 \le i \le m} \{\mu_{ij}\}, \left(1 - \left(\max_{1 \le i \le m} \{\mu_{ij}^t\} + \min_{1 \le i \le m} \{\nu_{ij}^t\} \right) \right)^{\frac{1}{t}}, \min_{1 \le i \le m} \{\nu_{ij}\} \right\rangle, \\
G_j^- &= \left\langle \min_{1 \le i \le m} \{\mu_{ij}\}, \min_{1 \le i \le m} \{\eta_{ij}\}, \max_{1 \le i \le m} \{\nu_{ij}\} \right\rangle.
\end{aligned} \tag{5.5}$$

Substitute the positive and negative ideal solutions into the collective T-spherical fuzzy interactional Aczél-Alsina operator to obtain two aggregated T-SFNs:

$$\begin{aligned}
G^+ &= \text{T-SFIAAWG}(G_1^+, G_2^+, \ldots, G_n^+) \triangleq \langle \mu^+, \eta^+, \nu^+ \rangle, \\
G^- &= \text{T-SFIAAWG}(G_1^-, G_2^-, \ldots, G_n^-) \triangleq \langle \mu^-, \eta^-, \nu^- \rangle.
\end{aligned} \tag{5.6}$$

Calculate the Manhattan distances between F_i^* and G^+, G^- based on Eq (3.1):

$$\begin{aligned}
d_i^+ &= d(F_i^*, G^+) = \left| (\mu_i^*)^t - (\mu^+)^t \right| + \left| (\eta_i^*)^t - (\eta^+)^t \right| + \left| (\nu_i^*)^t - (\nu^+)^t \right| + \left| (\pi_i^*)^t - (\pi^+)^t \right|, \\
d_i^- &= d(F_i^*, G^-) = \left| (\mu_i^*)^t - (\mu^-)^t \right| + \left| (\eta_i^*)^t - (\eta^-)^t \right| + \left| (\nu_i^*)^t - (\nu^-)^t \right| + \left| (\pi_i^*)^t - (\pi^-)^t \right|.
\end{aligned} \tag{5.7}$$

Then we have the relative closeness (RC) of the state H respect to the alternative z_i $(i = 1, 2, \ldots, m)$:

$$RC(z_i) = \frac{d_i^-}{d_i^+ + d_i^-}. \tag{5.8}$$

Here, the relative closeness $RC(z_i)$ represents the proximity of alternative z_i to the positive ideal solution while indicating its distance from the negative ideal solution; a higher coefficient value corresponds to a superior alternative. In this context, $RC(z_i)$ can be interpreted as the degrees to which z_i meets the DMs' requirements, i.e., the subjective probability or possibility that z_i belongs to the desirable state H. Accordingly, we regard $RC(z_i)$ as the conditional probability of H given z_i, denoted by $Pr(H|z_i) = RC(z_i)$, and consequently $Pr(\neg H|z_i) = 1 - RC(z_i)$.

Within the T-SF three-way multi-attribute decision framework, the relative closeness degree, as the alternative ranking index of the TOPSIS method, can be converted into a conditional probability for three-way decision-region division. This conversion helps DMs classify alternatives based on probabilities. The key to finalizing the classification decision is determining a pair of thresholds (α, β) that accurately separate the three decision regions. In subsequent research, we will focus on determining these thresholds.

5.3. The relative loss function based on RT

To convert T-SFNs into real-valued inputs suitable for utility functions, we map the score function values S_{ij} to the real number domain for computation by the regret function.

Definition 13. *Let $D = (S_{ij})_{m\times n}$ be the score matrix, where S_{ij} represents the score of F_{ij}. The current utility function of F_{ij} is defined as*

$$v(S_{ij}) = \frac{1 - \mathrm{e}^{-\gamma S_{ij}}}{\gamma}, \tag{5.9}$$

where $\gamma \in (0, 1)$ represents the risk aversion rate.

A smaller value of $v(S_{ij})$ indicates that the DM is more risk-averse. On this basis, to express the psychological effects experienced by the DM after making a choice, a regret function is further introduced to characterize the emotional reactions triggered by utility differences between alternatives. The function form is as follows:

$$h(\Delta v_{ij}) = 1 - \mathrm{e}^{-\delta \Delta v_{ij}}, \quad \Delta v_{ij} = v(S_{ij}) - v(S_j^*), \quad S_j^* = \max_{1\le i\le m}\{S_{ij}\}, \tag{5.10}$$

where $\delta \in [0, +\infty)$ is the regret avoidance coefficient used to describe the individual's sensitivity to regret or rejoicing psychology.

If $h(\Delta v_{ij}) = 0$, it indicates that the current alternative is superior to the compared alternative, and the DM experiences rejoicing psychology; if $h(\Delta v_{ij}) < 0$, it indicates that the selected alternative is inferior to others, and the DM tends to experience regret.

Utility functions in the literature often combine current utility and psychological utility in additive forms, which may not adequately capture the relative importance and interaction between these two components. To address this limitation, we propose the following definition.

Definition 14. *The RUF of F_{ij} is defined as follows:*

$$c_{ij} = \frac{\kappa v(S_{ij}) + (1-\kappa) h(\Delta v_{ij})}{v(S_{ij}) + \epsilon}, \tag{5.11}$$

where parameter $\kappa \in (0, 1]$ adjusts the relative weight between current utility and psychological utility, enabling DMs to express their preferences. Moreover, $\epsilon \ll 1$ is a small positive constant ensuring numerical stability when $v(S_{ij})$ approaches zero.

The term $\dfrac{h(\Delta v_{ij})}{v(S_{ij}) + \epsilon}$ captures the elasticity of psychological utility with respect to current utility. When $v(S_{ij})$ is small (i.e., poor outcomes), this elasticity is automatically amplified; when $v(S_{ij})$ is large, it is automatically reduced. Unlike the additive form $v(S_{ij}) + h(\Delta v_{ij})$, the ratio naturally encodes this relative intensity, making the model more behaviorally realistic. In this paper, we take $\epsilon = 10^{-6}$.

Loss functions that reflect the DMs' psychological behavior are constructed from the previously calculated relative utility matrix, providing a foundation for classification in TWD.

Let c_{ij} be the RUF of F_{ij}, risk aversion factor $\theta \in (0,1)$, and the attribute weight vector $\Omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$. Under state H and state $\neg H$, the relative loss functions corresponding to each decision action are defined as shown in Table 2.

Table 2. The relative loss function of object z_i under the j-th attribute.

	H	$\neg H$
action$_P$	$\widehat{\lambda}_{ij}^{PP} = 0$	$\widehat{\lambda}_{ij}^{PN} = \max\limits_{1\le i\le m}\{c_{ij}\} - c_{ij}$
action$_B$	$\widehat{\lambda}_{ij}^{BP} = \theta\Big(c_{ij} - \min\limits_{1\le i\le m}\{c_{ij}\}\Big)$	$\widehat{\lambda}_{ij}^{BN} = \theta\Big(\max\limits_{1\le i\le m}\{c_{ij}\} - c_{ij}\Big)$
action$_N$	$\widehat{\lambda}_{ij}^{NP} = c_{ij} - \min\limits_{1\le i\le m}\{c_{ij}\}$	$\widehat{\lambda}_{ij}^{NN} = 0$

Since the relative loss functions of F_{ij} have been calculated, the aggregated loss function based on multiple attributes is further computed, as shown in Table 3.

Table 3. The aggregated relative loss function of object z_i.

	H	$\neg H$
action$_P$	$\widehat{\lambda}_{i}^{PP} = 0$	$\widehat{\lambda}_{i}^{PN} = \sum\limits_{j=1}^{n} \omega_j \widehat{\lambda}_{ij}^{PN}$
action$_B$	$\widehat{\lambda}_{i}^{BP} = \sum\limits_{j=1}^{n} \omega_j \widehat{\lambda}_{ij}^{BP}$	$\widehat{\lambda}_{i}^{BN} = \sum\limits_{j=1}^{n} \omega_j \widehat{\lambda}_{ij}^{BN}$
action$_N$	$\widehat{\lambda}_{i}^{NP} = \sum\limits_{j=1}^{n} \omega_j \widehat{\lambda}_{ij}^{NP}$	$\widehat{\lambda}_{i}^{NN} = 0$

5.4. Three-way decision rules based on expected loss minimization

In this subsection, we present the three-way decision model based on the aggregated relative loss functions. Based on the aggregated relative loss functions of each alternative given in Table 3 and the conditional probability $Pr(H|z_i)$ obtained in Subsection 5.2, the expected losses $R(I_\star|z_i)$ ($\star = P, B, N$) for alternative z_i under the three decision actions (acceptance, deferment, rejection) are calculated as follows:

$$\begin{aligned}
R(I_P|z_i) &= \widehat{\lambda}_i^{PP} Pr(H|z_i) + \widehat{\lambda}_i^{PN} Pr(\neg H|z_i) = \widehat{\lambda}_i^{PN}(1 - Pr(H|z_i)),\\
R(I_B|z_i) &= \widehat{\lambda}_i^{BP} Pr(H|z_i) + \widehat{\lambda}_i^{BN} Pr(\neg H|z_i),\\
R(I_N|z_i) &= \widehat{\lambda}_i^{NP} Pr(H|z_i) + \widehat{\lambda}_i^{NN} Pr(\neg H|z_i) = \widehat{\lambda}_i^{NP} Pr(H|z_i).
\end{aligned}$$

Define

$$\begin{aligned}
\alpha_i &= \frac{(\widehat{\lambda}_i^{PN} - \widehat{\lambda}_i^{BN})}{(\widehat{\lambda}_i^{PN} - \widehat{\lambda}_i^{BN}) + (\widehat{\lambda}_i^{BP} - \widehat{\lambda}_i^{PP})} = \frac{(1-\theta)\sum\limits_{j=1}^{n}\omega_j(\max\limits_{1\le i\le m}\{c_{ij}\} - c_{ij})}{(1-\theta)\sum\limits_{j=1}^{n}\omega_j(\max\limits_{1\le i\le m}\{c_{ij}\} - c_{ij}) + \theta\sum\limits_{j=1}^{n}\omega_j(c_{ij} - \min\limits_{1\le i\le m}\{c_{ij}\})},\\
\beta_i &= \frac{(\widehat{\lambda}_i^{BN} - \widehat{\lambda}_i^{NN})}{(\widehat{\lambda}_i^{BN} - \widehat{\lambda}_i^{NN}) + (\widehat{\lambda}_i^{NP} - \widehat{\lambda}_i^{BP})} = \frac{\theta\sum\limits_{j=1}^{n}\omega_j(\max\limits_{1\le i\le m}\{c_{ij}\} - c_{ij})}{\theta\sum\limits_{j=1}^{n}\omega_j(\max\limits_{1\le i\le m}\{c_{ij}\} - c_{ij}) + (1-\theta)\sum\limits_{j=1}^{n}\omega_j(c_{ij} - \min\limits_{1\le i\le m}\{c_{ij}\})}.
\end{aligned} \tag{5.12}$$

Since $\widehat{\lambda}_i^{PP} \leq \widehat{\lambda}_i^{BP} \leq \widehat{\lambda}_i^{NP}, \widehat{\lambda}_i^{NN} \leq \widehat{\lambda}_i^{BN} \leq \widehat{\lambda}_i^{PN}$, and $Pr(H|z_i) + Pr(\neg H|z_i) = 1$. Then the decision rules can be reexpressed as:

(1) If $Pr(H|z_i) \geq \alpha_i$, decide $z_i \in POS(H)$;

(2) If $\beta_i < Pr(H|z_i) < \alpha_i$, decide $z_i \in BND(H)$;

(3) If $Pr(H|z_i) \leq \beta_i$, decide $z_i \in NEG(H)$.

5.5. Concrete algorithm of the novel TWMADM

In Algorithm 1, each step has a time complexity as follows: The time complexity for Steps 1–5 of the algorithm is $O(mn)$, while for Steps 6–8, it is $O(m)$. Finally, the overall time complexity of the algorithm is $O(mn)$.

Algorithm 1 T-SF TWMADM intergrated RT and interactional AA aggregation operators.

Input: A T-SFIS (Z, ϱ, Φ), and the parameters t, γ, δ, κ, θ, and M.
Output: The classification and ranking of all alternatives.
Step 1: Transform the T-SFN F_{ij} into a unified form $\widetilde{F}_{ij}$ as follows:

$$\widetilde{F}_{ij} = \begin{cases} F_{ij} = \langle \mu_{ij}, \eta_{ij}, \nu_{ij} \rangle, & \text{if } \varrho_j \text{ is a benefit attribute;} \\ F_{ij}^c = \langle \nu_{ij}, \eta_{ij}, \mu_{ij} \rangle, & \text{if } \varrho_j \text{ is a cost attribute.} \end{cases}$$

Step 2: Using Eq (3.3) calculate the score $S_{ij} = S(\widetilde{F}_{ij})$, and obtain the score matrix $D = \left(S_{ij}\right)_{m\times n}$.
Step 3: Calculate the attribute weights ω_j using Eqs. (5.1)–(5.3).
Step 4: Deduce the conditional probability $Pr(H|z_i)$ according with Eqs. (5.4)–(5.8).
Step 5: Based on Eqs.(5.9)–(5.11), compute RUF c_{ij} for each $\widetilde{F}_{ij}$.
Step 6: Based on Table 2, calculate the relative loss functions $\widehat{\lambda}_{ij}^{BP}$, $\widehat{\lambda}_{ij}^{NP}$, $\widehat{\lambda}_{ij}^{PN}$, and $\widehat{\lambda}_{ij}^{BN}$ for each alternative z_i under each attribute ϱ_j. Then, calculate aggregated relative loss functions $\widehat{\lambda}_i^{BP}$, $\widehat{\lambda}_i^{NP}$, $\widehat{\lambda}_i^{PN}$, and $\widehat{\lambda}_i^{BN}$ for each alternative according with Table 3.
Step 7: Compute each pair of the thresholds α_i, β_i for each z_i based on Eq (5.12).
Step 8: For each alternative z_i,
if $Pr(H|z_i) \geq \alpha_i$, $z_i \in POS(H)$, and calculate the total expected loss value

$$TC(z_i) = (1 - Pr(H|z_i))\widehat{\lambda}_i^{PN}; \tag{5.13}$$

else if $\beta_i < Pr(H|z_i) < \alpha_i$, $z_i \in BND(H)$, and calculate

$$TC(z_i) = Pr(H|z_i)\widehat{\lambda}_i^{BP} + (1 - Pr(H|z_i))\widehat{\lambda}_i^{BN}; \tag{5.14}$$

else, $z_i \in NEG(H)$ and deduce

$$TC(z_i) = Pr(H|z_i)\widehat{\lambda}_i^{NP}. \tag{5.15}$$

Step 9: Rank the priorities of each alternative in the three regions according to the following rules: $POS(H) > BNG(H) > NEG(H)$. Alternatives in $POS(H)$ and $BNG(H)$ are ranked in descending order of total expected loss values, while those in $NEG(H)$ are ranked in ascending order.

6. Application on novel T-SF TWMADM: A case study

In this section, we apply the proposed method to address the TWMADM problem and verify its feasibility, validity, and stability through a practical case.

6.1. Case background introduction

We apply the aforementioned method to the optimal design of medium-sized cruise ship cabins. This case study, derived from [41], is based on an actual project supported by the Ministry of Industry and Information Technology of China. Experts from a shipbuilding factory in Weihai, Shandong are selected, forming an evaluation and selection team to assess eight candidate cabin designs, constituting the alternative set $Z = \{z_1, z_2, \ldots, z_8\}$. The evaluation index system consists of five attributes, forming the attribute set $\varrho = \{\varrho_1, \varrho_2, \ldots, \varrho_5\}$, which correspond to Esthetic, Technology, Environment, Human-Machine, and Economy, respectively. All five attributes are benefit-type criteria, meaning that higher evaluation values indicate better performance in the corresponding dimension.

In this case, the decision environment is characterized by high uncertainty and severe consequences of selecting an inappropriate design. The traditional two-way decision framework is not directly applicable, as design schemes often require further refinement before a final decision can be made. The TWD framework aligns well with this mechanism: Designs that meet key requirements are directly accepted into the positive region, those with obvious deficiencies are rejected into the negative region, while promising but immature designs are placed into the boundary region, thereby providing a buffer for risk management. Moreover, incorporating RT captures DMs' psychological aversion to irreversible choices, making the model more behaviorally realistic and better suited to the practical needs of this case study.

The evaluation T-SF data for the technology attributes are derived from Table 3 in [41]. The original study employed PFSs, which correspond to the special case of T-SFSs with $t = 1$. To maintain consistency with the original data and facilitate direct comparison, we set $t = 1$ in this case study. The T-SFIS is denoted as $(Z, \varrho, (F_{ij})_{8\times5})$ with $t = 1$, and the corresponding evaluation values are summarized in Table 4.

Table 4. The T-SFIS $(Z, \varrho, (F_{ij})_{8\times5})$.

	ϱ_1	ϱ_2	ϱ_3	ϱ_4	ϱ_5
z_1	$\langle 0.302, 0.281, 0.210\rangle$	$\langle 0.350, 0.261, 0.262\rangle$	$\langle 0.348, 0.155, 0.207\rangle$	$\langle 0.304, 0.307, 0.247\rangle$	$\langle 0.307, 0.262, 0.247\rangle$
z_2	$\langle 0.303, 0.201, 0.233\rangle$	$\langle 0.302, 0.202, 0.296\rangle$	$\langle 0.402, 0.204, 0.194\rangle$	$\langle 0.374, 0.177, 0.162\rangle$	$\langle 0.268, 0.238, 0.266\rangle$
z_3	$\langle 0.379, 0.180, 0.274\rangle$	$\langle 0.336, 0.142, 0.342\rangle$	$\langle 0.200, 0.261, 0.359\rangle$	$\langle 0.307, 0.202, 0.230\rangle$	$\langle 0.337, 0.299, 0.221\rangle$
z_4	$\langle 0.304, 0.260, 0.436\rangle$	$\langle 0.307, 0.301, 0.234\rangle$	$\langle 0.200, 0.349, 0.251\rangle$	$\langle 0.478, 0.222, 0.141\rangle$	$\langle 0.341, 0.349, 0.183\rangle$
z_5	$\langle 0.328, 0.182, 0.289\rangle$	$\langle 0.306, 0.235, 0.274\rangle$	$\langle 0.449, 0.251, 0.200\rangle$	$\langle 0.403, 0.239, 0.231\rangle$	$\langle 0.236, 0.307, 0.246\rangle$
z_6	$\langle 0.335, 0.280, 0.200\rangle$	$\langle 0.369, 0.267, 0.163\rangle$	$\langle 0.415, 0.285, 0.200\rangle$	$\langle 0.203, 0.497, 0.300\rangle$	$\langle 0.391, 0.240, 0.223\rangle$
z_7	$\langle 0.305, 0.272, 0.219\rangle$	$\langle 0.307, 0.293, 0.255\rangle$	$\langle 0.504, 0.248, 0.148\rangle$	$\langle 0.446, 0.242, 0.212\rangle$	$\langle 0.405, 0.366, 0.129\rangle$
z_8	$\langle 0.276, 0.209, 0.345\rangle$	$\langle 0.236, 0.274, 0.363\rangle$	$\langle 0.301, 0.350, 0.350\rangle$	$\langle 0.435, 0.315, 0.150\rangle$	$\langle 0.437, 0.303, 0.161\rangle$

6.2. Case calculation process and result analysis

According to the algorithm flow described above, we calculate the instance of the eight cabin designs to be evaluated. When $t = 1$, $\gamma = 0.2$, $\delta = 0.6$, $\kappa = 0.5$, $\theta = 0.2$ and $M = 3$, the results are as follows:

Step 1: Since all attributes in this case study are benefit-type, the original T-SFNs remain unchanged.

Step 2: According to Table 4 and Eq (3.3), we obtain the score matrix D:

$$D = \left(S_{ij}\right)_{8\times 5} = \begin{pmatrix} 0.5460 & 0.5440 & 0.5705 & 0.5285 & 0.5300 \\ 0.5350 & 0.5030 & 0.6040 & 0.6060 & 0.5010 \\ 0.5525 & 0.4970 & 0.4205 & 0.5385 & 0.5580 \\ 0.4340 & 0.5365 & 0.4745 & 0.6685 & 0.5790 \\ 0.5195 & 0.5160 & 0.6245 & 0.5860 & 0.4950 \\ 0.5675 & 0.6030 & 0.6075 & 0.4515 & 0.5840 \\ 0.5430 & 0.5260 & 0.6780 & 0.6170 & 0.6380 \\ 0.4655 & 0.4365 & 0.4755 & 0.6425 & 0.6380 \end{pmatrix}.$$

Step 3: Calculate the attribute weight vector ω_j using Eqs (5.1)–(5.3):

$$\Omega = (0.1214, 0.1221, 0.3908, 0.2239, 0.1418)^T.$$

Step 4: The conditional probability $Pr(H|z_i)$ is calculated in accordance with Eqs. (5.4)–(5.8), as shown in Table 5.

Table 5. The conditional probabilities of all alternatives.

$Pr(H\|z_1)$	$Pr(H\|z_2)$	$Pr(H\|z_3)$	$Pr(H\|z_4)$	$Pr(H\|z_5)$	$Pr(H\|z_6)$	$Pr(H\|z_7)$	$Pr(H\|z_8)$
0.3595	0.3518	0.2469	0.4571	0.4646	0.6114	0.6337	0.4843

Step 5: The RUFs are calculated by Eqs. (5.9)–(5.11), resulting in the relative utility matrix C:

$$C = \left(c_{ij}\right)_{8\times 5} = \begin{pmatrix} 0.4888 & 0.4689 & 0.4457 & 0.4229 & 0.4410 \\ 0.4826 & 0.4423 & 0.4650 & 0.4706 & 0.4202 \\ 0.4923 & 0.4380 & 0.3159 & 0.4299 & 0.4588 \\ 0.4096 & 0.4644 & 0.3730 & 0.5000 & 0.4709 \\ 0.4735 & 0.4513 & 0.4757 & 0.4596 & 0.4155 \\ 0.5000 & 0.5000 & 0.4669 & 0.3570 & 0.4736 \\ 0.4871 & 0.4578 & 0.5000 & 0.4763 & 0.5000 \\ 0.4362 & 0.3873 & 0.3740 & 0.4886 & 0.5000 \end{pmatrix}.$$

Step 6: Aggregate relative loss functions for each alternative under different attributes is calculated, as shown in Table 6.

Table 6. The relative loss values of all alternatives.

Alternative	$\widehat{\lambda_i^{PP}}$	$\widehat{\lambda_i^{PN}}$	$\widehat{\lambda_i^{BP}}$	$\widehat{\lambda_i^{BN}}$	$\widehat{\lambda_i^{NP}}$	$\widehat{\lambda_i^{NN}}$
z_1	0.0000	0.0520	0.0177	0.0104	0.0886	0.0000
z_2	0.0000	0.0407	0.0200	0.0081	0.0999	0.0000
z_3	0.0000	0.1020	0.0077	0.0204	0.0387	0.0000
z_4	0.0000	0.0691	0.0143	0.0138	0.0716	0.0000
z_5	0.0000	0.0397	0.0202	0.0079	0.1010	0.0000
z_6	0.0000	0.0487	0.0184	0.0097	0.0920	0.0000
z_7	0.0000	0.0120	0.0257	0.0024	0.1286	0.0000
z_8	0.0000	0.0733	0.0135	0.0147	0.0673	0.0000

Step 7: Calculate the threshold values and the total expected loss values, which are shown in Tables 7 and 8, respectively.

Table 7. The thresholds of all alternatives.

Thresholds	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8
α_i	0.7013	0.6198	0.9134	0.7942	0.6113	0.6793	0.2722	0.8133
β_i	0.1280	0.0925	0.3972	0.1943	0.0895	0.1169	0.0228	0.2140

Table 8. The total expected loss values TC of all alternatives.

Regions	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8
$POS(H)$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0044	0.0000
$BND(H)$	0.0130	0.0123	0.0000	0.0140	0.0136	0.0150	0.0000	0.0141
$NEG(H)$	0.0000	0.0000	0.0096	0.0000	0.0000	0.0000	0.0000	0.0000

Step 8: Then, classify and rank the eight alternative solutions according to the classification criteria and ranking criteria. The results are as follows:

Classification results:

- $POS(H) = \{z_7\}$;
- $BND(H) = \{z_1, z_2, z_4, z_5, z_6, z_8\}$;
- $NEG(H) = \{z_3\}$.

The ranking result of the eight alternatives is: $z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$.

6.3. Parameter sensitivity analysis

The proposed method involves a total of six parameters: t, θ, γ, δ, κ, and M.

6.3.1. Sensitivity analysis of T-SF parameter t

First, a sensitivity analysis is conducted for parameter t, as shown in Figure 1. This figure visualizes the ranking changes of the eight cabin designs as t varies from 1 to 6 in steps of 1. The figure shows that the ranking order is highly stable over the tested range of t. Thus, the cabin design z_7 consistently maintains the top ranking across all values of t, confirming its stability as the optimal choice. Similarly, the worst design z_3 remains almost unchanged. Although the overall ranking structure is stable, minor fluctuations in the middle rankings are observed as t increases. For example, the ranking of z_5 and z_6 show slight crossings or proximity, indicating that their ranking is more sensitive to the specific value of parameter t. However, as the parameter gradually increases, the optimal cabin design remains unchanged, and the ranking results show slight fluctuations, verifying the stability of the method.

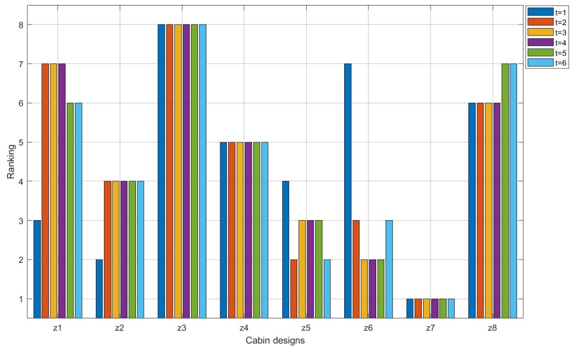

Figure 1. Ranking of alternatives under different t.

6.3.2. Sensitivity analysis of the TWD parameter θ

In TWD, conducting sensitivity analysis on the parameters in the utility function is extremely important. Regarding the influence of the value of θ on the positive, negative, and boundary regions, refer to Figure 2. The figure shows that classification changes with θ varying from 0.1 to 0.8 in steps of 0.1. When $\theta \in [0.1, 0.4]$, the decision results exhibit typical TWD characteristics, where alternatives in the boundary region gradually migrate to the positive or negative regions. Specifically, z_7 remains consistently in the positive region, while z_3 is classified into the negative region and stays unchanged. For $\theta \in [0.5, 0.8]$, all eight alternatives are divided into positive and negative regions, formally transitioning the decision problem from TWD to a two-way decision. In this range, z_7 consistently stays in the positive region, while z_3 is always in the negative region, with only minor shifts observed in the classification of other alternatives. The analysis demonstrates that increasing θ to relax the acceptance criterion leads to the migration of objects from the negative region to the positive region. This visually confirms θ's role as a key parameter for decision results.

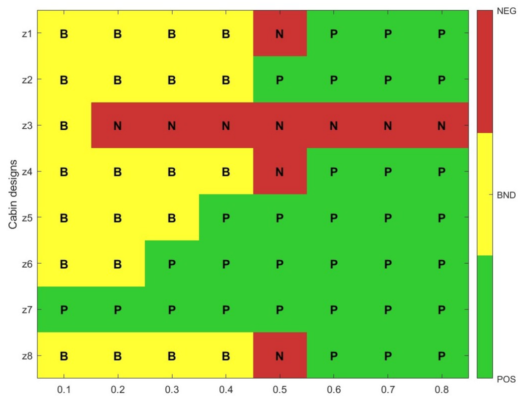

Figure 2. Three-way decision classification under different θ.

6.3.3. Sensitivity analysis of RT parameters γ, δ, and k

Next, a sensitivity analysis is conducted for parameters γ and δ in regret theory to explore their impact on the decision results. Tables 9 and 10 show the ranking changes of the eight decision alternatives under different values of γ and δ from 0.1 to 0.8 in steps of 0.1. From the tables, it can be seen that as they increase, the ranking remains largely as $z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$, showing strong stability. The results indicate that z_7, as the core alternative, remains the optimal solution across the range of γ and δ values, and the decision alternatives exhibit strong robustness throughout the analysis.

Table 9. Ranking of alternatives under different γ.

γ	Ranking	Optimal
$\gamma = 0.1$	$z_7 > z_2 > z_1 > z_5 > z_4 \approx z_8 > z_6 > z_3$	z_7
$\gamma = 0.2$	$z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$	z_7
$\gamma = 0.3$	$z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$	z_7
$\gamma = 0.4$	$z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$	z_7
$\gamma = 0.5$	$z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$	z_7
$\gamma = 0.6$	$z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$	z_7
$\gamma = 0.7$	$z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$	z_7
$\gamma = 0.8$	$z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$	z_7

Table 10. Ranking of alternatives under different δ.

δ	Ranking	Optimal
$\delta = 0.1$	$z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$	z_7
$\delta = 0.2$	$z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$	z_7
$\delta = 0.3$	$z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$	z_7
$\delta = 0.4$	$z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$	z_7
$\delta = 0.5$	$z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$	z_7
$\delta = 0.6$	$z_7 > z_5 > z_6 > z_2 > z_8 > z_4 > z_1 > z_3$	z_7
$\delta = 0.7$	$z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$	z_7
$\delta = 0.8$	$z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$	z_7

From Table 11, it can be seen that utility parameter k has a minimal effect on the ranking, which remains essentially unchanged under different κ values from 0.1 to 0.8 in steps of 0.1, with z_7 still being the optimal alternative, verifying the stability of the parameters.

Table 11. Ranking of alternatives under different κ.

κ	Ranking	Optimal
$\kappa = 0.1$	$z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$	z_7
$\kappa = 0.2$	$z_7 > z_2 > z_1 > z_5 > z_4 \approx z_8 > z_6 > z_3$	z_7
$\kappa = 0.3$	$z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$	z_7
$\kappa = 0.4$	$z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$	z_7
$\kappa = 0.5$	$z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$	z_7
$\kappa = 0.6$	$z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$	z_7
$\kappa = 0.7$	$z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$	z_7
$\kappa = 0.8$	$z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$	z_7

6.3.4. Sensitivity analysis of the AA operator parameter M

From Figure 3, it can be observed that the variation of parameter M reveals the transition of the decision model from a sensitive to a stable aggregation process, with the rankings of the eight alternatives remaining largely consistent. When the value of M is small, the operator is more sensitive to the aggregation results of the attribute values, leading to slight fluctuations in the rankings of the middle-ranked alternatives z_4 and z_8. As M increases, the influence on the differences in aggregation results is effectively mitigated, enabling the rankings of all alternatives to converge rapidly and remain stable. This process verifies that the T-SFIAAWG aggregation operator exhibits strong robustness within the interval where M is relatively large, providing reliable support for MADM.

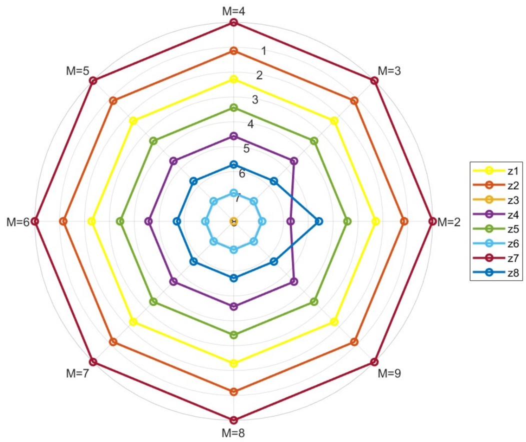

Figure 3. Ranking of alternative schemes under different M.

6.4. Bivariate sensitivity analysis of M, θ, and t

To investigate the interaction between parameters, we conduct bivariate sensitivity analyses for three parameter pairs, namely M and θ in Figure 4, M and t in Figure 5, and t and θ in Figure 6. Each subfigure corresponds to one alternative, and the color represents the ranking of that alternative under different parameter combinations: Lighter blue indicates a better rank, while darker red indicates a worse rank.

As shown in Figure 4 for M and θ, the optimal alternative z_7 and the worst alternative z_3 maintain constant colors over almost the entire parameter space, indicating that their rankings are almost unaffected by the parameters. For the middle alternatives, when θ is small, the colors show clear light-dark alternations, implying considerable ranking fluctuations. As for parameter M, when $M \geq 3$, the colors become essentially stable. This suggests that a larger M leads to more stable rankings.

As shown in Figure 5 for M and t, the variation of t has a very weak effect on colors. The colors of each alternative hardly change along the t axis, indicating that the fuzzy dimension parameter t has negligible influence on rankings. The effect of M is similar to that in Figure 4: Colors fluctuate when M is small and become stable when $M \geq 3$. Therefore, we recommend setting $M \geq 3$.

As shown in Figure 6 for t and θ, the colors of all alternatives show little change with t, but vary more noticeably with θ. The optimal and worst alternatives remain unchanged in color throughout. For the middle alternatives, slight color differences exist when θ is small, and the colors become stable as

θ increases, further confirming that a larger θ yields more stable rankings.

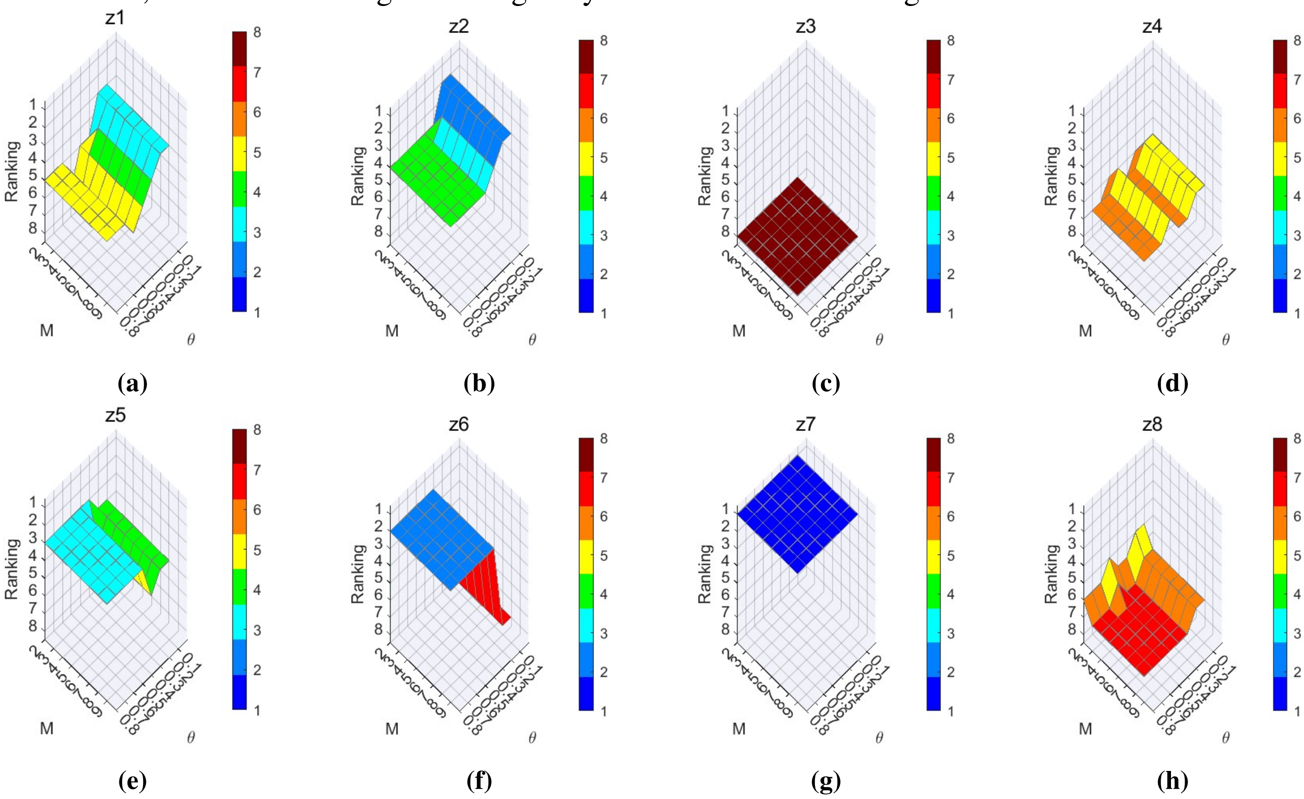

Figure 4. Bivariate analysis of rankings with respect to parameters M and θ.

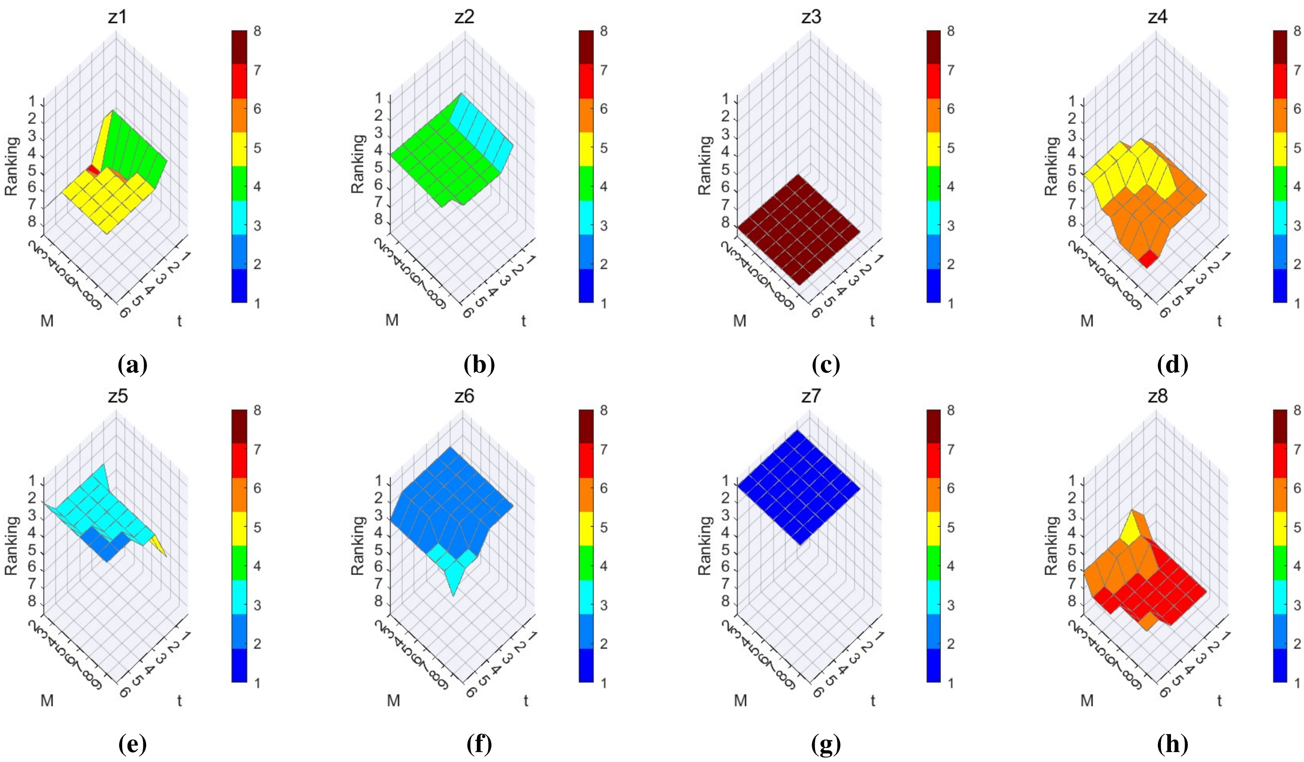

Figure 5. Bivariate analysis of rankings with respect to parameters M and t.

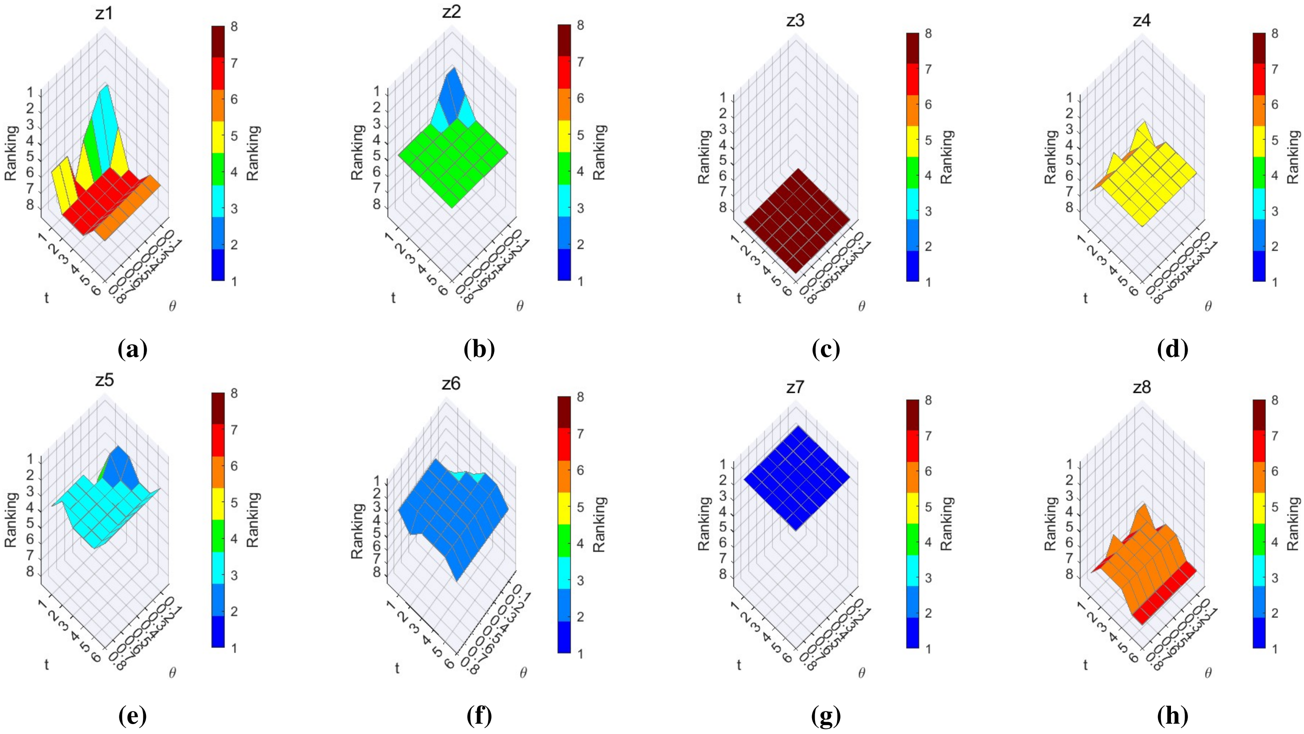

Figure 6. Bivariate analysis of rankings with respect to parameters t and θ.

These observations provide the basis for the practical parameter initialization guidelines given in the next subsection.

6.5. Practical guidelines for parameter initialization

To address the practical concern regarding the number of parameters, in this subsection, we provide initialization guidelines based on the sensitivity analysis. Table 12 summarizes the theoretical origin, practical meaning, suggested initial values, and stable ranges of the six parameters.

$M = 3$ is employed as the default setting, while $t = 1$ is adopted in the numerical example, and both can be adjusted within their stable ranges as required. For θ, values within $[0.1, 0.4]$ can be adopted for deferrable decision scenarios, while values in $[0.5, 0.8]$ are suitable for urgent decision situations. In contrast, γ, δ, and κ have negligible impacts on the final ranking over wide ranges, enabling DMs to adopt the suggested initial values or adjust them according to personal preferences. This guidance enhances the practical applicability of the proposed model.

Table 12. Practical guidelines for parameter initialization.

Parameter	Theoretical origin	Practical meaning	Initial value	Recommended range
t	T-SFS	Fuzziness dimensionality	1	$[1, 6]$
θ	TWD	Risk aversion factor	0.3	High-risk, deferrable: $[0.1, 0.4]$; urgent: $[0.5, 0.8]$
γ	RT	Risk aversion rate	0.3	$[0.1, 0.8]$
δ	RT	Regret avoidance coefficient	0.5	$[0.1, 0.8]$
κ	RUF	Utility weight parameter	0.5	$[0.1, 0.8]$
M	TSFIAA operator	Aggregation intensity	3	$[3, 10]$

Note: Initial values are based on common practice ($t = 1$, $M = 3$) and recommended ranges are the stable intervals derived from sensitivity analysis.

6.6. *Experimental analysis*

6.6.1. Comparison with MADM methods

To verify the effectiveness and rationality of the proposed method, we compare it with six MADM methods. The comparison results are shown in Table 13.

Table 13. Comparison of ranking results based on different methods.

Methods	Ranking results
Li et al.'s method [42]	$z_7 > z_2 > z_5 > z_6 > z_8 > z_4 > z_1 > z_3$
Fan et al.'s method [43]	$z_7 > z_5 > z_1 > z_2 > z_6 > z_4 > z_8 > z_3$
Hussain et al.'s method [8]	$z_2 > z_7 > z_1 > z_5 > z_3 > z_4 > z_6 > z_8$
Torun et al.'s method [44]	$z_7 > z_4 > z_8 > z_2 > z_3 > z_6 > z_1 > z_5$
Zhan et al.'s method [45]	$z_7 > z_4 > z_5 > z_2 > z_6 > z_1 > z_8 > z_3$
Wang et al.'s method [15]	$z_7 > z_2 > z_5 > z_1 > z_6 > z_3 > z_4 > z_8$
Our method	$z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$

Like other methods, our method consistently identifies z_7 as the optimal choice. In contrast, z_3 remains the worst choice. Overall, our method exhibits ranking trends similar to those of other methods. This result demonstrates the effectiveness of our method. Fan et al. [43] not only used RT but also combined it with the TODIM decision method in solving MADM problems. However, there are significant differences in rankings between their method and ours. This difference may be attributed to the fact that their information aggregation relies on traditional operational rules, which treat membership, neutrality, and non-membership degrees as independent components. In contrast, our method employs the proposed T-SFIAA aggregation operators, which introduce interactive operational laws to achieve more comprehensive information aggregation.

Additionally, Hussain et al.'s method [8] is inconsistent with most of the alternative rankings in this paper. The reason is that Hussain et al.'s method [8] not only neglects the influence of psychological factors on the DM, but also the T-spherical fuzzy Aczél-Alsina operators they define do not satisfy closure. In contrast, we propose T-SFIAAWA and T-SFIAAWG not only to rigorously demonstrate their mathematical properties, but also to model their interactions among internal dimensions, thereby establishing a more realistic foundation for information aggregation.

6.6.2. Comparison of T-SFIAAWA and T-SFIAAWG aggregation operators

To examine the influence of the aggregation operators on the decision outcome, we conduct a comparative experiment where the T-SFIAAWA aggregation operator is applied in Step 3 of the algorithm from Section 6.2, while all other steps remain unchanged. Table 1 presents the ranking results under both operators.

The results show that the optimal alternative z_7 and the worst alternative z_3 remain unchanged under both operators. The only differences occur among the middle-ranked alternatives z_5 and z_8. This indicates that the choice of aggregation operator does not affect the essential decision outcome. Therefore, the geometric operator is retained in the proposed model, as its insensitivity to extreme values and multiplicative structure align well with the TOPSIS methodology.

6.6.3. Ablation study

To isolate the contribution of the proposed T-SFIAAWG operator and the new RUF, we design two ablation experiments. We replace the T-SFIAAWG operator with the T-spherical fuzzy Aczél-Alsina weighted geometric(T-SFAAWG) operator [8] (G1), and replace the new RUF with the perceived utility function [37] (G2), while keeping all other steps unchanged.

As shown in Table 14, all three methods identify z_7 as the best alternative and z_3 as the worst. However, our method yields a strictly monotonic ranking among the middle alternatives ($z_1 > z_5 > z_4 > z_8$), while the middle ranking of G1 differs considerably from the other two methods, and G2 produces a tie ($z_4 \approx z_8$). This demonstrates that the proposed T-SFIAAWG operator and RUF provide better discriminability and ranking rationality.

Table 14. Ablation study.

Method	Aggregation operator	Utility function	Ranking
G1 [8]	T-SFAAWG	RUF	$z_7 > z_2 > z_8 > z_5 > z_1 > z_4 > z_6 > z_3$
G2 [37]	T-SFIAAWG	perceived utility function	$z_7 > z_2 > z_5 > z_1 > z_4 \approx z_8 > z_6 > z_3$
Our method	T-SFIAAWG	RUF	$z_7 > z_2 > z_1 > z_5 > z_4 > z_8 > z_6 > z_3$

6.6.4. Computational efficiency analysis

For computational efficiency comparison, we select several methods that can be directly implement in the same T-SF environment, including our method, Hussain et al.'s method [8], Wang et al.'s method [15], and Torun et al.'s method [44]. The other methods compared in Table 13 are not included in this analysis because they either operate under different fuzzy frameworks (e.g., PFS or q-ROFS) or rely on different decision procedures such as VIKOR or TODIM, making a fair runtime comparison difficult under the same T-SF setting.

We implement these methods in MATLAB (R2024b) and measure their running times on the same dataset (8 alternatives, 5 attributes). Each method is executed 20 times, and the mean execution time $\bar{t}$ and standard deviation σ are computed as

$$\bar{t} = \frac{1}{m}\sum_{i=1}^{m} t_i, \qquad \sigma = \sqrt{\frac{1}{m-1}\sum_{i=1}^{m}(t_i - \bar{t})^2}. \tag{6.1}$$

As shown in Table 15, among the four compared methods, Hussain et al. [8] achieves the shortest average execution time of 1.599 ms, which is expected since their operators are non-interactive and follow a simpler aggregation scheme. However, their AA operators do not satisfy the closure under the T-spherical fuzzy environment, which may lead to information loss. The classical TOPSIS method of Torun et al. [44] has an average time of 3.700 ms but exhibits a notably larger standard deviation of 4.726 ms, indicating higher instability across runs. Wang et al.'s method [15] requires 5.102 ms on average and shows a large standard deviation of 4.058 ms, reflecting higher computational cost and noticeable variability.

In contrast, our method requires 4.258 ms on average, which is higher than Hussain et al.'s method [8] and Torun et al.'s method [44], but lower than Wang et al.'s method [15]. More importantly, the standard deviation of the proposed method is 1.066 ms, which is significantly smaller than that of all three compared methods, indicating the best computational stability. Moreover, the time complexity of all methods is $O(mn)$, ensuring linear scalability with problem size. Therefore, the computational cost of our method increases slightly, but this is reasonable. It provides superior capabilities such as closed interactive aggregation, regret-theoretic psychological modeling, and TWD rules that the other compared methods lack.

Table 15. Computational efficiency comparison (20 runs).

Method	Mean time (ms)	Std dev (ms)	Min–Max (ms)
Hussain et al. 's method [8]	1.599	1.383	[0.641, 4.940]
Wang et al.'s method [15]	5.102	4.058	[2.259, 16.634]
Torun et al.'s method [44]	3.700	4.726	[1.602, 21.830]
Our method	4.258	1.066	[2.631, 6.572]

6.7. Discussion

Through the comparative analysis presented above, the effectiveness and feasibility of our new method in addressing MADM problems have been demonstrated. Next, we select seven distinct MADM methods for comparative analysis from various perspectives, as shown in Table 16.

Table 16. Differences between five MADM methods.

Methods	Psychological behavior	Loss/utility function	T-SF environment	Ranking	Classi-fication
Li et al.'s method [42]	×	×	×	✓	×
Fan et al.'s method [43]	✓	✓	×	✓	×
Hussain et al.'s method [8]	×	×	✓	✓	×
Torun et al.'s method [44]	×	×	×	✓	×
Zhan et al.'s method [45]	×	✓	×	✓	✓
Wang et al.'s method [15]	×	×	✓	✓	×
Our method	✓	✓	✓	✓	✓

The methods of Li et al. [42] and Torun et al. [44], which are based on VIKOR and TOPSIS for ranking alternatives, respectively, employ traditional weighted average operators for information aggregation, failing to consider the intrinsic correlations among attributes adequately. Hussain et al.'s method [8] introduces the T-spherical fuzzy Aczél-Alsina operator, which exhibits strong generalization capabilities and finds extensive applications in advanced decision-making contexts. However, the operator's operation on Φ fails to satisfy closure, thereby leading to information loss. In contrast, the T-SFIAA operator we propose uses triangular norms to model interactions between dimensions, reducing information loss when aggregating T-SF information.

Li et al. [42], Torun et al. [44], Hussain et al. [8], and Wang et al. [15] focus solely on ranking without classification. In contrast, our method and that of Zhan et al. [45] address a broader range of decision-making needs. However, the method of Zhan et al. [45] overlooks the influence of psychological behaviors, and its decision environment remains limited to traditional fuzzy information systems, without extending to the high-dimensional T-SF environment. In this paper, within the high-dimensional T-SF environment, psychological factors from RT are systematically integrated into the

TWD framework. By constructing a state set that incorporates psychological perceptions, the risk attitudes and emotional responses of DMs are expressed during the classification process.

Among researchers considering DMs' psychological behaviors, Fan et al.'s method [43] constructs a traditional utility function based on RT. The proposed relative utility loss function introduces a regulatory parameter, $\kappa \in (0, 1]$, enabling DMs to flexibly set the weight ratio between basic and psychological utility for specific decision scenarios. Furthermore, our method classifies alternatives before ranking them, enhancing the practicality and accuracy of the decision-making process.

Subsequently, we use the Spearman correlation coefficient (SCC) to assess the correlation between methods, thereby verifying the validity and reliability of our proposed method. As shown in Figure 7, our proposed method exhibits SCC values exceeding 0.66 compared with other MADM methods. This result explicitly confirms the validity and reliability of our proposed method.

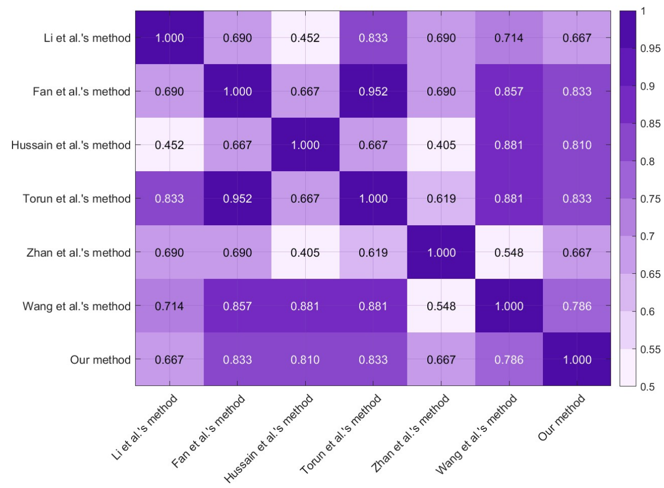

Figure 7. Spearman correlation coefficient.

7. Conclusion and outlook

This paper highlights the characteristics and limitations of MADM models while emphasizing the need for further exploration in combining the TWD theory with T-FSs. A novel TWD-MADM method is proposed, built on a T-SF environment, RT, and the T-SFIAA operator. The T-SFIAAWG operator aggregates T-SF information, the TOPSIS method calculates conditional probabilities, and loss functions are constructed based on a novel RUF for classification and ranking. A case study on cabin design evaluation verifies the model's effectiveness.

The proposed method offers several distinct advantages. First, it integrates RT into the T-SF environment with a novel RUF, enabling the model to capture real decision-making behavior under complex uncertainty. Second, the T-SF interactional AA aggregation operator achieves robust aggregation of multi-attribute uncertain information. Third, the dynamically adjustable TWD mechanism, combined with RT and the proposed operator, enables adaptive classification and ranking. Finally, comprehensive experiments confirm the model's strong robustness and generalization capability.

Although sensitivity analysis confirms the strong robustness of the parameters, the model contains numerous parameters, making it challenging to identify the optimal parameter combination in practical applications. In future studies, researchers could employ metaheuristic algorithms or machine learning methods to achieve automated parameter calibration based on historical decision data or expert feedback, thereby enhancing the model's applicability and decision-making efficiency. Moreover, our operators ensure closure but capture only partial interactions. In future work, we will extend them to achieve full pairwise interactions while preserving closure.

Author contributions

Liwen Sun: Investigation, writing original draft; Tingting Zheng: Conceptualization, Supervision, Writing–review and editing, Funding acquisition; Xinyi Sun: Investigation, Validation; Yibo Fan: Experimental support, Funding acquisition; Xinyu Ma: Experimental support, Funding acquisition.

Use of Generative-AI tools declaration

The authors declare that AI-assisted tool (DeepSeek) was used solely for language polishing and grammar improvement, and not for generating scientific ideas, data analysis, or conclusions.

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Conflict of interest

The authors declare that they have no conflict of interest.

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Appendix A. Proof of Theorem 2

Proof. (1) According to Definition 12, let $F_1 \oplus_{\mathrm{AA}} F_2 \triangleq \langle \mu_\oplus, \eta_\oplus, \nu_\oplus \rangle$, where

$$\mu_\oplus = \left(1 - \mathrm{e}^{-\left(\left(-\ln(1-\mu_1^t) \right)^M + \left(-\ln(1-\mu_2^t) \right)^M \right)^{\frac{1}{M}}} \right)^{\frac{1}{t}},$$

$$\eta_\oplus = \left(e^{-\left(\left(-\ln(\eta_1^t+\nu_1^t)\right)^M + \left(-\ln(\eta_2^t+\nu_2^t)\right)^M \right)^{\frac{1}{M}}} - e^{-\left(\left(-\ln(\nu_1^t)\right)^M + \left(-\ln(\nu_2^t)\right)^M \right)^{\frac{1}{M}}} \right)^{\frac{1}{t}},$$

$$\nu_\oplus = \left(e^{-\left(\left(-\ln(\nu_1^t)\right)^M + \left(-\ln(\nu_2^t)\right)^M \right)^{\frac{1}{M}}} \right)^{\frac{1}{t}}.$$

Thus, $0 \le \mu_\oplus, \eta_\oplus, \nu_\oplus \le 1$. Since $\left(1 - e^{-(-\ln(1-x))}\right)^{\frac{1}{t}}$ is increasing with respect to x, we have

$$\begin{aligned} \mu_\oplus^t + \eta_\oplus^t + \nu_\oplus^t &\le 1 - e^{-\left(\left(-\ln(1-\mu_1^t)\right)^M + \left(-\ln(1-\mu_2^t)\right)^M \right)^{\frac{1}{M}}} + e^{-\left(\left(-\ln(\eta_1^t+\nu_1^t)\right)^M + \left(-\ln(\eta_2^t+\nu_2^t)\right)^M \right)^{\frac{1}{M}}} \\ &\le 1 - e^{-\left(\left(-\ln(\eta_1^t+\nu_1^t)\right)^M + \left(-\ln(\eta_2^t+\nu_2^t)\right)^M \right)^{\frac{1}{M}}} + e^{-\left(\left(-\ln(\eta_1^t+\nu_1^t)\right)^M + \left(-\ln(\eta_2^t+\nu_2^t)\right)^M \right)^{\frac{1}{M}}} \\ &= 1. \end{aligned}$$

Therefore, $F_1 \oplus_{\mathrm{AA}} F_2 \in \Phi$. Similarly, we can prove that $F_1 \otimes_{\mathrm{AA}} F_2 \in \Phi$.

(2) Let

$$(\rho F)_{\mathrm{AA}} = \left\langle \left(1 - (1-\mu^t)^{\rho^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left((\eta^t + \nu^t)^{\rho^{\frac{1}{M}}} - (\nu^t)^{\rho^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left((\nu^t)^{\rho^{\frac{1}{M}}}\right)^{\frac{1}{t}} \right\rangle \triangleq \langle \mu_\rho, \eta_\rho, \nu_\rho \rangle,$$

where $\rho > 0$ and $M \ge 1$.

It is evident that $0 \le \mu_\rho, \eta_\rho, \nu_\rho \le 1$. Note that

$$\mu_\rho^t + \eta_\rho^t + \nu_\rho^t = 1 - (1-\mu^t)^{\rho^{\frac{1}{M}}} + (\eta^t + \nu^t)^{\rho^{\frac{1}{M}}} \le 1 - (\eta^t + \nu^t)^{\rho^{\frac{1}{M}}} + (\eta^t + \nu^t)^{\rho^{\frac{1}{M}}} = 1.$$

Hence, $(\rho F)_{\mathrm{AA}} \in \Phi$, completing the proof.

Similarly, we can prove that $(F^\rho)_{\mathrm{AA}} \in \Phi$. □

Appendix B. Proof of Theorem 3

Proof. Conclusions (1) and (2) are obvious.

(3) Since $F_1 \oplus_{\mathrm{AA}} F_2 = \Bigg\langle \left(1 - e^{-\left(\left(-\ln(1-\mu_1^t)\right)^M + \left(-\ln(1-\mu_2^t)\right)^M \right)^{\frac{1}{M}}}\right)^{\frac{1}{t}},$

$$\left(e^{-\left(\left(-\ln(\eta_1^t+\nu_1^t)\right)^M + \left(-\ln(\eta_2^t+\nu_2^t)\right)^M \right)^{\frac{1}{M}}} - e^{-\left(\left(-\ln(\nu_1^t)\right)^M + \left(-\ln(\nu_2^t)\right)^M \right)^{\frac{1}{M}}} \right)^{\frac{1}{t}}, \left(e^{-\left(\left(-\ln(\nu_1^t)\right)^M + \left(-\ln(\nu_2^t)\right)^M \right)^{\frac{1}{M}}} \right)^{\frac{1}{t}} \Bigg\rangle,$$

then $(\rho(F_1 \oplus_{\mathrm{AA}} F_2))_{\mathrm{AA}} = \Bigg\langle \left(1 - e^{-\left(\rho\left(-\ln(1-\mu_1^t)\right)^M + \rho\left(-\ln(1-\mu_2^t)\right)^M \right)^{\frac{1}{M}}}\right)^{\frac{1}{t}},$

$$\left(e^{-\left(\rho\left(-\ln(\eta_1^t+\nu_1^t)\right)^M + \rho\left(-\ln(\eta_2^t+\nu_2^t)\right)^M \right)^{\frac{1}{M}}} - e^{-\left(\rho\left(-\ln(\nu_1^t)\right)^M + \rho\left(-\ln(\nu_2^t)\right)^M \right)^{\frac{1}{M}}} \right)^{\frac{1}{t}}, \left(e^{-\left(\rho\left(-\ln(\nu_1^t)\right)^M + \rho\left(-\ln(\nu_2^t)\right)^M \right)^{\frac{1}{M}}} \right)^{\frac{1}{t}} \Bigg\rangle.$$

Moreover, since

$$\begin{aligned} (\rho F_j)_{\mathrm{AA}} &= \left\langle \left(1 - (1-\mu_j^t)^{\rho^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left((\eta_j^t + \nu_j^t)^{\rho^{\frac{1}{M}}} - (\nu_j^t)^{\rho^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left((\nu_j^t)^{\rho^{\frac{1}{M}}}\right)^{\frac{1}{t}} \right\rangle \\ &= \left\langle \left(1 - e^{-\rho^{\frac{1}{M}}\left(-\ln(1-\mu_j^t)\right)}\right)^{\frac{1}{t}}, \left(e^{-\rho^{\frac{1}{M}}\left(-\ln(\eta_j^t+\nu_j^t)\right)} - e^{-\rho^{\frac{1}{M}}\left(-\ln \nu_j^t\right)}\right)^{\frac{1}{t}}, \left(e^{-\rho^{\frac{1}{M}}\left(-\ln \nu_j^t\right)}\right)^{\frac{1}{t}} \right\rangle, \quad (j = 1, 2). \end{aligned}$$

So $(\rho F_1)_{\mathrm{AA}} \oplus_{\mathrm{AA}} (\rho F_2)_{\mathrm{AA}} = \Bigg\langle \left(1 - \mathrm{e}^{-\left(\rho(-\ln(1-\mu_1^t))^M + \rho(-\ln(1-\mu_2^t))^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}},$

$\left(\mathrm{e}^{-\left(\rho(-\ln(\eta_1^t+\nu_1^t))^M + \rho(-\ln(\eta_2^t+\nu_2^t))^M\right)^{\frac{1}{M}}} - \mathrm{e}^{-\left(\rho(-\ln \nu_1^t)^M + \rho(-\ln \nu_2^t)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\rho(-\ln \nu_1^t)^M + \rho(-\ln \nu_2^t)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}} \Bigg\rangle.$

Hence, $(\rho(F_1 \oplus_{\mathrm{AA}} F_2))_{\mathrm{AA}} = (\rho F_1)_{\mathrm{AA}} \oplus_{\mathrm{AA}} (\rho F_2)_{\mathrm{AA}}$, which completes the proof of (3).

Similarly, we can prove that $((F_1 \otimes_{\mathrm{AA}} F_2)^{\rho})_{\mathrm{AA}} = (F_1^{\rho})_{\mathrm{AA}} \otimes_{\mathrm{AA}} (F_2^{\rho})_{\mathrm{AA}}$. □

Appendix C. Proof of Theorem 4

Proof. (By mathematical induction) When $n = 2$, we have

T-SFIAAWA$(F_1, F_2) = (\omega_1 F_1)_{\mathrm{AA}} \oplus_{\mathrm{AA}} (\omega_2 F_2)_{\mathrm{AA}}$

$$= \Bigg\langle \left(1 - \mathrm{e}^{-\left(\omega_1(-\ln(1-\mu_1^t))^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\omega_1(-\ln(\eta_1^t+\nu_1^t))^M\right)^{\frac{1}{M}}} - \mathrm{e}^{-\left(\omega_1(-\ln(\nu_1^t))^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\omega_1(-\ln(\nu_1^t))^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}} \Bigg\rangle$$

$$\oplus_{\mathrm{AA}} \Bigg\langle \left(1 - \mathrm{e}^{-\left(\omega_2(-\ln(1-\mu_2^t))^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\omega_2(-\ln(\eta_2^t+\nu_2^t))^M\right)^{\frac{1}{M}}} - \mathrm{e}^{-\left(\omega_2(-\ln(\nu_2^t))^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\omega_2(-\ln(\nu_2^t))^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}} \Bigg\rangle$$

$$= \Bigg\langle \left(1 - \mathrm{e}^{-\left(\sum_{j=1}^{2} \omega_j(-\ln(1-\mu_j^t))^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\sum_{j=1}^{2} \omega_j(-\ln(\eta_j^t+\nu_j^t))^M\right)^{\frac{1}{M}}} - \mathrm{e}^{-\left(\sum_{j=1}^{2} \omega_j(-\ln(\nu_j^t))^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\sum_{j=1}^{2} \omega_j(-\ln(\nu_j^t))^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}} \Bigg\rangle.$$

Therefore, when $n = 2$, by Theorem 2, we know that the value of T-SFIAAWA(F_1, F_2) is a T-SFN. Now, assume that when $n = k$,

T-SFIAAWA$(F_1, F_2, \ldots, F_k)$

$$= \Bigg\langle \left(1 - \mathrm{e}^{-\left(\sum_{j=1}^{k} \omega_j\left(-\ln(1-\mu_j^t)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\sum_{j=1}^{k} \omega_j\left(-\ln(\eta_j^t+\nu_j^t)\right)^M\right)^{\frac{1}{M}}} - \mathrm{e}^{-\left(\sum_{j=1}^{k} \omega_j\left(-\ln(\nu_j^t)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\sum_{j=1}^{k} \omega_j\left(-\ln(\nu_j^t)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}} \Bigg\rangle.$$

Then T-SFIAAWA$(F_1, F_2, \ldots, F_{k+1}) = \left(\overset{n}{\underset{i=1}{\oplus}}{}_{\mathrm{AA}}\left(\omega_j F_j\right)_{\mathrm{AA}}\right) \oplus_{\mathrm{AA}} (\omega_{k+1} F_{k+1})_{\mathrm{AA}}$

$$= \Bigg\langle \left(1 - \mathrm{e}^{-\left(\sum_{j=1}^{k} \omega_j\left(-\ln(1-\mu_j^t)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\sum_{j=1}^{k} \omega_j\left(-\ln(\eta_j^t+\nu_j^t)\right)^M\right)^{\frac{1}{M}}} - \mathrm{e}^{-\left(\sum_{j=1}^{k} \omega_j\left(-\ln(\nu_j^t)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\sum_{j=1}^{k} \omega_j\left(-\ln(\nu_j^t)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}} \Bigg\rangle$$

$$\oplus_{\mathrm{AA}}\Bigg\langle\left(1-\mathrm{e}^{-\left(\omega_{k+1}\left(-\ln(1-\mu_{k+1}^t)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\omega_{k+1}\left(-\ln(\eta_{k+1}^t+\nu_{k+1}^t)\right)^M\right)^{\frac{1}{M}}} - \mathrm{e}^{-\left(\omega_{k+1}\left(-\ln(\nu_{k+1}^t)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\omega_{k+1}\left(-\ln(\nu_{k+1}^t)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}\Bigg\rangle$$

$$= \Bigg\langle \left(1 - \mathrm{e}^{-\left(\sum_{j=1}^{k+1} \omega_j\left(-\ln(1-\mu_j^t)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\sum_{j=1}^{k+1} \omega_j\left(-\ln(\eta_j^t+\nu_j^t)\right)^M\right)^{\frac{1}{M}}} - \mathrm{e}^{-\left(\sum_{j=1}^{k+1} \omega_j\left(-\ln(\nu_j^t)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\sum_{j=1}^{k+1} \omega_j\left(-\ln(\nu_j^t)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}} \Bigg\rangle.$$

By mathematical induction, we can obtain

T-SFIAAWA$(F_1, F_2, \ldots, F_n)$

$$= \Bigg\langle \left(1 - \mathrm{e}^{-\left(\sum_{j=1}^{n} \omega_j\left(-\ln(1-\mu_j^t)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\sum_{j=1}^{n} \omega_j\left(-\ln(\eta_j^t+\nu_j^t)\right)^M\right)^{\frac{1}{M}}} - \mathrm{e}^{-\left(\sum_{j=1}^{n} \omega_j\left(-\ln(\nu_j^t)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\sum_{j=1}^{n} \omega_j\left(-\ln(\nu_j^t)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}} \Bigg\rangle.$$

According to Theorem 3, the result of aggregating several T-SFNs using the T-SFIAAWA operator remains a T-SFN. □

Appendix D. Proof of Theorem 5

Proof. Since $F_j = \langle \mu, \eta, \nu \rangle$ for all j, substituting into the T-SFIAAWA operator from Theorem 4, we have

T-SFIAAWA$(F_1, F_2, \ldots, F_n)$

$$= \left\langle \left(1 - \mathrm{e}^{-\left(\sum_{j=1}^{n} \omega_j\left(-\ln(1-\mu^t)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\sum_{j=1}^{n} \omega_j\left(-\ln(\eta^t+\nu^t)\right)^M\right)^{\frac{1}{M}}} - \mathrm{e}^{-\left(\sum_{j=1}^{n} \omega_j\left(-\ln \nu^t\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\sum_{j=1}^{n} \omega_j\left(-\ln \nu^t\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}} \right\rangle$$

$$= \left\langle \left(1 - \mathrm{e}^{-\left(\left(-\ln(1-\mu^t)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\left(-\ln(\eta^t+\nu^t)\right)^M\right)^{\frac{1}{M}}} - \mathrm{e}^{-\left(\left(-\ln \nu^t\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\left(-\ln \nu^t\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}} \right\rangle$$

$$= \left\langle \left(1 - \mathrm{e}^{-\left(-\ln(1-\mu^t)\right)}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(-\ln(\eta^t+\nu^t)\right)} - \mathrm{e}^{-\left(-\ln \nu^t\right)}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(-\ln \nu^t\right)}\right)^{\frac{1}{t}} \right\rangle$$

$$= \left\langle (1 - (1 - \mu^t))^{\frac{1}{t}}, ((\eta^t + \nu^t) - \nu^t)^{\frac{1}{t}}, (\nu^t)^{\frac{1}{t}} \right\rangle = \langle \mu, \eta, \nu \rangle. \qquad \square$$

Appendix E. Proof of Theorem 6

Proof. According to Theorem 3, we have

T-SFIAAWA$(G_1, G_2, \ldots, G_n)$

$$= \left\langle \left(1-\mathrm{e}^{-\left(\sum_{j=1}^{n} \omega_j\left(-\ln\left(1-\mu_{G_j}^t\right)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\sum_{j=1}^{n} \omega_j\left(-\ln\left(\eta_{G_j}^t+\nu_{G_j}^t\right)\right)^M\right)^{\frac{1}{M}}} - \mathrm{e}^{-\left(\sum_{j=1}^{n} \omega_j\left(-\ln(\nu_{G_j}^t)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\sum_{j=1}^{n} \omega_j\left(-\ln(\nu_{G_j}^t)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}} \right\rangle$$

$\triangleq \langle \widetilde{\mu}, \widetilde{\eta}, \widetilde{\nu} \rangle$.

T-SFIAAWA$(L_1, L_2, \ldots, L_n)$

$$= \left\langle \left(1-\mathrm{e}^{-\left(\sum_{j=1}^{n} \omega_j\left(-\ln\left(1-\mu_{L_j}^t\right)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\sum_{j=1}^{n} \omega_j\left(-\ln\left(\eta_{L_j}^t+\nu_{L_j}^t\right)\right)^M\right)^{\frac{1}{M}}} - \mathrm{e}^{-\left(\sum_{j=1}^{n} \omega_j\left(-\ln(\nu_{L_j}^t)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}, \left(\mathrm{e}^{-\left(\sum_{j=1}^{n} \omega_j\left(-\ln(\nu_{L_j}^t)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}} \right\rangle$$

$\triangleq \langle \widehat{\mu}, \widehat{\eta}, \widehat{\nu} \rangle$.

Since $\mu_{G_j} \le \mu_{L_j}$, the function $\left(1 - \mathrm{e}^{-\left(\sum_{j=1}^{n} \omega_j\left(-\ln\left(1-\mu_j^t\right)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}$ is strictly monotonically increasing with respect to μ_j, so $\widetilde{\mu} \le \widehat{\mu}$. Similarly, since $\nu_{G_j} \ge \nu_{L_j}$, and $\left(\mathrm{e}^{-\left(\sum_{j=1}^{n} \omega_j\left(-\ln(\nu_j^t)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}}$ is strictly monotonically increasing with respect to ν_j, we have $\widetilde{\nu} \ge \widehat{\nu}$.

To prove $\langle \widetilde{\mu}, \widetilde{\eta}, \widetilde{\nu} \rangle \preccurlyeq_S \langle \widehat{\mu}, \widehat{\eta}, \widehat{\nu} \rangle$, we consider the following three cases:

(1) If there exists $1 \le j_0 \le n$ such that $\mu_{G_{j_0}} < \mu_{L_{j_0}}$, then,

$$\widetilde{\mu} = \left(1 - \mathrm{e}^{-\left(\sum_{j=1}^{n} \omega_j\left(-\ln\left(1-\mu_{G_j}^t\right)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}} < \left(1 - \mathrm{e}^{-\left(\sum_{j=1}^{n} \omega_j\left(-\ln\left(1-\mu_{L_j}^t\right)\right)^M\right)^{\frac{1}{M}}}\right)^{\frac{1}{t}} = \widehat{\mu},$$

which implies that $S(\langle\tilde{\mu},\widetilde{\eta},\widetilde{\nu}\rangle) = \widetilde{\mu} - \widetilde{\nu} < \widehat{\mu} - \widetilde{\nu} \leq \widehat{\mu} - \widehat{\nu} = S(\langle\widehat{\mu},\widehat{\eta},\widehat{\nu}\rangle)$ based on Eq (3.3). Therefore,

$$\text{T-SFIAAWA}(G_1, G_2, \ldots, G_n) = \langle\widetilde{\mu},\widetilde{\eta},\widetilde{\nu}\rangle \prec_S \langle\widehat{\mu},\widehat{\eta},\widehat{\nu}\rangle = \text{T-SFIAAWA}(L_1, L_2, \ldots, L_n).$$

(2) If there exists $1 \leq j_0 \leq n$ such that $\nu_{G_{j_0}} > \nu_{L_{j_0}}$, then,

$$\widetilde{\nu} = \left(e^{-\left(\sum_{j=1}^{n} \omega_j \left(-\ln(\nu^t_{G_j})\right)^M\right)^{\frac{1}{M}}} \right)^{\frac{1}{t}} > \left(e^{-\left(\sum_{j=1}^{n} \omega_j \left(-\ln(\nu^t_{L_j})\right)^M\right)^{\frac{1}{M}}} \right)^{\frac{1}{t}} = \widehat{\nu},$$

which implies that $S(\langle\widetilde{\mu},\widetilde{\eta},\widetilde{\nu}\rangle) = \widetilde{\mu} - \widetilde{\nu} \leq \widehat{\mu} - \widetilde{\nu} < \widehat{\mu} - \widehat{\nu} = S(\langle\widehat{\mu},\widehat{\eta},\widehat{\nu}\rangle)$. Therefore,

$$\text{T-SFIAAWA}(G_1, G_2, \ldots, G_n) = \langle\widetilde{\mu},\widetilde{\eta},\widetilde{\nu}\rangle \prec_S \langle\widehat{\mu},\widehat{\eta},\widehat{\nu}\rangle = \text{T-SFIAAWA}(L_1, L_2, \ldots, L_n).$$

(3) If $\mu_{G_j} = \mu_{L_j}$ and $\nu_{G_j} = \nu_{L_j}$, $(j = 1, 2, \ldots, n)$, then $\widetilde{\mu} = \widehat{\mu}$ and $\widetilde{\nu} = \widehat{\nu}$, therefore $S(\langle\widetilde{\mu},\widetilde{\eta},\widetilde{\nu}\rangle) = S(\langle\widehat{\mu},\widehat{\eta},\widehat{\nu}\rangle)$ and $A_c(\langle\widetilde{\mu},\widetilde{\eta},\widetilde{\nu}\rangle) = A_c(\langle\widehat{\mu},\widehat{\eta},\widehat{\nu}\rangle)$. Therefore, when $\eta_{G_j} \leq \eta_{L_j}$, we have

$$\left(e^{-\left(\sum_{j=1}^{n} \omega_j \left(-\ln(\eta_{G_j}+\nu^t_{G_j})\right)^M\right)^{\frac{1}{M}}} - e^{-\left(\sum_{j=1}^{n} \omega_j \left(-\ln(\nu^t_{G_j})\right)^M\right)^{\frac{1}{M}}} \right)^{\frac{1}{t}} \leq \left(e^{-\left(\sum_{j=1}^{n} \omega_j \left(-\ln(\eta_{L_j}+\nu^t_{L_j})\right)^M\right)^{\frac{1}{M}}} - e^{-\left(\sum_{j=1}^{n} \omega_j \left(-\ln(\nu^t_{L_j})\right)^M\right)^{\frac{1}{M}}} \right)^{\frac{1}{t}},$$

then $\widetilde{\eta} \leq \widehat{\eta}$. So, $A_d(\langle\widetilde{\mu},\widetilde{\eta},\widetilde{\nu}\rangle) = \widetilde{\mu} + \widetilde{\eta} + \widetilde{\nu} \leq \widehat{\mu} + \widehat{\eta} + \widehat{\nu} = A_d(\langle\widehat{\mu},\widehat{\eta},\widehat{\nu}\rangle)$. Hence,

$$\text{T-SFIAAWA}(G_1, G_2, \ldots, G_n) = \langle\widetilde{\mu},\widetilde{\eta},\widetilde{\nu}\rangle \preccurlyeq_S \langle\widehat{\mu},\widehat{\eta},\widehat{\nu}\rangle = \text{T-SFIAAWA}(L_1, L_2, \ldots, L_n).$$

□

Appendix F. Proof of Theorem 7

Proof. For convenience, let $F_{Min} = \left\langle \min_{1\leq j\leq n}\{\mu_j\}, \min_{1\leq j\leq n}\{\eta_j\}, \max_{1\leq j\leq n}\{\nu_j\} \right\rangle \triangleq \langle\widetilde{\mu}_1,\widetilde{\eta}_1,\widetilde{\nu}_1\rangle$, $F_{Max} = \left\langle \max_{1\leq j\leq n}\{\mu_j\}, \left(1 - \max_{1\leq j\leq n}\{\mu^t_j\} - \min_{1\leq j\leq n}\{\nu^t_j\}\right)^{\frac{1}{t}}, \min_{1\leq j\leq n}\{\nu_j\} \right\rangle \triangleq \langle\widetilde{\mu}_2,\widetilde{\eta}_2,\widetilde{\nu}_2\rangle$. Thus, both $F_{Min}, F_{Max} \in \Phi$.

Since $\widetilde{\mu}_1 \leq \mu_j$, $\widetilde{\eta}_1 \leq \eta_j$, and $\widetilde{\nu}_1 \geq \nu_j$, for any j. Then $F_{Min} \subseteq F_j$. Hence, based on Theorems 5 and 6

$$F_{Min} = \text{T-SFIAAWA}(F_{Min}, F_{Min}, \ldots, F_{Min}) \preccurlyeq_S \text{T-SFIAAWA}(F_1, F_2, \ldots, F_n).$$

In addition, $\mu_j \leq \widetilde{\mu}_2$ and $\nu_j \geq \widetilde{\nu}_2$ for any j.

If there exists $1 \leq j_0 \leq n$, such that $\mu_{j_0} < \widetilde{\mu}_2$, or $\nu_{j_0} > \widetilde{\nu}_2$, then,

$$\text{T-SFIAAWA}(F_1, F_2, \ldots, F_{j_0}, \ldots, F_n) \prec_S \text{T-SFIAAWA}(F_{Max}, F_{Max}, ..., F_{Max})$$

based on the proof of Theorem 6.

If $\mu_j = \widetilde{\mu}_2$ and $\nu_j = \widetilde{\nu}$, for any j, then $S(F_j) = S(F_{Max})$ and $A_c(F_j) = A_c(F_{Max})$. Since $\mu^t_j + \eta^t_j + \nu^t_j \leq 1$, it follows that $\eta_j \leq \left(1 - \mu^t_2 - \nu^t_2\right)^{\frac{1}{t}} = \left(1 - \widetilde{\mu}^t_2 - \widetilde{\nu}^t_2\right)^{\frac{1}{t}} = \widetilde{\eta}_2$, for any j. Therefore, $A_d(F_j) \leq A_d(F_{Max})$. Thus, $F_j \subseteq F_{Max}$. Hence,

$$\text{T-SFIAAWA}(F_1, F_2, \ldots, F_n) \preccurlyeq_S \text{T-SFIAAWA}(F_{Max}, F_{Max}, \ldots, F_{Max}) = F_{Max}.$$

□

Appendix G. Proof of Theorem 8

Proof. By the basic operations in Definition 12 and mathematical induction, we verify that Eq (4.6) holds directly for $n = 2$. Assume it holds for $n = k$. Then for $n = k + 1$,

$$\bigotimes_{j=1}^{k+1}{}_{\mathrm{AA}}(F_j^{\omega_j})_{\mathrm{AA}} = \left(\bigotimes_{j=1}^{k}{}_{\mathrm{AA}}(F_j^{\omega_j})_{\mathrm{AA}}\right) \otimes_{\mathrm{AA}} (F_{k+1}^{\omega_{k+1}})_{\mathrm{AA}}.$$

Substituting the induction hypothesis and the product operation from Definition 12 yields Eq (4.6). The closedness of the result follows directly from Theorem 2. □

Appendix H. Proof of Theorems 9–11

Proof. The proofs are analogous to those of Theorems 5–7 for the T-SFIAAWA operator, with the weighted average replaced by the weighted geometric. Idempotency holds because substituting identical T-SFNs $F_j = \langle \mu, \eta, \nu \rangle$ into Eq (4.6) returns the same triple. For monotonicity, if $G_j \subseteq L_j$ for all j, that is $\mu_{G_j} \leq \mu_{L_j}$, $\eta_{G_j} \leq \eta_{L_j}$, and $\nu_{G_j} \geq \nu_{L_j}$, then the aggregated μ and η components are increasing with respect to μ_j and η_j, and the aggregated ν component is decreasing with respect to ν_j. Hence, the order $\preccurlyeq_S$ is preserved. Boundedness follows directly from monotonicity and idempotency. □