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Research article

The Kuralay-Ma-Myrzakulov equation and the Akbota–Myrzakulov-Tolkynay-Zhaidary equation: Integrability, equivalence, reductions, and solutions

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Abstract: The Kuralay-Ma-Myrzakulov equation (KMME) and the Akbota- Myrzakulov-Tolkynay-Zhaidary equation (AMTZE) are studied. The integrability of these two equations is achieved by the existence of corresponding Lax pairs. The corresponding Lax representations are presented. Gauge equivalence between these two integrable equations is proved. It is shown that the KMME admits two integrable reductions, namely, the Manukure-Zhanbota equation and the Manukure-Zhaidary equation. Similarly, the AMTZE has two integrable reductions, namely, the Kairat-Kuralay-Myrzakulov-Shynaray equation (KKMSE) and the Wu-Zhang equation (WZE). From these results, it follows that the Manukure- Zhanbota equation is gauge equivalent to the KKMSE. At the same time, the Manukure-Zhaidary equation and the Wu-Zhang equation are gauge equivalent to each other. Some exact traveling wave solutions of the AMTZE are presented. These solutions demonstrate a variety of structures, including Jacobi elliptic, trigonometric, soliton, and rational forms. The results are illustrated through 3D and contour plots, which clearly depict the system's behavior during momentum propagation and help identify suitable parameter values. This graphical analysis offers important insights into the properties and dynamics of the soliton solutions derived from the integrable AMTZE equation.

Keywords: Kuralay-Ma-Myrzakulov equation; Akbota-Myrzakulov-Tolkynay-Zhaidar equation; Manukure-Zhanbota equation; Wu-Zhang equation; Kairat-Kuralay-Myrzakulov-Shynaray equation
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1. Introduction

There are many nonlinear physical phenomena in nature that are described by nonlinear systems of partial differential equations (PDEs) [1–3]. Some of these nonlinear PDEs are integrable but others nonintegrable [4–6]. For that reason in modern mathematical and theoretical physics, one of the most important topics is to find the integrable nonlinear PDEs [7–9]. Another interesting topic is to find the exact solutions of these differential equations. In modern mathematical and theoretical physics, one of the most important topics is to find the exact solutions of nonlinear differential equations [10–12]. A large number of useful methods have been proposed to construct soliton solutions and locally coherent structure solutions for nonlinear PDEs [13–15]. Some of the most important methods are the inverse scattering transformation, the bilinear form, symmetry reduction, the Darboux transformation, the Painlevė analysis method, the Backlund transformation [16–18], the separated variable method, etc. Nowadays, with rapid development of symbolic computation systems, the search for the exact solutions of nonlinear systems of PDEs has attracted a lot of attention; because the exact solutions make it possible to explore nonlinear physical phenomena comprehensively and facilitate testing the numerical schemes. In recent years, a variety of approaches have been proposed and applied to the nonlinear systems of PDEs, such as the modified extended *tanh* function method, first integral method, extended *tanh* function method, *exp*-function method, and so on [13, 14, 19].

The outline of the present paper is organized as follows. In Section 2, we present the KMME. In Section 3, the Akbota-Myrzakulov-Tolkynay-Zhaidary equation (AMTZE) and its Lax pair are considered. In Section 4, the gauge equivalence between the AMTZE and the KMME is established. Some reductions of the AMTZE and the KMME are found in Sections 5 and 6. The traveling wave solutions of the AMTZE are constructed in Section 7. The Hamiltonian structure of the AMTZE is considered in Section 8. The paper is concluded by some comments and remarks in Section 9.

2. The Kuralay-Ma-Myrzakulov equation

One of the most interesting and important integrable equations in 2 + 1 dimensions is the KMME. In this paper, we study some properties of this KMME.

2.1. Equation

The KMME is given by [20]

$$Z_t - \frac{\beta}{b} f_{xy} Z + \frac{\beta}{b} f_y Z_x = 0, \tag{2.1}$$

$$f_{xxy} + \frac{b}{4a^3\beta} tr\left([Z_{xt}, Z_{xx}] Z_x\right) = 0, \tag{2.2}$$

where $Z(x, y, t)$ is a 2×2 matrix function, $f(x, y, t)$ is a scalar function and (a, b, β) are some constants. Let us we introduce two new functions $q(x, y, t)$ and $u(x, y, t)$ as

$$\begin{aligned} q &= 2f_x, \\ u &= -\beta f_y. \end{aligned}$$

Then, the KMME (2.1) and (2.2) can be rewritten as

$$Z_t - \frac{\beta}{2b} q_y Z - \frac{1}{b} u Z_x = 0, \tag{2.3}$$

$$q_{xy} + \frac{b}{2a^3\beta} tr\left([Z_{xt}, Z_{xx}] Z_x\right) = 0, \tag{2.4}$$

$$u_x + 0.5\beta q_y = 0. \tag{2.5}$$

Thus, we have two equivalent forms of the KMME. One more, the third form of the KMME, is given by

$$\left(\frac{1}{\beta f_{xxy}}(S_t - \frac{u}{b} S_x)\right)_t - \frac{f_{xy}}{b f_{xxy}}(S_t - \frac{u}{b} S_x) + \frac{\beta}{b^2} f_y S = 0,$$

$$f_{xxy} - \frac{b}{4\beta} tr\left([S_t, S_x] S\right) = 0,$$

where (about our notations, see below)

$$S = g^{-1}\sigma_3 g.$$

These three forms of the KMME are equivalent to each other. As we can see in Section 4, the KMME and the AMTZE are gauge equivalent to each other.

2.2. *Lax representation*

As we mentioned above, the KMME (2.1) and (2.2) is integrable. The corresponding Lax representation is given by [20]

$$\Psi_x = U_1 \Psi, \tag{2.6}$$

$$\Psi_t = \beta\lambda \Psi_y + C\Psi, \tag{2.7}$$

where

$$U_1 = (b\lambda^2 + q\lambda) Z, \tag{2.8}$$

$$C = \lambda^2 C_2 + \lambda C_1. \tag{2.9}$$

Here,

$$C_2 = uZ, \quad C_1 = \frac{uq}{b} Z + \beta g^{-1} g_y.$$

The compatibility condition of the systems (2.6) and (2.7)

$$\Psi_{xt} = \Psi_{tx},$$

that is,

$$U_{1t} - C_x + [U_1, C] - \beta\lambda U_{1y} = 0, \tag{2.10}$$

is equivalent to the KMME (2.1) and (2.2). In fact, substituting expressions (2.8) and (2.9) into Eq (2.10) and equating the coefficients of the same power of λ to be zero, we can get the KMME (2.1) and (2.2).

3. The Akbota-Myrzakulov-Tolkynay-Zhaidary equation

The well-known AMTZE plays an important role among the integrable equations in 2+1 dimensions. We are now going to consider this AMTZE in more detail.

3.1. Equation

Consider the well-known AMTZE. The AMTZE reads as

$$2f_{xt} + \frac{2\beta}{b}f_y f_{xx} + \frac{4\beta}{b}f_x f_{xy} - \beta r_y = 0, \tag{3.1}$$

$$r_t + \frac{\beta}{b}f_y r_x + \frac{2\beta}{b}r f_{xy} - \frac{\beta}{2ab}f_{xxxy} = 0. \tag{3.2}$$

We can rewrite this AMTZE in the following equivalent form:

$$q_t - \frac{1}{b}uq_x + \frac{\beta}{b}qq_y - \beta r_y = 0, \tag{3.3}$$

$$r_t - \frac{1}{b}ur_x + \frac{\beta}{b}rq_y - \frac{\beta}{4ab}q_{xxy} = 0, \tag{3.4}$$

$$u_x + \frac{\beta}{2}q_y = 0, \tag{3.5}$$

where a, b, β are real constants and (q, r, u, f) are some functions of (x, t, y). Some exacts soliton solutions and other properties of the AMTZE were investigated in the paper [21].

3.2. Lax representation

Note that the AMTZE (3.3)–(3.5) is integrable. Its Lax representation is given by

$$\Phi_x = U_2\Phi, \tag{3.6}$$

$$\Phi_t = \beta\lambda\Phi_y + B\Phi, \tag{3.7}$$

where

$$U_2 = \begin{pmatrix} 0 & a \\ b\lambda^2 + q\lambda + r & 0 \end{pmatrix} = (b\lambda^2 + q\lambda)\Sigma + Q, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$B = B_2\lambda^2 + B_1\lambda + B_0, \quad \Sigma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & a \\ r & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} = u\Sigma, \quad B_1 = \begin{pmatrix} 0 & 0 \\ b^{-1}uq & 0 \end{pmatrix} = b^{-1}uq\Sigma,$$

$$B_0 = \begin{pmatrix} \frac{\beta}{4b}q_y & ab^{-1}u \\ b^{-1}ur + \frac{\beta}{4ab}q_{xy} & -\frac{\beta}{4b}q_y \end{pmatrix} = \frac{\beta}{4b}q_y\sigma_3 + \frac{u}{b}Q + \frac{\beta}{4ab}q_{xy}\Sigma.$$

The compatibility condition $\Phi_{xt} = \Phi_{tx}$ of the linear Eqs (3.6) and (3.7), that is,

$$U_{2t} - B_x + [U_2, B] - \beta\lambda U_{2y} = 0,$$

gives the AMTZE (3.3)–(3.5). As the integrable equation, the ATZME (3.3)–(3.5) has the N-soliton solution, infinite number of conservation laws, Hamiltonian structure, and so on.

4. Gauge equivalence between the KMME and the AMTZE

As integrable equations, the above considered KMME and AMTZE have the gauge equivalent counterparts. In this section, we try to prove that these two equations are gauge equivalent to each other. Consider the transformation

$$\Psi = g^{-1}\Phi,$$

where Φ is a solution of the set of Eqs (3.6) and (3.7) and $g(x, y, t) = \Phi|_{\lambda=0}$. Then the new matrix function $\Psi(x, y, t)$ satisfies the set of Eqs (2.6) and (2.7). We have

$$\begin{aligned} U_1 &= g^{-1}U_2 g - g^{-1}g_x, \\ C &= g^{-1}Bg - g^{-1}B_0 g + \beta\lambda g^{-1}g_y. \end{aligned}$$

This result means that the KMME and AMTZE are gauge equivalent to each other. So, we have proved that between these two equations, gauge equivalence takes place.

Let us now we present some useful formulas that follow from the gauge equivalence between the AMTZE and KMME. We have

$$\begin{aligned} \Psi &= g^{-1}\Phi, \quad Q = \begin{pmatrix} 0 & a \\ r & 0 \end{pmatrix}, \\ \Psi_x &= (g^{-1}\Phi)_x = g^{-1}(\Phi_x - g_x g^{-1}\Phi) = g^{-1}(U_2 - U_{20})\Phi = U_1\Psi, \\ \Psi_x &= (b\lambda^2 + \lambda q)Z\Psi, \quad U_{20} = U_2|_{\lambda=0} = Q, \\ \Psi_t &= \beta\lambda\Psi_y + g^{-1}(\beta\lambda g_y g^{-1} + B - B_0)g\Psi, \\ \Psi_t &= \beta\lambda\Psi_y + g^{-1}(\beta\lambda g_y g^{-1} + \lambda^2 B_2 + \lambda B_1)g\Psi, \\ \Psi_t &= \beta\lambda\Psi_y + C\Psi, \quad C_2 = g^{-1}B_2 g, \\ C_1 &= g^{-1}B_1 g + \beta g^{-1}g_y, \\ g_x &= U_{20}g = Qg, \\ g_t &= B_0 g. \end{aligned}$$

Here,

$$\begin{aligned} Z &= \frac{ab}{u_{xx}}\left(S_t - \frac{u}{b}S_x\right), \\ S_x &= 2g^{-1}\begin{pmatrix} 0 & a \\ -r & 0 \end{pmatrix}g, \\ S_t &= \frac{u}{b}S_x + \frac{u_{xx}}{ab}Z, \\ N_2 &= bZ, \quad N_1 = qZ, \quad E_2 = uZ, \\ E_1 &= \frac{uq}{b}Z - \frac{\beta u_x}{2b}S + \frac{\beta u}{4b}[S, S_x] - \frac{\beta u_{xx}}{2ab}Z, \end{aligned}$$

and

$$Z = g^{-1}\Sigma g, \quad [Z, S] = 2Z,$$

$$Z^2 = 0, \quad [Z, [S, S_x]] = -4S,$$

$$Z_x = -aS.$$

5. Reductions of the Kuralay-Ma-Myrzakulov equation

In this section, the KMME and the AMTZE are studied from the viewpoint of integrable nonlinear evolution equations.

Below, we present three reductions of the KMME.

5.1. The complex KMME

First, let us consider the complex KMME, when we assume that $b = i$. Then the KMME takes the form

$$\begin{aligned} iZ_t - \frac{\beta}{2} q_y Z - uZ_x &= 0, \\ q_{xy} + \frac{i}{2a^3\beta} tr\left([Z_{xt}, Z_{xx}]Z_x\right) &= 0, \\ u_x + 0.5\beta q_y &= 0. \end{aligned}$$

It is the so-called complex KMME.

5.2. The Manukure-Zhanbota equation

Let us consider the case when $y = t$. Then the KMME takes the form

$$Z_t - \frac{\beta}{b} f_{xt} Z + \frac{\beta}{b} f_t Z_x = 0, \tag{5.1}$$

$$f_{xxt} + \frac{b}{4a^3\beta} tr\left([Z_{xt}, Z_{xx}]Z_x\right) = 0. \tag{5.2}$$

It is the Manukure-Zhanbota equation. At the same time, Eqs (2.3)–(2.5) follow the other equivalent form of the Manukure-Zhanbota equation:

$$Z_t - \frac{\beta}{2b} q_t Z - \frac{1}{b} uZ_x = 0, \tag{5.3}$$

$$q_{xt} + \frac{b}{2a^3\beta} tr\left([Z_{xt}, Z_{xx}]Z_x\right) = 0, \tag{5.4}$$

$$u_x + 0.5\beta q_t = 0, \tag{5.5}$$

where

$$\begin{aligned} q &= 2f_x, \\ u &= -\beta f_t. \end{aligned}$$

Thus, in this subsection, two equivalent forms of the Manukure-Zhanbota equation are presented. Note that both of these forms are integrable.

5.3. The Manukure-Zhaidary equation

Our second reduction corresponds to the case when $y = x$. Then the KMME (2.1) and (2.2) takes the form

$$Z_t - \frac{\beta}{b} f_{xx} Z + \frac{\beta}{b} f_x Z_x = 0, \tag{5.6}$$

$$f_{xxx} + \frac{b}{4a^3\beta} tr\left([Z_{xt}, Z_{xx}]Z_x\right) = 0. \quad (5.7)$$

It is the so-called the Manukure-Zhaidary equation. The other equivalent form of the Manukure-Zhaidary equation reads as

$$Z_t - \frac{\beta}{2b} q_x Z - \frac{1}{b} u Z_x = 0, \quad (5.8)$$

$$q_{xx} + \frac{b}{2a^3\beta} tr\left([Z_{xt}, Z_{xx}]Z_x\right) = 0. \quad (5.9)$$

Thus, in this subsection, we presented two equivalent forms of the Manukure-Zhaidary equation.

6. Reductions of the Akbota-Myrzakulov-Tolkynay-Zhaidary equation

In this section, we consider two reductions of the AMTZE (3.1) and (3.2).

6.1. The complex AMTZE

One of the interesting reductions of the AMTZE we get is when we consider the case $b = i$. In this case, the AMTZE takes the form

$$iq_t - uq_x + \beta q q_y - i\beta r_y = 0,$$

$$ir_t - ur_x + \beta r q_y - \frac{\beta}{4a} q_{xxy} = 0,$$

$$u_x + \frac{\beta}{2} q_y = 0.$$

It is the complex AMTZE. Note that in this case, $b = i$, the functions q, r, u become the complex functions, and the AMTZE has the more complicated solutions with complex functions.

6.2. The Kairat-Kuralay-Myrzakulov-Shynaray equation

6.2.1. The real KKMSE

First, let us consider the case when $y = t$. Then the AMTZE takes the form

$$2f_{xt} + \frac{2\beta}{b} f_t f_{xx} + \frac{4\beta}{b} f_x f_{xt} - \beta r_t = 0, \quad (6.1)$$

$$r_t + \frac{\beta}{b} f_t r_x + \frac{2\beta}{b} r f_{xt} - \frac{\beta}{2ab} f_{xxxt} = 0. \quad (6.2)$$

It is the well-known KKMSE as

$$q_t - \frac{1}{b} u q_x + \frac{\beta}{b} q q_t - \beta r_t = 0, \quad (6.3)$$

$$r_t - \frac{1}{b} u r_x + \frac{\beta}{b} r q_t - \frac{\beta}{4ab} q_{xxt} = 0, \quad (6.4)$$

$$u_x + \frac{\beta}{2} q_t = 0. \quad (6.5)$$

6.2.2. The complex KKMSE

If we assume that $b = i$, then the KKMSE takes the form

$$iq_t - uq_x + \beta qq_t - i\beta r_t = 0,$$
$$ir_t - ur_x + \beta rq_t - \frac{\beta}{4a}q_{xxt} = 0,$$
$$u_x + \frac{\beta}{2}q_t = 0.$$

It is the complex KKMSE, since in this case, the functions q, r, u become complex functions.

6.2.3. Lax representation

Note that the KKMSE (6.1) and (6.2) is integrable. Its Lax representation is given by:

$$\Phi_x = U_2\Phi, \quad (6.6)$$
$$\Phi_t = \frac{1}{1-\beta\lambda}B\Phi, \quad (6.7)$$

where

$$U_2 = \begin{pmatrix} 0 & a \\ b\lambda^2 + q\lambda + r & 0 \end{pmatrix} = (b\lambda^2 + q\lambda)\Sigma + Q, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$B = B_2\lambda^2 + B_1\lambda + B_0, \quad \Sigma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & a \\ r & 0 \end{pmatrix},$$
$$B_2 = \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} = u\Sigma, \quad B_1 = \begin{pmatrix} 0 & 0 \\ b^{-1}uq & 0 \end{pmatrix} = b^{-1}uq\Sigma,$$
$$B_0 = \begin{pmatrix} \frac{\beta}{4b}q_t & ab^{-1}u \\ b^{-1}ur + \frac{\beta}{4ab}q_{xt} & -\frac{\beta}{4b}q_t \end{pmatrix} = \frac{\beta}{4b}q_t\sigma_3 + \frac{u}{b}Q + \frac{\beta}{4ab}q_{xt}\Sigma.$$

The compatibility condition $\Phi_{xt} = \Phi_{tx}$ of the linear Eqs (6.6) and (6.7) gives the KKMSE (6.3)–(6.5). As the integrable equation, the KKMSE (6.3)–(6.5) has the N-soliton solution, infinite number of conservation laws, Hamiltonian structure, and so on. The gauge equivalent of the KKMSE is the Manukure-Zhanbota equations (5.3)–(5.5) or (5.1)–(5.2).

6.3. The Wu-Zhang equation

6.3.1. The real WZE

The second reduction of the AMTZE follows from the case when $y = x$. Then the AMTZE (3.1) and (3.2) takes the form

$$2f_{xt} + \frac{2\beta}{b}f_xf_{xx} + \frac{4\beta}{b}f_xf_{xx} - \beta r_x = 0,$$
$$r_t + \frac{\beta}{b}f_xr_x + \frac{2\beta}{b}rf_{xx} - \frac{\beta}{2ab}f_{xxxx} = 0.$$

It is nothing but the so-called Wu-Zhang equation (WZE). We can rewrite this WZE in the following equivalent form:

$$q_t + \frac{3\beta}{2b}qq_x - \beta r_x = 0, \tag{6.8}$$

$$r_t + \frac{\beta}{2b}qr_x + \frac{\beta}{b}rq_x - \frac{\beta}{4ab}q_{xxx} = 0. \tag{6.9}$$

6.3.2. The complex WZE

One more interesting form of the WZE that we obtain is when $b = i$. Then the WZE becomes

$$iq_t + \frac{3\beta}{2}qq_x - i\beta r_x = 0,$$
$$ir_t + \frac{\beta}{2}qr_x + \beta rq_x - \frac{\beta}{4a}q_{xxx} = 0.$$

It is the complex WZE, since in this case, the functions q, r become complex functions.

6.3.3. The original WZE

We now assume that $a = b = \beta = 1$. Then the WZE (6.8) and (6.9) takes the form

$$q_t + \frac{3}{2}qq_x - r_x = 0,$$
$$r_t + \frac{1}{2}qr_x + rq_x - \frac{1}{4}q_{xxx} = 0.$$

The last set of equations exactly coincides with the original form of the WZE [22] after some simple renaming of $q \to v$ and $r \to u$:

$$v_t + \frac{3}{2}vv_x - u_x = 0,$$
$$u_t + \frac{1}{2}vu_x + uv_x - \frac{1}{4}v_{xxx} = 0.$$

Finally, we note that the WZE is the gauge equivalent counterpart of the Manukure-Zhaidary equation (5.6), (5.7) or (5.8),(5.9).

6.3.4. Lax representation

Note that the WZE (6.8)and (6.9) is integrable. Its Lax representation is given by [22]:

$$\Phi_x = U_2\Phi, \tag{6.10}$$

$$\Phi_t = (\beta\lambda U_2 + B)\Phi, \tag{6.11}$$

where

$$U_2 = \begin{pmatrix} 0 & a \\ b\lambda^2 + q\lambda + r & 0 \end{pmatrix} = (b\lambda^2 + q\lambda)\Sigma + Q, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6.12}$$

$$B = B_2\lambda^2 + B_1\lambda + B_0, \quad \Sigma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & a \\ r & 0 \end{pmatrix}, \tag{6.13}$$

$$B_2 = \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} = u\Sigma, \quad B_1 = \begin{pmatrix} 0 & 0 \\ b^{-1}uq & 0 \end{pmatrix} = b^{-1}uq\Sigma, \tag{6.14}$$

$$B_0 = \begin{pmatrix} \frac{\beta}{4b}q_x & ab^{-1}u \\ b^{-1}ur + \frac{\beta}{4ab}q_{xx} & -\frac{\beta}{4b}q_x \end{pmatrix} = \frac{\beta}{4b}q_x\sigma_3 + \frac{u}{b}Q + \frac{\beta}{4ab}q_{xx}\Sigma. \tag{6.15}$$

The compatibility condition $\Phi_{xt} = \Phi_{tx}$ of the linear Eqs (6.10) and (6.11) gives the WZE (6.8) and (6.9). As the integrable equation, the WZE (6.8) and (6.9) has the N-soliton solution, infinite number of conservation laws, Hamiltonian structure, and so on.

Thus, it has been shown that the KMME admits two important integrable reductions, namely, the Manukure-Zhanbota equation and the Manukure-Zhaidary equation. Similarly, the AMTZE possesses two integrable reductions given by the KKMSE and the WZE. Through the gauge transformation procedure, we have demonstrated that the Manukure-Zhanbota equation is gauge equivalent to the KKMSE, while the Manukure-Zhaidary equation is gauge equivalent to the WZE. To improve the clarity of the analysis, the main steps of the gauge equivalence construction and the reduction procedures have been discussed in a more explicit and systematic manner.

7. Exact solutions

In modern mathematical and theoretical physics, one of the most important topics is to find the exact solutions of nonlinear differential equations. In this section, we present several classes of exact traveling-wave solutions for the AMTZE, including Jacobi elliptic, trigonometric, soliton, and rational solutions. These solutions describe different nonlinear wave structures and reveal the rich dynamical behavior of the system under various parameter regimes. All these integrable equations have different types of exact solutions including N-soliton solutions. To find the traveling wave solutions of the AMTZE (3.3)–(3.5), we consider the following transformations:

$$q(x, y, t) = q(\zeta), \quad r(x, y, t) = r(\zeta), \quad u(x, y, t) = u(\zeta), \quad \zeta = \mu x + \nu t + \eta y,$$

where μ, ν, η are some real constants. Then we have

$$q_x = \mu q', q_{xx} = \mu^2 q'', q_{xxy} = \mu^2\eta q''', r_t = \nu r', r_y = \eta r', r_x = \mu r', u_x = \mu u',$$

where $f' = \frac{df}{d\zeta}$. Plugging these expressions into the AMTZE (3.3)–(3.5), we obtain

$$\nu q' - \frac{\mu}{b}uq' + \frac{\beta\eta}{b}qq' - \beta\eta r' = 0, \tag{7.1}$$

$$\nu r' - \frac{\mu}{b}ur' + \frac{\beta\eta}{b}rq' - \frac{\beta\mu^2\eta}{4ab}q''' = 0, \tag{7.2}$$

$$\mu u' + \frac{\beta\eta}{2}q' = 0. \tag{7.3}$$

From Eq (7.3), we get

$$u = c_1 - \frac{\beta\eta}{2\mu}q,$$

where $c_1 = const$ is an integration constant. Then Eqs (7.1) and (7.2) take the forms

$$\nu q^{'} - \frac{\mu}{b}(c_1 - \frac{\beta\eta}{2\mu}q)q^{'} + \frac{\beta\eta}{b}qq^{'} - \beta\eta r^{'} = 0, \tag{7.4}$$

$$\nu r^{'} - \frac{\mu}{b}(c_1 - \frac{\beta\eta}{2\mu}q)r^{'} + \frac{\beta\eta}{b}rq^{'} - \frac{\beta\mu^2\eta}{4ab}q^{'''} = 0. \tag{7.5}$$

From (7.4), we obtain

$$c_2\beta\eta + \nu q - \frac{\mu}{b}c_1 q + \frac{\beta\eta}{4b}q^2 + \frac{\beta\eta}{2b}q^2 - \beta\eta r = 0,$$

where $c_2 = const$ is an integration constant. Hence, we get

$$c_2\beta\eta + (\nu - \frac{\mu}{b}c_1)q + (\frac{\beta\eta}{4b} + \frac{\beta\eta}{2b})q^2 - \beta\eta r = 0,$$

or

$$c_2\beta\eta + c_3\beta\eta q + c_4\beta\eta q^2 - \beta\eta r = 0,$$

where

$$c_3 = \frac{1}{\beta\eta}(\nu - \frac{\mu}{b}c_1), \quad c_4 = \frac{1}{\beta\eta}(\frac{\beta\eta}{4b} + \frac{\beta\eta}{2b}) = \frac{3}{4b}, \quad c_2 = const.$$

The last equation gives

$$r = c_2 + c_3 q + c_4 q^2. \tag{7.6}$$

Hence, we have

$$r^{'} = c_3 q^{'} + 2c_4 qq^{'}. \tag{7.7}$$

Let us rewrite Eq (7.5) as

$$c_3\beta\eta r^{'} + \frac{\beta\eta}{2b}qr^{'} + \frac{\beta\eta}{b}rq^{'} - \frac{\beta\mu^2\eta}{4ab}q^{'''} = 0.$$

Hence, using (7.6) and (7.7), finally, we get

$$c_3\beta\eta(c_3 q^{'} + 2c_4 qq^{'}) + \frac{\beta}{2b}q(c_3 q^{'} + 2c_4 qq^{'}) + \frac{\beta\eta}{b}(c_2 + c_3 q + c_4 q^2)q^{'} - \frac{\beta\mu^2\eta}{4ab}q^{'''} = 0.$$

Integrating this equation one we obtain the following equation:

$$q^{''} = c_5 q^3 + c_6 q^2 + c_7 q + c_8, \tag{7.8}$$

where $c_8 = const$ is a constant of integration and

$$c_5 = \frac{4ab}{3\beta\mu^2\eta}\left(\frac{c_4\beta}{b} + \frac{\beta\eta c_4}{b}\right) = \frac{4ac_4}{3\mu^2\eta}(1+\eta),$$

$$
\begin{aligned}
c_6 &= \frac{4ab}{\beta\mu^2\eta}(c_3c_4\beta\eta + \frac{c_3\beta}{4b} + \frac{\beta\eta c_3}{2b}) = \frac{ac_3}{\mu^2\eta}(4bc_4\beta\eta + 1 + 2\eta),\\
c_7 &= \frac{4ab}{\beta\mu^2\eta}(c_3^2\beta\eta + \frac{\beta\eta c_2}{b}) = \frac{4a}{\mu^2}(c_3^2 b + c_2),\\
c_8 &= const.
\end{aligned}
$$

Hence, we have

$$2q^{'}q^{''} = (c_5q^3 + c_6q^2 + c_7q + c_8)2q^{'},$$

so that

$$q^{'2} = c_9q^4 + c_{10}q^3 + c_{11}q^2 + c_{12}q + c_{13},$$

where

$$c_9 = \frac{1}{2}c_5, \quad c_{10} = \frac{2}{3}c_6, \quad c_{11} = c_7, \quad c_{12} = 2c_8, \quad c_{13} = const.$$

Let us introduce a new function $p(x, y, t) = p(\zeta)$ as

$$q = c_{14}p + c_{15},$$

where c_{14} and c_{15} are new constants. Then Eq (7.8) takes the form

$$c_{14}p^{''} = c_5(c_{14}p + c_{15})^3 + c_6(c_{14}p + c_{15})^2 + c_7(c_{14}p + c_{15}) + c_8,$$

or

$$p^{''} = c_{16}p^3 + c_{17}p^2 + c_{18}p + c_{19},$$

where

$$
\begin{aligned}
c_{16} &= \frac{c_5c_{14}^3}{c_{14}} = c_5c_{14}^2,\\
c_{17} &= \frac{3c_5c_{14}^2 + c_6c_{14}^2}{c_{14}} = (3c_5 + c_6)c_{14},\\
c_{18} &= \frac{3c_5c_{14}c_{15}^2 + 2c_6c_{14}c_{15} + c_7c_{14}}{c_{14}} = 3c_5c_{15}^2 + 2c_6c_{15} + c_7,\\
c_{19} &= \frac{c_5c_{15}^3 + c_6c_{15}^2 + c_7c_{15} + c_8}{c_{14}}.
\end{aligned}
$$

Now we assume that

$$p^{''} = 2mp^3 - (1 + m)p, \tag{7.9}$$

or

$$p^{'2} = (1 - p^2)(1 - mp^2), \tag{7.10}$$

where

$$m = 0.5c_{16} = 0.5c_5c_{14}^2,\ c_{17} = c_{19} = 0,\ c_{18} = -(1 + m) = -(1 + 0.5c_5c_{14}^2).$$

These expressions give us

$$c_6 = -3c_5, \quad c_8 = -c_{15}(c_5c_{15}^2 - 3c_5c_{15} + c_7).$$

It is well-known that Eqs (7.9)–(7.10) have the following solution:

$$p(\zeta, m) = sn(\zeta, m),$$

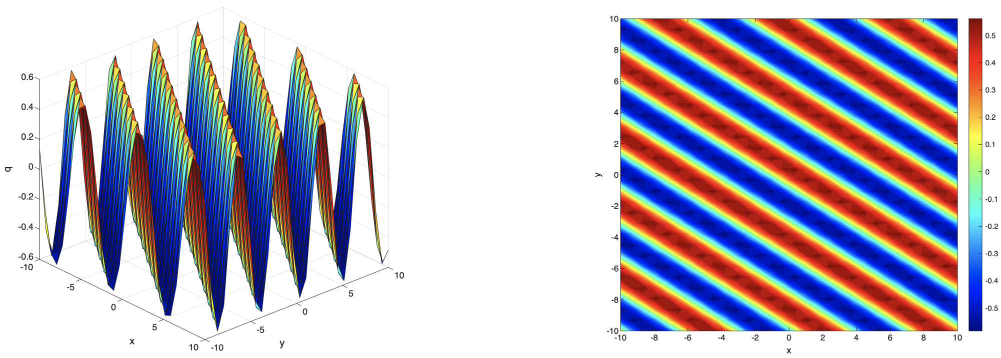
where $sn(\zeta, m)$ is the Jacobi elliptic function.

7.1. *Jacobi elliptic function solution*

The previous results give us the following solution of the AMTZE in terms of the Jacobi elliptic function as

$$\begin{aligned} q &= c_{14}sn(\zeta, m) + c_{15}, \\ r &= c_2 + c_3[c_{14}sn(\zeta, m) + c_{15}] + c_4[c_{14}sn(\zeta, m) + c_{15}]^2, \\ u &= c_1 - \frac{\beta\eta}{2\mu}[c_{14}sn(\zeta, m) + c_{15}]. \end{aligned} \tag{7.11}$$

Figure 1 illustrates the plots of the obtained solution to Eq (7.11).



(a) **(b)**

Figure 1. (a) The 3D plot and (b) contour plot of the function $q(x, y, t)$ with the parameters $\mu = 0.6,\ \nu = 0.3,\ \eta = 0.9,\ c_{14} = 1.0$, and $c_{15} = 0.0$.

7.2. *Trigonometric function solution*

Using the well-known properties of the Jacobi elliptic function

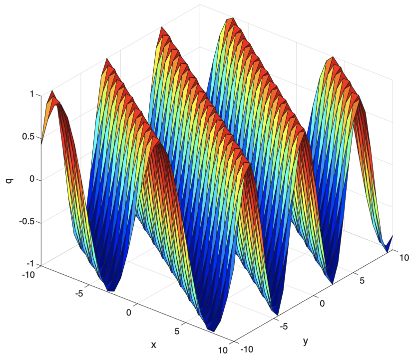
$$sn(\zeta, m) : sn(\zeta, 0) = \sin(\zeta),$$

we obtain the following trigonometric function solution of the AMTZE:

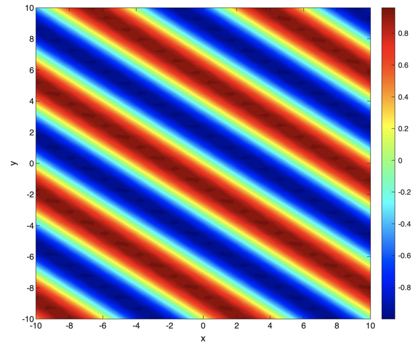
$$q = c_{14}\sin(\zeta) + c_{15}, \tag{7.12}$$

$$
\begin{aligned}
r &= c_2 + c_3[c_{14}\sin(\zeta) + c_{15}] + c_4[c_{14}\sin(\zeta) + c_{15}]^2, \\
u &= c_1 - \frac{\beta\eta}{2\mu}[c_{14}\sin(\zeta) + c_{15}].
\end{aligned}
$$

Figure 2 illustrates the plots of the obtained solution to (7.12).



(a)



(b)

Figure 2. (a) The 3D plot and (b) contour plot of the function $q(x, y, t)$ with the parameters $\mu = 0.6,\ \nu = 0.3,\ \eta = 0.9,\ c_{14} = 1.0$, and $c_{15} = 0.0$

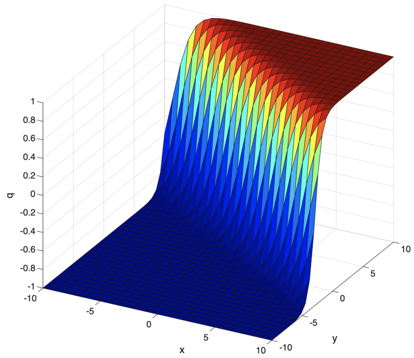
.

7.3. Soliton solution

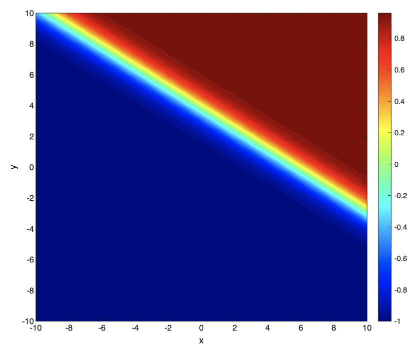
Now we use the following properties of the Jacobi elliptic function: $sn(\zeta, m) : sn(\zeta, 1) = \tanh(\zeta)$. The corresponding trigonometric function solution of the AMTZE reads as:

$$
\begin{aligned}
q &= c_{14}\tanh(\zeta) + c_{15}, \\
r &= c_2 + c_3[c_{14}\tanh(\zeta) + c_{15}] + c_4[c_{14}\tanh(\zeta) + c_{15}]^2, \\
u &= c_1 - \frac{\beta\eta}{2\mu}[c_{14}\tanh(\zeta) + c_{15}].
\end{aligned}
\tag{7.13}
$$

Figure 3 illustrates the plots of the obtained solution to Eq (7.13).



(a)



(b)

Figure 3. (a) The 3D plot and (b) contour plot of the function $q(x, , yt)$ with the parameters $\mu = 0.6,\ \nu = 0.3,\ \eta = 0.9,\ c_{14} = 1.0$, and $c_{15} = 0.0$.

7.4. Rational solution

To find the rational solution of the AMTZE, we consider the following expression for the function $q = \frac{k}{\zeta} + l$. Then the rational solution of the AMTZE is given by

$$\begin{aligned} q &= \frac{k}{\zeta} + l, \\ r &= c_2 + c_3[\frac{k}{\zeta} + l] + c_4[\frac{k}{\zeta} + l]^2, \\ u &= c_1 - \frac{\beta\eta}{2\mu}[\frac{k}{\zeta} + l], \end{aligned} \tag{7.14}$$

where k, l are some new constants. To fix constants k, l, we use Eq (7.8), that is,

$$q'' = c_5 q^3 + c_6 q^2 + c_7 q + c_8. \tag{7.15}$$

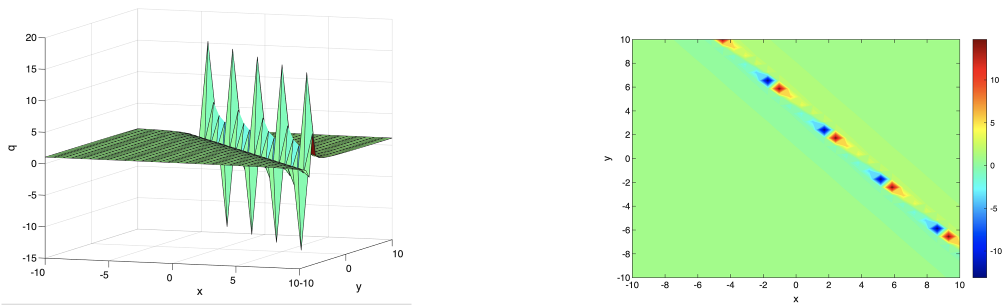
Substituting (7.14) into Eq (7.15), we get

$$\begin{aligned} y^{-3} &: \ 2k = c_5 k^3, \\ y^{-2} &: \ 0 = 3c_5 k^2 l + c_6 k^2, \\ y^{-1} &: \ 0 = 3c_5 k l^2 + 2klc_6 + c_7 k, \\ y^{-0} &: \ 0 = c_5 l^3 + c_6 l^2 + c_7 l + c_8. \end{aligned}$$

Hence, we get

$$\begin{aligned} k &= \pm\sqrt{\frac{2}{c_5}}, \\ l &= -\frac{c_6}{3c_5}, \\ c_7 &= -(3c_5 l^2 + 2lc_6), \\ c_8 &= -(c_5 l^3 + c_6 l^2 + c_7 l). \end{aligned}$$

Figure 4 illustrates the plots of the obtained solution to Eq (7.14).



(a) **(b)**

Figure 4. (a) The 3D plot and (b) contour plot of the function $q(x, y, t)$ with the parameters $\mu = 0.6,\ \nu = 0.3,\ \eta = 0.9,\ \alpha = 0.2,$ and $\beta = 0.3$

.

3D and contour plots illustrate the propagation characteristics and parameter-dependent features of the resulting waves. A comparative analysis of various families of solutions also provides additional insight into the transition between periodic, localized, and rational wave patterns.

8. Hamiltonian structure

From (7.9) and (7.15), it follows that

$$q' = z,\ z' = c_5 q^3 + c_6 q^2 + c_7 q + c_8,$$

or

$$q' = \frac{\delta H}{\delta z},$$
$$z' = -\frac{\delta H}{\delta q}.$$

Here, H has the form

$$H = \int [\frac{1}{2}z^2 - (\frac{1}{4}c_5 q^4 + \frac{1}{3}c_6 q^3 + \frac{1}{2}c_7 q^2 + c_8 q)]ds.$$

9. Conclusions

In this paper, the KMME and the AMTZE are studied. These two equations are integrable. The corresponding Lax representations are presented. Gauge equivalence between these two integrable equations is proved. It is shown that the KMME admits two integrable reductions, namely, the Manukure-Zhanbota equation and the Manukure-Zhaidary equation. Similarly, the AMTZE has two integrable reductions: the KKMSE and the Wu-Zhang equation. From these results follow that the Manukure-Zhanbota equation is gauge equivalent to the KKMSE. At the same time, the Manukure-Zhaidary equation and the the WZE are gauge equivalent to each other. Finally, we would like to note that it is very interesting to study the relations between AMTZE, the Broer-Kaup equations, and the Ablowitz- Kaup-Newell-Segur system following the paper.

From a physical perspective, integrable equations that admit exact solutions in the form of traveling waves are of considerable interest. The results obtained in this paper can be applied to the study of nonlinear wave propagation processes arising in hydrodynamics, plasma physics, nonlinear optics, and other related areas of mathematical physics, as well as to the modeling of stable wave interactions, energy transfer mechanisms, and the formation of coherent structures in nonlinear media.

Our future studies may include numerical simulations of KMME and AMTZE under different initial and boundary conditions, which would further validate the analytical solutions and help to clarify the stability and interaction properties of the resulting wave structures.

Author contributions

Ratbay Myrzakulov: Conceptualization, Methodology, Supervision, Writing–review and editing; Gulgassyl Nugmanova: Formal analysis, Investigation, Validation, Writing-original draft; Aidana Azhikhan: Investigation, Formal analysis, Methodology, Writing-original draft; Kuralay Yesmakhanova: Investigation, Formal analysis, Validation, Writing–original draft; Zhanbota Myrzakul: Investigation, Formal analysis, Visualization. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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