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Research article

New results on free and simple hypermodules

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Abstract: In this paper, we investigate the relationship between free hypermodules and normal projective hypermodules. In particular, we prove that every cyclic free R-hypermodule is normal projective and establish a criterion characterizing the normal projectivity of subhypermodules of the free hypermodule $H(R)$, where R is a hyperring. In addition, by means of quotient hypermodule constructions, we provide an example of a free hypermodule arising from a free module. Furthermore, we study the normal injectivity of simple hypermodules and obtain a fundamental characterization of hyperrings whose simple hypermodules are normal injective.

Keywords: free hypermodule; simple hypermodule; normal projective (injective) hypermodule
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1. Introduction

The theory of hypercompositional algebra, a sub-branch of abstract algebra, is based on hypergroups introduced by the French mathematician Marty at the eighth Congress of Scandinavian Mathematicians held in Stockholm in 1934 [1]. This theory generalizes classical algebraic systems by allowing the image of an operation to be a set rather than a single element. Krasner studied various classes of hypercompositional algebra, including hypergroups, hyperrings, and hypermodules, each providing a natural generalization of their classical analogs [2–4].

The concept of a hypermodule over a hyperring extends the notion of a module over a ring by replacing the ring and module operations with multivalued operations that satisfy compatibility conditions analogous to those of the classical case. Despite sharing many similarities with the category of modules, the category **HypMod**(R) of hypermodules exhibits several structural differences, where

R is a hyperring. Among these, the most important is that the category **HypMod**(R) of hypermodules is an exact category [5, Theorem 4.4]. The category **HypMod**(R) admits well-defined short exact sequences, stable kernels and cokernels, and categorical constructions that behave analogously to those in the category Mod(R) of modules over a ring R. This study allows homological and categorical tools to be extended to hypercompositional structures, paving the way for further developments in lattice-valued generalizations.

In this paper, we prove that every cyclic free hypermodule over a hyperring is normal projective. In addition, we provide an example of a free hypermodule corresponding to each free module. Moreover, we characterize hyperrings whose simple hypermodules are normal injective.

2. Preliminaries

Hypercompositional algebra is founded on algebraic structures whose operations are set-valued. More precisely, the map is defined as a mapping

$$\circ : H \times H \longrightarrow \mathcal{P}(H)$$

rather than as a mapping from $H \times H$ into H.

The term *magma* was later introduced into the literature of classical algebra by Bourbaki. Within the framework of hypercompositional algebra, a generalized version of this concept was subsequently formulated by Massouros and is defined as follows.

Definition 1. [6] Let H be a nonempty set. A map $\circ : H \times H \longrightarrow \mathcal{P}(H)$ is called a *hypercomposition* on H. The pair $(H, \circ)$ is called a *magma* (according to [7], *hypermagma*). For any two nonempty subsets $A, B \subseteq H$, we define

$$A \circ B = \bigcup \{\, a \circ b \mid a \in A,\ b \in B \,\}.$$

If $A = \{a\}$ or $B = \{b\}$, we write $a \circ B$ instead of $\{a\} \circ B$ and $A \circ b$ instead of $A \circ \{b\}$.

Definition 2. [1,6] A magma $(H, \circ)$ is called a *hypergroup* if it satisfies the following axioms:

(H1) $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in H$ (associativity),

(H2) $a \circ H = H \circ a = H$ for all $a \in H$ (reproduction).

Let $(H, \circ)$ be a hypergroup, and let N be a nonempty subset of H. The set N is called a *subhypergroup* of H if $a \circ N = N \circ a = N$ for all $a \in N$.

Definition 3. [3,4] Let $(R, +, \cdot)$ be a hypercompositional structure, where $+$ is a hypercomposition on R, and $\cdot$ is a single-valued operation on R. The structure R is called a *Krasner hyperring* (briefly, hyperring) if it satisfies the following conditions:

(I) **Additive axioms.**

(1) $a + (b + c) = (a + b) + c$ for all $a, b, c \in R$,

(2) $a + b = b + a$ for all $a, b \in R$,

(3) There exists an identity element $0 \in R$ such that, for every $a \in R$, there is a unique element $a' \in R$, denoted by $-a$, satisfying $0 \in a + (-a)$ and $0 \in (-a) + a$,

(4) (Reversibility law) If $a \in b + c$, then $b \in a - c$ for all $a, b, c \in R$.

(II) **Multiplicative axioms.**
$(R, \cdot)$ is a monoid in which the identity 0 of $(R, +)$ is a bilaterally absorbing element.

(III) **Axioms of distributivity.**
Multiplication distributes over the hypercomposition: $z(x + y) = zx + zy$ and $(x + y)z = xz + yz$ for all $x, y, z \in R$.

If, in addition,

(IV) $a \cdot b = b \cdot a$ for all $a, b \in R$,

then R is called a *commutative hyperring*. A commutative hyperring $(R, +, \cdot)$ is called a *hyperfield* if $(R \setminus \{0\}, \cdot)$ is a group.

The additive part of hyperring or hyperfield was first studied independently by Mittas in [8–10], where they were called *canonical hypergroups*. Later, Roth applied canonical hypergroups to the character theory of finite groups [11]. Join spaces lying between canonical hypergroups and hypergroups have been studied in W. Prenowitz's research on geometry [12, 13]. Join spaces with an identity element are closely related to canonical hypergroups; in fact, these two structures essentially coincide.

Let $(H, +)$ be a canonical hypergroup with identity element 0. By [14],

$$0 + a = a \quad \text{and} \quad -(-a) = a$$

for all $a \in H$. Furthermore, [15, Theorem 5] proves that

$$-(a + b) = -a - b$$

for all $a, b \in H$, and shows that this property is equivalent to the reversibility law.

According to a construction from Krasner [4], hyperring structures can be derived from rings through suitable quotient-type operations. Let R be a ring with identity and let G be a subgroup of the multiplicative monoid of R such that

$$xG = Gx$$

for every element $x \in R$.

Under this condition, the multiplicative cosets $[x] = xG$ form a partition of R, and the product of two such classes, considered as subsets of R, is again a class:

$$[x][y] = xGyG = xyG = [xy].$$

Let $R(G)$ denote the set of all such cosets. Then $R(G)$ becomes a hyperring if the hypercomposition is defined by

$$[x] \oplus [y] = \{\, [z] \mid z \in [x] + [y] \,\}.$$

Krasner further showed that if R is a field, then $R(G)$ is a hyperfield. These structures were introduced by Krasner under the names *quotient hyperring* and *quotient hyperfield*, respectively. Later, in [16], Massouros obtained the most general form of this construction as follows. Let $(R, +, \cdot)$ be a ring with identity and let G be a subgroup of the monoid R satisfying

$$xGyG = xyG$$

for all $x, y \in R$. Under these assumptions, the collection of cosets

$$R(G) = \{xG \mid x \in R\}$$

admits a structure of a hyperring. The multiplication is defined by

$$xGyG = xyG,$$

whereas the hypercomposition is given by

$$xG \oplus yG = \{(xp + yq)G \ : \ p, q \in G\}.$$

He showed that $R(G)$ is a hyperring. Moreover, when R is a field, the resulting hyperring $R(G)$ satisfies the axioms of a hyperfield.

In analogy with classical algebraic structures such as abelian groups, rings, and fields, the concept of a *hypermodule* arises in hypercompositional algebra in connection with canonical hypergroups and hyperrings.

Definition 4. [3, 17] Let R be a hyperring. A *left hypermodule* over R (or left R-hypermodule) is a canonical hypergroup $(M, +)$ together with a map $R \times M \longrightarrow M, \quad (r, m) \mapsto rm$ such that for all $r, s \in R$ and $m, m_1, m_2 \in M$, the following conditions hold:

(1) $r(m_1 + m_2) = rm_1 + rm_2$,

(2) $(r + s)m = rm + sm$,

(3) $(r \cdot s)m = r(sm)$,

(4) $1_R m = m$ and $r0_M = 0_R m = 0_M$.

Throughout this paper, the term hypermodule will mean a left hypermodule. A nonempty subset N of an R-hypermodule M is called a *subhypermodule* of M, denoted $N \leq M$, if N is itself an R-hypermodule under the same hypercomposition as M. Clearly, M and $\{0_M\}$ are *trivial subhypermodules* of M. A nonempty subset N of M is a subhypermodule if and only if $a - b \subseteq N$ and $ra \in N$ for all $a, b \in N$ and $r \in R$. In particular, for a hyperring R, the structure R itself is an R-hypermodule, denoted by $H(R)$. The left hyperideals of R are precisely the subhypermodules of $H(R)$.

Let M be a hypermodule over a hyperring R and let K be a subhypermodule of M. Define the relation "$x \cong_K y$ if and only if $(x + K) \cap (y + K) \neq \varnothing$". Therefore $\cong_K$ is an equivalence relation on M and for $m \in M$, the equivalence class of $m \in M$ is of the form $m + K$. We denote by $\frac{M}{K}$ the set of all equivalence classes of the relation $\cong_K$. By [2, Theorem 2.4.9], $(\frac{M}{K}, \boxplus)$ is a canonical hypergroup where

$$(a + K) \boxplus (b + K) = \{\, c + K \mid b \in a + b \,\} = \bigcup_{c \in a+b} (c + K).$$

The *factor hypermodule* of M modulo K is the canonical hypergroup $\frac{M}{K} = \{a + K \mid a \in M\}$ equipped with external scalar multiplication $\odot$ defined by $r \odot (a + K) = ra + K$ for all $a \in M$ and $r \in R$ [5] or [18, Proposition 3].

Let R be a hyperring and M an R-hypermodule. For a family of subhypermodules $\{M_i\}_{i\in I}$ of M, the *sum* of this family, denoted by $\sum_{i\in I} M_i$, is defined as the set of all elements $x \in M$ such that $x \in \sum_{i\in I_0} m_i, \quad m_i \in M_i$ for some finite subset $I_0 \subseteq I$. It is well-known that $\sum_{i\in I} M_i$ is itself a subhypermodule of M.

Definition 5. [19] Let M be a hypermodule and $\{M_i\}_{i\in I}$ a nonempty collection of subhypermodules of M. The hypermodule M is said to be an *internal direct sum* of the subhypermodules $\{M_i\}_{i\in I}$ if the following conditions hold:

(1) $M = \sum_{i\in I} M_i$,

(2) $M_i \cap \left(\sum_{k\in I, k\neq i} M_k\right) = \{0_M\}$ for all $i \in I$.

The following theorem is taken from [19, Theorem 1] and will be used frequently throughout this work without further reference.

Theorem 1. *A hypermodule M is an internal direct sum of the subhypermodules $\{M_i\}_{i\in I}$ if and only if every element $m \in M$ can be written as $m \in m_{i_1} + m_{i_2} + \cdots + m_{i_n}$, where $m_{i_j} \in M_{i_j}$ are uniquely determined for distinct indices $i_1, \ldots, i_n \in I$.*

Analogous to the internal direct sum, one may consider the *external direct sum* of hypermodules. However, the external direct sum of two given hypermodules does not, in general, preserve the structure of a hypermodule [20, Theorem 16]. This provides another difference between the categories hypermodules and modules.

Definition 6. [17] Let R be a hyperring. Given two R-hypermodules M and N, a map $f : M \longrightarrow \mathcal{P}(N)$ is called *homomorphism* if

(1) $f(m_1 +_M m_2) \subseteq f(m_1) +_N f(m_2)$ for all $m_1, m_2 \in M$,

(2) $f(rm) = rf(m)$ for all $r \in R$ and for all $m \in M$.

f is called *strong homomorphism* if equality holds in the condition (1) of the definition of homomorphism. A single-valued map $f : M \longrightarrow N$ is called *good homomorphism* if f satisfies the above conditions (1) and (2). $f : M \longrightarrow N$ is called *normal homomorphism* if instead of (1), we have $f(m_1 +_M m_2) = f(m_1) +_N f(m_2)$.

Let R be a hyperring. We denote by **HypMod**(R) the category whose objects are all R-hypermodules and whose morphisms are all normal homomorphisms. It is shown in [5, Theorem 4.4] that **HypMod**(R) is an exact category. Therefore, by [21, 15. Exact Categories], exact sequences in this category are defined. It also follows from the same section that a sequence

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact if and only if f is a normal monomorphism, g is a normal epimorphism, and $\ker(g) = \operatorname{Im}(f)$. In this case, $\mathbb{E}$ is called a *short exact sequence*. We denote by $\mathcal{H}Abs$ the class of all short exact sequences in **HypMod**(R).

3. On free hypermodules

In [17], Massouros introduced free hypermodules and established several of their properties. Moreover, he provided a construction of a free hypermodule. In this section, we present an example of a free hypermodule corresponding to each free module, and we investigate the relationship between free hypermodules and normal projective hypermodules.

Definition 7. [17] Let F be any left R-hypermodule. An R-hypermodule F is called a free hypermodule if there exists a nonempty set X, called a basis of F, satisfying the following conditions:

(1) X generates F, and

(2) for every function ψ of X into an R-hypermodule M, there exists a homomorphism $\phi : F \to P(M)$ such that $\phi(x) = \{\psi(x)\}$ for every $x \in X$.

Massouros characterized hypermodules with a finite basis in [17, Proposition 3]. Using the same argument as in Massouros' proof, combined with Theorem 1, we extend this characterization to hypermodules with an arbitrary basis.

Theorem 2. *Let R be a hyperring and F be an R-hypermodule. Then the following statements are equivalent for a nonempty subset X of F:*

(1) X is a basis of F,

(2) X is linearly independent and generates F,

(3) F is the internal direct sum of the cyclic R-hypermodules Rx ($x \in X$), where each Rx is isomorphic to $H(R)$.

Following [17], Massouros constructed quotient hypermodules by a method analogous to that for quotient hyperrings. Let M be an R-module, where R is a unitary ring, and let G be a subgroup of the monoid of R, which satisfies the condition $xGyG = xyG$ for all $x, y \in R$. Define an equivalence relation on M by

$$x \sim y \iff x = ty \quad \text{for some } t \in G,$$

and denote by $[x]$ the equivalence class of $x \in M$. Let $\overline{M}$ be the set of all such classes. For $[x], [y] \in \overline{M}$, define

$$[x] \oplus [y] = \{[w] \in \overline{M} \ : \ w \in [x] + [y]\}.$$

Then, $(\overline{M}, \oplus)$ is a canonical hypergroup with identity $[0]$. Moreover, defining

$$\overline{r}\,[x] = [rx] \quad \text{for } \overline{r} \in R(G),\ [x] \in \overline{M}$$

turns $\overline{M}$ into an $R(G)$-hypermodule, called the *quotient hypermodule*. It follows from [17] that every Euclidean spherical geometry can be realized as a quotient hypermodule.

Now by means of quotient hypermodules, we construct free hypermodules associated with a free R-module and a multiplicative subgroup G of the ring R such that $xGyG = xyG$ for all $x, y \in R$. Moreover, for different multiplicative subgroups G of R such that $xGyG = xyG$ for all $x, y \in R$, the resulting quotient structures give rise to mutually nonisomorphic free hypermodules.

Example 1. Let R be a ring with identity, and let F be a nonzero free left R-module. Assume that G is a subgroup of the monoid of R satisfying

$$rGsG = rsG$$

for all $r, s \in R$. Then, $\overline{F}$ becomes a free left $R(G)$-hypermodule. Let X be a basis of F and consider the set $\overline{X} = \{[x] \in \overline{F} \mid x \in X\}$. We show that $\overline{X}$ is linearly independent and generates the hypermodule $\overline{F}$. Because $X \neq \emptyset$, it follows that $\overline{X} \neq \emptyset$. Observe that $[0] = \{0\}$, and $0 \in [m]$ if and only if $m = 0$. Let $S = \{[x_1], [x_2], \ldots, [x_k]\}$ be a finite subset of $\overline{X}$ and suppose that

$$0 = [0] \in \overline{r}_1[x_1] + \overline{r}_2[x_2] + \cdots + \overline{r}_k[x_k],$$

where $\overline{r}_1, \overline{r}_2, \ldots, \overline{r}_k \in R(G)$. Then $0 \in [r_1x_1] + [r_2x_2] + \cdots + [r_kx_k]$. Hence there exist elements $p_1, p_2, \ldots, p_k \in G$ such that $0 = p_1(r_1x_1) + p_2(r_2x_2) + \cdots + p_k(r_kx_k)$. Because X is a linearly independent subset of F, it follows that $p_1r_1 = p_2r_2 = \cdots = p_kr_k = 0$. Because G is a group, for each $i \in \{1, 2, \ldots, k\}$ there exists $t_i \in G$ such that $t_ip_i = 1_R$. Multiplying the equality $p_ir_i = 0$ by t_i, we obtain

$$0 = t_i(p_ir_i) = (t_ip_i)r_i = 1_Rr_i = r_i.$$

Thus, $\overline{r}_i = \overline{0}$ for all i, and therefore $\overline{X}$ is linearly independent. Now let $[m]$ be an arbitrary element of $\overline{F}$. Since F is free with basis X, there exist $r_1, r_2, \ldots, r_k \in R$ and $x_1, x_2, \ldots, x_k \in X$ such that

$$m = r_1x_1 + r_2x_2 + \cdots + r_kx_k = 1_R(r_1x_1) + 1_R(r_2x_2) + \cdots + 1_R(r_kx_k).$$

Hence,

$$[m] \in [r_1x_1] + [r_2x_2] + \cdots + [r_kx_k] = \overline{r}_1[x_1] + \overline{r}_2[x_2] + \cdots + \overline{r}_k[x_k].$$

Thus $\overline{X}$ generates $\overline{F}$. Consequently, $\overline{F}$ is a free $R(G)$-hypermodule with basis $\overline{X}$ by Theorem 2.

Definition 8. [22, 23] Let M be a hypermodule. If for every normal epimorphism $f : A \to B$ of hypermodules and every normal homomorphism $g : M \to B$ of hypermodules, there exists a normal homomorphism $h : M \to A$ such that $fh = g$, then, M is called *normal projective*.

Characterizations of normal projective hypermodules in terms of short exact sequences were obtained in [24]. In particular, hyperrings over which every hypermodule is normal projective were completely characterized. On the other hand, the relationship between free hypermodules and normal projective hypermodules has not yet been investigated. Moreover, it is currently unknown whether the category **HypMod**(R) has enough projectives. Because every free R-hypermodule is an internal direct sum of cyclic free hypermodules, it is natural to begin with cyclic free hypermodules and examine whether they possess the normal projectivity property. The following theorem provides a first step in this direction by showing that the cyclic free hypermodule is always normal projective.

Theorem 3. *Let R be a hyperring. Then, every cyclic free R-hypermodule F is normal projective.*

Proof. Because F is a cyclic free R-hypermodule, there exists a free generator $x \in F$ such that $F = Rx$. By Theorem 2, the cyclic hypermodule Rx is isomorphic to the hypermodule $H(R)$. On the other hand, it follows from [19, Theorem 2] that $H(R)$ is normal projective. Because normal projectivity is preserved under isomorphisms, we conclude that F is normal projective. □

Although every cyclic free hypermodule is normal projective, the following example shows that the converse does not hold; that is, a normal projective cyclic hypermodule need not be free. Recall from [19] that an R-hypermodule M is semisimple if it is the sum of simple subhypermodules of M. It is proven in [19, Theorem 9] that a module M is semisimple if and only if every subhypermodule of M is a direct summand of M.

Example 2. Consider the quotient hyperring $\mathbb{Z}_6(G) = \{[0],[1],[2],[3]\}$, where $G = \{1,5\}$ is the multiplicative group of the ring $\mathbb{Z}_6$. The equivalence classes are

$$[0] = 0G = \{0\}, \qquad [1] = 1G = \{1,5\},$$

$$[2] = 2G = \{2,10\} = \{2,4\} = [4], \qquad [3] = 3G = \{3,15\} = \{3\}.$$

We have the following Tables 1 and 2:

Table 1. The hypercomposition table of the canonical hypergroup $(\mathbb{Z}_6(G), \oplus)$.

$\oplus$	[0]	[1]	[2]	[3]
[0]	{[0]}	{[1]}	{[2]}	{[3]}
[1]	{[1]}	{[0],[2]}	{[1],[3]}	{[2]}
[2]	{[2]}	{[1],[3]}	{[0],[2]}	{[1]}
[3]	{[3]}	{[2]}	{[1]}	{[0]}

Table 2. The operation table of the monoid $(\mathbb{Z}_6(G), \cdot)$.

$\cdot$	[0]	[1]	[2]	[3]
[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]
[2]	[0]	[2]	[2]	[0]
[3]	[0]	[3]	[0]	[3]

Put $X = \{[1]\}$. Then X is a generating set of $\mathbb{Z}_6(G)$ and is linearly independent. Therefore, $\mathbb{Z}_6(G)$ is a free hypermodule with basis $\{[1]\}$. Thus, the only proper subhypermodules of $\mathbb{Z}_6(G)$ are $I_1 = \{[0],[2]\}$ and $I_2 = \{[0],[3]\}$. Moreover, we have

$$\mathbb{Z}_6(G) = I_1 \oplus I_2.$$

It follows that $H(\mathbb{Z}_6(G))$ is semisimple. Hence, by [24, Theorem 3], I_1 and I_2 are normal projective hypermodules. On the other hand, because

$$[3][2] = [2][3] = [0],$$

neither I_1 nor I_2 admits a basis. Therefore, I_1 and I_2 are not free hypermodules.

Remark 1. **Corrigendum of [24, Example 6].** Unfortunately, we observed a minor error in the hypercomposition used in the construction of the hyperring given in [24, Example 6], which was presented as an example of a hyperring whose every hypermodule is normal projective. The detected mistake affects the verification of the canonical hypergroup condition for the underlying set. Therefore, the construction in [24, Example 6] cannot be used as stated.

As a replacement, the hyperring constructed in the above example satisfies the semisimplicity condition and hence provides the correct example in this context.

Let R be a hyperring and $\mathbb{E} : 0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \longrightarrow 0$ be a short exact sequence of R-hypermodules. Following [24], $\mathbb{E}$ is called *splitting* in case $Im(\psi) = Ker(\phi)$ is a direct summand of B. By $\mathcal{H}Split$ we denote the class of all splitting short exact sequences of left R-hypermodules.

Corollary 1. *Let* $\mathbb{E} : 0 \longrightarrow M \longrightarrow N \longrightarrow F \longrightarrow 0$ *be a short exact sequence of R-hypermodules. If F is a cyclic free R-hypermodule, then the sequence splits.*

Proof. Let F be a cyclic free R-hypermodule. It follows from Theorem 3 that F is normal projective. Hence, by [24, Theorem 2], $\mathbb{E}$ is splitting. □

In [17, Theorem 8], it was proved that every subhypermodule of a finitely generated free hypermodule over a principal hyperideal domain is free. In what follows, we develop an approach through normal injectivity in order to analyze the normal projectivity of subhypermodules of the free hypermodule $H(R)$. Let M be a hypermodule. If for every normal monomorphism $f : A \to B$ and every normal homomorphism $g : A \to M$, there exists a normal homomorphism $h : B \to M$ such that $hf = g$, then M is called *normal injective* [22,23,25].

Theorem 4. *Let R be a hyperring. Then, the following statements are equivalent.*

(1) Every factor hypermodule of any normal injective R-hypermodule is normal injective,

(2) Every subhypermodule of the free hypermodule $H(R)$ *is normal projective with respect to any short exact sequence* $\mathbb{E}:\ 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$*, where B is normal injective.*

Proof. (1) ⇒ (2). Let I be any subhypermodule of $H(R)$. Consider a short exact sequence $\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$, where B is normal injective. Let $\lambda : I \to C$ be any normal homomorphism. Given the inclusion map $i : I \to H(R)$, we have the following Figure 1.:

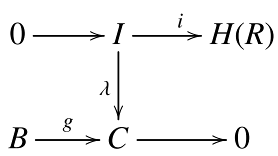

Figure 1. Normal injectivity.

Becaues C is normal injective by assumption, there exists a normal homomorphism $\tau : R \longrightarrow C$ such that $\tau i = \lambda$. Because $H(R)$ is normal projective, there exists a normal homomorphism $\alpha : H(R) \longrightarrow B$ such that $\tau = g\alpha$. Let $\Phi = \lambda|_I$ denote the restriction of λ to subhypermodule I. Then for every $a \in I$ we have,

$$g(\Phi(a)) = g(\alpha(a)) = \tau(a) = (\tau i)(a) = \lambda(a).$$

Hence, $g\Phi = \lambda$, which shows that I is normal projective with respect to the short exact sequence $\mathbb{E}$.

(2) $\Rightarrow$ (1). Let M be a normal injective hypermodule, and let N be any factor hypermodule of M. For any left hyperideal I of R, let $f : I \longrightarrow N$ be a normal homomorphism. We show that f extends to R. Consider the short exact sequence $0 \longrightarrow \ker(\pi) \longrightarrow M \xrightarrow{\pi} N \longrightarrow 0$, where $\pi : M \to N$ is the canonical epimorphism. By the assumption (2), I is normal projective with respect to this short exact sequence. Hence, there exists a normal homomorphism $h : I \longrightarrow M$ such that $\pi h = f$. Because M is normal injective, there exists a normal homomorphism $\lambda : R \longrightarrow M$ such that $h = \lambda i$, where $i : I \to R$ is the inclusion map. Put $\tau = \pi\lambda : R \longrightarrow N$. Then $\tau i = (\pi\lambda)i = \pi(\lambda i) = \pi h = f$. Thus, f extends to R, which proves that N is normal injective by [23, Theorem 4]. □

Example 3. Let $\mathbb{N}$ denote the set of all non-negative integers. Let $+$ and $\cdot$ denote the usual addition and multiplication in $\mathbb{N}$. By [26], we define the hypercomposition "$\oplus$" on $\mathbb{N}$ as follows: For any $m, n \in \mathbb{N}$,

$$\begin{aligned} m \oplus n &= \{m+n,\ k \mid \min\{m,n\} + k = \max\{m,n\} \text{ for some } k \in \mathbb{N}\} \\ &= \{m+n,\ |m-n|\}. \end{aligned}$$

Thus the hypercompositional structure $(\mathbb{N}, \oplus, \cdot)$ is a principal hyperideal domain by [26, Proposition 3.1]. So $H(\mathbb{N})$ is a cyclic free hypermodule. It follows [17, Theorem 8] that every subhypermodule of $H(\mathbb{N})$ is free. As a consequence of Theorem 3, every subhypermodule of $H(\mathbb{N})$ is normal projective. Thus, the class of normal injective R-hypermodules over the hyperring $H(\mathbb{N})$ is closed under factor hypermodules by Theorem 4.

The above example also demonstrates the existence of hyperrings whose every hyperideal is normal projective. This observation naturally leads to the introduction of the notion of a hereditary hypermodule (hyperring).

Definition 9. Let R be a hypeerring and M be a normal projective left R-hypermodule. M is called *hereditary* if every subhypermodule of M is normal projective. R is called *a left hereditary hyperring* if $H(R)$ is hereditary.

We now provide several illustrative examples.

Example 4. (1) Let R be a hyperring. If $H(R)$ is semisimple, it follows from [24, Theorem 3] that R is a left hereditary.

(2) Let R be a principal hyperideal domain. It follows from [17, Theorem 8] that every submodule of $H(R)$ is free. By Theorem 3, we obtain that $H(R)$ is hereditary.

A factor hyperring of a left hereditary hyperring is not necessarily left hereditary. Let R be any hyperring and M be a left R-hypermodule. Following [19], for any subhypermodule N of M, define $\operatorname{ann}(N) = \{ r \in R \mid rN = 0 \}$ which is a hyperideal of R. For any hyperideal I of R, we have $I \subseteq \operatorname{ann}(\frac{R}{I})$.

If M is an R-hypermodule with $I \subseteq \operatorname{ann}(M)$, then M becomes an $\frac{R}{I}$-hypermodule via $(r+I)m = rm$. In particular, M is an $\frac{R}{\operatorname{ann}(M)}$-hypermodule. Conversely, any $\frac{R}{I}$-hypermodule M_1 becomes an R-hypermodule by $r \cdot m_1 = (r+I)m_1$ and satisfies $I \subseteq \operatorname{ann}(M_1)$. Hence, R-hypermodules with annihilator containing I coincide with $\frac{R}{I}$-hypermodules. In particular, $\frac{M}{IM}$ is also an $\frac{R}{I}$-hypermodule.

Example 5. As in Example 3, consider the hereditary hyperring $\mathbb{N}$ and take the hyperideal $I = 4\mathbb{N}$. Then, the set $\frac{\mathbb{N}}{4\mathbb{N}} = \{ m \oplus 4\mathbb{N} \mid m \in \mathbb{N} \}$. For each $m \in \mathbb{N}$, we have $m \oplus 4\mathbb{N} = \bigcup_{n\in\mathbb{N}}(m \oplus 4n)$. In particular,

$$0 \oplus 4\mathbb{N} = \bigcup_{n\in\mathbb{N}}(0 \oplus 4n) = \{0, 4, 8, 12, \ldots\} = 4\mathbb{N},$$

$$1 \oplus 4\mathbb{N} = \bigcup_{n\in\mathbb{N}}(1 \oplus 4n) = \{1, 3, 5, 7, 9, 11, 13, \ldots\} = 3 \oplus 4\mathbb{N},$$

and

$$2 \oplus 4\mathbb{N} = \bigcup_{n\in\mathbb{N}}(2 \oplus 4n) = \{2, 6, 10, 14, \ldots\}.$$

Hence,

$$\frac{\mathbb{N}}{4\mathbb{N}} = \{4\mathbb{N},\ 1 \oplus 4\mathbb{N},\ 2 \oplus 4\mathbb{N}\}.$$

Now, we compute the hypercomposition in $\frac{\mathbb{N}}{4\mathbb{N}}$ as follows:

$$(1 \oplus 4\mathbb{N}) \boxplus (1 \oplus 4\mathbb{N}) = \bigcup_{c\in 1\oplus 1} (c \oplus 4\mathbb{N}) = \bigcup_{c\in\{0,2\}} (c \oplus 4\mathbb{N}) = \{ 4\mathbb{N},\ 2 \oplus 4\mathbb{N} \},$$

$$(1 \oplus 4\mathbb{N}) \boxplus (2 \oplus 4\mathbb{N}) = \bigcup_{c\in 1\oplus 2} (c \oplus 4\mathbb{N}) = \bigcup_{c\in\{1,3\}} (c \oplus 4\mathbb{N}) = \{ 1 \oplus 4\mathbb{N},\ 3 \oplus 4\mathbb{N} \} = \{ 1 \oplus 4\mathbb{N} \},$$

and

$$(2 \oplus 4\mathbb{N}) \boxplus (2 \oplus 4\mathbb{N}) = \bigcup_{c\in 2\oplus 2} (c \oplus 4\mathbb{N}) = \bigcup_{c\in\{0,4\}} (c \oplus 4\mathbb{N}) = \{ 4\mathbb{N},\ 0 \oplus 4\mathbb{N} \} = \{ 4\mathbb{N} \}.$$

Hence, the hypercomposition table of $\frac{\mathbb{N}}{4\mathbb{N}} = \{4\mathbb{N},\ 1 \oplus 4\mathbb{N},\ 2 \oplus 4\mathbb{N}\}$ is given in Table 3, and $\frac{\mathbb{N}}{4\mathbb{N}}$ is the $\mathbb{N}$-hypermodule with external scalar multiplication $\odot$ defined by $n \odot (m \oplus 4\mathbb{N}) = nm \oplus 4\mathbb{N}$ for all $n,\ m \in \mathbb{N}$. Also, the hypermodule structure of $\frac{\mathbb{N}}{4\mathbb{N}}$ coincides with the $\mathbb{N}$-hypermodule structure. Observe that the only proper subhypermodule of $\frac{\mathbb{N}}{4\mathbb{N}}$ is $S = \{ 4\mathbb{N},\ 2 \oplus 4\mathbb{N} \}$. In particular, S is a maximal subhypermodule of $\frac{\mathbb{N}}{4\mathbb{N}}$. Let M denote the factor hypermodule of $\frac{\mathbb{N}}{4\mathbb{N}}$ with respect to S. Then,

$$M = \{ S,\ (1 \oplus 4\mathbb{N}) \boxplus S \}.$$

Define a map

$$f : S \longrightarrow M$$

by

$$f(4\mathbb{N}) = S \qquad \text{and} \qquad f(2 \oplus 4\mathbb{N}) = (1 \oplus 4\mathbb{N}) \boxplus S.$$

Then, f is a normal isomorphism. Hence, $M \cong S$. Now, consider the short exact sequence

$$0 \longrightarrow S \xrightarrow{i} \frac{\mathbb{N}}{4\mathbb{N}} \xrightarrow{f\pi} S \longrightarrow 0,$$

where $i : S \to \frac{\mathbb{N}}{4\mathbb{N}}$ is the inclusion map and $\pi : \frac{\mathbb{N}}{4\mathbb{N}} \to M$ is the canonical projection. Assume that S is normal projective. Then, it follows by [24, Proposition 1] that the above short exact sequence splits. Therefore, S is a direct summand of $\frac{\mathbb{N}}{4\mathbb{N}}$. This is impossible because S is the only proper subhypermodule of $\frac{\mathbb{N}}{4\mathbb{N}}$. Hence, S is not normal projective. Therefore, the hyperring $\frac{\mathbb{N}}{4\mathbb{N}}$ is not hereditary.

Table 3. The hypercomposition table of $\frac{\mathbb{N}}{4\mathbb{N}}$.

$\boxplus$	$4\mathbb{N}$	$1 \oplus 4\mathbb{N}$	$2 \oplus 4\mathbb{N}$
$4\mathbb{N}$	$\{4\mathbb{N}\}$	$\{1 \oplus 4\mathbb{N}\}$	$\{2 \oplus 4\mathbb{N}\}$
$1 \oplus 4\mathbb{N}$	$\{1 \oplus 4\mathbb{N}\}$	$\{4\mathbb{N}, 2 \oplus 4\mathbb{N}\}$	$\{1 \oplus 4\mathbb{N}\}$
$2 \oplus 4\mathbb{N}$	$\{2 \oplus 4\mathbb{N}\}$	$\{1 \oplus 4\mathbb{N}\}$	$\{4\mathbb{N}\}$

4. Hyperrings whose simple hypermodules are normal injective

It is shown in [24, Theorem 3] that a hyperring R, $H(R)$ is semisimple if and only if every simple R-hypermodule is normal projective. As a dual version of this characterization, it is natural to investigate hyperrings whose simple hypermodules are normal injective. In this section, we characterize hyperrings whose simple hypermodules are normal injective. Firstly, in the category **HypMod**(R), we study the concept of relative normal injectivity by means of short exact sequences and obtain some of their basic properties. The motivation for this concept comes from the works presented in [27–29]. Furthermore, for detailed information on simple hypermodules, we refer the reader to references [19, 30].

Definition 10. Let M be an R-hypermodule and $\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a short exact sequence of R-hypermodules. M is called $\mathbb{E}$-*injective* if the following Figure 2 can be extended by a normal homomorphism $g : B \longrightarrow M$ with $g\psi = f$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\psi} & B & \xrightarrow{\phi} & C & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle f} & & & & & & \\ & & M & & & & & & \end{array}$$

Figure 2. Normal injective with respect to $\mathbb{E}$.

The result follows from [23, Theorem 3].

Corollary 2. *An R-hypermodule M is normal injective if and only if M is $\mathbb{E}$-injective for every $\mathbb{E} \in \mathcal{H}Abs$.*

Let M be a hypermodule. By $i^{-1}(M)$ we denote the class of all short exact sequences $\mathbb{E}$, where M is $\mathbb{E}$-injective. The motivation for this notation comes from the works [28] carried out in the category of modules.

Proposition 1. *Let R be a hyperring and M be any R-hypermodule. Then, $\mathcal{H}Split \subseteq i^{-1}(M)$.*

Proof. Let $\mathbb{E} : 0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \longrightarrow 0 \in \mathcal{H}Split$ given any normal homomorphism $\tau : A \longrightarrow M$. Because $\mathbb{E}$ is splitting, by [26, Theorem 1], there exists a normal homomorphism $\lambda : B \longrightarrow A$ with $\lambda\psi = I_A$, where $I_A : A \longrightarrow A$ is the identity map.

Put $h = \tau\lambda$. Therefore, $h\psi = (\tau\lambda)\psi = \tau(\lambda\psi) = \tau I_A = \tau$. Thus, M is $\mathbb{E}$-injective. Consequently, we obtain that $\mathcal{H}Split \subseteq i^{-1}(M)$. □

Proposition 2. *Let M be an R-hypermodule and $\mathbb{E} : 0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \longrightarrow 0$ be a short exact sequence of R-hypermodules. If M is $\mathbb{E}$-injective, then every direct summand of M is $\mathbb{E}$-injective.*

Proof. Let N be a direct summand of M. To show that N is $\mathbb{E}$-injective, take any normal homomorphism $f : A \longrightarrow N$. Put $\iota f : A \longrightarrow M$, where $\iota : N \longrightarrow M$ is the inclusion map. Because M is $\mathbb{E}$-injective, there exists a normal homomorphism $g : B \longrightarrow M$ such that $g\psi = \iota f$.

Consider the canonical homomorphism $\pi : M \longrightarrow N$, and define $h := \pi g : B \longrightarrow N$. It follows that, for all $a \in A$, $(h\psi)(a) = \pi((g\psi)(a)) = \pi((\iota f)(a)) = (\pi\iota)(f(a)) = \pi(f(a)) = f(a)$. Thus, N is $\mathbb{E}$-injective. □

Proposition 3. *Let M be an R-hypermodule and $\mathbb{E} : 0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \longrightarrow 0$ be a short exact sequence of R-hypermodules. If M is $\mathbb{E}$-injective, M is $\mathbb{E}_A$-injective for every short exact sequence of the form $\mathbb{E}_A : 0 \longrightarrow * \longrightarrow A \longrightarrow * \longrightarrow 0$.*

Proof. Assume that the R-hypermodule M is $\mathbb{E}$-injective. Let $\mathbb{E}_A : \quad 0 \longrightarrow X \xrightarrow{\alpha} A \xrightarrow{\beta} Y \longrightarrow 0$ be a short exact sequence of hypermodules, and let $f : X \to M$ be a normal homomorphism. Because M is $\mathbb{E}$-injective, there exists a normal homomorphism $\lambda : B \to M$ such that $\lambda(\psi\alpha) = f$. Put $h = \lambda\psi$. Then, we obtain $h\alpha = f$. Therefore, the hypermodule M is $\mathbb{E}_A$-injective. □

As stated at the beginning of this section, our aim is to characterize hyperrings whose simple hypermodules are normal injective. For this purpose, we introduce the notion of a *co-semisimple hypermodule*, inspired by the classical concept of a co-semisimple module (or, V-modules) in module theory. The definition is motivated by the role that co-semisimple modules play in the characterization of *V-rings*, that is, rings over which every simple left module is injective [31–33]. We shall show that this hypercompositional analog provides a natural framework for characterizing hyperrings whose simple hypermodules are normal injective.

Definition 11. A hypermodule M is called a co-semisimple if every simple hypermodule S is $\mathbb{E}$-injective with respect to every short exact sequence of the form

$$\mathbb{E} : \quad 0 \longrightarrow * \longrightarrow \frac{M}{N} \longrightarrow * \longrightarrow 0$$

for all $N \leq M$.

Proposition 4. *The class of co-semisimple hypermodules is closed under subhypermodules and factor hypermodules.*

Proof. Let M be a co-semisimple hypermodule, and let $N \leq M$. First, we show that N is an H(V)-hypermodule. Let A be any subhypermodule of N. Consider the short exact sequence

$$\mathbb{E} : \quad 0 \longrightarrow \frac{N}{A} \xrightarrow{i} \frac{M}{A} \longrightarrow \frac{M}{N} \longrightarrow 0.$$

Let S be any simple R-hypermodule. By assumption, S is $\mathbb{E}$-injective. It follows from Proposition 3 that S is $\mathbb{E}_N$-injective for every short exact sequence of the form $\mathbb{E}_N : \quad 0 \longrightarrow * \longrightarrow \frac{N}{A} \longrightarrow * \longrightarrow 0$. Therefore, N is a co-semisimple hypermodule.

Because any factor hypermodule of $\frac{M}{N}$ is also a factor hypermodule of M, it is clear that $\frac{M}{N}$ is a co-semisimple hypermodule. □

The following theorem characterizes co-semisimple hypermodules. Let Rad(M) denote the Jacobson radical of a given hypermodule M. Note that $Rad(M)$ is the sum of all small subhypermodules of M [34].

Theorem 5. *For a hypermodule M, the following conditions are equivalent:*

(1) M is a co-semisimple hypermodule;

(2) $\mathrm{Rad}\left(\frac{M}{N}\right) = 0$ *(that is, radical-free) for every subhypermodule N of M;*

(3) Every subhypermodule N $(\neq M)$ *of M is the intersection of some maximal subhypermodules of M.*

Proof. (1) ⇒ (2) Let N be a subhypermodule of M and assume that $\frac{A}{N} \neq 0$ is a cyclic subhypermodule of $\mathrm{Rad}\left(\frac{M}{N}\right)$. Therefore $\frac{A}{N}$ is a small subhypermodule of $\frac{M}{N}$. Let $\frac{K}{N}$ be any maximal subhypermodule of $\frac{A}{N}$. Then $\frac{\frac{A}{N}}{\frac{K}{N}} \cong \frac{A}{K}$ is a simple hypermodule of $\frac{M}{K}$. Consider the short exact sequence

$$\mathbb{E}: \quad 0 \longrightarrow \frac{A}{K} \xrightarrow{\iota} \frac{M}{K} \longrightarrow \frac{M}{A} \longrightarrow 0.$$

Because $\frac{A}{K}$ is $\mathbb{E}$-injective, there exists a normal homomorphism $\lambda : \frac{M}{K} \longrightarrow \frac{A}{K}$ such that $\lambda\iota = \mathrm{id}_{A/K}$. Hence, by [19, Theorem 2], the sequence $\mathbb{E}$ splits and we obtain $\frac{M}{K} = \frac{A}{K} \oplus \frac{B}{K}$ for some subhypermodule B of M containing K. Observe from [35, Proposition 2.6] that $\frac{A}{K}$ is small in $\frac{M}{K}$. This implies $\frac{A}{K} = 0$, which is a contradiction. Therefore, $\mathrm{Rad}\left(\frac{M}{N}\right) = 0$.

(2) ⇒ (3) It is clear.

(3) ⇒ (1) Let S be any simple hypermodule and N any subhypermodule of M. Consider the short exact sequence

$$\mathbb{E}: \quad 0 \longrightarrow A \xrightarrow{\psi} \frac{M}{N} \longrightarrow K \longrightarrow 0.$$

Let $f : A \to S$ be a nonzero normal homomorphism. Because ψ is a normal monomorphism, we have $\overline{a} \in \psi(A) \backslash \psi(\ker f)$. Note that our assumption (3) is preserved under factor hypermodules. Hence, $\psi(\ker f)$ is contained in a maximal subhypermodule $\frac{K}{N}$ of $\frac{M}{N}$ such that $\overline{a} \notin \frac{K}{N}$. Therefore, $\psi(\ker f) \subseteq \frac{K}{N} \cap \psi(A)$ and $\frac{K}{N} + \psi(A) = \frac{M}{N}$. Because $\ker f$ is a maximal subhypermodule of A, it follows that $\psi(\ker f) = \frac{K}{N} \cap \psi(A)$.

Now, let $\overline{m} \in \frac{M}{N}$. Because $\frac{M}{N} = \frac{K}{N} + \psi(A)$, there exist $\overline{b} \in \frac{K}{N}$ and $a \in A$ such that $\overline{m} \in \overline{b} + \psi(a)$. Define a map

$$\lambda : \frac{M}{N} \longrightarrow S, \quad \lambda(\overline{m}) = f(a).$$

We show that λ is well-defined. Suppose $\overline{m} = \overline{n}$. It follows that $\overline{m} \in \overline{b} + \psi(a_1)$, $\overline{n} \in \overline{c} + \psi(a_2)$ with $\overline{b}, \overline{c} \in \frac{K}{N}$ and $a_1, a_2 \in A$. Then $0 \in \overline{m} - \overline{n} \subseteq (\overline{b} - \overline{c}) + (\psi(a_1) - \psi(a_2))$. Hence, $0 \in (\overline{b} - \overline{c}) + \psi(a_1 - a_2)$. Thus, by the reversibility law, there exists x such that $x \in \psi(a_2 - a_1) \cap (\overline{b} - \overline{c}) \subseteq \psi(\ker f)$. Therefore, $x = \psi(y)$ for some $y \in \ker f$. Since ψ is a normal monomorphism, we obtain that $y \in a_2 - a_1$. This implies

$$0 = f(y) \in f(a_2 - a_1) = f(a_2) - f(a_1),$$

so $f(a_1) = f(a_2)$. Thus, λ is well-defined.

Let $\overline{m}, \overline{n} \in \frac{M}{N}$ be arbitrary elements. Then there exist $\overline{b}, \overline{c} \in \frac{K}{N}$ and $a_1, a_2 \in A$ such that $\overline{m} \in \overline{b} + \psi(a_1)$ and $\overline{n} \in \overline{c} + \psi(a_2)$. Let $\overline{x} \in \overline{m} + \overline{n}$. Then $\overline{x} \in \overline{m} + \overline{n} \subseteq (\overline{b} + \overline{c}) + \psi(a_1 + a_2)$. Hence, there exist $\overline{y} \in \overline{b} + \overline{c}$ and $a \in a_1 + a_2$ such that $\overline{x} \in \overline{y} + \psi(a)$. Then $\lambda(\overline{x}) = f(a)$. Therefore,

$$\lambda(\overline{m} + \overline{n}) = \bigcup_{\overline{x} \in \overline{m} + \overline{n}} \lambda(\overline{x}) = \bigcup_{a \in a_1 + a_2} f(a) = f(a_1 + a_2).$$

Because f is a normal homomorphism, we obtain $f(a_1 + a_2) = f(a_1) + f(a_2)$ and hence $\lambda(\overline{m} + \overline{n}) = \lambda(\overline{m}) + \lambda(\overline{n})$. Moreover, for all $r \in R$, we have $\lambda(r\overline{m}) = r\lambda(\overline{m})$. Therefore, λ is a normal homomorphism.

Moreover, it is clear that $\lambda\psi = f$. Hence, S is $\mathbb{E}$-injective. Therefore, M is a co-semisimple hypermodule. □

Lemma 1. *Let R be a hyperring. Then, $H(R)$ is a co-semisimple hypermodule if and only if every simple left R-hypermodule is normal injective.*

Proof. It follows from [23, Theorem 4]. □

We now provide a characterization of hyperrings over which every simple hypermodule is normal injective.

Corollary 3. *For a hyperring R, the following statements are equivalent:*

(1) $H(R)$ is a co-semisimple hypermodule;

(2) Every simple left R-hypermodule is normal injective;

(3) Every left R-hypermodule M is a co-semisimple hypermodule;

(4) $\mathrm{Rad}(M) = 0$ for every left R-hypermodule M;

(5) Every left hyperideal I $(I \neq R)$ of R is the intersection of some maximal left hyperideals of R.

Proof. (1) ⇔ (2) It follows from Lemma 1.

(2) ⇒ (3) Let M be a hypermodule, and let X be a factor hypermodule of M. Let S be a simple hypermodule, and let $\mathbb{E}$ denote an arbitrary short exact sequence whose middle term is X. Then, by Corollary 2, S is $\mathbb{E}$-injective. Consequently, M is a co-semisimple hypermodule.

(3) ⇒ (4), (4) ⇒ (5), and (5) ⇒ (2) follow from Theorem 5. □

We now construct a co-semisimple hypermodule by means of V-modules. The study of the injectivity of simple modules, which plays a fundamental role in module theory, naturally leads to the notions of V-modules and left V-rings. For more information on these concepts, we refer the reader to [36–38].

Example 6. Let R be a unital ring, and let M be a left R-module which is a V-module. Assume that G is a subgroup of the monoid of R such that $xGyG = xyG$ for all $x, y \in R$. Now, we consider $\overline{M}$ as an $R(G)$-hypermodule. Observe that there is a one-to-one correspondence between the maximal submodules of M and the maximal subhypermodules of $\overline{M}$. Because this correspondence exists, every subhypermodule of $\overline{M}$ can be expressed as an intersection of maximal subhypermodules. Therefore, by Theorem 5, it follows that $\overline{M}$ is a co-semisimple hypermodule.

In particular, if R is a left V-ring which is not semisimple, then the hypermodule $H(R(G))$ is a co-semisimple hypermodule, which is not semisimple.

Let us consider the hypermodule $H(\mathbb{N})$ given in Example 3. The maximal subhypermodules of $H(\mathbb{N})$ are of the form $p\mathbb{N}$, where $p \in \mathbb{P}$ is a positive prime integer. Hence, $\mathrm{Rad}(H(\mathbb{N})) = \bigcap_{p\in\mathbb{P}} p\mathbb{N} = 0$. On the other hand, the factor hypermodule $\frac{\mathbb{N}}{4\mathbb{N}}$ has a unique maximal subhypermodule. Therefore,

$$\pi(\mathrm{Rad}(\mathbb{N})) = 0 \subseteq \frac{2\mathbb{N}}{4\mathbb{N}} = \mathrm{Rad}(\frac{\mathbb{N}}{4\mathbb{N}}) = \mathrm{Rad}(\pi(\mathbb{N})),$$

where $\pi : \mathbb{N} \to \frac{\mathbb{N}}{4\mathbb{N}}$ is the canonical projection. Consequently, the Jacobson radical is not necessarily preserved under normal epimorphisms. We now demonstrate, by means of co-semisimple hypermodules, that this equality holds (see [39, 40]).

Proposition 5. *Let M be a hypermodule. Then, the following statements are equivalent:*

(1) $\varphi(\mathrm{Rad}(M)) = \mathrm{Rad}(\varphi(M))$ *for every normal homomorphism* $\varphi : M \to N$*,*

(2) $\frac{M}{\mathrm{Rad}(M)}$ *is a co-semisimple hypermodule.*

Proof. (1) $\Rightarrow$ (2) By Theorem 5, it is sufficient to show that every factor hypermodule of $\frac{M}{\mathrm{Rad}(M)}$ is radical-free. Let A be any factor hypermodule of $\frac{M}{\mathrm{Rad}(M)}$. Consider the canonical projection $\pi : M \longrightarrow \frac{M}{\mathrm{Rad}(M)}$. Let $\varphi = f\pi$, where $f : \frac{M}{\mathrm{Rad}(M)} \longrightarrow A$ is any normal epimorphism. Then, $\mathrm{Rad}(M) \subseteq \ker(\varphi)$, and hence

$$\mathrm{Rad}(A) = \mathrm{Rad}(\varphi(M)) = \varphi(\mathrm{Rad}(M)) = 0.$$

Thus, A is radical-free.

(2) $\Rightarrow$ (1) Let $\varphi : M \to N$ be any normal homomorphism. Then, we consider $\frac{\varphi(M)}{\varphi(\mathrm{Rad}(M))}$ as a factor hypermodule of $\frac{M}{\mathrm{Rad}(M)}$. By Theorem 5, we have $\mathrm{Rad}\left(\frac{\varphi(M)}{\varphi(\mathrm{Rad}(M))}\right) = 0$, and hence

$$\mathrm{Rad}(\varphi(M)) = \varphi(\mathrm{Rad}(M)).$$

□

Theorem 6. *Let R be a hyperring. Then, the following statements are equivalent:*

(1) $\frac{R}{\mathrm{Rad}(R)}$ *is a co-semisimple hypermodule;*

(2) $\mathrm{Rad}(M) = \mathrm{Rad}(R)M$ *for every left R-hypermodule M;*

(3) For every normal epimorphism $\varphi\colon M \to N$ of left R-hypermodules, $\mathrm{Rad}(N) = \varphi(\mathrm{Rad}(M))$.

Proof. (1) $\Rightarrow$ (2) Let M be a left R-hypermodule and consider the canonical projection $\pi\colon M \to \frac{M}{\mathrm{Rad}(R)M}$. Because $Rad(R)$ is contained in the annihilator of the hypermodule $\frac{M}{\mathrm{Rad}(R)M}$, $\frac{M}{\mathrm{Rad}(R)M}$ is naturally a left $\frac{R}{\mathrm{Rad}(R)}$-hypermodule. By (1), $\mathrm{Rad}\left(\frac{M}{\mathrm{Rad}(R)M}\right) = 0$. Hence, $\pi(\mathrm{Rad}(M)) = 0$, and so

$$\mathrm{Rad}(M) \subseteq \ker(\pi) = \mathrm{Rad}(R)M.$$

On the other hand, for every maximal subhypermodule L of M, we have $\mathrm{Rad}(R)M \subseteq L$. Taking intersection over all such L, we get $\mathrm{Rad}(R)M \subseteq \mathrm{Rad}(M)$. Thus, $\mathrm{Rad}(M) = \mathrm{Rad}(R)M$.

(2) $\Rightarrow$ (3) Let $\varphi : M \to N$ be a normal epimorphism. Then, $\varphi(\mathrm{Rad}(M)) \subseteq \mathrm{Rad}(N)$. By (2), we have $\mathrm{Rad}(N) = \mathrm{Rad}(R)N$. Because $N = \varphi(M)$, $\mathrm{Rad}(N) = \mathrm{Rad}(R)\varphi(M) = \varphi(\mathrm{Rad}(R)M)$. Again by (2), $\mathrm{Rad}(R)M = \mathrm{Rad}(M)$; hence, $\mathrm{Rad}(N) = \varphi(\mathrm{Rad}(M))$.

(3) $\Rightarrow$ (1) Consider the canonical projection $\pi\colon R \to \frac{R}{\mathrm{Rad}(R)}$. Then, by (3), $\mathrm{Rad}\left(\frac{R}{\mathrm{Rad}(R)}\right) = \pi(\mathrm{Rad}(R)) = 0$. Hence, $\frac{R}{\mathrm{Rad}(R)}$ is a co-semisimple hypermodule. □

5. Discussion

The results obtained in this study reveal fundamental structural differences between the category **HypMod**$(\cdot)$ of hypermodules and the category of modules. Although cyclic free hypermodules are shown to be normal projective, the absence of a fully additive structure necessitates the use of alternative techniques, such as internal direct sum decompositions. Furthermore, normal projective cyclic hypermodules need not be free. In addition, the characterization of hyperrings whose simple hypermodules are normal injective provides further insight into injectivity-type properties in this setting. These results suggest several directions for future research, including the investigation of flatness in the category of hypermodules.

6. Conclusions

In this paper, we investigated the structure of free hypermodules over a hyperring and their relationship with normal projectivity. It is shown that every cyclic free hypermodule is normal projective. We also constructed free hypermodules via quotient hypermodules associated with normal multiplicative subgroups of a ring. In addition, we characterized hyperrings whose simple hypermodules are normal injective, providing a dual perspective to normal projectivity. This characterization contributes to the development of relative homological methods in the category of hypermodules. Overall, the results of this study contribute to a deeper understanding of homological properties in the category of hypermodules and reveal several directions for further research, including the study of flatness, the existence of enough projective and injective objects in hypercompositional algebra.

Author contributions

All authors contributed equally in this work. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors used ChatGPT (OpenAI, GPT-5.5) solely for language editing and improvement of grammar, readability, and presentation. No AI tools were used in the development of the mathematical results, proofs, analysis, or scientific conclusions of this work. The authors take full responsibility for the content, interpretation, and conclusions of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. F. Marty, Sur une généralisation de la notion de groupe, *8th Congress of Scandinavian Mathematicians*, Stockholm, 1934, 45–49.
2. B. Davvaz, V. Leoreanu-Fotea, *Hyperring theory and applications*, International Academic Press, 2007.
3. M. Krasner, Approximation des corps valués complets de caractéristique $p \neq 0$ par ceux de caractéristique 0, *Colloque d' Algèbre Supérieure*, Centre Belge de Recherches Mathématiques, 1957, 129–206.
4. M. Krasner, A class of hyperrings and hyperfields, *Int. J. Math. Math. Sci.*, **6** (1983), 307–311. https://doi.org/10.1155/S0161171283000265
5. A. Madanshekaf, Exact category of hypermodules, *Int. J. Math. Math. Sci.*, **2006** (2006), 31368. https://doi.org/10.1155/IJMMS/2006/31368
6. C. Massouros, C. G. Massouros, An overview of the foundations of the hypergroup theory, *Mathematics*, **9** (2021), 1054. https://doi.org/10.3390/math9091014
7. S. Nakamura, M. L. Reyes, Categories of hypermagmas, hypergroups, and related hyperstructures, *J. Algebra*, **676** (2025), 408–474. https://doi.org/10.1016/j.jalgebra.2025.03.056
8. J. Mittas, Hypergroupes canoniques valués et hypervalués, *Math. Balk.*, **1** (1971), 181–185.
9. J. Mittas, Hypergroupes canoniques, *Math. Balk.*, **2** (1972), 165–179.
10. J. Mittas, Sur les hyperanneaux et les hypercorps, *Math. Balk.*, **3** (1973), 368–382.
11. R. L. Roth, Character and conjugacy class hypergroups of a finite group, *Ann. Mat. Pura Appl.*, **105** (1975), 295–311. https://doi.org/10.1007/BF02414935
12. W. Prenowitz, Projective geometries as multigroups, *Amer. J. Math.*, **65** (1943), 235–256. https://doi.org/10.2307/2371812
13. W. Prenowitz, Descriptive geometries as multigroups, *Trans. Amer. Math. Soc.*, **59** (1946), 333–380. https://doi.org/10.1090/S0002-9947-1946-0015126-6
14. C. G. Massouros, Methods of constructing hyperfields, *Int. J. Math. Math. Sci.*, **8** (1985), 725–728. https://doi.org/10.1155/S0161171285000813
15. C. Massouros, G. Massouros, On the borderline of fields and hyperfields, part II – Enumeration and classification of the hyperfields of order 7, *AIMS Math.*, **10** (2025), 21287–21421. https://doi.org/10.3934/math.2025951
16. C. G. Massouros, On the theory of hyperrings and hyperfields, *Algebra Logika*, **24** (1985), 728–742.
17. C. G. Massouros, Free and cyclic hypermodules, *Ann. Mat. Pura Appl.*, **150** (1988), 153–166. https://doi.org/10.1007/BF01761468

18. H. Shojaei, D. Fasino, Isomorphism theorems in the primary categories of Krasner hypermodules, *Symmetry*, **11** (2019), 687. https://doi.org/10.3390/sym11050687

19. E. Türkmen, B. Nişancı Türkmen, H. Bordbar, A hyperstructural approach to semisimplicity, *Axioms*, **13** (2024), 81. https://doi.org/10.3390/axioms13020081

20. G. Massouros, C. Massouros, Hypercompositional algebra, computer sciences and geometry, *Mathematics*, **8** (2020), 1338. https://doi.org/10.3390/math8081338

21. B. Mitchell, *Theory of categories*, Academic Press, 1965.

22. R. Ameri, H. Shojaci, Projective and injective Krasner hypermodules, *J. Algebra Appl.*, **20** (2021), 2150186. https://doi.org/10.1142/S0219498821501863

23. H. Bordbar, I. Cristea, About the normal projectivity and injectivity of Krasner hypermodules, *Axioms*, **10** (2021), 83. https://doi.org/10.3390/axioms10020083

24. E. Türkmen, B. Nişancı Türkmen, H. Bordbar, A characterization of normal injective and normal projective hypermodules, *Axioms*, **13** (2024), 410. https://doi.org/10.3390/axioms13060410

25. H. Bordbar, Torsion elements and torsionable hypermodules, *Mathematics*, **11** (2023), 4525. https://doi.org/10.3390/math11214525

26. E. Kaynar, B. Nişancı Türkmen, E. Türkmen, A note on hyperrings and hypermodules, *Ital. J. Pure Appl. Math.*, **53** (2025), 104–116.

27. R. Alizade, M. Diril, Y. Durğun, Subinjective portfolios and rings with a linearly ordered subinjective profile, *Commun. Algebra*, **53** (2025), 408–416. https://doi.org/10.1080/00927872.2024.2377807

28. P. Aydoğdu, Y. Durğun, The opposite of injectivity by proper classes, *Quaest. Math.*, **46** (2023), 1547–1570. https://doi.org/10.2989/16073606.2022.2109221

29. Y. M. Demirci, B. Nişancı Türkmen, E. Türkmen, Rings with modules having a restricted injectivity domain, *São Paulo J. Math. Sci.*, **14** (2020), 312–326. https://doi.org/10.1007/s40863-019-00153-4

30. K. T. Howel, A. Goswami, B. Davvaz, Primitive hyperideals and hyperstructure spaces of hyperrings, *Categ. Gen. Algebr. Struct. Appl.*, **22** (2025), 157–173. https://doi.org/10.48308/cgasa.2023.234185.1460

31. K. R. Fuller, Relative projectivity and injectivity classes determined by simple modules, *J. London Math. Soc.*, **s2-5** (1972), 423–431. https://doi.org/10.1112/jlms/s2-5.3.423

32. Y. Hirano, Regular modules and V-modules, *Hiroshima Math. J.*, **11** (1981), 125–142.

33. G. O. Michler, O. E. Villamayor, On rings whose simple modules are injective, *J. Algebra*, **25** (1973), 185–201. https://doi.org/10.1016/0021-8693(73)90088-4

34. A. R. Moniri Hamzekolaee, M. Norouzi, V. Leoreanu-Fotea, A new approach to smallness in hypermodules, *Algebraic Struct. Appl.*, **8** (2021), 131–145. https://doi.org/10.22034/as.2020.1962

35. B. Talaee, Small subhypermodules and their applications, *Rom. J. Math. Computer Sci.*, **3** (2013), 5–14.

36. Y. Durğun, S. Özdemir, On subinjectivity domains of finitely generated modules, *Commun. Algebra*, **53** (2025), 5170–5182. https://doi.org/10.1080/00927872.2025.2507144

37. Y. Durğun, Sa-supplement submodules, *Bull. Korean Math. Soc.*, **58** (2021), 147–161. https://doi.org/10.4134/BKMS.b200128

38. E. Büyükaşık, Y. Durğun, Coneat submodules and coneat-flat modules, *J. Korean Math. Soc.*, **51** (2021), 1305–1319. https://doi.org/10.4134/JKMS.2014.51.6.1305

39. W. Pinsan, H. Xianhui, Some new characterizations of V-rings, *Processing of the Second Japan-China Symposium on Ring Theory*, 1995, 147–149.

40. R. Wisbauer, *Foundations of module and ring theory*, Gordon and Breach Science Publishers, 1991.