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Research article

Study of complex-valued differential equations with the Hilfer fractional derivative

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Abstract: In this paper, a class of nonlinear impulsive pantograph-type Hilfer complex-valued systems (IPH-CVSes) is investigated. Two cases corresponding to the fractional orders $\rho \in (0,1)$ and $\rho \in (1,2)$ are studied. Explicit solution representations for the considered models are derived. Moreover, existence and uniqueness results are established using Krasnoselskii's fixed-point theorem and the Banach contraction principle. Finally, an application arising from an aerodynamic flow model is provided to demonstrate the applicability of the obtained theoretical results.

Keywords: Hilfer fractional derivative; impulsive pantograph equations; complex-valued systems; Banach contraction principle; Krasnoselskii's fixed-point theorem
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1. Introduction

Fractional calculus (FC) traces its origin from the correspondence between Gottfried Wilhelm Leibniz and Guillaume de l'Hôpital in 1695 regarding derivatives of noninteger order. Since then, FC has attracted significant attention due to its effectiveness in describing systems with memory and hereditary properties that cannot be adequately modeled using classical integer-order operators [1, 2].

Among the various fractional operators, the Riemann-Liouville (R-L) [3] and Liouville-Caputo (L-C) [4] derivatives are widely used in theoretical and applied studies. However, these operators possess certain limitations regarding the treatment of initial conditions and memory characterization. To overcome these drawbacks, the Hilfer fractional derivative (HFD) [5] was introduced as an interpolating operator between the R-L and L-C derivatives, combining the advantages of both formulations through the additional type parameter σ. A comparison of the fundamental characteristics of the R-L, L-C, and HFDs is provided in Table 1.

Table 1. Key distinctions between R-L, L-C, and HFD.

Characteristics	R-L ([5], Eq (2.1.5))	L-C ([6], Definition 3)	HFD ([7], Definition 18)
Memory representation	Effectively captures nonlocal and hereditary memory effects.	Provides memory description with clear physical interpretation.	Enables adjustable memory behavior through order and type parameters.
Initial conditions	Requires fractional-order initial conditions.	Admits classical integer-order initial conditions.	Allows flexible handling of both types via interpolation.
Analytical tractability	Suitable for rigorous theoretical analysis.	Convenient for numerical and applied analysis.	Balances analytical and numerical tractability.

Motivated by these advantages, several authors have employed generalized fractional operators in different applications. Zahra et al. [8] investigated weighted FC with respect to functions. The applicability of FC has also been explored in diverse settings, where Dawuken et al. [9] utilized fractional equations in image processing, and Liao et al. [10] established stability criteria for fractional-order switched systems via Lyapunov-based methods. Zhang et al. [11] proposed an exponential Euler approach for multidelay Caputo-Fabrizio fractional equations and subsequently examined global exponential stability in discrete-time Caputo-Fabrizio neural network models [12]. In addition, Singh et al. [13] studied nonlinear equations involving Hilfer-Prabhakar derivatives, and Wang et al. [14] investigated distributed-order diffusion equations with Hilfer-Prabhakar operators. Furthermore, Nieto et al. [15] discussed recent advances in fixed-point theory and their applications to nonlinear systems.

In addition to memory effects, many dynamical systems experience abrupt changes at specific moments, which are effectively modeled using impulsive differential equations (IDEs). Table 2 presents a comparative analysis of instantaneous and noninstantaneous impulses in dynamical systems. Compared with classical continuous systems, impulsive systems provide a more realistic formulation for describing instantaneous perturbations arising in applications such as population dynamics [16], biological systems [17], pharmacokinetics [18], industrial robotics [19], and optimal control [20].

Table 2. Distinction between instantaneous and noninstantaneous impulses in dynamical systems.

Features	Instantaneous impulses ([21], Section 1.1)	Noninstantaneous impulses ([22], Section 1.1)
Duration	Very short (point-in-time).	Finite.
Modeling focus	Occurs at a point t_i.	Occurs over an interval $(t_i, s_i]$.
State effect	Discontinuity in state.	Continuous changes over an interval.

Consequently, impulsive systems involving HFDs have received considerable attention in recent years. Xu et al. [23] investigated the stability of discrete-time impulsive stochastic systems with hybrid delays, and Lu et al. [24] analyzed stability criteria for nonlinear systems involving periodic impulses.

From the viewpoint of existence theory, Lan [25] established existence and uniqueness results for nonlinear fractional Cauchy-type problems, whereas Arjunan et al. [26] extended such investigations to impulsive systems with generalized R-L derivatives. Further developments involving HFDs includes the following. Trivedi et al. [27] studied noninstantaneous impulsive integro-differential evolution systems including HFD; Radhakrishnan et al. [28] addressed the existence and uniqueness of random Hilfer pantograph IDEs; and Shah et al. [29] investigated the existence and uniqueness of Hilfer IDEs under nonlocal conditions, indicating a gradual progression from classical impulsive models to generalized HFDs.

On the other hand, many practical models naturally involve complex-valued quantities. This leads to the study of complex-valued differential systems (CVDSes), where the unknown state variable is complex-valued, namely $z(t) = x(t) + iy(t)$. Due to their ability to simultaneously represent amplitude and phase information, CVDSes arise in laser systems [30], robotics [31], magnetic resonance imaging [32], and AI-assisted wearable devices [33]. Fang et al. [34] studied the existence and uniqueness of nonlinear impulsive CVDSes, and Phan et al. [35] analyzed existence, uniqueness, and boundedness results for CVDSes. Related studies on complex-valued fractional equations were carried out by Higgins et al. [36], who examined approximation techniques for fractional derivatives of complex analytic functions. Afassinou et al. [37] employed fixed-point techniques to investigate differential and R-L equations, and Gissy et al. [38] analyzed R-L fractional integral equations in CVDSes. Extending these developments, Zahed [39] established existence and uniqueness results for fractional differential equations in complex-valued suprametric systems. Furthermore, Verma et al. [40] investigated complex diffusion equations involving Hilfer-Hadamard derivatives. These investigations reflect the increasing interest in fractional CVDSes and their analytical developments.

In view of the above discussion, it may be observed that although considerable progress has been achieved in the study of fractional and CDSes, comparatively few works have considered HFDs with CVDSes involving both impulsive effects and pantograph-type delays. Moreover, most existing investigations are confined to real-valued settings. Motivated by this observation, the present work investigates a class of impulsive pantograph-type Hilfer complex-valued systems (IPH-CVSes) corresponding to the cases $\rho \in (0, 1)$ and $\rho \in (1, 2)$, and it establishes the associated solution representations together with existence and uniqueness results. Accordingly, the mathematical formulation of the proposed systems is given in Section 3.

In addition, the novelty of this work lies in the investigation of a class of IPH-CVSes for both $\rho \in (0, 1)$ and $\rho \in (1, 2)$. The analysis incorporates impulsive conditions and proportional delay arguments governed by HFDs in CVDSes, where only a few authors have addressed such problems in the existing literature. By employing standard fixed-point techniques, sufficient conditions for the existence and uniqueness of solutions are established. Furthermore, the effectiveness of the obtained theoretical results is supported through an application arising from airflow dynamics around an aircraft wing under delayed and impulsive effects.

We now highlight the main contributions of the proposed work.

(1) A class of IPH-CVSes is investigated for both fractional orders $\rho \in (0, 1)$ and $\rho \in (1, 2)$, providing a unified formulation for the study of lower-and higher-order dynamics.

(2) The incorporation of the pantograph term $z(\kappa t)$ together with impulsive conditions enables the proposed model to describe proportional delay effects and instantaneous perturbations simultaneously.

(3) For the higher-order case, both function-type and derivative-type impulsive conditions are considered, which extends several existing results available in the literature.

The remainder of this paper is organized as follows. Section 2 presents the necessary preliminaries, including basic definitions and auxiliary lemmas required for the subsequent analysis. Section 3 formulates the proposed impulsive pantograph systems corresponding to the cases $\rho \in (0,1)$ and $\rho \in (1,2)$. Section 4 is divided into two subsections devoted to these two classes of systems, where the corresponding solution representations, weighted spaces, assumptions, and existence and uniqueness results are established. Section 5 presents an application illustrating the effectiveness of the obtained theoretical results through an aerodynamic flow model. Finally, Section 6 concludes the paper and discusses possible future research directions.

2. Preliminaries

In this section, we introduce the basic notations, function spaces, and preliminary concepts used throughout the paper. Our approach is based on the fundamental ideas presented in [34], Section 2.

Let $C(J,\mathbb{R})$ denote the Banach space of all continuous real-valued functions defined on the interval J, endowed with the norm

$$\|z\| = \sup_{t\in J} |z(t)|.$$

Further, let $C(J,\mathbb{R}^n)$ denote the Banach space of all continuous vector-valued functions from J into $\mathbb{R}^n$, equipped with the norm

$$\|z\|_{JR} = \sup_{t\in J}\left\{\sum_{l=1}^{n} |z_l(t)|^2\right\}^{1/2},$$

where $z_l(t)$ is the lth component of the vector function $z(t)$.

Next, we denote by $C(J,\mathbb{C}^n)$ the Banach space of all complex-valued functions defined on J, given by

$$C(J,\mathbb{C}^n) = \Big\{z(t) = \mathrm{Re}(z(t)) + i\ \mathrm{Im}(z(t))\ :\ \mathrm{Re}(z(t)), \mathrm{Im}(z(t)) \in C(J,\mathbb{R}^n)\Big\},$$

where $i = \sqrt{-1}$, $Re(z(t))$ and $Im(z(t))$ denote the real and imaginary parts of $z(t)$, respectively. This space is equipped with the norm

$$\|z\|_{JC} = (\|\mathrm{Re}(z)\|_{JR}^2 + \|\mathrm{Im}(z)\|_{JR}^2)^{1/2}.$$

Finally, we denote by $PC(J,\mathbb{C}^n)$ the Banach space of all complex-valued piecewise continuous functions defined by

$$\begin{aligned} PC(J,\mathbb{C}^n) = \Big\{z : J \to \mathbb{C}^n \ \Big|\ & \\ & z(t) = Re(z(t)) + iIm(z(t)) : \\ & Re(z(t_l^{\pm})),\ Im(z(t_l^{\pm})) \text{ exist}, \\ & Re(z(t_l^{-})) = Re(z(t_l)),\ Im(z(t_l^{-})) = Im(z(t_l)), \\ & Re(z(t)), Im(z(t)) \in C(J_k,\mathbb{R}^n)\Big\}, \end{aligned}$$

where $l = 0,1,2,\ldots,m,\ t \in J = [t_0,T],\ t_0 < t_1 < \cdots < t_m < T$ with subintervals $J_0 = [t_0,t_1],\ J_l = (t_l,t_{l+1}],\ l = 1,2,\ldots,m-1,\ J_m = (t_m,T]$.

The space $PC(J, \mathbb{C}^n)$ equipped with the norm

$$\|z\|_{PC} = \max_{0 \le l \le m} \sup_{t \in J_l} \Big\{ \|z(t)\|_{JC} \Big\}.$$

We now recall the fundamental definitions and lemmas that will be required in the subsequent sections.

Definition 2.1. *([5], Eq (2.1.1)) The R-L fractional integral of order* $\rho > 0$ *of a function* $z(t)$ *is defined by*

$$I^{\rho}_{t_0^+} z(t) = \frac{1}{\Gamma(\rho)} \int_{t_0}^{t} (t-s)^{\rho-1} z(s)\, ds, \quad t > t_0.$$

Definition 2.2. *([5], Eq (2.1.5)) Let* $\rho \in [n-1, n)$ *with* $n \in \mathbb{Z}^+$*. The left-sided R-L fractional derivative of order* ρ *of a function* $z(t)$ *is defined by*

$$(D^{\rho}_{t_0^+} z)(t) = \frac{1}{\Gamma(n-\rho)} \left(\frac{d}{dt}\right)^n \int_{t_0}^{t} (t-s)^{n-\rho-1} z(s)\, ds, \quad t > t_0.$$

Definition 2.3. *([7], Definition 18) The HFD of order* $\rho \in (n-1, n)$ *and type* $\sigma \in [0, 1]$ *of a function* $z(t)$ *is defined by*

$$D^{\rho,\sigma}_{t_0^+} z(t) = I^{\sigma(n-\rho)}_{t_0^+} \left(\frac{d^n}{dt^n}\right) I^{(1-\sigma)(n-\rho)}_{t_0^+} z(t), \quad t > t_0.$$

Lemma 2.4. *([5], Property 2.1, Lemma 2.19) Let* $\rho > 0$*,* $\sigma > 0$*, and* $\nu > 0$*. Then the following properties hold:*

$$I^{\rho}_{t_0^+} I^{\sigma}_{t_0^+} z(t) = I^{\rho+\sigma}_{t_0^+} z(t)$$

and

$$I^{\rho}_{t_0^+} (t-t_0)^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\rho+\nu)} (t-t_0)^{\rho+\nu-1}. \tag{2.1}$$

Remark 2.5.

(i) In addition to Definition 2.3, the HFD of a function $z(t)$ *can also be represented in the form*

$$D^{\rho,\sigma}_{t_0^+} z(t) = I^{\sigma(n-\rho)}_{t_0^+} D^{\nu}_{t_0^+} z(t), \quad t > t_0, \tag{2.2}$$

where

$$\nu = \rho + n\sigma - \rho\sigma.$$

(ii) From Definition 2.3 and Eq (2.1)*, we obtain the identity*

$$D^{\rho,\sigma}_{t_0^+} (t-t_0)^{\tau-1} = 0. \tag{2.3}$$

Lemma 2.6. *[**Krasnoselskii**]([41], Theorem 1.8) Let* $\mathcal{S}$ *be a nonempty, closed, and convex subset of a Banach space* $\mathscr{X}$*. Let* $\mathscr{M}$ *and* $\mathscr{N}$ *be operators satisfying the following conditions:*

(1) $\mathscr{M}p + \mathscr{N}q \in \mathcal{S}$ *for all* $p, q \in \mathcal{S}$*;*

(2) $\mathscr{M}$ *is continuous and compact;*

(3) $\mathscr{N}$ *is a contraction mapping.*

Then there exists $r \in \mathcal{S}$ *such that*

$$r = \mathscr{M}r + \mathscr{N}r.$$

3. Problem formulation

Following the above discussion, we now present the mathematical formulation of the proposed system. The model is constructed using the HFD and incorporates both impulsive effects and proportional delay. For completeness, the formulation is divided into two cases corresponding to $\rho \in (0,1)$ and $\rho \in (1,2)$. The formulation and analysis of the considered CVDSes are inspired by the methodology and ideas developed in [34].

Type I: Fractional order $\rho \in (0,1)$. Consider the following IPH-CVS:

$$\begin{cases} D^{\rho,\sigma}_{t_0^+} z(t) = g(t, z(t), z(\kappa t)), & t \in J \setminus \{t_1, t_2, \ldots, t_m\}, \\ \Delta I^{1-\tau}_{t_0^+} z(t_l) = I_l(z(t_l^-)), & l = 1, 2, \ldots, m, \\ I^{1-\tau}_{t_0^+} z(t_0) = \eta_0, & \eta_0 \in \mathbb{C}^n, \end{cases} \tag{3.1}$$

where $\rho \in (0,1)$, $\sigma \in [0,1]$, $\tau = \rho+\sigma-\rho\sigma$, $\kappa \in (0,1)$, and $J = [t_0, T]$ with $t_0 < t_1 < \ldots < t_m < t_{m+1} = T$. Here, $D^{\rho,\sigma}_{t_0^+}(\cdot)$ denotes the HFD of order $\rho \in (0,1)$ and type σ, and $I^{1-\tau}_{t_0^+}$ denotes the R-L fractional integral operator. Furthermore, $z \in PC(J, \mathbb{C}^n)$, $g : J \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$, $I_l \in C(\mathbb{C}^n, \mathbb{C}^n)$, $l = 1, 2, \ldots, m$.

The impulsive condition is defined by

$$\Delta I^{1-\tau}_{t_0^+} z(t_l) = I^{1-\tau}_{t_0^+} z(t_l^+) - I^{1-\tau}_{t_0^+} z(t_l^-),$$

where

$$I^{1-\tau}_{t_0^+} z(t_l^+) = \lim_{\epsilon \to 0^+} I^{1-\tau}_{t_0^+} z(t_l + \epsilon),$$

and

$$I^{1-\tau}_{t_0^+} z(t_l^-) = \lim_{\epsilon \to 0^-} I^{1-\tau}_{t_0^+} z(t_l + \epsilon).$$

Type II: Fractional order $\rho \in (1,2)$. Consider the following IPH-CVS:

$$\begin{cases} D^{\rho,\sigma}_{t_0^+} z(t) = g(t, z(t), z(\kappa t)), & t \in J \setminus \{t_1, t_2, \ldots, t_m\}, \\ \Delta I^{(1-\sigma)(2-\rho)}_{t_0^+} z(t_l) = I_l(z(t_l^-)), & l = 1, 2, \ldots, m, \\ \Delta (I^{(1-\sigma)(2-\rho)}_{t_0^+} z(t_l))' = \bar{I}_l(z(t_l^-)), & l = 1, 2, \ldots, m, \\ I^{(1-\sigma)(2-\rho)}_{t_0^+} z(t_0) = \eta_1, & \eta_1 \in \mathbb{C}^n, \\ (I^{(1-\sigma)(2-\rho)}_{t_0^+} z(t))' \Big|_{t=t_0} = \eta_2, & \eta_2 \in \mathbb{C}^n. \end{cases} \tag{3.2}$$

Here, $D^{\rho,\sigma}_{t_0^+}(\cdot)$ denotes the HFD of order $\rho \in (1,2)$, and $I^{(1-\sigma)(2-\rho)}_{t_0^+}(\cdot)$ denotes the left-sided R-L fractional integral operator of order $(1-\sigma)(2-\rho)$, $\kappa \in (0,1)$. Let $\vartheta = \rho + 2\sigma - \rho\sigma$. Then $2 - \vartheta = (1-\sigma)(2-\rho) \geq 0$, and $\rho \leq \vartheta \leq 2$. Moreover, I_l, $\bar{I}_l \in C(\mathbb{C}^n, \mathbb{C}^n)$, $g : J \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ is assumed to be continuous and uniformly bounded. The jump conditions are given by

$$\Delta I^{2-\vartheta}_{t_0^+} z(t_l) = I^{2-\vartheta}_{t_0^+} z(t_l^+) - I^{2-\vartheta}_{t_0^+} z(t_l^-),$$
$$\Delta (I^{2-\vartheta}_{t_0^+} z(t_l))' = (I^{2-\vartheta}_{t_0^+} z(t_l^+))' - (I^{2-\vartheta}_{t_0^+} z(t_l^-))'.$$

Furthermore,

$$\begin{aligned} I_{t_0^+}^{2-\vartheta} z(t_l^+) &= \lim_{\epsilon \to 0^+} I_{t_0^+}^{2-\vartheta} z(t_l + \epsilon), \\ I_{t_0^+}^{2-\vartheta} z(t_l^-) &= \lim_{\epsilon \to 0^+} I_{t_0^+}^{2-\vartheta} z(t_l - \epsilon), \\ (I_{t_0^+}^{2-\vartheta} z(t_l^+))' &= \lim_{\epsilon \to 0^+} (I_{t_0^+}^{2-\vartheta} z(t_l + \epsilon))', \\ (I_{t_0^+}^{2-\vartheta} z(t_l^-))' &= \lim_{\epsilon \to 0^+} (I_{t_0^+}^{2-\vartheta} z(t_l - \epsilon))'. \end{aligned}$$

4. Main results

This section presents the main theoretical results concerning the existence and uniqueness of solutions for the proposed systems. The results are established under appropriate conditions using fixed-point methods in suitable function spaces.

4.1. Existence and uniqueness of solutions for type I system

We begin by defining the weighted piecewise continuous space together with its norm. This is followed by the necessary assumptions and lemmas through which the corresponding Volterra integral equation (VIE) is derived, and the existence and uniqueness of its solution are subsequently investigated.

Consider the weighted space associated with the system of order $(0, 1)$ [42] defined by

$$C_{1-\tau}(J, \mathbb{C}^n) = \left\{ z : (t_0, T] \to \mathbb{C}^n : (t - t_0)^{1-\tau} z(t) \in C(J, \mathbb{C}^n) \right\}.$$

Next, define the weighted space of piecewise continuous functions by

$$\begin{aligned} PC_{1-\tau}(J, \mathbb{C}^n) = \Big\{ z : (t_0, T] \to \mathbb{C}^n \ \Big| \ & \\ & (t - t_0)^{1-\tau} z(t) \in C((t_l, t_{l+1}), \mathbb{C}^n), \quad l = 0, 1, 2, \ldots, m, \\ & I_{t_0^+}^{1-\tau} z(t_l^+),\ I_{t_0^+}^{1-\tau} z(t_l^-) \text{ exist}, \\ & I_{t_0^+}^{1-\tau} z(t_l^-) = I_{t_0^+}^{1-\tau} z(t_l), \quad l = 1, 2, \ldots, m \Big\}. \end{aligned}$$

Clearly, $PC_{1-\tau}(J, \mathbb{C}^n)$ is a Banach space endowed with the norm

$$\|z\|_{PC_{1-\tau}} = \sup_{t \in J} \left\{ (t - t_0)^{1-\tau} \|z(t)\| \right\}.$$

Assumptions for type I. Let the function

$$g : J \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n,$$

be continuous and satisfy the following assumptions.

($\mathscr{A}_1$) For any $z \in PC_{1-\tau}(J, \mathbb{C}^n)$,

$$g(t, z(t), z(\kappa t)) \in PC_{1-\tau}(J, \mathbb{C}^n).$$

($\mathscr{A}_2$) There exists a function $\mathscr{L}(t) \in L^1(J, \mathbb{R}^+)$ such that

$$\|g(t, x, u) - g(t, y, v)\| \leq \mathscr{L}(t)(\|x - y\| + \|u - v\|),$$

for all $u, v, x, y \in \mathbb{C}^n$ and $t \in J$. Moreover,

$$\|\mathscr{L}\|_{L^1} = \int_{t_0}^{T} \mathscr{L}(t)\, dt.$$

($\mathcal{A}_3$) There exist constants $\theta_l \geq 0,\ l = 1, 2, \ldots, m$ such that

$$\|I_l(z_1) - I_l(z_2)\| \leq \theta_l \|z_1 - z_2\|,$$

for all $z_1, z_2 \in \mathbb{C}^n$, and

$$\sum_{l=1}^{m} \theta_l < 1,$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{C}^n$.

Lemma 4.1. *([7], Lemma 22 and Eqs (6) and (7)) Let* $\rho \in (0, 1)$, *and* $\sigma \in [0, 1]$. *Define* $\tau = \rho + \sigma - \rho\sigma$. *Then the following relations hold:*

(1)

$$I^{\rho}_{t_0^+} D^{\rho,\sigma}_{t_0^+} y(t) = y(t) - \frac{(t - t_0)^{\tau-1}}{\Gamma(\tau)} I^{(1-\sigma)(1-\rho)}_{t_0^+} y(t_0^+).$$

(2)

$$D^{\rho,\sigma}_{t_0^+} I^{\rho}_{t_0^+} y(t) = y(t).$$

Lemma 4.2. *([41], Lemma 1.3) Let* $\mathscr{X}$ *be a Banach space, and let* $\mathscr{T}_{1-\tau} \subset PC_{1-\tau}(J, \mathscr{X})$. *Suppose that the following conditions are satisfied:*

(1) $\mathscr{T}_{1-\tau}$ *is a uniformly bounded subset of* $PC_{1-\tau}(J, \mathscr{X})$;
(2) $\mathscr{T}_{1-\tau}$ *is equicontinuous on each interval* $(t_l, t_{l+1}),\ l = 0, 1, \ldots, m$, *where* $t_{m+1} = T$;
(3) The sets

$$\begin{aligned} \mathscr{T}_{1-\tau} &= \Big\{z(t) : z \in \mathscr{T}_{1-\tau}\Big\},\ t \in J \setminus \{t_1, t_2, \ldots, t_m\}, \\ \mathscr{T}_{1-\tau}(t_l^+) &= \Big\{z(t_l^+) : z \in \mathscr{T}_{1-\tau}\Big\}, \\ \mathscr{T}_{1-\tau}(t_l^-) &= \Big\{z(t_l^-) : z \in \mathscr{T}_{1-\tau}\Big\} \end{aligned}$$

are relatively compact subsets of $\mathscr{X}$.

Then $\mathscr{T}_{1-\tau}$ *is a relatively compact subset of* $PC_{1-\tau}(J, \mathscr{X})$.

Lemma 4.3. *([43], Eq (2.4)) Let* $g : J \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ *be continuous. Then a function* $z \in PC_{1-\tau}(J, \mathbb{C}^n)$ *is a solution of* (3.1) *if and only if z satisfies the VIE*

$$\begin{cases} \frac{(t-t_0)^{\tau-1}}{\Gamma(\tau)}\eta_0 + \frac{1}{\Gamma(\rho)} \int_{t_0}^{t} (t - v)^{\rho-1} g(v, z(v), z(\kappa v))\, dv, & t \in [t_0, t_1], \\ \frac{(t-t_0)^{\tau-1}}{\Gamma(\tau)}\Big(\eta_0 + \sum_{t_l<t} I_l(z(t_l^-))\Big) + \frac{1}{\Gamma(\rho)} \int_{t_0}^{t} (t - v)^{\rho-1} g(v, z(v), z(\kappa v))\, dv, & t \in (t_l, t_{l+1}],\ l = 1, 2, \ldots, m. \end{cases} \tag{4.1}$$

Proof. Assume that $z \in PC_{1-\tau}(J, \mathbb{C}^n)$ satisfies System (3.1).

Step 1. For $t \in [t_0, t_1]$, consider the problem

$$\begin{cases} D^{\rho,\sigma}_{t_0^+} z(t) = g\big(t, z(t), z(\kappa t)\big), & t \in [t_0, t_1], \\ I^{1-\tau}_{t_0^+} z(t_0) = \eta_0, & \eta_0 \in \mathbb{C}^n. \end{cases} \tag{4.2}$$

Then Problem (4.2) is equivalent to the following VIE [44]:

$$z(t) = \frac{(t-t_0)^{\tau-1}}{\Gamma(\tau)}\eta_0 + \frac{1}{\Gamma(\rho)}\int_{t_0}^{t}(t-v)^{\rho-1}g(v,z(v),z(\kappa v))\,dv, \quad t\in[t_0,t_1]. \tag{4.3}$$

Step 2. For $t\in(t_1,t_2]$, we have

$$D_{t_0^+}^{\rho,\sigma}z(t) = g(t,z(t),z(\kappa t)),\ t\in(t_1,t_2], \tag{4.4}$$

together with the impulsive condition

$$I_{t_0^+}^{1-\tau}z(t_1^+) - I_{t_0^+}^{1-\tau}z(t_1^-) = I_1(z(t_1^-)).$$

Applying the operator $I_{t_0^+}^{\rho}$ to both sides of Eq (4.4) and using Lemma 4.1, we obtain

$$z(t) = \frac{(t-t_0)^{\tau-1}}{\Gamma(\tau)}I_{t_0^+}^{1-\tau}z(t_0^+) + \frac{1}{\Gamma(\rho)}\int_{t_0}^{t}(t-v)^{\rho-1}g(v,z(v),z(\kappa v))\,dv. \tag{4.5}$$

The impulsive condition at $t=t_1$ is given by

$$I_{t_0^+}^{1-\tau}z(t_1^+) = I_{t_0^+}^{1-\tau}z(t_1^-) + I_1(z(t_1^-)). \tag{4.6}$$

Substituting Eq (4.6) into Eq (4.5), we obtain

$$z(t) = \frac{(t-t_0)^{\tau-1}}{\Gamma(\tau)}\Big(\eta_0 + I_1(z(t_1^-))\Big) + \frac{1}{\Gamma(\rho)}\int_{t_0}^{t}(t-v)^{\rho-1}g(v,z(v),z(\kappa v))\,dv,\ t\in(t_1,t_2].$$

Similarly, the impulsive condition at $t=t_2$ is given by

$$I_{t_0^+}^{1-\tau}z(t_2^+) = I_{t_0^+}^{1-\tau}z(t_2^-) + I_2(z(t_2^-)). \tag{4.7}$$

Proceeding in the same way, we obtain

$$z(t) = \frac{(t-t_0)^{\tau-1}}{\Gamma(\tau)}\Big(\eta_0 + I_1(z(t_1^-)) + I_2(z(t_2^-))\Big) + \frac{1}{\Gamma(\rho)}\int_{t_0}^{t}(t-v)^{\rho-1}g(v,z(v),z(\kappa v))\,dv.$$

Continuing this process, we obtain

$$z(t) = \frac{(t-t_0)^{\tau-1}}{\Gamma(\tau)}\Big(\eta_0 + \sum_{i=1}^{l}I_i(z(t_i^-))\Big) + \frac{1}{\Gamma(\rho)}\int_{t_0}^{t}(t-v)^{\rho-1}g(v,z(v),z(\kappa v))\,dv,$$

where $t\in(t_l,t_{l+1}]$, and $l=1,2,\dots,m$.

Conversely, assume that $z\in PC_{1-\tau}(J,\mathbb{C}^n)$ satisfies Eq (4.1). For $t\in[t_0,t_1]$, we have

$$z(t) = \frac{(t-t_0)^{\tau-1}}{\Gamma(\tau)}\eta_0 + \frac{1}{\Gamma(\rho)}\int_{t_0}^{t}(t-v)^{\rho-1}g(v,z(v),z(\kappa v))\,dv.$$

Applying the operator $D_{t_0^+}^{\rho,\sigma}$ to both sides and using Lemma 4.1 together with Eq (2.3), we get

$$D_{t_0^+}^{\rho,\sigma}z(t) = g(t,z(t),z(\kappa t)),\ t\in[t_0,t_1].$$

Now, for $t \in (t_l, t_{l+1}],\ l = 1, 2, \ldots, m$ from Eq (4.1), we have

$$z(t) = \frac{(t-t_0)^{\tau-1}}{\Gamma(\tau)}\left(\eta_0 + \sum_{i=1}^{l} I_i(z(t_i^-))\right) + \frac{1}{\Gamma(\rho)}\int_{t_0}^{t}(t-v)^{\rho-1}g(v, z(v), z(\kappa v))\, dv.$$

Applying the operator $D_{t_0^+}^{\rho,\sigma}$ to both sides and using Eq (2.3) together with Lemma 4.1, we obtain

$$\begin{aligned} D_{t_0^+}^{\rho,\sigma} z(t) &= \left(\eta_0 + \sum_{i=1}^{l} I_i(z(t_i^-))\right) D_{t_0^+}^{\rho,\sigma}\left(\frac{(t-t_0)^{\tau-1}}{\Gamma(\tau)}\right) + D_{t_0^+}^{\rho,\sigma} I_{t_0^+}^{\rho} g(t, z(t), z(\kappa t)) \\ &= g(t, z(t), z(\kappa t)). \end{aligned}$$

Thus, the function $z(t)$ satisfies the differential equation in Eq (3.1). Next, we prove that z satisfies the initial and impulse conditions of Eq (3.1). Applying the R-L fractional operator $I_{t_0^+}^{1-\tau}(\cdot)$ to both sides of Eq (4.3), we obtain

$$I_{t_0^+}^{1-\tau} z(t) = \eta_0 I_{t_0^+}^{1-\tau}\left(\frac{(t-t_0)^{\tau-1}}{\Gamma(\tau)}\right) + \frac{1}{\Gamma(\rho+1-\tau)}\int_{t_0}^{t}(t-v)^{\rho-\tau}g(v, z(v), z(\kappa v))\, dv.$$

Evaluating at $t = t_0$, we obtain

$$I_{t_0^+}^{1-\tau} z(t_0) = \eta_0.$$

Next, we prove that z satisfies the impulse condition of Eq (3.1). Applying $I_{t_0^+}^{1-\tau}(\cdot)$ to both sides of Eq (4.1), for $t \in (t_l, t_{l+1}]$, we obtain

$$I_{t_0^+}^{1-\tau} z(t) = \left(\eta_0 + \sum_{i=1}^{l} I_i(z(t_i^-))\right) I_{t_0^+}^{1-\tau}\left(\frac{(t-t_0)^{\tau-1}}{\Gamma(\tau)}\right) + \frac{1}{\Gamma(\rho+1-\tau)}\int_{t_0}^{t}(t-v)^{\rho-\tau}g(v, z(v), z(\kappa v))\, dv, \tag{4.8}$$

and for $t \in (t_{l-1}, t_l]$, we have

$$I_{t_0^+}^{1-\tau} z(t) = \left(\eta_0 + \sum_{i=1}^{l-1} I_i(z(t_i^-))\right) I_{t_0^+}^{1-\tau}\frac{(t-t_0)^{\tau-1}}{\Gamma(\tau)} + \frac{1}{\Gamma(\rho+1-\tau)}\int_{t_0}^{t}(t-v)^{\rho-\tau}g(v, z(v), z(\kappa v))\, dv. \tag{4.9}$$

From Eqs (4.8) and (4.9), we obtain

$$I_{t_0^+}^{1-\tau} z(t_l^+) - I_{t_0^+}^{1-\tau} z(t_l^-) = \sum_{i=1}^{l} I_i(z(t_i^-)) - \sum_{i=1}^{l-1} I_i(z(t_i^-)) = I_l(z(t_l^-)).$$

Thus, the impulsive condition of System (3.1) is satisfied.

This completes the proof. □

Theorem 4.4. *Let the assumptions* $(\mathscr{A}_1)$–$(\mathscr{A}_3)$ *hold. Then the system* (3.1) *admits at least one solution in* $PC_{1-\tau}(J, \mathbb{C}^n)$.

Proof. By Lemma 4.3, the system (3.1) is equivalent to the VIE

$$z(t) = \frac{(t-t_0)^{\tau-1}}{\Gamma(\tau)}\left(\eta_0 + \sum_{t_0<t_l<t} I_l(z(t_l^-))\right) + \frac{1}{\Gamma(\rho)}\int_{t_0}^{t}(t-v)^{\rho-1}g(v, z(v), z(\kappa v))\, dv, \quad t \in J. \tag{4.10}$$

Define the set Λ_a by

$$\Lambda_a = \Big\{z \in PC_{1-\tau}(J, \mathbb{C}^n) : I_{t_0^+}^{1-\tau} z(t_0) = \eta_0,\ \|z\|_{PC_{1-\tau}(J,\mathbb{C}^n)} \leq a\Big\},$$

where

$$a \geq 2\left(\frac{1}{\Gamma(\tau)}\Big(\|\eta_0\| + \sum_{l=1}^{m} \mathcal{M}_l\Big) + \frac{\mathscr{M}(T-t_0)^{1-\tau+\rho}}{\Gamma(\rho+1)}\right),$$

and

$$\begin{aligned} \mathcal{M}_l &= \sup_{z\in\Lambda_a} \|I_l(z(t_l^-))\|, \\ \mathscr{M} &= \sup_{t\in J} \|g(v,0,0)\|. \end{aligned}$$

Define

$$\mathcal{W} := \frac{2}{\Gamma(\rho)} \|\mathscr{L}\|_{L^1} (T-t_0)^{\rho-1}, \tag{4.11}$$

and assume

$$\mathcal{W} \leq \frac{1}{2}. \tag{4.12}$$

Define the operators $\mathcal{P},\ \mathcal{Q} : \Lambda_a \to PC_{1-\tau}(J, \mathbb{C}^n)$ by

$$\begin{aligned} \mathcal{P}z(t) &= \frac{(t-t_0)^{\tau-1}}{\Gamma(\tau)} \eta_0, \\ \mathcal{Q}z(t) &= \frac{(t-t_0)^{\tau-1}}{\Gamma(\tau)} \sum_{t_0<t_l<t} I_l(z(t_l^-)) + \frac{1}{\Gamma(\rho)} \int_{t_0}^{t} (t-v)^{\rho-1} g(v, z(v), z(\kappa v))\, dv. \end{aligned}$$

Eq (4.10) can then be written as

$$z = \mathcal{P}z + \mathcal{Q}z,\ z \in \Lambda_a.$$

Step 1. We show that $\mathcal{P}r + \mathcal{Q}s \in \Lambda_a$ for all $r, s \in \Lambda_a$.

Let $r, s \in \Lambda_a$. Using Assumption $(\mathscr{A}_1)$, for $t \in J$, we have

$$\begin{aligned} &(t-t_0)^{1-\tau} \|(\mathcal{P}r(t) + \mathcal{Q}s(t))\| \\ &= (t-t_0)^{1-\tau} \left\| \frac{(t-t_0)^{\tau-1}}{\Gamma(\tau)} \eta_0 + \frac{(t-t_0)^{\tau-1}}{\Gamma(\tau)} \sum_{t_0<t_l<t} I_l(s(t_l^-)) + \frac{1}{\Gamma(\rho)} \int_{t_0}^{t} (t-v)^{\rho-1} g(v, s(v), s(\kappa v))\, dv \right\| \\ &\leq \frac{1}{\Gamma(\tau)}\left(\|\eta_0\| + \sum_{l=1}^{m} \|I_l(s(t_l^-))\|\right) + \frac{(t-t_0)^{1-\tau}}{\Gamma(\rho)} \int_{t_0}^{t} (t-v)^{\rho-1} \|g(v, s(v), s(\kappa v))\|\, dv \\ &\leq \frac{1}{\Gamma(\tau)}\left(\|\eta_0\| + \sum_{l=1}^{m} \|I_l(s(t_l^-))\|\right) + \frac{(t-t_0)^{1-\tau}}{\Gamma(\rho)} \int_{t_0}^{t} (t-v)^{\rho-1} \|g(v, s(v), s(\kappa v)) - g(v,0,0)\| dv \\ &\quad + \frac{(t-t_0)^{1-\tau}}{\Gamma(\rho)} \int_{t_0}^{t} (t-v)^{\rho-1} \|g(v,0,0)\|\, dv \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\tau)}\left(\|\eta_0\| + \sum_{l=1}^{m} \|I_l(s(t_l^-))\|\right) + \frac{(t-t_0)^{1-\tau}}{\Gamma(\rho)} \int_{t_0}^{t} (t-v)^{\rho-1} \mathscr{L}(v)(\|s(v)\| + \|s(\kappa v)\|)\, dv \\
&\quad + \frac{\mathscr{M}(t-t_0)^{1-\tau}}{\Gamma(\rho)} \int_{t_0}^{t} (t-v)^{\rho-1}\, dv \\
&\leq \frac{1}{\Gamma(\tau)}\left(\|\eta_0\| + \sum_{l=1}^{m} \|I_l(s(t_l^-))\|\right) + \frac{2\|s\|_{PC_{1-\tau}(J,\mathbb{C}^n)}}{\Gamma(\rho)} \int_{t_0}^{t} (t-v)^{\rho-1} \mathscr{L}(v)\, dv + \frac{\mathscr{M}(t-t_0)^{1-\tau}}{\Gamma(\rho)} \int_{t_0}^{t} (t-v)^{\rho-1}\, dv \\
&\leq \frac{1}{\Gamma(\tau)}\left(\|\eta_0\| + \sum_{l=1}^{m} \mathcal{M}_l\right) + \frac{2a}{\Gamma(\rho)}\|\mathscr{L}\|_{L^1}(T-t_0)^{\rho-1} + \frac{\mathscr{M}(T-t_0)^{1-\tau+\rho}}{\Gamma(\rho+1)} \\
&\leq \frac{a}{2} + \frac{2a}{\Gamma(\rho)}\|\mathscr{L}\|_{L^1}(T-t_0)^{\rho-1}.
\end{aligned}
$$

Thus,

$$\|\mathcal{P}r + Qs\|_{PC_{1-\tau}(J,\mathbb{C}^n)} \leq a, \quad I_{t_0^+}^{1-\tau}(\mathcal{P}r + Qs)(t_0) = \eta_0,$$

which shows

$$\mathcal{P}r + Qs \in \Lambda_a.$$

Step 2. It follows that the operator $\mathcal{P}$ is a contraction with contraction constant zero.
Step 3. We now show that the operator Q is compact and continuous.
(i) Continuity: Because g is continuous, the integral operator Q is continuous on Λ_a.
(ii) Uniform boundedness: From Step 1, Q is uniformly bounded on Λ_a.
(iii) Equicontinuity: Let $z \in \Lambda_a$, and take $t_1, t_2 \in J$ with $t_1 < t_2$. Then, by Assumption $(\mathcal{A}_2)$,

$$
\begin{aligned}
\|Qz(t_2) - Qz(t_1)\| &= \Bigg\| \frac{(t_2-t_0)^{\tau-1}}{\Gamma(\tau)} \sum_{t_0<t_l<t_2} I_l(z(t_l^-)) + \frac{1}{\Gamma(\rho)} \int_{t_0}^{t_2} (t_2-v)^{\rho-1} g(v, z(v), z(\kappa v))\, dv \\
&\qquad - \frac{(t_1-t_0)^{\tau-1}}{\Gamma(\tau)} \sum_{t_0<t_l<t_1} I_l(z(t_l^-)) - \frac{1}{\Gamma(\rho)} \int_{t_0}^{t_1} (t_1-v)^{\rho-1} g(v, z(v), z(\kappa v))\, dv \Bigg\| \\
&\leq \frac{1}{\Gamma(\tau)} \Bigg\| (t_2-t_0)^{\tau-1} \sum_{t_0<t_l<t_2} I_l(z(t_l^-)) - (t_1-t_0)^{\tau-1} \sum_{t_0<t_l<t_1} I_l(z(t_l^-)) \Bigg\| \\
&\qquad + \frac{\|g\|_{PC_{1-\tau}(J,\mathbb{C}^n)}}{\Gamma(\rho+1)} (t-t_0)^{\tau-1} [(t_2-t_0)^{\rho} - (t_1-t_0)^{\rho}].
\end{aligned}
$$

As t_2 approaches t_1 the above difference tends to zero. Hence, the family $\left\{Qz : z \in \Lambda_a\right\}$ is equicontinuous. Because Q is continuous, uniformly bounded, and equicontinuous, it follows that Q is relatively compact on Λ_a. By Lemma 4.2, the operator Q is compact. Therefore, all assumptions of Lemma 2.6 are satisfied for the operators $\mathcal{P}$ and Q. Hence, the system (3.1) admits at least one solution on J.

This completes the proof. □

Theorem 4.5. *Assume that the function g is continuous and satisfies conditions* $(\mathcal{A}_1)$–$(\mathcal{A}_3)$. *Then the ystem* (3.1) *admits a unique solution in the weighted space* $PC_{1-\tau}(J, \mathbb{C}^n)$.

Proof. Consider the set Λ_a as defined in Theorem 4.4. Define the operator $\mathcal{T} : \Lambda_a \to PC_{1-\tau}(J, \mathbb{C}^n)$ by

$$\mathcal{T}z(t) = \frac{(t-t_0)^{\tau-1}}{\Gamma(\tau)}\left(\eta_0 + \sum_{t_0<t_l<t} I_l(z(t_l^-))\right) + \frac{1}{\Gamma(\rho)}\int_{t_0}^{t}(t-v)^{\rho-1}g(v, z(v), z(\kappa v))dv,\ t \in J.$$

Assume that

$$\mathscr{U} := \frac{(T-t_0)^{\tau-1}}{\Gamma(\tau)}\sum_{l=1}^{m}\theta_l + \frac{2\|\mathscr{L}\|_{L^1}}{\Gamma(\rho)}(T-t_0)^{\rho-1} < 1.$$

Next, we show that $z = \mathcal{T}z$ is a fixed point.

Step 1. We prove that $\mathcal{T}\Lambda_a \subset \Lambda_a$.

For any $z \in \Lambda_a$ and $t \in J$, using assumption $(\mathcal{A}_2)$, we have

$$\begin{aligned}
&\|\mathcal{T}z(t)\|\\
&\leq \frac{(t-t_0)^{\tau-1}}{\Gamma(\tau)}\left(\|\eta_0\| + \sum_{t_0<t_l<t}\|I_l(z(t_l^-))\|\right) + \frac{1}{\Gamma(\rho)}\int_{t_0}^{t}(t-v)^{\rho-1}\|g(v, z(v), z(\kappa v))\|\,dv\\
&\leq \frac{(t-t_0)^{\tau-1}}{\Gamma(\tau)}\left(\|\eta_0\| + \sum_{t_0<t_l<t}\|I_l(z(t_l^-))\|\right) + \frac{2}{\Gamma(\rho)}\int_{t_0}^{t}(t-v)^{\rho-1}\mathscr{L}(v)\|z(v)\|\,dv + \frac{\mathscr{M}}{\Gamma(\rho+1)}(t-t_0)^{\rho}\\
&\leq \frac{(t-t_0)^{\tau-1}}{\Gamma(\tau)}\left(\|\eta_0\| + \sum_{t_0<t_l<t}\|I_l(z(t_l^-))\|\right) + \frac{2\|\mathscr{L}\|_{L^1}\|z\|_{PC_{1-\tau}(J,\mathbb{C}^n)}}{\Gamma(\rho)}(t-t_0)^{\tau-1}(T-t_0)^{\rho-1} + \frac{\mathscr{M}}{\Gamma(\rho+1)}(t-t_0)^{\rho}.
\end{aligned}$$

Multiplying both sides by $(t-t_0)^{1-\tau}$, we obtain

$$\begin{aligned}
\|(t-t_0)^{1-\tau}\mathcal{T}z(t)\| &\leq \frac{1}{\Gamma(\tau)}\left(\|\eta_0\| + \sum_{t_0<t_l<t}\mathcal{M}_l\right) + \frac{2a\|\mathscr{L}\|_{L^1}}{\Gamma(\rho)}(T-t_0)^{\rho-1} + \frac{\mathscr{M}}{\Gamma(\rho+1)}(T-t_0)^{1-\tau+\rho}\\
&\leq \frac{a}{2} + \frac{2a\|\mathscr{L}\|_{L^1}}{\Gamma(\rho)}(T-t_0)^{\rho-1}.
\end{aligned}$$

Hence,

$$\|Tz\|_{PC_{1-\tau}(J,\mathbb{C}^n)} \leq a,$$

which shows that

$$\mathcal{T}\Lambda_a \subset \Lambda_a.$$

Step 2. We prove that the operator $\mathcal{T}$ is a contraction on Λ_a.

Let $r, s \in \Lambda_a$ and $t \in J$. Then, by Assumptions $(\mathcal{A}_2)$ and $(\mathcal{A}_3)$, we obtain

$$\begin{aligned}
\|(t-t_0)^{1-\tau}(\mathcal{T}r(t) - \mathcal{T}s(t))\| &\leq (t-t_0)^{1-\tau}\left\|\frac{(t-t_0)^{\tau-1}}{\Gamma(\tau)}\sum_{t_0<t_l<t}\left(I_l(r(t_l^-)) - I_l(s(t_l^-))\right)\right\|\\
&\quad + \frac{(t-t_0)^{1-\tau}}{\Gamma(\rho)}\int_{t_0}^{t}(t-v)^{\rho-1}\|g(v, r(v), r(\kappa v)) - g(v, s(v), s(\kappa v))\|\,dv\\
&\leq \frac{1}{\Gamma(\tau)}\sum_{l=1}^{m}\theta_l\|r(t_l^-) - s(t_l^-)\| + \frac{2(t-t_0)^{1-\tau}}{\Gamma(\rho)}\int_{t_0}^{t}(t-v)^{\rho-1}\mathscr{L}(v)\|r-s\|\,dv
\end{aligned}$$

$$\leq \frac{(T-t_0)^{\tau-1}}{\Gamma(\tau)} \sum_{l=1}^{m} \theta_l \|r-s\|_{PC_{1-\tau}(J,\mathbb{C}^n)} + \frac{2\|\mathscr{L}\|_{L^1}}{\Gamma(\rho)}(T-t_0)^{\rho-1}\|r-s\|_{PC_{1-\tau}(J,\mathbb{C}^n)}$$
$$= \mathscr{U}\|r-s\|_{PC_{1-\tau}(J,\mathbb{C}^n)}.$$

Because $\mathscr{U} < 1$, the operator $\mathcal{T}$ is a contraction mapping on Λ_a. Therefore, by the Banach contraction principle, $\mathcal{T}$ has a unique fixed point in Λ_a. This fixed point is the unique solution of the system (3.1) in $PC_{1-\tau}(J,\mathbb{C}^n)$. □

4.2. Existence and uniqueness of solutions for type II system

This section addresses the existence and uniqueness of solutions for Problem (3.2). First, we define the piecewise continuous weighted space for systems of order $(1,2)$.

Consider the following weighted space ([45], Section 2):

$$C_{2-\vartheta}(J,\mathbb{C}^n) = \left\{ z : (t_0,T] \to \mathbb{C}^n : (t-t_0)^{2-\vartheta} z(t) \in C(J,\mathbb{C}^n) \right\}.$$

For the piecewise continuous weighted space ([45], Section 2), let $PC_{2-\vartheta}(J,\mathbb{C}^n)$ denote the set of functions $z : (t_0,T] \to \mathbb{C}^n$ such that

(1) $(t-t_0)^{2-\vartheta} z(t)$ is continuous on each interval (t_l,t_{l+1}) for $l = 0,1,2,\ldots,m$.

(2) The limits $\lim_{t\to t_l^+}(t-t_0)^{2-\vartheta}z(t)$ and $\lim_{t\to t_l^-}(t-t_0)^{2-\vartheta}z(t)$ exist for $l = 1,2,\ldots,m$.

This space is a Banach space equipped with the norm,

$$\|z\|_{PC_{2-\vartheta}(J,\mathbb{C}^n)} = \sup_{t\in J} \left\{ (t-t_0)^{2-\vartheta} \|z(t)\| \right\}.$$

Assumptions for type II.

($\mathscr{A}_4$) For any $z \in PC_{2-\vartheta}(J,\mathbb{C}^n)$,

$$g(t,z(t),z(\kappa t)) \in PC_{2-\vartheta}(J,\mathbb{C}^n).$$

($\mathscr{A}_5$) There exist constants $\lambda_l \geq 0$ and $\bar{\lambda}_l \geq 0$ for $l = 1,2,\ldots,m$, such that

$$\|I_l(z_1) - I_l(z_2)\| \leq \lambda_l \|z_1 - z_2\|,$$
$$\|\bar{I}_l(z_1) - \bar{I}_l(z_2)\| \leq \bar{\lambda}_l \|z_1 - z_2\|,$$

for all $z_1, z_2 \in \mathbb{C}^n$. Moreover, the following conditions hold:

$$\sum_{l=1}^{m} \lambda_l < 1, \quad \sum_{l=1}^{m} \bar{\lambda}_l < 1,$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{C}^n$.

Lemma 4.6. *([5], Corollary 2.1) Let* $\nu > 0$*, and let* $n = [\nu] + 1$*. Then the equality* $D^{\nu}_{t_0^+} z(t) = 0$ *holds if and only if*

$$z(t) = \sum_{k=1}^{n} b_k (t-t_0)^{\nu-k},$$

where $b_k \in \mathbb{C}^n$*, for* $k = 1,2,\ldots,n$ *are arbitrary constants.*

Lemma 4.7. *([7], Lemma 22) Let* $\rho \in (1, 2)$, *and* $\sigma \in [0, 1]$. *Then the following relation holds:*

$$D^{\rho;\sigma}_{t_0^+} \, I^{\rho}_{t_0^+} g(t, z(t), z(\kappa t)) = g(t, z(t), z(\kappa t)).$$

Lemma 4.8. *([41], Lemma 1.3) Let* $\mathscr{X}$ *be a Banach space, and let* $\mathscr{T}_{2-\vartheta} \subset PC_{2-\vartheta}(J, \mathscr{X})$. *If the following conditions are satisfied:*

(1) $\mathscr{T}_{2-\vartheta}$ *is a uniformly bounded subset of* $PC_{2-\vartheta}(J, \mathscr{X})$;

(2) $\mathscr{T}_{2-\vartheta}$ *is equicontinuous on each interval* (t_l, t_{l+1}), *for* $l = 0, 1, \ldots, m$, *where* $t_{m+1} = T$;

(3) The sets

$$\begin{aligned} \mathscr{T}_{2-\vartheta} &= \{z(t) : z \in \mathscr{T}_{2-\vartheta}\}, \quad t \in J \setminus \{t_1, t_2, \ldots, t_m\}, \\ \mathscr{T}_{2-\vartheta}(t_l^+) &= \{z(t_l^+) : z \in \mathscr{T}_{2-\vartheta}\}, \\ \mathscr{T}_{2-\vartheta}(t_l^-) &= \{z(t_l^-) : z \in \mathscr{T}_{2-\vartheta}\}, \end{aligned}$$

are relatively compact subsets of $\mathscr{X}$, *then* $\mathscr{T}_{2-\vartheta}$ *is a relatively compact subset of* $PC_{2-\vartheta}(J, \mathscr{X})$.

Lemma 4.9. *([45], Lemma 9) Let g be a continuous function. Then a function* $z \in PC_{2-\vartheta}(J, \mathbb{C}^n)$ *is a solution of the system* (3.2) *if and only if z satisfies the following VIE:*

$$z(t) = \begin{cases} \dfrac{(t-t_0)^{\vartheta-1}}{\Gamma(\vartheta)}\eta_2 + \dfrac{(t-t_0)^{\vartheta-2}}{\Gamma(\vartheta-1)}\eta_1 + \dfrac{1}{\Gamma(\rho)}\displaystyle\int_{t_0}^{t}(t-v)^{\rho-1} g(v, z(v), z(\kappa v))\,dv, & t \in [t_0, t_1], \\ \dfrac{(t-t_0)^{\vartheta-1}}{\Gamma(\vartheta)}\eta_2 + \dfrac{(t-t_0)^{\vartheta-2}}{\Gamma(\vartheta-1)}\eta_1 + \displaystyle\sum_{t_0<t_l<t} \dfrac{\bar{I}_l(z(t_l^-))}{\Gamma(\vartheta)}(t-t_0)^{\vartheta-2}((t-t_0) - (\vartheta-1)(t_l - t_0)) & \\ + \displaystyle\sum_{t_0<t_l<t} \dfrac{I_l(z(t_l^-))}{\Gamma(\vartheta-1)}(t-t_0)^{\vartheta-2} + \dfrac{1}{\Gamma(\rho)}\displaystyle\int_{t_0}^{t}(t-v)^{\rho-1} g(v, z(v), z(\kappa v))\,dv, & t \in (t_l, t_{l+1}]. \end{cases} \tag{4.13}$$

Proof. Let z be a solution of the system (3.2). We show that z satisfies the VIE (4.13). Applying the fractional integral operator $I^{\rho}_{t_0^+}$ to both sides of Eq (3.2) and using Eq (2.2), we obtain

$$I^{\vartheta}_{t_0^+} D^{\vartheta}_{t_0^+} z(t) = I^{\rho}_{t_0^+} g(t, z(t), z(\kappa t)).$$

(i) Solution on the first interval $[t_0, t_1]$.

From Lemma 4.6, the general solution can be written as

$$z(t) = I^{\rho}_{t_0^+} g(t, z(t), z(\kappa t)) + m_1 (t-t_0)^{\vartheta-1} + m_2 (t-t_0)^{\vartheta-2}. \tag{4.14}$$

Applying the fractional integral operator $I^{2-\vartheta}_{t_0^+}$ to Eq (4.14), we obtain

$$I^{2-\vartheta}_{t_0^+} z(t) = I^{2-\vartheta+\rho}_{t_0^+} g(t, z(t), z(\kappa t)) + m_1 I^{2-\vartheta}_{t_0^+} (t-t_0)^{\vartheta-1} + m_2 I^{2-\vartheta}_{t_0^+} (t-t_0)^{2-\vartheta} (t-t_0)^{\vartheta-2}.$$

Using the property of the R-L fractional integral in Lemma 2.4, we obtain

$$I^{2-\vartheta}_{t_0^+} z(t) = I^{2-\vartheta+\rho}_{t_0^+} g(t, z(t), z(\kappa t)) + m_1 (t-t_0)\Gamma(\vartheta) + m_2 \Gamma(\vartheta-1). \tag{4.15}$$

Differentiating Eq (4.15) with respect to t gives

$$\left(I_{t_0^+}^{2-\vartheta} z(t)\right)' = I_{t_0^+}^{1-\vartheta+\rho} g(t, z(t), z(\kappa t)) + m_1 \Gamma(\vartheta). \tag{4.16}$$

Using the initial conditions at $t = t_0$, we obtain

$$m_1 = \frac{\eta_2}{\Gamma(\vartheta)}, \quad m_2 = \frac{\eta_1}{\Gamma(\vartheta - 1)}.$$

Thus, for $t \in [t_0, t_1]$, we get

$$z(t) = \frac{(t-t_0)^{\vartheta-1}}{\Gamma(\vartheta)} \eta_2 + \frac{(t-t_0)^{\vartheta-2}}{\Gamma(\vartheta-1)} \eta_1 + \frac{1}{\Gamma(\rho)} \int_{t_0}^{t} (t-v)^{\rho-1} g(v, z(v), z(\kappa v))\, dv.$$

(ii) Solution after the first impulse ($t > t_1$).

Let $n_1,\ n_2$ be constants on $(t_1, t_2]$. According to Lemma 4.6, we obtain

$$z(t) = I_{t_0^+}^{\rho} g(t, z(t), z(\kappa t)) + n_1 (t-t_0)^{\vartheta-1} + n_2 (t-t_0)^{\vartheta-2}. \tag{4.17}$$

Before the impulse ($t < t_1$), we have

$$z^-(t) = \frac{(t-t_0)^{\vartheta-1}}{\Gamma(\vartheta)} \eta_2 + \frac{(t-t_0)^{\vartheta-2}}{\Gamma(\vartheta-1)} \eta_1 + \frac{1}{\Gamma(\rho)} \int_{t_0}^{t} (t-v)^{\rho-1} g(v, z(v), z(\kappa v))\, dv. \tag{4.18}$$

Apply $I_{t_0^+}^{2-\vartheta}$, and differentiating gives

$$I_{t_0^+}^{2-\vartheta} z^-(t) = \eta_2 (t-t_0) + \eta_1 + \frac{1}{\Gamma(2-\vartheta+\rho)} \int_{t_0}^{t} (t-v)^{1-\vartheta+\rho} g(v, z(v), z(\kappa v))\, dv. \tag{4.19}$$

Differentiating Eq (4.19) with respect to t, we obtain

$$(I_{t_0^+}^{2-\vartheta} z^-(t))' = \eta_2 + \frac{1}{\Gamma(1-\vartheta+\rho)} \int_{t_0}^{t} (t-v)^{\rho-\vartheta} g(v, z(v), z(\kappa v))\, dv. \tag{4.20}$$

After the impulse ($t > t_1$), we have

$$z^+(t) = n_1 (t-t_0)^{\vartheta-1} + n_2 (t-t_0)^{\vartheta-2} + \frac{1}{\Gamma(\rho)} \int_{t_0}^{t} (t-v)^{\rho-1} g(v, z(v), z(\kappa v))\, dv. \tag{4.21}$$

Applying the fractional integral operator $I_{t_0^+}^{2-\vartheta}$, we obtain

$$I_{t_0^+}^{2-\vartheta} z^+(t) = n_1 (t-t_0) \Gamma(\vartheta) + n_2 \Gamma(\vartheta-1) + \frac{1}{\Gamma(2-\vartheta+\rho)} \int_{t_0}^{t} (t-v)^{1-\vartheta+\rho} g(v, z(v), z(\kappa v))\, dv.$$

Differentiating with respect to t, we get

$$(I_{t_0^+}^{2-\vartheta} z^+(t))' = n_1 \Gamma(\vartheta) + \frac{1}{\Gamma(1-\vartheta+\rho)} \int_{t_0}^{t} (t-v)^{\rho-\vartheta} g(v, z(v), z(\kappa v))\, dv.$$

Apply jump conditions at $t = t_1$ to obtain

$$I_{t_0^+}^{2-\vartheta} z(t_1^+) - I_{t_0^+}^{2-\vartheta} z(t_1^-) = I_1\, z(t_1^-), \qquad (I_{t_0^+}^{2-\vartheta} z(t_1^+))' - (I_{t_0^+}^{2-\vartheta} z(t_1^-))' = \bar{I}_1\, z(t_1^-).$$

Solving these equations gives,

$$n_1 = \frac{\eta_2 + \bar{I}_1 z(t_1^-)}{\Gamma(\vartheta)},$$

and

$$n_2 \Gamma(\vartheta - 1) = \frac{\eta_1 + I_1 z(t_1^-)}{\Gamma(\vartheta - 1)} - \frac{\bar{I}_1 z(t_1^-)}{\Gamma(\vartheta - 1)}(t - t_0).$$

Therefore, Eq (4.17) becomes

$$\begin{aligned} z(t) = {} & \frac{\eta_2}{\Gamma(\vartheta)}(t - t_0)^{\vartheta-1} + \frac{\eta_1}{\Gamma(\vartheta - 1)}(t - t_0)^{\vartheta-2} + \frac{\bar{I}_1(z(t_1^-))}{\Gamma(\vartheta)}(t - t_0)^{\vartheta-2}\Big[(t - t_0) - (\vartheta - 1)(t_1 - t_0)\Big] \\ & + \frac{I_1(z(t_1^-))}{\Gamma(\vartheta - 1)}(t - t_0)^{\vartheta-2} + \frac{1}{\Gamma(\rho)} \int_{t_0}^{t} (t - v)^{\rho-1} g(v, z(v), z(\kappa v))\, dv. \end{aligned}$$

Similarly, for $t \in (t_l, t_{l+1}],$ with $l = 1, 2, \ldots, m$, we obtain

$$\begin{aligned} z(t) = {} & \frac{(t - t_0)^{\vartheta-1}}{\Gamma(\vartheta)}\eta_2 + \frac{(t - t_0)^{\vartheta-2}}{\Gamma(\vartheta - 1)}\eta_1 + \sum_{t_0<t_l<t} \frac{\bar{I}_l(z(t_l^-))}{\Gamma(\vartheta)}(t - t_0)^{\vartheta-2}((t - t_0) - (\vartheta - 1)(t_l - t_0)) \\ & + \sum_{t_0<t_l<t} \frac{I_l(z(t_l^-))}{\Gamma(\vartheta - 1)}(t - t_0)^{\vartheta-2} + \frac{1}{\Gamma(\rho)} \int_{t_0}^{t} (t - v)^{\rho-1} g(v, z(v), z(\kappa v))\, dv. \end{aligned}$$

Thus, $z(t)$ satisfies the VIE (4.13).

Conversely, assume that z satisfies Eq (4.13). Using Definition 2.3 and Lemma 4.6, by direct computation, we conclude that z satisfies the system (3.2).

This completes the proof. □

Theorem 4.10. *Assume the conditions* ($\mathcal{A}_2$), ($\mathcal{A}_4$), *and* ($\mathcal{A}_5$) *are satisfied. Then the system* (3.2) *admits at least one solution in* $PC_{2-\vartheta}(J, \mathbb{C}^n)$

Proof. By Lemma 4.9, the system (3.2) is equivalent to the VIE

$$\begin{aligned} z(t) = {} & \frac{(t - t_0)^{\vartheta-1}}{\Gamma(\vartheta)}\eta_2 + \frac{(t - t_0)^{\vartheta-2}}{\Gamma(\vartheta - 1)}\eta_1 + \sum_{t_0<t_l<t} \frac{\bar{I}_l(z(t_l^-))}{\Gamma(\vartheta)}(t - t_0)^{\vartheta-2}\Big[(t - t_0) - (\vartheta - 1)(t_l - t_0)\Big] \\ & + \sum_{t_0<t_l<t} \frac{I_l(z(t_l^-))}{\Gamma(\vartheta - 1)}(t - t_0)^{\vartheta-2} + \frac{1}{\Gamma(\rho)} \int_{t_0}^{t} (t - v)^{\rho-1} g(v, z(v), z(\kappa v))\, dv. \qquad (4.22) \end{aligned}$$

Let

$$\mathcal{B}_b = \Big\{z \in PC_{2-\vartheta}(J, \mathbb{C}^n) : I_{t_0^+}^{2-\vartheta} z(t_0) = \eta_1,\ \Big(I_{t_0^+}^{2-\vartheta} z(t_0)\Big)' = \eta_2,\ \|z\|_{PC_{2-\vartheta}(J,\mathbb{C}^n)} \le b\Big\},$$

where $b > 0$ is chosen sufficiently large such that

$$b \ge 2\bigg(\frac{\|\eta_2\|}{\Gamma(\vartheta)}(T - t_0) + \frac{\|\eta_1\|}{\Gamma(\vartheta - 1)} + \frac{1}{\Gamma(\vartheta)} \sum_{l=1}^{m} \mathcal{E}_l[\vartheta(T - t_0)] + \frac{1}{\Gamma(\vartheta - 1)} \sum_{l=1}^{m} \mathcal{F}_l + \frac{\mathscr{M}}{\Gamma(\rho + 1)}(T - t_0)^{2-\vartheta+\rho}\bigg),$$

and

$$\mathcal{E}_l = \sup_{z\in\mathcal{B}_b} \|\bar{I}_l z(t_l^-)\|, \ \mathcal{F}_l = \sup_{z\in\mathcal{B}_b} \|I_l z(t_l^-)\|, \ \mathscr{M} = \sup_{t\in J} \|g(v,0,0)\|.$$

Assume Eqs (4.11) and (4.12) as in Theorem 4.4, and define the operators $\mathcal{P}_1$ and $\mathcal{Q}_1$ on $\mathcal{B}_b$ by

$$\begin{aligned}
(\mathcal{P}_1 z)(t) &= \frac{(t-t_0)^{\vartheta-1}}{\Gamma(\vartheta)}\eta_2 + \frac{(t-t_0)^{\vartheta-2}}{\Gamma(\vartheta-1)}\eta_1,\\
(\mathcal{Q}_1 z)(t) &= \sum_{t_0<t_l<t} \frac{\bar{I}_l(z(t_l^-))}{\Gamma(\vartheta)}(t-t_0)^{\vartheta-2}[(t-t_0)-(\vartheta-1)(t_l-t_0)] + \sum_{t_0<t_l<t}\frac{I_l(z(t_l^-))}{\Gamma(\vartheta-1)}(t-t_0)^{\vartheta-2}\\
&\quad + \frac{1}{\Gamma(\rho)}\int_{t_0}^{t}(t-v)^{\rho-1} g(v,z(v),z(\kappa v))\,dv.
\end{aligned}$$

Then the integral equation can be written in the operator form

$$z = \mathcal{P}_1 z + \mathcal{Q}_1 z, \ z \in PC_{2-\vartheta}(J,\mathbb{C}^n).$$

Step 1. We show that $\mathcal{P}_1 r + \mathcal{Q}_1 s \in \mathcal{B}_b$ for $r, s \in \mathcal{B}_b$.

Let $r, s \in \mathcal{B}_b$. Using Assumption ($\mathscr{A}_4$) and the properties of g, for $t \in J$, we estimate

$$\begin{aligned}
&(t-t_0)^{2-\vartheta}\|(\mathcal{P}_1 r + \mathcal{Q}_1 s)(t)\|\\
&\le \frac{\|\eta_2\|}{\Gamma(\vartheta)}(t-t_0) + \frac{\|\eta_1\|}{\Gamma(\vartheta-1)} + \frac{1}{\Gamma(\vartheta)}\sum_{l=1}^{m}\|\bar{I}_l\|\left|(t-t_0)-(\vartheta-1)(t_l-t_0)\right| + \frac{1}{\Gamma(\vartheta-1)}\sum_{l=1}^{m}\|I_l\|\\
&\quad + \frac{(t-t_0)^{2-\vartheta}}{\Gamma(\rho)}\int_{t_0}^{t}(t-v)^{\rho-1}\|g(v,s(v),s(\kappa(v)))\|\,dv\\
&\le \frac{\|\eta_2\|}{\Gamma(\vartheta)}(T-t_0) + \frac{\|\eta_1\|}{\Gamma(\vartheta-1)} + \frac{1}{\Gamma(\vartheta)}\sum_{l=1}^{m}\|\bar{I}_l\|\left[|t-t_0| + (\vartheta-1)|t_l-t_0|\right] + \frac{1}{\Gamma(\vartheta-1)}\sum_{l=1}^{m}\|I_l\|\\
&\quad + \frac{(t-t_0)^{2-\vartheta}}{\Gamma(\rho)}\int_{t_0}^{t}(t-v)^{\rho-1}\|g(v,s(v),s(\kappa(v))) - g(v,0,0)\|\,dv\\
&\quad + \frac{(t-t_0)^{2-\vartheta}}{\Gamma(\rho)}\int_{t_0}^{t}(t-v)^{\rho-1}\|g(v,0,0)\|\,dv\\
&\le \frac{\|\eta_2\|}{\Gamma(\vartheta)}(T-t_0) + \frac{\|\eta_1\|}{\Gamma(\vartheta-1)} + \frac{1}{\Gamma(\vartheta)}\sum_{l=1}^{m}\mathcal{E}_l\,[\vartheta(T-t_0)] + \frac{1}{\Gamma(\vartheta-1)}\sum_{l=1}^{m}\mathcal{F}_l\\
&\quad + \frac{2b\|\mathscr{L}\|_{L^1}}{\Gamma(\rho)}(T-t_0)^{\rho-1} + \frac{\mathscr{M}}{\Gamma(\rho+1)}(T-t_0)^{2-\vartheta+\rho}\\
&\le \frac{b}{2} + \frac{2b\|\mathscr{L}\|_{L^1}}{\Gamma(\rho)}(T-t_0)^{\rho-1}.
\end{aligned}$$

Therefore,

$$\|\mathcal{P}_1 r + \mathcal{Q}_1 s\|_{PC_{2-\vartheta}(J,\mathbb{C}^n)} \le b.$$

Moreover, from the definitions of $\mathcal{P}_1$ and $\mathcal{Q}_1$,

$$I_{t_0^+}^{2-\vartheta}(\mathcal{P}_1 r + \mathcal{Q}_1 s)(t_0) = \eta_1,$$

and

$$\left(I_{t_0^+}^{2-\vartheta}(\mathcal{P}_1 r + Q_1 s)(t_0)\right)' = \eta_2.$$

Hence,

$$\mathcal{P}_1 r + Q_1 s \in \mathcal{B}_b.$$

Step 2. It follows that the operator $\mathcal{P}_1$ is a contraction with contraction constant zero.
Step 3. We show that Q_1 is compact and continuous.
(i) Continuity: Because g is continuous, the integral part of Q_1 is continuous. The sums involving I_l, $\bar{I}_l$ are also continuous. Hence, Q_1 is continuous on $\mathcal{B}_b$.
(ii) Uniform boundedness: From Step 1, it is clear that Q_1 is uniformly bounded on $\mathcal{B}_b$.
(iii) Equicontinuity: Let $z \in \mathcal{B}_b$ and $t_1, t_2 \in J$ with $t_1 < t_2$. Then

$$\begin{aligned}
&\|Q_1 z(t_2) - Q_1 z(t_1)\| \\
&\leq \left\| \sum_{t_0<t_l<t_2} \frac{\bar{I}_l}{\Gamma(\vartheta)}(t_2 - t_0)^{\vartheta-2}((t_2 - t_0) - (\vartheta - 1)(t_l - t_0)) - \sum_{t_0<t_l<t_1} \frac{\bar{I}_l}{\Gamma(\vartheta)}(t_1 - t_0)^{\vartheta-2}((t_1 - t_0) - (\vartheta - 1)(t_l - t_0)) \right\| \\
&\quad + \left\| \sum_{t_0<t_l<t_2} \frac{I_l}{\Gamma(\vartheta - 1)}(t_2 - t_0)^{\vartheta-2} - \sum_{t_0<t_l<t_1} \frac{I_l}{\Gamma(\vartheta - 1)}(t_1 - t_0)^{\vartheta-2} \right\| \\
&\quad + \frac{1}{\Gamma(\rho)} \left\| \int_{t_0}^{t_2} (t_2 - v)^{\rho-1} g(v, z(v), z(\kappa v))\, dv - \int_{t_0}^{t_1} (t_1 - v)^{\rho-1} g(v, z(v), z(\kappa v))\, dv \right\|.
\end{aligned}$$

By the continuity of g and the fact that the impulses I_l, $\bar{I}_l$ are finite, the right-hand side tends to zero as t_2 approaches t_1. Therefore, Q_1 is equicontinuous. Because Q_1 is continuous, uniformly bounded, and equicontinuous on $\mathcal{B}_b$, by Lemma 4.8, Q_1 is relatively compact on $\mathcal{B}_b$. Hence, Q_1 is compact on $\mathcal{B}_b$. All the assumptions of Lemma 2.6 are satisfied. Therefore, the system (3.2) admits at least one solution in $PC_{2-\vartheta}(J, \mathbb{C}^n)$.

This completes the proof. □

Theorem 4.11. *Assume that the conditions* $(\mathcal{A}_2)$, $(\mathcal{A}_4)$, *and* $(\mathcal{A}_5)$ *are satisfied. Then the system* (3.2) *admits a unique solution* $z \in PC_{2-\vartheta}(J, \mathbb{C}^n)$.

Proof. Consider the same set $\mathcal{B}_b$ as in Theorem 4.10, and define an operator $\mathcal{T}$ on $\mathcal{B}_b$ by,

$$\begin{aligned}
\mathcal{T} z(t) = &\frac{(t - t_0)^{\vartheta-1}}{\Gamma(\vartheta)}\eta_2 + \frac{(t - t_0)^{\vartheta-2}}{\Gamma(\vartheta - 1)}\eta_1 + \sum_{t_0<t_l<t} \frac{\bar{I}_l(z(t_l^-))}{\Gamma(\vartheta)}(t - t_0)^{\vartheta-2}((t - t_0) - (\vartheta - 1)(t_l - t_0)) \\
&+ \sum_{t_0<t_l<t} \frac{I_l(z(t_l^-))}{\Gamma(\vartheta - 1)}(t - t_0)^{\vartheta-2} + \frac{1}{\Gamma(\rho)} \int_{t_0}^{t} (t - v)^{\rho-1} g(v, z(v), z(\kappa v))\, dv.
\end{aligned}$$

Define the constant

$$\mathcal{K} := \sum_{l=1}^{m} \lambda_l \frac{(T - t_0)^{\vartheta-2}}{\Gamma(\vartheta - 1)} + \sum_{l=1}^{m} \bar{\lambda}_l \frac{\vartheta(T - t_0)^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{2\|\mathcal{L}\|_{L^1}}{\Gamma(\rho)}(T - t_0)^{\rho-1}.$$

Assume $\mathcal{K} < 1$.

Step 1. Show $\mathscr{T}\mathcal{B}_b \subset \mathcal{B}_b$.

For any $z \in \mathcal{B}_b,\ t \in J$, and by using $(\mathcal{A}_2)$, we have

$$\begin{aligned}
\|\mathscr{T}z(t)\| \leq{}& \frac{(t-t_0)^{\vartheta-1}}{\Gamma(\vartheta)}\|\eta_2\| + \frac{(t-t_0)^{\vartheta-2}}{\Gamma(\vartheta-1)}\|\eta_1\| + \sum_{t_0<t_l<t}\frac{\|\bar{I}_l\|}{\Gamma(\vartheta)}(t-t_0)^{\vartheta-2}((t-t_0)-(\vartheta-1)(t_l-t_0)) \\
&+ \sum_{t_0<t_l<t}\frac{\|I_l\|}{\Gamma(\vartheta-1)}(t-t_0)^{\vartheta-2} + \frac{1}{\Gamma(\rho)}\int_{t_0}^{t}(t-v)^{\rho-1}\|g(v,z(v),z(\kappa v))-g(v,0,0)\|\,dv \\
&+ \frac{1}{\Gamma(\rho)}\int_{t_0}^{t}(t-v)^{\rho-1}\|g(v,0,0)\|\,dv \\
\leq{}& \frac{(t-t_0)^{\vartheta-1}}{\Gamma(\vartheta)}\|\eta_2\| + \frac{(t-t_0)^{\vartheta-2}}{\Gamma(\vartheta-1)}\|\eta_1\| + \sum_{t_0<t_l<t}\frac{\|\bar{I}_l\|}{\Gamma(\vartheta)}(t-t_0)^{\vartheta-2}((t-t_0)-(\vartheta-1)(t_l-t_0)) \\
&+ \sum_{t_0<t_l<t}\frac{\|I_l\|}{\Gamma(\vartheta-1)}(t-t_0)^{\vartheta-2} + \frac{2\|\mathscr{L}\|_{L^1}\|z\|_{PC_{2-\vartheta}(J,\mathbb{C}^n)}}{\Gamma(\rho)}(t-t_0)^{\vartheta-1}(T-t_0)^{\rho-1} + \frac{\mathscr{M}}{\Gamma(\rho+1)}(t-t_0)^{\rho}.
\end{aligned}$$

Multiplying both sides by $(t-t_0)^{2-\vartheta}$, we get

$$\begin{aligned}
\|(t-t_0)^{2-\vartheta}\mathscr{T}z(t)\| \leq{}& \frac{(t-t_0)\|\eta_2\|}{\Gamma(\vartheta)} + \frac{\|\eta_1\|}{\Gamma(\vartheta-1)} + \sum_{t_0<t_l<t}\frac{\mathcal{E}_l}{\Gamma(\vartheta)}\Big|(t-t_0)-(\vartheta-1)(t_l-t_0)\Big| \\
&+ \sum_{t_0<t_l<t}\frac{\mathcal{F}_l}{\Gamma(\vartheta-1)} + \frac{2\|\mathscr{L}\|_{L^1}\|z\|_{PC_{2-\vartheta}(J,\mathbb{C}^n)}}{\Gamma(\rho)}(t-t_0)(T-t_0)^{\rho-1} + \frac{\mathscr{M}}{\Gamma(\rho+1)}(t-t_0)^{2-\vartheta+\rho} \\
\leq{}& \frac{b}{2} + \frac{2b\|\mathscr{L}\|_{L^1}}{\Gamma(\rho)}(T-t_0)^{\rho}.
\end{aligned}$$

Hence, we obtain

$$\|\mathscr{T}z(t)\|_{PC_{2-\vartheta}(J,\mathbb{C}^n)} \leq b,$$

which implies

$$\mathscr{T}\mathcal{B}_b \subset \mathcal{B}_b.$$

Step 2. Show that $\mathscr{T}$ is a contraction on $\mathcal{B}_b$.

For any $r, s \in \mathcal{B}_b$ and by using $(\mathcal{A}_2)$ and $(\mathcal{A}_5)$, we estimate

$$\begin{aligned}
\|(t-t_0)^{2-\vartheta}(\mathscr{T}r(t)-\mathscr{T}s(t))\| \leq{}& \sum_{l=1}^{m}(t-t_0)^{2-\vartheta}\lambda_l\|r-s\|\frac{(T-t_0)^{\vartheta-2}}{\Gamma(\vartheta-1)} + \sum_{l=1}^{m}(t-t_0)^{2-\vartheta}\bar{\lambda}_l\|r-s\|\frac{\vartheta(T-t_0)^{\vartheta-1}}{\Gamma(\vartheta)} \\
&+ \frac{(t-t_0)^{2-\vartheta}}{\Gamma(\rho)}\int_{t_0}^{t}(t-v)^{\rho-1}\|g(v,r(v),r(\kappa v))-g(v,s(v),s(\kappa v))\|\,dv \\
\leq{}& \sum_{l=1}^{m}\lambda_l\frac{(T-t_0)^{\vartheta-2}}{\Gamma(\vartheta-1)}\|r-s\|_{PC_{2-\vartheta}(J,\mathbb{C}^n)} + \sum_{l=1}^{m}\bar{\lambda}_l\frac{\vartheta(T-t_0)^{\vartheta-1}}{\Gamma(\vartheta)}\|r-s\|_{PC_{2-\vartheta}(J,\mathbb{C}^n)} \\
&+ \frac{2\|\mathscr{L}\|_{L^1}}{\Gamma(\rho)}(T-t_0)^{\rho-1}\|r-s\|_{PC_{2-\vartheta}(J,\mathbb{C}^n)} \\
={}& \mathscr{K}\|r-s\|_{PC_{2-\vartheta}(J,\mathbb{C}^n)}.
\end{aligned}$$

Because $\mathscr{K} < 1$, $\mathscr{T}$ is a contraction, by the Banach contraction principle, $\mathscr{T}$ has a unique fixed point in $\mathcal{B}_b$. This fixed point is the unique solution of the system (3.2).

This completes the proof. □

5. An application

In many physical systems, accurately describing dynamic behavior is essential for understanding stability and performance under varying conditions. Such systems often involve complex interactions, uncertainties, and time-dependent effects that cannot be fully captured using classical modeling approaches. A prominent example arises in aerodynamic systems, where the behavior of airflow around an aircraft wing plays a crucial role in determining flight stability and performance [46, 47]. During flight, the airflow is affected by multiple interacting phenomena such as vortex formation , wake interaction [48, 49], and external disturbances including atmospheric gusts and turbulence [50, 51]. These effects produce complex flow patterns that evolve over time and significantly influence the aerodynamic response of the wing. Consequently, the airflow depends on both present and past states, making accurate prediction more challenging. To effectively capture these characteristics, fractional calculus is employed to model memory and hereditary effects, allowing a more realistic representation of the persistence of vortex interactions over time. In addition, complex functions provide a convenient model to describe the multidirectional nature of airflow within a unified representation. Together, this approach captures both the memory-dependent behavior and the structural complexity of the flow. Recent studies and survey works [52–54] demonstrate that fractional-order models and complex-valued formulations offer improved representation of aerodynamic dynamics under uncertainty. Figure 1 represents the aerodynamic phenomena underlying the proposed model. The vortex wake illustrates memory effects, the delayed wake interaction corresponds to the pantograph term $z(\kappa t)$, and the impulsive gust depicts sudden perturbations. Consequently, two distinct aerodynamic regimes arise depending on the intensity of interactions and disturbances.

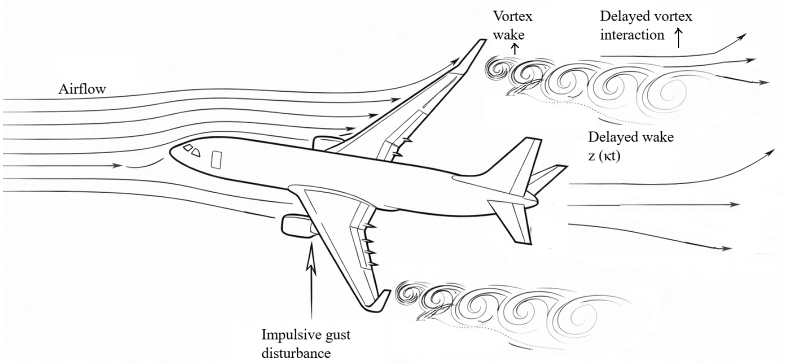

Figure 1. Schematic representation of airflow around an aircraft wing.

Case 1: Mild memory aerodynamic flow.

Under relatively stable flight conditions, aerodynamic disturbances are moderate, and the airflow exhibits smooth temporal evolution with limited fluctuations. External disturbances such as gust loads are small and introduce only slight deviations from the steady-state behavior. In this regime, the airflow dynamics reflect a balanced interaction between current flow conditions and weak historical effects, resulting in stable and bounded behavior over time. The evolution of the airflow velocity is therefore governed by

$$D_{0^+}^{0.6,0.5}z(t) = (0.1 + 0.2i)z(t) + 0.1e^{-t}z(0.5t),$$

where each component of the model has a clear physical interpretation. The fractional order $\rho = 0.6$

represents the moderate memory effect arising from viscous interactions and weak wake influence, whereas the parameter $\sigma = 0.5$ reflects a balanced contribution between the current airflow state and its past behavior. The term $z(0.5t)$ describes the delayed aerodynamic influence of previously generated vortex structures, with $\kappa = 0.5$ indicating that the past flow state affects the present dynamics after a proportional delay. The coefficient $(0.1 + 0.2i)$ characterizes the combined horizontal and vertical aerodynamic response of the flow, whereas the exponentially decaying term e^{-t} models the gradual dissipation of vortex-induced effects over time. Sudden disturbances in the airflow, such as small gust loads, are incorporated through the impulsive condition

$$\Delta I_{0^+}^{0.2} z(0.5) = 0.1 z(0.5^-), \quad t_1 = 0.5,$$

which represents an instantaneous perturbation occurring at a specific time. The initial airflow configuration

$$I_{0^+}^{0.2} z(0) = 1 + i,$$

describes the initial state of the velocity field at the beginning of observation. Next, we verify that the proposed model satisfies the required assumptions.

According to Assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_5)$, consider

$$g(t, z(t), z(\kappa t)) = (0.1 + 0.2i) z(t) + 0.1 e^{-t} z(0.5t).$$

Because $z(t),\ z(0.5t) \in PC_{1-\tau}((0, 1], \mathbb{C})$ and e^{-t} is continuous, it follows that

$$g(t, z(t), z(0.5t)) \in PC_{1-\tau}((0, 1], \mathbb{C}).$$

Thus, Assumption $(\mathcal{A}_1)$ is satisfied.

For any $x, y, u, v \in \mathbb{C}$,

$$\begin{aligned} \|g(t, x, u) - g(t, y, v)\| &\leq |0.1 + 0.2i|\, \|x - y\| + 0.1 e^{-t} \|u - v\| \\ &\leq 0.2236\, \|x - y\| + 0.1\, \|u - v\|. \end{aligned}$$

Hence, g satisfies a Lipschitz condition $(\mathcal{A}_2)$ with

$$\mathscr{L}(t) = 0.3236, \quad \|\mathscr{L}\|_{L^1} = 0.3236.$$

For any $z_1,\ z_2 \in \mathbb{C}$,

$$\|I_1(z_1) - I_1(z_2)\| = \|0.1(z_1 - z_2)\| = 0.1 \|z_1 - z_2\|.$$

Thus, $\theta_1 = 0.1 < 1$, and Assumption $(\mathcal{A}_3)$ holds.

We compute

$$\mathcal{W} = \frac{2}{\Gamma(\rho)} \|\mathscr{L}\|_{L^1} (T - t_0)^{\rho - 1}.$$

Substituting values,

$$\mathcal{W} = \frac{2}{1.4891} (0.3236) \approx 0.4346 < 0.5.$$

Hence, all conditions of Theorem 4.4 are satisfied, and the system admits at least one solution. Next,

$$\mathscr{U} = \frac{(T-t_0)^{\tau-1}}{\Gamma(\tau)}\sum_{l=1}^{m}\theta_l + \frac{2\|\mathscr{L}\|_{L^1}}{\Gamma(\rho)}(T-t_0)^{\rho-1}.$$

Substituting values,

$$\mathscr{U} = \frac{1(0.1)}{1.1642} + \frac{2}{1.4891}(0.3236) \approx 0.5205 < 1.$$

Therefore, by Theorem 4.5, the solution is unique.

By Lemma 4.3, the airflow velocity function $z(t)$ is given by

$$z(t) = \begin{cases} \dfrac{t^{-0.2}}{1.1642}(1+i) + \dfrac{1}{1.4891}\displaystyle\int_0^t (t-v)^{-0.4}[(0.1+0.2i)z(v) + 0.1e^{-v}z(0.5v)]\,dv, & t \in [0, 0.5], \\ \dfrac{t^{-0.2}}{1.1642}(1+i+0.1(z(0.5^-))) + \dfrac{1}{1.4891}\displaystyle\int_0^t (t-v)^{-0.4}[(0.1+0.2i)z(v) + 0.1e^{-v}z(0.5v)]\,dv, & t \in (0.5, 1]. \end{cases}$$

The results in Table 3 are consistent with the graphical behavior of $Re\,z(t)$ and $Im\,z(t)$. For lower values of ρ, smaller values of $\mathscr{U}$ indicate high stability, which is reflected in the graphs by smooth decay and a negligible impulse effect at $t = 0.5$. Figures 2 and 3 illustrate the real and imaginary parts of the solution $z(t)$ for different values of ρ in the mild-memory regime. The obtained profiles illustrate the effect of the fractional order on the system dynamics and reveal the presence of the impulse at $t = 0.5$. As ρ increases, the rise in $\mathscr{U}$ leads to reduced stability, visible as a noticeable jump and slower decay. For higher values of ρ, the system exhibits mild oscillations with a strong impulse effect and delayed stabilization.

Table 3. Distinction between instantaneous and noninstantaneous impulses in dynamical systems.

ρ	$\mathscr{W}$	$\mathscr{U}$	Impulse effect at $t = 0.5$
0.2	0.1410	0.2082	Very small jump
0.4	0.2918	0.3688	Small jump
0.6	0.4346	0.5205	Clear visible jump
0.8	0.5559	0.6495	Strong jump
0.9	0.6057	0.7026	Significant jump

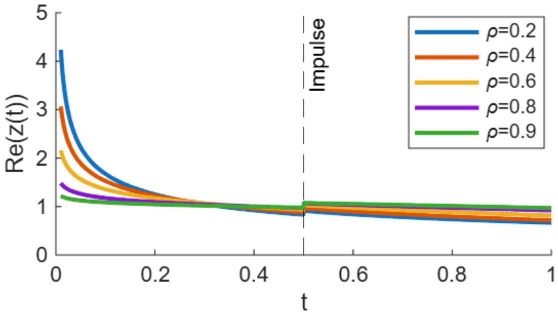

Figure 2. Real part of the solution $z(t)$ for different values of ρ.

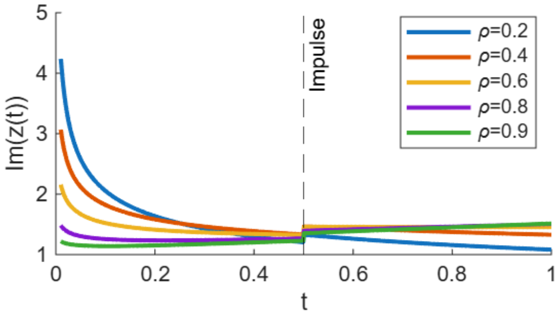

Figure 3. Imaginary part of the solution $z(t)$ for different values of ρ.

Case 2: Strong memory and acceleration effects.

In contrast, during highly dynamic flight conditions such as rapid maneuvering, strong turbulence, or intense wake interactions, the airflow exhibits significantly more complex behavior. These effects result in noticeable oscillations and enhanced sensitivity of the airflow to both past and present conditions. In this regime, the airflow evolution is described by

$$D_{0^+}^{1.5,0.5}z(t) = 0.01e^{-t} + (0.02 + 0.01i)z(t) + 0.02z(0.5t),$$

where the fractional order $\rho = 1.5$ reflects stronger memory and acceleration effects in the airflow, capturing both velocity and higher-order dynamic behavior. The parameter $\sigma = 0.5$ continues to balance the influence of present and past states, and the delayed term $z(0.5t)$ again represents vortex-induced feedback with $\kappa = 0.5$. The forcing term $0.01e^{-t}$ accounts for external aerodynamic excitation that gradually diminishes over time, and the smaller complex coefficient $(0.02 + 0.01i)$ represents a more refined but sensitive aerodynamic response under strong dynamic conditions. The impulsive disturbances

$$\Delta I_{0^+}^{0.25}z(0.5) = 0.1z(0.5^-), \quad \Delta\Big(I_{0^+}^{0.25}z(t)\Big)'_{t=0.5} = 0.05z(0.5^-),$$

model sudden and more intense aerodynamic events, such as strong gusts or wake interference from nearby aircraft, which not only affect the velocity but also its rate of change. The initial airflow configurations

$$I_{0^+}^{0.25}z(0) = 1 + i, \quad \Big(I_{0^+}^{0.25}z(t)\Big)'_{t=0} = 2 - i,$$

describe both the initial airflow velocity and its initial rate of variation. We now verify that the proposed model satisfies the required assumptions.

Consider

$$g(t, z(t), z(\kappa t)) = 0.01e^{-t} + (0.02 + 0.01i)z(t) + 0.02z(0.5t).$$

Because $z(t),\ z(0.5t) \in PC_{2-\vartheta}(J, \mathbb{C})$, and e^{-t} is continuous, it follows that

$$g \in PC_{2-\vartheta}(J, \mathbb{C}).$$

Thus, Assumption $(\mathcal{A}_4)$ is satisfied.

For any $x, y, u, v \in \mathbb{C}$,

$$\begin{aligned}\|g(t, x, u) - g(t, y, v)\| &\leq |0.02 + 0.01i|\, \|x - y\| + 0.02\, \|u - v\| \\ &\leq 0.02236\, \|x - y\| + 0.02\, \|u - v\|.\end{aligned}$$

Hence, g satisfies a Lipschitz condition $(\mathcal{A}_2)$ with

$$\mathscr{L}(t) = 0.04236, \quad \|\mathscr{L}\|_{L^1} = 0.04236.$$

Because

$$I_1(z) = 0.1z, \quad \bar{I}_1(z) = 0.05z,$$

we obtain

$$\lambda_1 = 0.1 < 1, \quad \bar{\lambda}_1 = 0.05 < 1.$$

Thus, Assumption $(\mathcal{A}_5)$ holds.

We compute

$$\mathcal{W} = \frac{2}{\Gamma(\rho)} \|\mathscr{L}\|_{L^1} (T - t_0)^{\rho-1}.$$

Substituting values

$$\mathcal{W} = \frac{2(0.04236)}{0.8862} \approx 0.0956 < 0.5.$$

Hence, by Theorem 4.10, the system admits at least one solution. Next,

$$\mathscr{K} = \sum_{l=1}^{m} \lambda_l \frac{(T - t_0)^{\vartheta-2}}{\Gamma(\vartheta - 1)} + \sum_{l=1}^{m} \bar{\lambda}_l \frac{\vartheta(T - t_0)^{\vartheta-1}}{\Gamma(\vartheta)} + \frac{2\|\mathscr{L}\|_{L^1}}{\Gamma(\rho)} (T - t_0)^{\rho-1}.$$

Substituting values,

$$\mathscr{K} = \frac{0.1}{1.2254} + \frac{0.05\,(1.75)}{0.9190} + \frac{2\,(0.04236)}{0.8862} \approx 0.2724 < 1.$$

Therefore, by Theorem 4.11, the solution is unique.

By Lemma 4.9, the airflow velocity function $z(t)$ is given by

$$z(t) = \begin{cases} \dfrac{2-i}{0.9190} t^{0.75} + \dfrac{1+i}{1.2254} t^{-0.25} + 1.12839 \displaystyle\int_0^t (t - v)^{0.5} [0.01e^{-v} + (0.02 + 0.01i)z(v) + 0.02z(0.5v)]\, dv, & t \in [0, 0.5], \\ \dfrac{2-i}{0.9190} t^{0.75} + \dfrac{1+i}{1.2254} t^{-0.25} + \dfrac{0.1(z(0.5^-))}{1.2254} t^{-0.25} + \dfrac{0.05(z(0.5^-))}{0.9190} t^{-0.25} [t - 0.375] & \\ + 1.12839 \displaystyle\int_0^t (t - v)^{0.5} [0.01e^{-v} + (0.02 + 0.01i)z(v) + 0.02z(0.5v)]\, dv, & t \in (0.5, 1]. \end{cases}$$

The data summarized in Table 4 closely aligns with the graphical profiles of $\operatorname{Re} z(t)$ and $\operatorname{Im} z(t)$. For smaller values of ρ, the corresponding lower values of $\mathscr{K}$ ensure stable behavior, as evidenced by moderate impulse effects at $t = 0.5$ and smooth decay in the solution. Figures 4 and 5 depict the real and imaginary parts of the solution $z(t)$ for various values of ρ in the strong-memory regime. As ρ increases, stronger oscillations and more pronounced impulsive effects are observed, highlighting the influence of memory and acceleration effects on the airflow dynamics. As ρ increases, the gradual increase in $\mathscr{K}$ leads to a reduction in damping, which is reflected in the graphs through enhanced oscillatory patterns and a more pronounced jump at the impulse point. For larger values of ρ, the solution exhibits significant oscillations accompanied by stronger impulse effects and delayed stabilization, thereby emphasizing the role of pronounced memory and acceleration effects in the system.

Table 4. Effect of fractional order ρ on impulse behavior (strong memory case).

ρ	$\mathcal{W}$	$\mathscr{K}$	Impulse effect at $t = 0.5$
1.1	0.0890	0.2381	Moderate jump
1.3	0.0944	0.2583	Noticeable jump
1.5	0.0956	0.2724	Strong jump
1.7	0.0932	0.2810	Significant jump

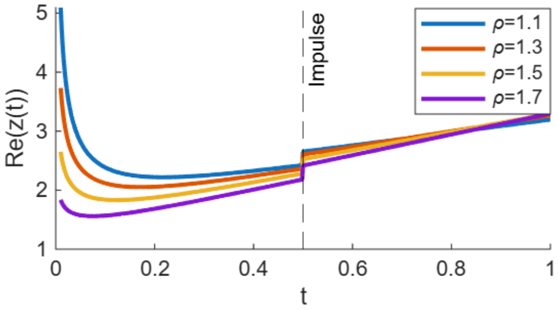

Figure 4. Real part of the solution $z(t)$ for various ρ.

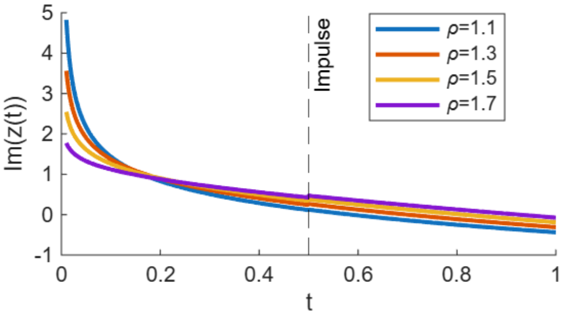

Figure 5. Imaginary part of the solution $z(t)$ for various ρ.

The obtained formulations describe the evolution of airflow around the aircraft wing under different operating conditions. In the mild memory regime, the airflow remains stable, with limited fluctuations, whereas in the strong memory regime, increased vortex interactions and disturbances lead to more complex and transient behavior. Overall, the model effectively captures the influence of delayed aerodynamic interactions and sudden disturbances, demonstrating stable and bounded airflow behavior under both moderate and highly dynamic conditions.

6. Conclusions

This study examined a class of CVSes with impulsive effects and pantograph-type arguments in the sense of the HFD. Standard fixed-point methods were used to prove the existence and uniqueness of solutions for two types of systems. The validity of the theoretical findings was illustrated by an application relevant to an aerodynamics model. Future research may focus on the investigation of stability analysis and other qualitative properties of the proposed systems. Moreover, motivated by the concept of fractional derivatives of imaginary order introduced by E. R. Love [55], the present model can be further extended by incorporating Hilfer complex-valued fractional derivatives. In addition, inspired by the work of S. Harikrishnan et al. [56] on Hilfer-Katugampola-type pantograph equations, such extensions can be developed to include more generalized fractional and complex-order operators.

Author contributions

N. Keerthana: Writing—original draft, validation, conceptualization, methodology; D. Vivek: Writing—original draft, conceptualization, methodology, validation, writing—review and editing, supervision; E. M. Elsayed: Writing—review, validation and guidance; A. A. Alghamdi, M. M. El-Dessoky: Review, validation, methodology. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors acknowledge the use of OpenAI's image generation tools in creating the illustrative image presented in Figure 1. The image was used solely for visualization purposes and does not represent experimental, numerical, or computational results.

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Conflict of interest

This work does not have any conflicts of interest.

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