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Research article

Coupled Langevin system with generalized proportional fractional derivatives relative to a function and Riemann-Stieltjes integral boundary conditions

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Abstract: We investigate a coupled Langevin system driven by the generalized proportional fractional derivative with respect to a function, closed by Riemann-Stieltjes integral boundary data that intertwine the two unknowns at the right endpoint. Under the assumption that the sum of the two fractional orders within each component exceeds one, the problem is recast as an equivalent system of Volterra-type integral equations. Existence is then established via Krasnoselskii's fixed point theorem and, alternatively, the Leray-Schauder nonlinear alternative; uniqueness follows from the Banach contraction principle. Three numerical examples illustrate the results, with explicit verification of every hypothesis.

Keywords: generalized proportional fractional derivative; coupled Langevin system; Riemann-Stieltjes integral; fixed point theory; existence and uniqueness
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1. Introduction

The mathematical theory of fractional differential equations has, over the past several decades, evolved into a powerful modelling language for processes that exhibit memory effects, hereditary phenomena, and long-range correlations, phenomena that classical integer-order calculus is unable to render in a satisfactory way. Such equations have proved indispensable in disciplines as diverse as continuum mechanics, biology, economics, electrical engineering, control theory, and the science of complex media; the standard treatises [1–3] collect the foundations and many applications (see also [4]). The companion subject of coupled fractional systems has emerged as particularly relevant whenever two or more interacting populations, compartments, or fields are described by different but coupled fractional dynamics; representative settings include multi-compartment pharmacokinetics [5], stabilization and control of fractional-order chaotic systems [6], and anomalous transport in heterogeneous porous media [7]. A growing literature treats the existence and qualitative theory of coupled fractional systems;

see, e.g., [8, 9].

The Langevin equation, introduced by P. Langevin in his celebrated note of 1908 [10], is a cornerstone of statistical physics: It provides an effective mesoscopic description of the velocity of a particle subject to a random force inside a viscous fluid, and underlies modern accounts of Brownian motion [11, 12]. For complex environments, where a single relaxation rate is no longer adequate, Kubo's reformulation of the fluctuation-dissipation theorem [13] pointed the way to a generalized Langevin equation with a fractional memory kernel, and the fractional Langevin equation in its present form was systematically developed by Mainardi and his collaborators [14, 15]. This formulation is now standard for capturing both subdiffusive and superdiffusive transport.

The applicability of fractional Langevin models extends far beyond their original physical setting. They have been employed to model long-range memory in financial markets [16], multifractal stride-interval variability in human gait dynamics [17], anomalous diffusion in complex biological media [18], anomalous transport in systems with incomplete Hamiltonian chaos [19], and aging and ergodicity-breaking phenomena in far-from-equilibrium dynamics [20]. We refer the reader to these works and the references therein for these and related developments.

On the analytical side, the qualitative theory of fractional Langevin equations under various nonlocal data has received considerable attention. Salem, Alzahrani, and the present author [21] established existence results for nonlinear Langevin equations involving two fractional orders with nonlocal integral and three-point boundary conditions; Salem, Alshehri, and the author [22] subsequently treated infinite systems of fractional Langevin equations in suitable sequence spaces by means of Hausdorff measure of noncompactness; the author, together with Salem [23], examined fractional Langevin equations with infinite-point boundary conditions, with explicit application to the fractional harmonic oscillator; and most recently, the author, with Alghamdi and Ghanmi [24], addressed a ψ-Hilfer Langevin equation through a variational mountain-pass approach. Together, these developments provide a strong incentive for a rigorous mathematical theory of fractional Langevin equations and of their coupled variants.

Within the rapidly expanding family of fractional operators, Anderson and Ulness [25] introduced the proportional conformable derivative, and shortly afterwards Jarad et al. [26, 27] developed its more general form, the generalized proportional fractional derivative with respect to another function η. The operator depends on a proportionality parameter $\varrho \in (0, 1]$ and incorporates the exponential weight $\exp(\frac{\varrho-1}{\varrho}[\eta(t) - \eta(s)])$ inside its kernel. By specializing η and ϱ, one recovers, among others, the classical Riemann–Liouville fractional operator, the Hadamard operator, and the Katugampola operator. This unifying capability is the principal reason for the present interest in the proportional framework.

In the scalar setting, Barakat, Soliman, and Hyder [28] examined the single Langevin equation

$$ {}_a\mathcal{D}^{\alpha,\varrho,\eta}\Big({}_a\mathcal{D}^{\beta,\varrho,\eta} + \lambda\Big)w(t) = G(t, w(t)), \qquad t \in [a, b], $$

in the Hadamard-Caputo setting (corresponding to $\eta(t) = \ln t$ in the present notation), under nonlocal integral and nonperiodic boundary data, and obtained existence and uniqueness through Krasnoselskii's theorem and the Banach contraction principle. Their treatment was, however, limited to a single equation. Concerning coupled Langevin systems with nonlocal boundary conditions, Salem and the present author [29] considered a coupled Caputo Langevin system equipped with Riemann-Stieltjes integral boundary data; in subsequent work, the author and collaborators [30] analyzed a coupled μ-Caputo Langevin system whose two equations carry different fractional orders and live on a more general domain; and most recently, the author, with Alghamdi and Ghanmi [31], treated a coupled ψ-Hilfer

Langevin system equipped with Riemann-Stieltjes integral boundary conditions, employing Schauder's fixed point theorem, the Leray-Schauder nonlinear alternative, and the Banach contraction principle. The qualitative theory of more general coupled fractional problems with Riemann-Stieltjes integral data has also been developed in parallel: Alruwaily, Aljoudi, the author, and collaborators [32] proved existence and uniqueness for coupled fractional integro-differential systems of different orders with anti-periodic nonlocal integral boundary conditions, while Alruwaily, the author, and collaborators [33] addressed a coupled fractional system mixing Riemann-Liouville and Caputo derivatives under coupled Riemann-Stieltjes integro-multipoint boundary conditions. To the best of the author's knowledge, no contribution in the existing literature addresses a coupled Langevin system formulated through the generalized proportional fractional derivative with respect to another function and equipped with Riemann-Stieltjes integral boundary conditions. Closing this gap is the objective of the present paper.

The choice of Riemann-Stieltjes integral boundary conditions is driven by a well-established mathematical rationale: The Stieltjes framework provides a unified formulation that subsumes, as particular instances, classical multi point conditions, weighted-integral conditions, and combinations thereof, simply by specializing the integrator function ζ. This unifying capability was systematically exploited by Webb and Infante [34, 35], who developed a general theory of positive solutions for boundary value problems under Stieltjes-type nonlocal data; their formulation has since become a standard reference in this area. In the fractional setting, Ahmad, Alghanmi, Ntouyas, and Alsaedi [36] employed closely related Stieltjes-type boundary conditions within a generalized fractional framework, providing an analytically related framework comparable to ours. The classical texts of Podlubny [4], Diethelm [37], and Kilbas, Srivastava, and Trujillo [1] provide the foundational background on fractional calculus, anomalous diffusion, and memory-dependent processes, within which nonlocal integral conditions naturally fit.

Main contributions. The novelties of this work can be summarized under the following items:

(i) The investigation, presented here for the first time, concerns the coupled Langevin system

$$ {}_a\mathcal{D}^{p_i,\varrho,\eta}\left({}_a\mathcal{D}^{q_i,\varrho,\eta}+\mu_i\right)z_i(t)=f_i(t,z_1(t),z_2(t)), \qquad t\in[a,b],\ i=1,2, \tag{1.1} $$

subject to the nonlocal Riemann-Stieltjes integral boundary conditions

$$ {}_a\mathcal{J}^{1-q_i,\varrho,\eta}z_i(a)=0, \qquad z_i(b)=\nu_i z_i(\xi_i)+\int_a^b z_j(s)\,d\zeta_i(s), \qquad i,j\in\{1,2\},\ i\neq j, \tag{1.2} $$

where $\varrho\in(0,1]$, $0<q_i<1$, $0<p_i<1$ with the regularity assumption $p_i+q_i>1$, $\mu_i,\nu_i\in\mathbb{R}$, $a<\xi_i<b$, and each ζ_i is a function of bounded variation on $[a,b]$.

(ii) The exponential factor $\exp(\frac{\varrho-1}{\varrho}[\eta(t)-\eta(s)])$ in the kernel of the proportional operator introduces a number of analytical subtleties that do not appear in the ψ-Hilfer treatment of [31] or in the μ-Caputo analysis of [30]. These subtleties affect both the structure of the boundary terms and the explicit form of the working bounds; we keep track of every such factor and obtain explicit constants A_1–A_3, B_1–B_3 for the existence and uniqueness estimates.

(iii) By suitable choices of η and ϱ, our findings recover, as special cases, several existing results in the literature, thus unifying and extending earlier contributions.

The structural foundation of the model considered in this work is the algebraic structure ${}_a\mathcal{D}^{p_i,\varrho,\eta}({}_a\mathcal{D}^{q_i,\varrho,\eta}+\mu_i)z_i=f_i$, which is a characteristic structure frequently employed in deterministic fractional Langevin equations. The classical Langevin equation belongs to the class of stochastic differential equations, where additive stochastic terms (typically Wiener/Brownian noise) must satisfy the fluctuation–dissipation relation to ensure thermodynamic consistency. The framework adopted here, however, is deterministic and corresponds to a fractional Langevin-type structure: Stochastic forcing is not explicitly incorporated; instead, the model employs a general nonlinear function $f_i(t, z_1, z_2)$ that captures state-dependent and time-dependent interactions characteristic of fractional Langevin-type systems in the deterministic regime, rather than stochastic excitations. Such deterministic forcing behavior naturally appears within the proposed framework and is explicitly illustrated in Example 6.1, where $f_i(t, 0, 0)$ recovers an explicit forcing term depending only on t, acting within the generalized Langevin structure adopted here. The present deterministic framework constitutes the analytical foundation upon which subsequent stochastic extensions, incorporating explicit fluctuation-dissipation constraints, can be naturally built in future work.

From a physical standpoint, the proposed model is motivated by the dynamics of fractional oscillators with memory effects and fluctuating frequencies, which arise naturally in viscoelastic media, anomalous diffusion, and complex stochastic environments. In particular, Du, Meng, Lin, and Wang [38] have recently investigated the resonant behavior of two coupled fluctuating-frequency oscillators with a tempered Mittag-Leffler memory kernel, demonstrating the role of tempered, power-law memory in such systems. The generalized proportional fractional derivative with respect to a function adopted here may be viewed as providing a memory-bearing operator suitable for such systems: the exponential tempering factor governed by ϱ, the power-law factor of order p_i (resp. q_i), and the time-deformation η collectively encode the tempered, long-range, and time-deformed memory effects relevant to the physical context outlined above (see Remark 2.3 below).

The remainder of the paper is organized as follows. Section 2 collects the necessary definitions and auxiliary results. Section 3 derives the equivalent integral formulation of the problem and identifies the boundary terms. Section 4 establishes the two existence theorems. Section 5 contains the uniqueness theorem. Three illustrative examples are worked out in Section 6, with the hypotheses of every theorem fully verified. A short conclusion closes the paper.

2. Preliminaries

We collect in this section the definitions and lemmas that the rest of the paper will rely upon. Throughout, $a < b$ are fixed real numbers, and $\eta\colon [a, b] \to \mathbb{R}$ denotes a function of class C^1 that is strictly increasing on $[a, b]$; equivalently $\eta'(t) > 0$ for every $t \in (a, b)$. The shifted weight is denoted by

$$H(t) := \eta(t) - \eta(a), \qquad H := H(b) = \eta(b) - \eta(a).$$

The notation that follows is consistent with that of [26, 27].

Definition 2.1 ([26]). For $\varrho \in (0, 1]$, the proportional derivative of order ϱ of a function u relative to η is defined by

$$\mathfrak{D}^{\varrho,\eta}u(t) = (1 - \varrho)u(t) + \varrho\frac{u'(t)}{\eta'(t)}.$$

Definition 2.2 ([26]). Let $\varrho \in (0, 1]$ and $\theta > 0$. The left-sided generalized proportional fractional integral, of order θ and with respect to η, of a function u is given by

$${}_a\mathcal{J}^{\theta,\varrho,\eta}u(t) = \frac{1}{\varrho^{\theta}\,\Gamma(\theta)}\int_a^t \exp\Big(\frac{\varrho - 1}{\varrho}[\eta(t) - \eta(s)]\Big)\,[\eta(t) - \eta(s)]^{\theta-1}\,u(s)\,\eta'(s)\,ds. \tag{2.1}$$

Remark 2.3 (Memory-kernel interpretation)**.** The integral (2.1) can be recast in convolution form as

$$
{}_a\mathcal{J}^{\theta,\varrho,\eta}u(t) = \int_a^t \mathcal{K}_{\theta,\varrho,\eta}(t,s)\,u(s)\,ds,
$$

where the memory kernel of the operator is

$$
\mathcal{K}_{\theta,\varrho,\eta}(t,s) = \frac{1}{\varrho^{\theta}\,\Gamma(\theta)}\,\exp\Big(\frac{\varrho-1}{\varrho}[\eta(t)-\eta(s)]\Big)\,[\eta(t)-\eta(s)]^{\theta-1}\,\eta'(s). \tag{2.2}
$$

This kernel may be interpreted as comprising three physically meaningful components: (i) an exponential tempering factor $\exp(\frac{\varrho-1}{\varrho}[\eta(t)-\eta(s)])$ governed by ϱ, which controls the strength of memory damping (cf. [38]); (ii) a power-law memory factor $[\eta(t)-\eta(s)]^{\theta-1}$ encoding long-range temporal correlations of order θ (cf. [4]); and (iii) a time-deformation factor $\eta'(s)$ associated with the choice of the function η, accommodating non uniform temporal scales (cf. [26]). The degenerate forms of the operator correspond to well-known memory regimes:

- As $\varrho \to 1^-$, the exponential tempering vanishes and the operator reduces to the η-Riemann–Liouville fractional integral with pure power-law memory.
- For $\eta(t) = t$, one recovers the standard tempered power-law kernel of proportional type, characteristic of systems with truncated memory.
- For $\eta(t) = \ln t$, the operator reduces to a Hadamard-type kernel suited to logarithmic time scales.

This kernel-based interpretation places the proposed operator within the broader physical framework of fractional systems with tempered, power-law, and time-deformed memory effects, and provides a transparent physical reading of the various special cases recovered by our results.

Definition 2.4 ([27])**.** Let $\varrho \in (0,1]$ and $\theta > 0$. The left-sided generalized proportional fractional derivative of order θ relative to η of a function u is defined by

$$
{}_a\mathcal{D}^{\theta,\varrho,\eta}u(t) = \mathcal{D}^{n,\varrho,\eta}({}_a\mathcal{J}^{n-\theta,\varrho,\eta}u)(t), \qquad n = [\theta]+1,
$$

where $\mathcal{D}^{n,\varrho,\eta}$ stands for the n-fold composition of $\mathcal{D}^{\varrho,\eta}$ with itself.

Lemma 2.5 (Semigroup property, [26])**.** *Let* $\varrho \in (0,1]$ *and* $\theta_1, \theta_2 > 0$. *Then for every continuous function u defined on* $[a,b]$ *one has*

$$
{}_a\mathcal{J}^{\theta_1,\varrho,\eta}({}_a\mathcal{J}^{\theta_2,\varrho,\eta}u)(t) = {}_a\mathcal{J}^{\theta_1+\theta_2,\varrho,\eta}u(t).
$$

Lemma 2.6 (Inversion formula, [27])**.** *Suppose* $0 < \theta < 1$. *For sufficiently regular u, the identity*

$$
{}_a\mathcal{J}^{\theta,\varrho,\eta}\,{}_a\mathcal{D}^{\theta,\varrho,\eta}u(t) = u(t) - \exp\Big(\frac{\varrho-1}{\varrho}H(t)\Big)\,\frac{H(t)^{\theta-1}}{\varrho^{\theta-1}\Gamma(\theta)}\,({}_a\mathcal{J}^{1-\theta,\varrho,\eta}u)(a^+) \tag{2.3}
$$

holds.

Lemma 2.7 (Action on weighted power functions, [27])**.** *For* $\theta, \nu > 0$ *and* $\varrho > 0$, *one has*

$$
{}_a\mathcal{J}^{\theta,\varrho,\eta}\Big[\exp\Big(\frac{\varrho-1}{\varrho}H(\cdot)\Big)H(\cdot)^{\nu-1}\Big](t) = \frac{\Gamma(\nu)}{\varrho^{\theta}\,\Gamma(\nu+\theta)}\,\exp\Big(\frac{\varrho-1}{\varrho}H(t)\Big)H(t)^{\theta+\nu-1}. \tag{2.4}
$$

Remark 2.8. Because $\varrho \in (0,1]$ implies $(\varrho-1)/\varrho \le 0$, and because $\eta(t)-\eta(s) \ge 0$ whenever $t \ge s \ge a$ owing to the monotonicity of η, the exponential factor satisfies

$$\Big|\exp(\frac{\varrho-1}{\varrho}[\eta(t)-\eta(s)])\Big| \le 1, \qquad t \ge s \ge a. \tag{2.5}$$

This bound will be used systematically in the integral estimates that follow.

Definition 2.9. A real-valued function $\zeta\colon [a,b] \to \mathbb{R}$ is said to be of bounded variation on $[a,b]$ provided

$$V_a^b(\zeta) := \sup_P \sum_{k=1}^{n} |\zeta(t_k)-\zeta(t_{k-1})| < \infty,$$

with the supremum being taken over all finite partitions $P : a = t_0 < t_1 < \cdots < t_n = b$ of $[a,b]$.

Lemma 2.10 ([39–41]). *Let $w \in C([a,b],\mathbb{R})$, and let ζ be of bounded variation on $[a,b]$. Then the Riemann-Stieltjes integral $\int_a^b w(s)\,d\zeta(s)$ exists and obeys the estimate*

$$\Big|\int_a^b w(s)\,d\zeta(s)\Big| \le \max_{s\in[a,b]} |w(s)|\; V_a^b(\zeta).$$

For the existence and basic properties of the Riemann-Stieltjes integral for continuous integrands and integrators of bounded variation, we refer the reader to the classical treatments in [39–41]; the application of the above estimate within the context of generalized fractional boundary value problems with Stieltjes-type nonlocal conditions closely follows the methodology of [36].

We now recall the fixed-point principles that we shall employ.

Lemma 2.11 (Schauder, [42]). *Let S be a closed, bounded, convex subset of a Banach space, and assume $T : S \to S$ is compact. Then T admits at least one fixed point in S.*

Lemma 2.12 (Krasnoselskii, [42]). *Let S be a closed, bounded, convex, non-empty subset of a Banach space E. Suppose T_1 and T_2 are operators defined on S with values in E such that*

(a) $T_1(u) + T_2(v) \in S$; for every pair $u, v \in S$;
(b) T_1 is continuous and compact;
(c) T_2 is a contraction.

Then there exists $w \in S$ with $w = T_1(w) + T_2(w)$.

Lemma 2.13 (Leray-Schauder nonlinear alternative, [42]). *Let C be a closed convex subset of a Banach space E and $U \subset C$ an open set with $0 \in U$. Assume that $T : \overline{U} \to C$ is continuous and compact. Then, either*

(a) T has a fixed point in $\overline{U}$, or
(b) there exist $u \in \partial U$ and $\nu \in (0,1)$ with $u = \nu T(u)$.

Lemma 2.14 (Banach contraction principle, [43]). *Every contraction mapping defined on a non-empty complete metric space admits a unique fixed point.*

3. Equivalent integral formulation

The first step of our analysis is to recast the boundary value problem (1.1) and (1.2) as an equivalent system of integral equations. The construction below is performed for an arbitrary forcing pair (ϕ_1, ϕ_2) and identifies the structural quantities that will underpin the estimates of Sections 4 and 5.

Remark 3.1. Throughout the remainder of the paper, all solutions z_i are sought in the Banach space $C([a,b],\mathbb{R})$ equipped with the supremum norm. Within this functional setting, the generalized proportional fractional derivatives ${}_a\mathcal{D}^{q_i,\varrho,\eta}z_i$ and ${}_a\mathcal{D}^{p_i+q_i,\varrho,\eta}z_i$ are well-defined for $z_i \in C([a,b],\mathbb{R})$, ensuring the consistency of the differential and integral formulations of the problem. This functional setting follows the methodology adopted in [36] for generalized fractional problems with Stieltjes-type boundary conditions.

Lemma 3.2. *Let* $\varrho \in (0,1]$, $0 < q_i < 1$, $0 < p_i < 1$ *with* $p_i + q_i > 1$, $\mu_i, \nu_i \in \mathbb{R}$, $a < \xi_i < b$, *and let* $\phi_i \in C([a,b],\mathbb{R})$ *for* $i = 1, 2$. *For solutions* $z_i \in C([a,b],\mathbb{R})$, *the linear coupled problem*

$$ {}_a\mathcal{D}^{p_i,\varrho,\eta}({}_a\mathcal{D}^{q_i,\varrho,\eta} + \mu_i)z_i(t) = \phi_i(t), \qquad t \in [a,b], \tag{3.1}$$

together with the boundary conditions

$$ {}_a\mathcal{J}^{1-q_i,\varrho,\eta}z_i(a) = 0, \qquad z_i(b) = \nu_i z_i(\xi_i) + \int_a^b z_j(s)\,d\zeta_i(s), \qquad i \neq j, \tag{3.2}$$

is equivalent to the integral representation

$$ z_i(t) = {}_a\mathcal{J}^{p_i+q_i,\varrho,\eta}\phi_i(t) - \mu_i {}_a\mathcal{J}^{q_i,\varrho,\eta}z_i(t) + \frac{K_i(t)}{\Sigma_i^*}\mathcal{G}_i, \tag{3.3}$$

where, with the abbreviations

$$ K_i(t) = \exp\Big(\frac{\varrho-1}{\varrho}H(t)\Big)\frac{H(t)^{p_i+q_i-1}}{\varrho^{p_i+q_i-1}\Gamma(p_i+q_i)}, \tag{3.4}$$

$$ \Sigma_i = K_i(b) = \exp\Big(\frac{\varrho-1}{\varrho}H\Big)\frac{H^{p_i+q_i-1}}{\varrho^{p_i+q_i-1}\Gamma(p_i+q_i)}, \tag{3.5}$$

$$ \Sigma_i^* = \Sigma_i - \nu_i K_i(\xi_i) \neq 0, \tag{3.6}$$

the quantity $\mathcal{G}_i$ *is given by*

$$ \begin{aligned} \mathcal{G}_i = {} & \nu_i {}_a\mathcal{J}^{p_i+q_i,\varrho,\eta}\phi_i(\xi_i) + \int_a^b z_j(s)\,d\zeta_i(s) \\ & - {}_a\mathcal{J}^{p_i+q_i,\varrho,\eta}\phi_i(b) + \mu_i {}_a\mathcal{J}^{q_i,\varrho,\eta}z_i(b) - \nu_i\mu_i {}_a\mathcal{J}^{q_i,\varrho,\eta}z_i(\xi_i). \end{aligned} \tag{3.7}$$

Proof. Fix $i \in \{1,2\}$ and apply the operator ${}_a\mathcal{J}^{p_i,\varrho,\eta}$ to both sides of (3.1). Owing to Lemma 2.6 (applicable since $0 < p_i < 1$), one obtains

$$ ({}_a\mathcal{D}^{q_i,\varrho,\eta} + \mu_i)z_i(t) = {}_a\mathcal{J}^{p_i,\varrho,\eta}\phi_i(t) + \exp\Big(\frac{\varrho-1}{\varrho}H(t)\Big)\frac{H(t)^{p_i-1}}{\varrho^{p_i-1}\Gamma(p_i)}c_i, \tag{3.8}$$

where $c_i := {}_a\mathcal{J}^{1-p_i,\varrho,\eta}\Big[({}_a\mathcal{D}^{q_i,\varrho,\eta} + \mu_i)z_i\Big](a^+)$ is a constant of integration produced by the inversion formula.

We next apply ${}_a\mathcal{J}^{q_i,\varrho,\eta}$ to (3.8). Combining Lemmas 2.5, 2.6 (used now for q_i), and 2.7 (with $\nu = p_i$ for the H^{p_i-1}-term), one arrives at

$$\begin{aligned} z_i(t) &= {}_a\mathcal{J}^{p_i+q_i,\varrho,\eta}\phi_i(t) - \mu_{i\,a}\mathcal{J}^{q_i,\varrho,\eta}z_i(t) \\ &\quad + \exp\Big(\frac{\varrho-1}{\varrho}H(t)\Big)\frac{H(t)^{p_i+q_i-1}}{\varrho^{p_i+q_i-1}\Gamma(p_i+q_i)}\,c_i + \exp\Big(\frac{\varrho-1}{\varrho}H(t)\Big)\frac{H(t)^{q_i-1}}{\varrho^{q_i-1}\Gamma(q_i)}\,d_i, \end{aligned} \tag{3.9}$$

where $d_i := {}_a\mathcal{J}^{1-q_i,\varrho,\eta}z_i(a^+)$ is the constant produced by the second use of Lemma 2.6. Each component therefore contributes two integration constants, c_i and d_i.

Recalling the abbreviation (3.4), formula (3.9) reads

$$z_i(t) = {}_a\mathcal{J}^{p_i+q_i,\varrho,\eta}\phi_i(t) - \mu_{i\,a}\mathcal{J}^{q_i,\varrho,\eta}z_i(t) + c_iK_i(t) + d_i\exp\Big(\frac{\varrho-1}{\varrho}H(t)\Big)\frac{H(t)^{q_i-1}}{\varrho^{q_i-1}\Gamma(q_i)}. \tag{3.10}$$

The first boundary condition $({}_a\mathcal{J}^{1-q_i,\varrho,\eta}z_i)(a^+) = 0$ (written as a right-sided limit, since the term involving $H(t)^{q_i-1}$ in (3.10) is singular at $t = a$) is now imposed. By Lemma 2.7, applied to each summand on the right-hand side of (3.10), all the contributions vanish at $t = a$ except the one involving d_i, which produces d_i itself in the limit $t \to a^+$ (after cancellation of the prefactors). Hence $d_i = 0$ for $i = 1, 2$. With $d_i = 0$, Eq (3.10) simplifies to

$$z_i(t) = {}_a\mathcal{J}^{p_i+q_i,\varrho,\eta}\phi_i(t) - \mu_{i\,a}\mathcal{J}^{q_i,\varrho,\eta}z_i(t) + c_iK_i(t). \tag{3.11}$$

Evaluating (3.11) at $t = b$ and at $t = \xi_i$ produces, respectively,

$$z_i(b) = {}_a\mathcal{J}^{p_i+q_i,\varrho,\eta}\phi_i(b) - \mu_{i\,a}\mathcal{J}^{q_i,\varrho,\eta}z_i(b) + c_i\Sigma_i, \tag{3.12}$$

$$z_i(\xi_i) = {}_a\mathcal{J}^{p_i+q_i,\varrho,\eta}\phi_i(\xi_i) - \mu_{i\,a}\mathcal{J}^{q_i,\varrho,\eta}z_i(\xi_i) + c_iK_i(\xi_i). \tag{3.13}$$

Substituting (3.13) into the second boundary condition of (3.2) and equating the resulting expression to (3.12), after rearrangement the constant c_i satisfies

$$\begin{aligned} c_i[\Sigma_i - \nu_iK_i(\xi_i)] &= \nu_{i\,a}\mathcal{J}^{p_i+q_i,\varrho,\eta}\phi_i(\xi_i) - \nu_i\mu_{i\,a}\mathcal{J}^{q_i,\varrho,\eta}z_i(\xi_i) + \int_a^b z_j(s)\,d\zeta_i(s) \\ &\quad - {}_a\mathcal{J}^{p_i+q_i,\varrho,\eta}\phi_i(b) + \mu_{i\,a}\mathcal{J}^{q_i,\varrho,\eta}z_i(b). \end{aligned}$$

Since $\Sigma_i^* = \Sigma_i - \nu_iK_i(\xi_i) \neq 0$, we conclude

$$c_i = \frac{\mathcal{G}_i}{\Sigma_i^*}, \qquad i = 1, 2,\ i \neq j. \tag{3.14}$$

Putting (3.14) back into (3.11) yields exactly (3.3), which completes the proof. □

Remark 3.3. The integral representation (3.3) unifies several existing formulations:

(a) Setting $\varrho = 1$ collapses every exponential factor in (2.1) and (3.4) to unity, while ${}_a\mathcal{J}^{\theta,1,\eta}$ becomes the standard Riemann-Liouville fractional integral with respect to η. This recovers, in the scalar case, the framework of [28].

(b) Specializing to $\eta(t) = t$, the formulation reduces to the classical Riemann-Liouville coupled Langevin problem on $[a, b]$.

(c) Choosing $\eta(t) = \ln t$ together with $\varrho = 1$ (provided $a > 0$, so that η is well-defined on $[a, b]$) produces a Hadamard-type coupled Langevin system.

This breadth of reduction illustrates the unifying nature of the generalized proportional setting.

We conclude this section by introducing a quantity that will play a pivotal role in the subsequent estimates. The hypothesis $p_i + q_i > 1$ is essential at this point: It forces the exponent $p_i + q_i - 1 > 0$ in (3.4), so that $K_i(t) \to 0$ as $t \to a^+$, and K_i extends continuously to the closed interval $[a, b]$ with $K_i(a) = 0$. Combined with the standing requirement $\Sigma_i^* \neq 0$, this ensures that the ratio $K_i(\cdot)/\Sigma_i^*$ is a continuous function on $[a, b]$, hence bounded. We therefore set

$$M_i := \sup_{t\in[a,b]} \left| \frac{K_i(t)}{\Sigma_i^*} \right|, \qquad i = 1, 2, \tag{3.15}$$

which is a finite positive constant. Were $p_i + q_i \leq 1$ permitted, the function K_i would be unbounded in a neighbourhood of a and the ratio above would fail to be defined uniformly on $[a, b]$.

Lemma 3.4. *Let*

$$\beta_i := p_i + q_i - 1 > 0, \qquad \lambda := \frac{1-\varrho}{\varrho} > 0, \qquad u_i^* := \frac{\beta_i}{\lambda} = \frac{(p_i + q_i - 1)\varrho}{1-\varrho}. \tag{3.16}$$

Then the supremum defining M_i in (3.15) *admits the following closed-form expressions.*

(i) *If $u_i^* \geq H$, then*

$$\sup_{t\in[a,b]} K_i(t) = K_i(b) = \Sigma_i, \qquad M_i = \frac{\Sigma_i}{|\Sigma_i^*|}. \tag{3.17}$$

(ii) *If $u_i^* < H$, then the supremum is attained at the unique interior point $t_i^* \in (a, b)$ defined by $H(t_i^*) = u_i^*$, and*

$$\sup_{t\in[a,b]} K_i(t) = K_i(t_i^*) = \frac{e^{-\beta_i}}{\Gamma(p_i + q_i)} \Big(\frac{\beta_i}{1-\varrho} \Big)^{\beta_i}, \qquad M_i = \frac{K_i(t_i^*)}{|\Sigma_i^*|}. \tag{3.18}$$

Proof. The change of variables $u = H(t)$ is a monotone increasing bijection from $[a, b]$ onto $[0, H]$ by the standing assumption $\eta' > 0$; hence it suffices to study

$$g_i(u) := e^{-\lambda u} u^{\beta_i}, \qquad u \in [0, H], \tag{3.19}$$

which differs from K_i only by the positive multiplicative constant $\varrho^{\beta_i}\Gamma(p_i + q_i)$. Differentiation gives $g_i'(u) = e^{-\lambda u} u^{\beta_i - 1}(\beta_i - \lambda u)$, so g_i is strictly increasing on $[0, u_i^*]$ and strictly decreasing on $[u_i^*, \infty)$.

In case (i), $u_i^* \geq H$, so g_i is monotone increasing on $[0, H]$ and the supremum is attained at $u = H$, that is, at $t = b$; this yields (3.17).

In case (ii), $u_i^* < H$, and the supremum is attained at $u = u_i^*$. Since H is continuous and strictly increasing on $[a, b]$, there is a unique $t_i^* \in (a, b)$ with $H(t_i^*) = u_i^*$. Substituting $u = u_i^* = \beta_i/\lambda$ in K_i and simplifying $1/(\varrho\lambda) = 1/(1-\varrho)$ yields (3.18). □

4. Existence results

The Banach space underlying our analysis is $Y := C([a,b],\mathbb{R})$ endowed with the supremum norm $\|u\| = \sup_{t\in[a,b]}|u(t)|$, and the Cartesian product space $Y \times Y$ is equipped with the additive norm $\|(u,v)\| = \|u\| + \|v\|$, which makes it a Banach space as well.

In view of Lemma 3.2, a pair $(z_1,z_2) \in Y \times Y$ solves (1.1) and (1.2) if and only if it is a fixed point of the operator $\mathcal{F} = (\mathcal{P},Q)$ defined by $\mathcal{F}(z_1,z_2)(t) = (\mathcal{P}(z_1,z_2)(t), Q(z_1,z_2)(t))$, where

$$\begin{aligned}\mathcal{P}(z_1,z_2)(t) = {}& {}_a\mathcal{J}^{p_1+q_1,\varrho,\eta}\widehat{f_1}(t) - \mu_1{}_a\mathcal{J}^{q_1,\varrho,\eta}z_1(t)\\ &+ \frac{K_1(t)}{\Sigma_1^*}\Bigg[\nu_1{}_a\mathcal{J}^{p_1+q_1,\varrho,\eta}\widehat{f_1}(\xi_1) + \int_a^b z_2(s)\,d\zeta_1(s)\\ &- {}_a\mathcal{J}^{p_1+q_1,\varrho,\eta}\widehat{f_1}(b) + \mu_1{}_a\mathcal{J}^{q_1,\varrho,\eta}z_1(b) - \nu_1\mu_1{}_a\mathcal{J}^{q_1,\varrho,\eta}z_1(\xi_1)\Bigg],\end{aligned} \tag{4.1}$$

$$\begin{aligned}Q(z_1,z_2)(t) = {}& {}_a\mathcal{J}^{p_2+q_2,\varrho,\eta}\widehat{f_2}(t) - \mu_2{}_a\mathcal{J}^{q_2,\varrho,\eta}z_2(t)\\ &+ \frac{K_2(t)}{\Sigma_2^*}\Bigg[\nu_2{}_a\mathcal{J}^{p_2+q_2,\varrho,\eta}\widehat{f_2}(\xi_2) + \int_a^b z_1(s)\,d\zeta_2(s)\\ &- {}_a\mathcal{J}^{p_2+q_2,\varrho,\eta}\widehat{f_2}(b) + \mu_2{}_a\mathcal{J}^{q_2,\varrho,\eta}z_2(b) - \nu_2\mu_2{}_a\mathcal{J}^{q_2,\varrho,\eta}z_2(\xi_2)\Bigg],\end{aligned} \tag{4.2}$$

and where, for brevity, we have set $\widehat{f_i}(t) := f_i(t, z_1(t), z_2(t))$. The continuity of $\mathcal{F}$ on $Y \times Y$ follows from the continuity of the proportional fractional integrals together with the hypothesis that each f_i is continuous; see, e.g., [26].

It will be convenient throughout this section to use the structural constants

$$\begin{aligned}A_1 &= \alpha_1\frac{H^{p_1+q_1}}{\varrho^{p_1+q_1}\,\Gamma(p_1+q_1+1)}, & A_2 &= \alpha_1\frac{H^{q_1}}{\varrho^{q_1}\,\Gamma(q_1+1)}, & A_3 &= M_1V_a^b(\zeta_1),\\ B_1 &= \alpha_2\frac{H^{p_2+q_2}}{\varrho^{p_2+q_2}\,\Gamma(p_2+q_2+1)}, & B_2 &= \alpha_2\frac{H^{q_2}}{\varrho^{q_2}\,\Gamma(q_2+1)}, & B_3 &= M_2V_a^b(\zeta_2),\end{aligned} \tag{4.3}$$

together with the abbreviation

$$\alpha_i := 1 + (1+|\nu_i|)\,M_i, \qquad i = 1,2. \tag{4.4}$$

A direct estimate, based on Lemma 2.10, the bound (2.5), the elementary inequality $|{}_a\mathcal{J}^{\theta,\varrho,\eta}u(t)| \le \|u\|\,H^\theta/(\varrho^\theta\Gamma(\theta+1))$, and the definition (3.15), yields the following pointwise bound for the components of $\mathcal{F}$:

$$\begin{aligned}\left|\mathcal{P}(z_1,z_2)(t)\right| &\le \|\widehat{f_1}\|\,A_1 + |\mu_1|\,\|z_1\|\,A_2 + \|z_2\|\,A_3,\\ \left|Q(z_1,z_2)(t)\right| &\le \|\widehat{f_2}\|\,B_1 + |\mu_2|\,\|z_2\|\,B_2 + \|z_1\|\,B_3.\end{aligned} \tag{4.5}$$

The detailed derivation of (4.5) is incorporated into the proofs that follow.

Finally, we set

$$\vartheta := \max\{\,|\mu_1|\,A_2 + B_3,\ A_3 + |\mu_2|\,B_2\,\}. \tag{4.6}$$

The motivation for the precise grouping of terms in (4.6) will be transparent inside the contraction estimate of Theorem 4.1.

4.1. Existence via Krasnoselskii's theorem

Theorem 4.1. *Assume that:*

(H$_1$) *Each* $f_i\colon [a,b]\times\mathbb{R}^2\to\mathbb{R}$ *is continuous.*

(H$_2$) *There exist* $\phi_i\in C([a,b],\mathbb{R}^+)$ *such that* $|f_i(t,u,v)|\le\phi_i(t)$ *for every* $t\in[a,b]$ *and every* $u,v\in\mathbb{R}$, $i=1,2$.

(H$_3$) $\vartheta<1$, *with* ϑ *as in* (4.6).

Then problem (1.1) *and* (1.2) *admits at least one solution in* $Y\times Y$.

Proof. Choose a positive radius

$$r\ge\frac{\|\phi_1\|\,A_1+\|\phi_2\|\,B_1}{1-\vartheta},\tag{4.7}$$

which is well defined by hypothesis (H$_3$), and consider the closed ball $\mathcal{B}_r:=\{(z_1,z_2)\in Y\times Y:\|z_1\|+\|z_2\|\le r\}$. We split $\mathcal{F}$ as $\mathcal{F}=\mathcal{F}_1+\mathcal{F}_2$, where the operator $\mathcal{F}_1=(\mathcal{P}_1,\mathcal{Q}_1)$ collects the terms involving $\widehat{f_i}$:

$$\begin{aligned}\mathcal{P}_1(z_1,z_2)(t)&={}_a\mathcal{J}^{p_1+q_1,\varrho,\eta}\widehat{f_1}(t)+\frac{K_1(t)}{\Sigma_1^*}\Big[\nu_1\,{}_a\mathcal{J}^{p_1+q_1,\varrho,\eta}\widehat{f_1}(\xi_1)-{}_a\mathcal{J}^{p_1+q_1,\varrho,\eta}\widehat{f_1}(b)\Big],\\ \mathcal{Q}_1(z_1,z_2)(t)&={}_a\mathcal{J}^{p_2+q_2,\varrho,\eta}\widehat{f_2}(t)+\frac{K_2(t)}{\Sigma_2^*}\Big[\nu_2\,{}_a\mathcal{J}^{p_2+q_2,\varrho,\eta}\widehat{f_2}(\xi_2)-{}_a\mathcal{J}^{p_2+q_2,\varrho,\eta}\widehat{f_2}(b)\Big],\end{aligned}\tag{4.8}$$

and $\mathcal{F}_2=(\mathcal{P}_2,\mathcal{Q}_2)$ collects the remaining linear-in-z terms:

$$\begin{aligned}\mathcal{P}_2(z_1,z_2)(t)&=-\mu_1{}_a\mathcal{J}^{q_1,\varrho,\eta}z_1(t)+\frac{K_1(t)}{\Sigma_1^*}\Big[\int_a^b z_2(s)\,d\zeta_1(s)+\mu_1{}_a\mathcal{J}^{q_1,\varrho,\eta}z_1(b)-\nu_1\mu_1{}_a\mathcal{J}^{q_1,\varrho,\eta}z_1(\xi_1)\Big],\\ \mathcal{Q}_2(z_1,z_2)(t)&=-\mu_2{}_a\mathcal{J}^{q_2,\varrho,\eta}z_2(t)+\frac{K_2(t)}{\Sigma_2^*}\Big[\int_a^b z_1(s)\,d\zeta_2(s)+\mu_2{}_a\mathcal{J}^{q_2,\varrho,\eta}z_2(b)-\nu_2\mu_2{}_a\mathcal{J}^{q_2,\varrho,\eta}z_2(\xi_2)\Big].\end{aligned}\tag{4.9}$$

We verify in turn the three hypotheses of Lemma 2.12 (Krasnoselskii's theorem).

Step 1: $\mathcal{F}_1(u_1,u_2)+\mathcal{F}_2(v_1,v_2)\in\mathcal{B}_r$ whenever $(u_1,u_2),(v_1,v_2)\in\mathcal{B}_r$.

For arbitrary $(u_1,u_2),(v_1,v_2)\in\mathcal{B}_r$ and any $t\in[a,b]$, hypothesis (H$_2$) delivers $\|f_i(\cdot,u_1,u_2)\|\le\|\phi_i\|$. Combining the bound (2.5) with the standard estimate

$$|{}_a\mathcal{J}^{\theta,\varrho,\eta}w(t)|\le\|w\|\,\frac{H^{\theta}}{\varrho^{\theta}\,\Gamma(\theta+1)},$$

the definition (3.15) of M_i, and Lemma 2.10, we obtain

$$\begin{aligned}&\Big|\mathcal{P}_1(u_1,u_2)(t)+\mathcal{P}_2(v_1,v_2)(t)\Big|\\ &\quad\le\|\phi_1\|\,\frac{H^{p_1+q_1}}{\varrho^{p_1+q_1}\,\Gamma(p_1+q_1+1)}+|\mu_1|\,\|v_1\|\,\frac{H^{q_1}}{\varrho^{q_1}\,\Gamma(q_1+1)}\\ &\qquad+M_1\Big[|\nu_1|\,\|\phi_1\|\,\frac{H^{p_1+q_1}}{\varrho^{p_1+q_1}\Gamma(p_1+q_1+1)}+\|v_2\|\,V_a^b(\zeta_1)+\|\phi_1\|\,\frac{H^{p_1+q_1}}{\varrho^{p_1+q_1}\Gamma(p_1+q_1+1)}\\ &\qquad+|\mu_1|\,\|v_1\|\,\frac{H^{q_1}}{\varrho^{q_1}\Gamma(q_1+1)}+|\nu_1|\,|\mu_1|\,\|v_1\|\,\frac{H^{q_1}}{\varrho^{q_1}\Gamma(q_1+1)}\Big].\end{aligned}$$

Collecting all ϕ_1-terms, all v_1-terms, and the single v_2-term, and recalling (4.3) and (4.4), we rewrite the right-hand side as

$$\left|\mathcal{P}_1(u_1,u_2)(t)+\mathcal{P}_2(v_1,v_2)(t)\right| \le \|\phi_1\|\,A_1+|\mu_1|\,\|v_1\|\,A_2+A_3\,\|v_2\|. \tag{4.10}$$

The same argument applied to the $\mathcal{Q}$-component produces

$$\left|\mathcal{Q}_1(u_1,u_2)(t)+\mathcal{Q}_2(v_1,v_2)(t)\right| \le \|\phi_2\|\,B_1+|\mu_2|\,\|v_2\|\,B_2+B_3\,\|v_1\|. \tag{4.11}$$

Taking the supremum over t in (4.10) and (4.11) and adding the two inequalities yields

$$\begin{aligned}\left\|\mathcal{F}_1(u_1,u_2)+\mathcal{F}_2(v_1,v_2)\right\| &\le \|\phi_1\|\,A_1+\|\phi_2\|\,B_1+(|\mu_1|\,A_2+B_3)\|v_1\|+(A_3+|\mu_2|\,B_2)\|v_2\|\\ &\le \|\phi_1\|\,A_1+\|\phi_2\|\,B_1+\vartheta\,(\|v_1\|+\|v_2\|)\\ &\le \|\phi_1\|\,A_1+\|\phi_2\|\,B_1+\vartheta\,r\\ &\le (1-\vartheta)\,r+\vartheta\,r=r,\end{aligned} \tag{4.12}$$

with the last inequality being a consequence of (4.7). We have therefore shown that $\mathcal{F}_1(u_1,u_2)+\mathcal{F}_2(v_1,v_2)\in\mathcal{B}_r$, as required.

Step 2: $\mathcal{F}_2$ is a contraction on $\mathcal{B}_r$.

For $(z_1,z_2),(z_1^*,z_2^*)\in\mathcal{B}_r$ and $t\in[a,b]$, the same chain of inequalities used above gives

$$\left|\mathcal{P}_2(z_1,z_2)(t)-\mathcal{P}_2(z_1^*,z_2^*)(t)\right| \le |\mu_1|\,A_2\,\|z_1-z_1^*\|+A_3\,\|z_2-z_2^*\|, \tag{4.13}$$

$$\left|\mathcal{Q}_2(z_1,z_2)(t)-\mathcal{Q}_2(z_1^*,z_2^*)(t)\right| \le |\mu_2|\,B_2\,\|z_2-z_2^*\|+B_3\,\|z_1-z_1^*\|. \tag{4.14}$$

Passing to the supremum over t and summing (4.13) and (4.14),

$$\left\|\mathcal{F}_2(z_1,z_2)-\mathcal{F}_2(z_1^*,z_2^*)\right\| \le (|\mu_1|\,A_2+B_3)\,\|z_1-z_1^*\|+(A_3+|\mu_2|\,B_2)\,\|z_2-z_2^*\|.$$

Bounding each parenthesis by the maximum, we conclude that

$$\left\|\mathcal{F}_2(z_1,z_2)-\mathcal{F}_2(z_1^*,z_2^*)\right\| \le \vartheta\,(\|z_1-z_1^*\|+\|z_2-z_2^*\|), \tag{4.15}$$

and the assumption $\vartheta<1$ tells us that $\mathcal{F}_2$ is a strict contraction on $\mathcal{B}_r$.

Step 3: $\mathcal{F}_1$ is continuous and compact on $\mathcal{B}_r$.

Continuity of $\mathcal{F}_1$ follows from the continuity of each f_i combined with the dominated convergence theorem: If $(z_1^{(n)},z_2^{(n)})\to(z_1,z_2)$ uniformly on $[a,b]$, then by uniform continuity of f_i on the corresponding compact set, $f_i(\cdot,z_1^{(n)},z_2^{(n)})\to f_i(\cdot,z_1,z_2)$ uniformly on $[a,b]$; since the proportional fractional integral ${}_a\mathcal{J}^{p_i+q_i,\varrho,\eta}$ is a bounded linear operator on $C([a,b],\mathbb{R})$ in view of (2.1), the corresponding integral expressions converge uniformly as well.

For uniform boundedness, hypothesis (H$_2$) together with (2.5) gives, for every $(z_1,z_2)\in\mathcal{B}_r$ and every $t\in[a,b]$,

$$\left|\mathcal{P}_1(z_1,z_2)(t)\right|\le\|\phi_1\|\,A_1,\qquad \left|\mathcal{Q}_1(z_1,z_2)(t)\right|\le\|\phi_2\|\,B_1,$$

so that $\|\mathcal{F}_1(\mathcal{B}_r)\|\le\|\phi_1\|\,A_1+\|\phi_2\|\,B_1$, a finite constant.

We turn to equicontinuity. Pick $\tau_1,\tau_2\in[a,b]$ with $\tau_1<\tau_2$, and let $(z_1,z_2)\in\mathcal{B}_r$. Set $\theta_1:=p_1+q_1>1$ and define

$$G_1(t,s):=\exp\Big(\tfrac{\varrho-1}{\varrho}[\eta(t)-\eta(s)]\Big)[\eta(t)-\eta(s)]^{\theta_1-1},\qquad a\le s\le t\le b. \tag{4.16}$$

Since $\eta \in C^1([a,b])$ and $\theta_1 - 1 > 0$, the kernel G_1 extends continuously to the compact triangle

$$\Delta := \{(t,s) \in [a,b]^2 : \ a \le s \le t \le b\},$$

with $G_1(t,t) = 0$. Hence G_1 is uniformly continuous on Δ.

By definition,

$$\begin{aligned}&\left|\mathcal{P}_1(z_1,z_2)(\tau_2) - \mathcal{P}_1(z_1,z_2)(\tau_1)\right| \\ &\le \left|{}_a\mathcal{J}^{\theta_1,\varrho,\eta}\widehat{f_1}(\tau_2) - {}_a\mathcal{J}^{\theta_1,\varrho,\eta}\widehat{f_1}(\tau_1)\right| + |K_1(\tau_2) - K_1(\tau_1)|\,\frac{|\nu_1|+1}{|\Sigma_1^*|}\,\|\phi_1\|\,\frac{H^{\theta_1}}{\varrho^{\theta_1}\Gamma(\theta_1+1)}.\end{aligned}$$

Because $\theta_1 > 1$, the function K_1 is uniformly continuous on $[a,b]$, and therefore $|K_1(\tau_2) - K_1(\tau_1)| \to 0$ as $\tau_2 \to \tau_1$, uniformly in $(z_1,z_2) \in \mathcal{B}_r$.

For the fractional-integral term, we split at τ_1:

$$\begin{aligned}{}_a\mathcal{J}^{\theta_1,\varrho,\eta}\widehat{f_1}(\tau_2) - {}_a\mathcal{J}^{\theta_1,\varrho,\eta}\widehat{f_1}(\tau_1) = &\frac{1}{\varrho^{\theta_1}\Gamma(\theta_1)}\int_a^{\tau_1}[G_1(\tau_2,s) - G_1(\tau_1,s)]\widehat{f_1}(s)\eta'(s)\,\mathrm{d}s \\ &+ \frac{1}{\varrho^{\theta_1}\Gamma(\theta_1)}\int_{\tau_1}^{\tau_2} G_1(\tau_2,s)\widehat{f_1}(s)\eta'(s)\,\mathrm{d}s.\end{aligned}$$

Using (H$_2$), $|\widehat{f_1}(s)| \le \|\phi_1\|$, we obtain

$$\begin{aligned}\left|{}_a\mathcal{J}^{\theta_1,\varrho,\eta}\widehat{f_1}(\tau_2) - {}_a\mathcal{J}^{\theta_1,\varrho,\eta}\widehat{f_1}(\tau_1)\right| \le &\frac{\|\phi_1\|}{\varrho^{\theta_1}\Gamma(\theta_1)}\int_a^{\tau_1}\left|G_1(\tau_2,s) - G_1(\tau_1,s)\right|\eta'(s)\,\mathrm{d}s \\ &+ \frac{\|\phi_1\|}{\varrho^{\theta_1}\Gamma(\theta_1)}\int_{\tau_1}^{\tau_2}|G_1(\tau_2,s)|\,\eta'(s)\,\mathrm{d}s.\end{aligned}$$

Since G_1 is uniformly continuous on the compact triangle Δ, the first integral tends to zero as $\tau_2 \to \tau_1$, uniformly in $(z_1,z_2) \in \mathcal{B}_r$. For the second integral, the bound (2.5) of Remark 2.8 yields $|G_1(\tau_2,s)| \le [\eta(\tau_2) - \eta(s)]^{\theta_1-1}$, and hence

$$\int_{\tau_1}^{\tau_2}|G_1(\tau_2,s)|\,\eta'(s)\,\mathrm{d}s \le \int_{\tau_1}^{\tau_2}[\eta(\tau_2) - \eta(s)]^{\theta_1-1}\eta'(s)\,\mathrm{d}s = \frac{[\eta(\tau_2) - \eta(\tau_1)]^{\theta_1}}{\theta_1} \to 0,$$

as $\tau_2 \to \tau_1$, again uniformly in (z_1,z_2). Consequently, $|\mathcal{P}_1(z_1,z_2)(\tau_2) - \mathcal{P}_1(z_1,z_2)(\tau_1)| \to 0$ uniformly in $(z_1,z_2) \in \mathcal{B}_r$. The same argument applies to $\mathcal{Q}_1$. Hence $\mathcal{F}_1(\mathcal{B}_r)$ is equicontinuous.

By the Arzelà-Ascoli theorem, $\mathcal{F}_1(\mathcal{B}_r)$ is relatively compact in $Y \times Y$, that is, $\mathcal{F}_1$ is compact.

The three steps above check the hypotheses of Lemma 2.12, which thus produces a fixed point of $\mathcal{F}_1 + \mathcal{F}_2 = \mathcal{F}$ inside $\mathcal{B}_r$. By the equivalence supplied by Lemma 3.2, this fixed point is a solution of (1.1) and (1.2), and the proof is complete. □

4.2. Existence via the Leray-Schauder nonlinear alternative

We replace the boundedness assumption (H$_2$) by a sublinear growth condition.

Theorem 4.2. *Suppose that:*

(H$_1$) *Each* $f_i\colon [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous.*

(H$_4$) *There exist non-decreasing functions* $\Psi_1, \Psi_2, \Xi_1, \Xi_2 \colon \mathbb{R}^+ \to \mathbb{R}^+$ *and non-negative continuous weights* $\phi_1, \phi_2, \chi_1, \chi_2 \in C([a,b], \mathbb{R}^+)$ *such that, for every* $t \in [a,b]$ *and every* $u, v \in \mathbb{R}$,

$$|f_1(t,u,v)| \le \phi_1(t)\,\Psi_1(|u|) + \phi_2(t)\,\Psi_2(|v|), \quad |f_2(t,u,v)| \le \chi_1(t)\,\Xi_1(|u|) + \chi_2(t)\,\Xi_2(|v|).$$

(H$_5$) *There exists a constant* $r > 0$ *such that*

$$r > \frac{(\|\phi_1\|\,\Psi_1(r) + \|\phi_2\|\,\Psi_2(r))A_1 + (\|\chi_1\|\,\Xi_1(r) + \|\chi_2\|\,\Xi_2(r))B_1}{1-\vartheta}, \tag{4.17}$$

with $\vartheta < 1$ *as in* (4.6).

Then problem (1.1) *and* (1.2) *has at least one solution in* $Y \times Y$.

Proof. Define the open ball $U_r := \{(z_1, z_2) \in Y \times Y : \|z_1\| + \|z_2\| < r\}$, so that its closure is $\overline{U_r} = \{(z_1, z_2) \in Y \times Y : \|z_1\| + \|z_2\| \le r\}$. Continuity of $\mathcal{F} : \overline{U_r} \to Y \times Y$ together with compactness of $\mathcal{F}(\overline{U_r})$ are established exactly as in Step 3 of the proof of Theorem 4.1 (recall that the equicontinuity argument relies only on $p_i + q_i > 1$, which is part of the standing hypotheses). The compactness argument applies verbatim to $\mathcal{F}_2$ as well: The proportional fractional integral ${}_a\mathcal{J}^{q_i,\varrho,\eta}$ is a compact operator on $C([a,b], \mathbb{R})$ by the Arzelà-Ascoli theorem, while the boundary terms involving $K_i(t)/\Sigma_i^*$ are of finite rank. Hence $\mathcal{F}$ is completely continuous on $\overline{U_r}$. To apply Lemma 2.13, we shall verify that no solution of the parametrized equation $(z_1, z_2) = \lambda\mathcal{F}(z_1, z_2)$ with $\lambda \in (0,1)$ can lie on the boundary ∂U_r.

Suppose, on the contrary, that $(z_1, z_2) = \lambda\mathcal{F}(z_1, z_2)$ for some $\lambda \in (0,1)$ and that $(z_1, z_2) \in \partial U_r$, that is, $\|z_1\| + \|z_2\| = r$. Then for every $t \in [a,b]$,

$$\begin{aligned}
|z_1(t)| &= \lambda\,|\mathcal{P}(z_1,z_2)(t)| \le |\mathcal{P}(z_1,z_2)(t)| \\
&\le (\phi_1(t)\,\Psi_1(\|z_1\|) + \phi_2(t)\,\Psi_2(\|z_2\|))\frac{H^{p_1+q_1}}{\varrho^{p_1+q_1}\Gamma(p_1+q_1+1)} + |\mu_1|\,\|z_1\|\,\frac{H^{q_1}}{\varrho^{q_1}\Gamma(q_1+1)} \\
&\quad + M_1\Big[|\nu_1|(\|\phi_1\|\Psi_1(\|z_1\|) + \|\phi_2\|\Psi_2(\|z_2\|))\frac{H^{p_1+q_1}}{\varrho^{p_1+q_1}\Gamma(p_1+q_1+1)} \\
&\quad + \|z_2\|\,V_a^b(\zeta_1) + (\|\phi_1\|\Psi_1(\|z_1\|) + \|\phi_2\|\Psi_2(\|z_2\|))\frac{H^{p_1+q_1}}{\varrho^{p_1+q_1}\Gamma(p_1+q_1+1)} \\
&\quad + |\mu_1|\,\|z_1\|\,\frac{H^{q_1}}{\varrho^{q_1}\Gamma(q_1+1)} + |\nu_1\mu_1|\,\|z_1\|\,\frac{H^{q_1}}{\varrho^{q_1}\Gamma(q_1+1)}\Big].
\end{aligned}$$

Following the same regrouping that produced (4.10), the right-hand side is bounded by

$$(\|\phi_1\|\,\Psi_1(\|z_1\|) + \|\phi_2\|\,\Psi_2(\|z_2\|))A_1 + |\mu_1|\,\|z_1\|\,A_2 + A_3\,\|z_2\|.$$

Taking the supremum over t produces

$$\|z_1\| \le (\|\phi_1\|\,\Psi_1(\|z_1\|) + \|\phi_2\|\,\Psi_2(\|z_2\|))A_1 + |\mu_1|\,\|z_1\|\,A_2 + A_3\,\|z_2\|. \tag{4.18}$$

The analogous argument for the $\mathcal{Q}$-component gives

$$\|z_2\| \le (\|\chi_1\|\,\Xi_1(\|z_1\|) + \|\chi_2\|\,\Xi_2(\|z_2\|))B_1 + |\mu_2|\,\|z_2\|\,B_2 + B_3\,\|z_1\|. \tag{4.19}$$

Adding (4.18) and (4.19), monotonicity of Ψ_i, Ξ_i together with the bound $\|z_i\| \le r$ yields

$$\begin{aligned}\|z_1\| + \|z_2\| &\le (\|\phi_1\|\Psi_1(r) + \|\phi_2\|\Psi_2(r))A_1 + (\|\chi_1\|\Xi_1(r) + \|\chi_2\|\Xi_2(r))B_1 \\ &\quad + (|\mu_1|A_2 + B_3)\|z_1\| + (A_3 + |\mu_2|B_2)\|z_2\| \\ &\le (\|\phi_1\|\Psi_1(r) + \|\phi_2\|\Psi_2(r))A_1 + (\|\chi_1\|\Xi_1(r) + \|\chi_2\|\Xi_2(r))B_1 \\ &\quad + \vartheta(\|z_1\| + \|z_2\|).\end{aligned}$$

Solving for $\|z_1\| + \|z_2\|$,

$$\|z_1\| + \|z_2\| \le \frac{(\|\phi_1\|\Psi_1(r) + \|\phi_2\|\Psi_2(r))A_1 + (\|\chi_1\|\Xi_1(r) + \|\chi_2\|\Xi_2(r))B_1}{1 - \vartheta}.$$

Hypothesis (H_5) implies $\|z_1\|+\|z_2\| < r$, which is in strict contradiction with the assumption $(z_1, z_2) \in \partial U_r$, i.e., $\|z_1\| + \|z_2\| = r$. The nonlinear alternative of Leray-Schauder (Lemma 2.13) thus delivers a fixed point of $\mathcal{F}$ inside $\overline{U_r}$, equivalently a solution of (1.1) and (1.2). □

5. Uniqueness result

We now establish a uniqueness theorem under a standard Lipschitz condition on the nonlinearities, by means of Banach's contraction principle.

Theorem 5.1. *Suppose that:*

(H_6) *Each* $f_i\colon [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous and there exist constants* $L_i > 0$, $i = 1, 2$, *such that*

$$\left|f_i(t, u_1, u_2) - f_i(t, v_1, v_2)\right| \le L_i(|u_1 - v_1| + |u_2 - v_2|), \qquad t \in [a, b],\ u_1, u_2, v_1, v_2 \in \mathbb{R}.$$

(H_7) *The structural inequality*

$$\Lambda := \vartheta + L_1 A_1 + L_2 B_1 < 1 \tag{5.1}$$

holds, with ϑ *as in* (4.6).

Then problem (1.1) *and* (1.2) *has a unique solution in* $Y \times Y$.

Proof. Set $W_i := \sup_{t\in[a,b]} |f_i(t, 0, 0)|$, $i = 1, 2$, which is finite because $f_i(\cdot, 0, 0)$ is continuous on the compact interval $[a, b]$, and choose

$$r \ge \frac{W_1 A_1 + W_2 B_1}{1 - \Lambda}. \tag{5.2}$$

Define the closed ball $\mathcal{B}_r := \{(z_1, z_2) \in Y \times Y : \|z_1\| + \|z_2\| \le r\}$. We shall show that $\mathcal{F} : \mathcal{B}_r \to \mathcal{B}_r$ is a contraction mapping; the existence and uniqueness of a fixed point of $\mathcal{F}$ will then follow from Lemma 2.14, and the equivalence of fixed points of $\mathcal{F}$ with solutions of (1.1) and (1.2) (provided by Lemma 3.2) will conclude the proof.

Step 1: $\mathcal{F}$ maps $\mathcal{B}_r$ into itself.

The Lipschitz condition (H_6) provides, for any $(z_1, z_2) \in \mathcal{B}_r$ and any $t \in [a, b]$,

$$|f_i(t, z_1(t), z_2(t))| \le L_i(|z_1(t)| + |z_2(t)|) + W_i \le L_i(\|z_1\| + \|z_2\|) + W_i \le L_i r + W_i.$$

Inserting this into the bound (4.5) and recalling (4.3), we obtain

$$\begin{aligned}\left|\mathcal{P}(z_1,z_2)(t)\right| &\le (L_1 r + W_1)\,A_1 + |\mu_1|\,\|z_1\|\,A_2 + A_3\,\|z_2\|,\\ \left|\mathcal{Q}(z_1,z_2)(t)\right| &\le (L_2 r + W_2)\,B_1 + |\mu_2|\,\|z_2\|\,B_2 + B_3\,\|z_1\|.\end{aligned}$$

Adding these bounds, taking the supremum over t, and using $\|z_1\| + \|z_2\| \le r$ together with the same regrouping that produced ϑ,

$$\|\mathcal{F}(z_1,z_2)\| \le (L_1A_1 + L_2B_1)\,r + W_1A_1 + W_2B_1 + \vartheta\,r = \Lambda\,r + W_1A_1 + W_2B_1 \le \Lambda\,r + (1-\Lambda)\,r = r,$$

where the last step uses (5.2). Hence $\mathcal{F}(\mathcal{B}_r) \subseteq \mathcal{B}_r$.

Step 2: $\mathcal{F}$ is a contraction on $\mathcal{B}_r$.

Let $(z_1,z_2), (z_1^*,z_2^*) \in \mathcal{B}_r$ and $t \in [a,b]$. The Lipschitz hypothesis yields

$$\left|f_i(t,z_1(t),z_2(t)) - f_i(t,z_1^*(t),z_2^*(t))\right| \le L_i(|z_1(t)-z_1^*(t)| + |z_2(t)-z_2^*(t)|) \le L_i(\|z_1-z_1^*\| + \|z_2-z_2^*\|).$$

Combining this Lipschitz estimate (which applied to the integral $\widehat{f_i}$-terms produces a contribution $L_1A_1(\|z_1-z_1^*\| + \|z_2-z_2^*\|)$ in the $\mathcal{P}$-component) with the previously derived bounds (4.13) and (4.14), we obtain

$$\begin{aligned}\left|\mathcal{P}(z_1,z_2)(t) - \mathcal{P}(z_1^*,z_2^*)(t)\right| &\le L_1A_1(\|z_1-z_1^*\| + \|z_2-z_2^*\|) + |\mu_1|A_2\,\|z_1-z_1^*\| + A_3\,\|z_2-z_2^*\|,\\ \left|\mathcal{Q}(z_1,z_2)(t) - \mathcal{Q}(z_1^*,z_2^*)(t)\right| &\le L_2B_1(\|z_1-z_1^*\| + \|z_2-z_2^*\|) + |\mu_2|B_2\,\|z_2-z_2^*\| + B_3\,\|z_1-z_1^*\|.\end{aligned}$$

Summing the two preceding inequalities and taking the supremum in t,

$$\begin{aligned}&\|\mathcal{F}(z_1,z_2) - \mathcal{F}(z_1^*,z_2^*)\|\\ &\le (L_1A_1 + L_2B_1)(\|z_1-z_1^*\| + \|z_2-z_2^*\|) + (|\mu_1|A_2 + B_3)\,\|z_1-z_1^*\| + (A_3 + |\mu_2|B_2)\,\|z_2-z_2^*\|\\ &\le (L_1A_1 + L_2B_1 + \vartheta)(\|z_1-z_1^*\| + \|z_2-z_2^*\|)\\ &= \Lambda\left\|(z_1,z_2) - (z_1^*,z_2^*)\right\|.\end{aligned}$$

The hypothesis $\Lambda < 1$ shows that $\mathcal{F}$ is a contraction on $\mathcal{B}_r$.

By Banach's fixed point theorem (Lemma 2.14), $\mathcal{F}$ admits a unique fixed point in $\mathcal{B}_r$, which by Lemma 3.2 is the unique solution of (1.1) and (1.2). □

6. Examples

In this section we present three explicit examples that illustrate the applicability of the theorems established in Sections 4 and 5. In every example, we make the standing assumption $p_i + q_i > 1$ explicit, verify the non-vanishing of Σ_i^*, evaluate M_i numerically, and check that the structural inequality in the corresponding theorem is satisfied. All numerical values were obtained with a fine uniform grid of 20,000 points on $[0,1]$ and rounded to four decimal places.

Example 6.1. Take $a = 0$, $b = 1$, $\eta(t) = t^3 + t$ (so that $H = 2$), and $\varrho = 9/10$. Consider the coupled Langevin system

$${}_a\mathcal{D}^{p_i,\varrho,\eta}({}_a\mathcal{D}^{q_i,\varrho,\eta} + \mu_i)z_i(t) = f_i(t,z_1(t),z_2(t)), \qquad t \in [0,1],\ i = 1,2, \tag{6.1}$$

with the following parameter choices:

- Component 1: $p_1 = \frac{4}{5}$, $q_1 = \frac{1}{2}$ (so that $p_1 + q_1 = \frac{13}{10} > 1$), $\mu_1 = \frac{1}{10}$, $\nu_1 = \frac{1}{8}$, $\xi_1 = \frac{1}{3}$;
- Component 2: $p_2 = \frac{7}{10}$, $q_2 = \frac{2}{5}$ (so that $p_2 + q_2 = \frac{11}{10} > 1$), $\mu_2 = \frac{1}{12}$, $\nu_2 = \frac{1}{7}$, $\xi_2 = \frac{1}{4}$.

The boundary measures are taken as $\zeta_1(t) = \frac{t}{7+t}$ and $\zeta_2(t) = \sin(\frac{\pi t}{8})$, for which

$$V_0^1(\zeta_1) = \zeta_1(1) - \zeta_1(0) = \tfrac{1}{8}, \qquad V_0^1(\zeta_2) = \zeta_2(1) - \zeta_2(0) = \sin\tfrac{\pi}{8} \approx 0.3827.$$

The right-hand sides are taken to be

$$\begin{aligned} f_1(t,u,v) &= \frac{e^{-2t}}{4}\,\frac{|u|+2}{|u|+3} + \frac{t^2}{7\,(1+v^2)}, \\ f_2(t,u,v) &= \frac{\cos(2t)}{5\,(1+|u|)} + \frac{t\,e^{-t}}{6+\sin^2(v)}. \end{aligned} \tag{6.2}$$

Verification. The functions f_1, f_2 are obviously continuous in their arguments, so (H_1) holds. Concerning (H_2),

$$|f_1(t,u,v)| \le \frac{e^{-2t}}{4}\cdot 1 + \frac{t^2}{7}\cdot 1 \le \frac{1}{4} + \frac{1}{7} = \phi_1^0, \qquad |f_2(t,u,v)| \le \frac{1}{5} + \frac{t\,e^{-t}}{6} \le \frac{1}{5} + \frac{1}{6e} = \phi_2^0,$$

so we may take $\phi_1(t) \equiv \phi_1^0$, $\phi_2(t) \equiv \phi_2^0$, with $\|\phi_1\| = \frac{1}{4} + \frac{1}{7} \approx 0.3929$ and $\|\phi_2\| = \frac{1}{5} + \frac{1}{6e} \approx 0.2613$.

Before computing the structural constants, we apply Lemma 3.4 to identify the location of the supremum in M_i for each component. With $\lambda = (1-\varrho)/\varrho = 1/9$ and $H = 2$, the threshold (3.16) gives

$$u_1^* = \frac{p_1 + q_1 - 1}{\lambda} = 9 \cdot 0.3 = 2.7, \qquad u_2^* = \frac{p_2 + q_2 - 1}{\lambda} = 9 \cdot 0.1 = 0.9.$$

Since $u_1^* = 2.7 \ge H = 2$, Case (i) of Lemma 3.4 applies to the first component, so $\sup_t K_1(t) = K_1(b) = \Sigma_1$, and the supremum is attained at the right endpoint $t = 1$. Conversely, $u_2^* = 0.9 < H = 2$, so Case (ii) applies to the second component: The supremum is attained at the unique interior point $t_2^* \in (0,1)$ defined by $H(t_2^*) = 0.9$, namely $t_2^* \approx 0.6390$, and (3.18) yields the explicit value

$$\sup_t K_2(t) = \frac{e^{-0.1}}{\Gamma(1.1)}\Big(\frac{0.1}{0.1}\Big)^{0.1} = \frac{e^{-0.1}}{\Gamma(1.1)} \approx \frac{0.9048}{0.9514} \approx 0.9511.$$

This single example thus illustrates both regimes of Lemma 3.4 simultaneously.

We now compute the structural quantities. Using $\Sigma_1 = K_1(b)$, $\Sigma_2 = K_2(b)$, and (3.4)–(3.6) together with (3.15)–(4.3)–(4.4), a direct numerical evaluation gives

$$\begin{aligned} &\Sigma_1 = 1.1337, \quad K_1(\xi_1) = 0.8193, \quad \Sigma_1^* = 1.0313 \neq 0, \quad M_1 = 1.0993, \quad \alpha_1 = 2.2367, \\ &A_1 = 5.4134, \quad A_2 = 3.7623, \quad A_3 = 0.1374, \\ &\Sigma_2 = 0.9117, \quad K_2(\xi_2) = 0.9033, \quad \Sigma_2^* = 0.7826 \neq 0, \quad M_2 = 1.2153, \quad \alpha_2 = 2.3889, \\ &B_1 = 5.4946, \quad B_2 = 3.7057, \quad B_3 = 0.4651. \end{aligned}$$

The constant ϑ from (4.6) evaluates to

$$\vartheta = \max\{\,|\mu_1|A_2 + B_3,\ A_3 + |\mu_2|B_2\,\} = \max\{\,0.1000 \cdot 3.7623 + 0.4651,\ 0.1374 + 0.0833 \cdot 3.7057\,\}.$$

Carrying out the arithmetic carefully,

$$|\mu_1|A_2 + B_3 = 0.3762 + 0.4651 = 0.8413, \qquad A_3 + |\mu_2|B_2 = 0.1374 + 0.3088 = 0.4462,$$

and hence $\vartheta = 0.8413 < 1$, so hypothesis (H_3) is fulfilled.

All hypotheses of Theorem 4.1 therefore hold, and we conclude that the boundary value problem (1.2) to (6.1), with the parameters and nonlinearities specified above, admits at least one solution in $Y \times Y$.

Example 6.2. Set $a = 0$, $b = 1$, $\eta(t) = (t^2 + 1)/2$ (so that $H = 1/2$), and $\varrho = 4/5$. Consider the system (1.1) together with the boundary conditions (1.2), where

- Component 1: $p_1 = \frac{7}{10}$, $q_1 = \frac{2}{5}$ (so that $p_1 + q_1 = \frac{11}{10} > 1$), $\mu_1 = \frac{1}{9}$, $\nu_1 = \frac{1}{4}$, $\xi_1 = \frac{2}{3}$;
- Component 2: $p_2 = \frac{4}{5}$, $q_2 = \frac{1}{2}$ (so that $p_2 + q_2 = \frac{13}{10} > 1$), $\mu_2 = \frac{1}{12}$, $\nu_2 = \frac{1}{3}$, $\xi_2 = \frac{1}{2}$.

We choose $\zeta_1(t) = \arctan(t)/3$ and $\zeta_2(t) = t/5$, with

$$V_0^1(\zeta_1) = \frac{\arctan(1)}{3} = \frac{\pi}{12} \approx 0.2618, \qquad V_0^1(\zeta_2) = \frac{1}{5} = 0.2000.$$

The right-hand sides are

$$f_1(t,u,v) = \frac{\sin t}{20}\,\frac{u}{1+|u|} + \frac{\cos^2(t)}{30}\,\frac{v}{1+|v|}, \qquad f_2(t,u,v) = \frac{e^{-2t}}{15}\,\frac{u}{1+|u|} + \frac{\sin t}{20}\,\frac{v}{1+|v|}. \tag{6.3}$$

Verification. Each f_i is plainly continuous on $[0,1] \times \mathbb{R}^2$, so ($H_1$) holds. Using the elementary bound $|s/(1+|s|)| \le |s|$ valid for every $s \in \mathbb{R}$, we obtain

$$|f_1(t,u,v)| \le \frac{\sin t}{20}|u| + \frac{\cos^2(t)}{30}|v|, \qquad |f_2(t,u,v)| \le \frac{e^{-2t}}{15}|u| + \frac{\sin t}{20}|v|.$$

Hence (H_4) is fulfilled with the choices

$$\phi_1(t) = \frac{\sin t}{20}, \quad \phi_2(t) = \frac{\cos^2(t)}{30}, \quad \chi_1(t) = \frac{e^{-2t}}{15}, \quad \chi_2(t) = \frac{\sin t}{20}, \quad \Psi_i(s) = \Xi_i(s) = s \ (i = 1,2),$$

and

$$\|\phi_1\| = \frac{\sin 1}{20} \approx 0.0421, \quad \|\phi_2\| = \frac{1}{30} \approx 0.0333, \quad \|\chi_1\| = \frac{1}{15} \approx 0.0667, \quad \|\chi_2\| = \frac{\sin 1}{20} \approx 0.0421.$$

The structural constants are

$$\begin{aligned}
&\Sigma_1 = 0.8850, \quad K_1(\xi_1) = 0.8748, \quad \Sigma_1^* = 0.6663 \neq 0, \quad M_1 = 1.3318, \quad \alpha_1 = 2.6647,\\
&A_1 = 1.5184, \quad A_2 = 2.4886, \quad A_3 = 0.3487,\\
&\Sigma_2 = 0.8540, \quad K_2(\xi_2) = 0.6188, \quad \Sigma_2^* = 0.6477 \neq 0, \quad M_2 = 1.3184, \quad \alpha_2 = 2.7579,\\
&B_1 = 1.2831, \quad B_2 = 2.4602, \quad B_3 = 0.2637.
\end{aligned}$$

For ϑ from (4.6), we obtain

$$|\mu_1|A_2 + B_3 = \frac{1}{9} \cdot 2.4886 + 0.2637 = 0.5402, \qquad A_3 + |\mu_2|B_2 = 0.3487 + \frac{1}{12} \cdot 2.4602 = 0.5537,$$

so $\vartheta = 0.5537 < 1$.

We next verify (H_5). Because each Ψ_i and Ξ_i is the identity, the right-hand side of (4.17) is linear in r with multiplier

$$\begin{aligned}\kappa &:= \frac{(\|\phi_1\| + \|\phi_2\|)\,A_1 + (\|\chi_1\| + \|\chi_2\|)\,B_1}{1-\vartheta}\\ &= \frac{(0.0421 + 0.0333)\cdot 1.5184 + (0.0667 + 0.0421)\cdot 1.2831}{0.4463} \approx 0.5692.\end{aligned}$$

Since $\kappa < 1$, the inequality $r > \kappa\, r$ holds for every $r > 0$, which means that hypothesis (H_5) is satisfied for any positive radius. Theorem 4.2 therefore guarantees the existence of at least one solution of (1.1) and (1.2) (with the present specifications) in $Y \times Y$.

Example 6.3. Set $a = 0$, $b = 1$, $\eta(t) = e^t - 1$ (so that $H = e - 1 \approx 1.7183$), and $\varrho = 7/8$. Consider (1.1) together with (1.2), where

- Component 1: $p_1 = \frac{4}{5}$, $q_1 = \frac{3}{10}$ (so that $p_1 + q_1 = \frac{11}{10} > 1$), $\mu_1 = \frac{1}{14}$, $\nu_1 = \frac{1}{8}$, $\xi_1 = \frac{1}{4}$;
- Component 2: $p_2 = \frac{7}{10}$, $q_2 = \frac{2}{5}$ (so that $p_2 + q_2 = \frac{11}{10} > 1$), $\mu_2 = \frac{1}{15}$, $\nu_2 = \frac{1}{9}$, $\xi_2 = \frac{1}{3}$.

We pick $\zeta_1(t) = t^2/9$ and $\zeta_2(t) = (1 - e^{-t})/5$, with

$$V_0^1(\zeta_1) = \frac{1}{9} \approx 0.1111, \qquad V_0^1(\zeta_2) = \frac{1-e^{-1}}{5} \approx 0.1264.$$

The right-hand sides are taken as

$$f_1(t,u,v) = \frac{\sin t}{40}\tanh(u) + \frac{e^{-t}}{60}\arctan(v), \qquad f_2(t,u,v) = \frac{\cos t}{100}\frac{u}{1+u^2} + \frac{t}{120}\sin(v). \tag{6.4}$$

Verification. The functions f_1, f_2 are continuous on $[0,1]\times\mathbb{R}^2$. We compute their Lipschitz constants in (u,v). Since the partial derivatives satisfy

$$\left|\frac{\partial f_1}{\partial u}\right| = \left|\frac{\sin t}{40}\operatorname{sech}^2(u)\right| \le \frac{|\sin t|}{40} \le \frac{1}{40}, \qquad \left|\frac{\partial f_1}{\partial v}\right| = \left|\frac{e^{-t}}{60}\frac{1}{1+v^2}\right| \le \frac{1}{60},$$

hypothesis (H_6) is fulfilled for f_1 with $L_1 = 1/40 + 1/60 = 1/24$, on account of the inequality $|f_1(t,u_1,u_2) - f_1(t,v_1,v_2)| \le L_{1,u}|u_1 - v_1| + L_{1,v}|u_2 - v_2| \le (L_{1,u} + L_{1,v})(|u_1 - v_1| + |u_2 - v_2|)$.

A similar computation for f_2 uses $|d/du(u/(1+u^2))| \le 1$ and $|\cos v| \le 1$:

$$\left|\frac{\partial f_2}{\partial u}\right| \le \frac{1}{100}, \qquad \left|\frac{\partial f_2}{\partial v}\right| \le \frac{1}{120},$$

yielding $L_2 = 1/100 + 1/120 = 11/600$.

With the parameters of the present example, the structural constants evaluate to

$$\begin{aligned}&\Sigma_1 = 0.8798, \quad K_1(\xi_1) = 0.9019, \quad \Sigma_1^* = 0.7670 \neq 0, \quad M_1 = 1.2126, \quad \alpha_1 = 2.3642,\\ &A_1 = 4.7463, \quad A_2 = 3.2255, \quad A_3 = 0.1347,\\ &\Sigma_2 = 0.8798, \quad K_2(\xi_2) = 0.9176, \quad \Sigma_2^* = 0.7778 \neq 0, \quad M_2 = 1.1958, \quad \alpha_2 = 2.3287,\\ &B_1 = 4.6749, \quad B_2 = 3.4379, \quad B_3 = 0.1512.\end{aligned}$$

For ϑ,

$$|\mu_1|A_2 + B_3 = \frac{1}{14} \cdot 3.2255 + 0.1512 = 0.3816, \qquad A_3 + |\mu_2|B_2 = 0.1347 + \frac{1}{15} \cdot 3.4379 = 0.3639,$$

and hence $\vartheta = 0.3816$. Finally,

$$L_1A_1 = \frac{1}{24} \cdot 4.7463 \approx 0.1978, \qquad L_2B_1 = \frac{11}{600} \cdot 4.6749 \approx 0.0857,$$

and we obtain

$$\Lambda = \vartheta + L_1A_1 + L_2B_1 \approx 0.3816 + 0.1978 + 0.0857 \approx 0.6650 < 1.$$

Hence both (H_6) and (H_7) are met, and Theorem 5.1 guarantees the existence of a unique solution of (1.1) and (1.2) (with the present specifications) in $Y \times Y$.

7. Conclusions

We have analyzed a coupled Langevin system formulated through the generalized proportional fractional derivative with respect to a prescribed increasing function η, supplemented with Riemann-Stieltjes integral boundary conditions that intertwine the two unknowns at the right endpoint. Working under the regularity assumption $p_i + q_i > 1$ in each component, we transformed the boundary value problem into an equivalent system of integral equations and identified the structural constants A_1, A_2, A_3, B_1, B_2, and B_3, as well as the boundary correction M_i. Three solvability statements were established: an existence theorem based on the Krasnoselskii fixed point theorem, an alternative existence result derived from the Leray-Schauder nonlinear alternative, and a uniqueness theorem produced by Banach's contraction principle. The general framework was illustrated by three explicit examples in which every hypothesis was verified numerically.

Because the operator ${}_a\mathcal{D}^{\theta,\varrho,\eta}$ specializes, by suitable choices of η and ϱ, to the Riemann-Liouville, Hadamard, and Katugampola fractional derivatives, our results unify and extend several existing contributions to the theory of coupled Langevin systems with nonlocal boundary couplings.

Several promising directions of further investigation remain open. One natural extension is to incorporate a p-Laplacian operator into the system, which would lead to a quasilinear coupling of the two equations. A second direction is the qualitative theory of stability: In particular, the Ulam-Hyers, generalized Ulam-Hyers, Ulam-Hyers-Rassias, and generalized Ulam-Hyers-Rassias stability of the system constitute a natural sequel to the present existence and uniqueness results. A third direction, complementary to the $C([a,b],\mathbb{R})$ framework adopted here, concerns the regularity of solutions: The Hölder-space framework enables more natural and, in certain respects, sharper conditions for the uniqueness of solutions of fractional equations, and the development of analogous results for the present system within suitable Hölder spaces $C^{0,\gamma}([a,b],\mathbb{R})$ represents an important research direction in its own right. Closely related to this is the construction of generalized Hölder spaces adapted to the proportional fractional operators ${}_a\mathcal{D}^{\theta,\varrho,\eta}$ and ${}_a\mathcal{J}^{\theta,\varrho,\eta}$ considered here; such spaces are likely to reflect the precise regularity properties of solutions more faithfully than the classical Hölder scale, and their study constitutes an independent research program that goes beyond the scope of the present paper and will be pursued in forthcoming work. Finally, the development of efficient numerical methods, for example collocation or spectral schemes adapted to the proportional kernel, would complement the analytical theory established here. We plan to pursue these directions in forthcoming work.

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares that she has no conflict of interest.

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