



Research article

Variable-order Bessel–Riesz operators and their boundedness on variable Lebesgue spaces

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Abstract: In this article, we introduce a variable-order Bessel–Riesz integral operator and investigate its boundedness properties on variable Lebesgue spaces. We first establish the boundedness in the diagonal case under suitable regularity assumptions on the exponent function. For the nondiagonal setting, we derive a sharp pointwise estimate, which enables us to obtain boundedness results from $L^{p(\cdot)}(\mathbb{R}_+)$ to $L^{q(\cdot)}(\mathbb{R}_+)$ under appropriate relations between the exponent functions. Moreover, we establish the boundedness through an analog of Young’s inequality. Because the classical Young’s inequality generally fails in variable Lebesgue spaces due to the lack of translation invariance, we develop an alternative framework adapted to this setting. In particular, we identify sufficient conditions under which the associated kernel belongs to an appropriate variable Lebesgue space and use these conditions to derive the corresponding boundedness results. To the best of our knowledge, this class of operators has not been studied previously in the literature, and the study provides a systematic way to develop the boundedness of variable-order fractional integral operators in variable Lebesgue spaces.

Keywords: variable Lebesgue spaces; variable exponent spaces; fractional integral operators; Bessel–Riesz operator; variable-order Bessel–Riesz operators; boundedness; maximal operators

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1. Introduction

Boundedness of integral operators in various function spaces is one of the most important topics in harmonic analysis. Over the last two decades, particularly in the last ten years, there has been a rapid expansion in the research on the boundedness of operators in variable Lebesgue $L^{p(\cdot)}$ spaces. These spaces were introduced in 1931 by Orlicz in [1]. These spaces generalize the idea of classical Lebesgue L^p spaces by substituting the constant exponent p with a measurable function $p(\cdot)$. Although

$L^{p(\cdot)}$ spaces possess many properties in common with the classical L^p spaces, they are also distinguished by several significant differences. These spaces are of intrinsic mathematical interest and are essential for applications involving partial differential equations (PDEs) and variational integrals exhibiting nonstandard growth conditions. For background on variable Lebesgue spaces, we refer to [2, 3]. The boundedness of the Hardy–Littlewood maximal operator on these spaces was established in [4–7], and the study of fractional-type operators was further developed in [8, 9]. Recent developments in variable exponent harmonic analysis have significantly advanced the study of fractional-type operators on generalized function spaces. In particular, Gürbüz [10–12] established important boundedness results for fractional rough commutators in variable exponent Herz–Triebel–Lizorkin spaces and developed the theory of variable exponent vanishing Morrey-type spaces on unbounded domains.

In mathematical analysis, the Hardy–Littlewood maximal operator plays a central role in controlling singular integrals. Bessel–Riesz operators are singular integral operators defined through convolution and are used to estimate the solution of the Schrödinger equation, the quantum mechanical analogous form of the Newton’s law of motion. Researchers have conducted extensive studies on these operators in different spaces. K. Kurata, S. Nishigaki, and S. Sugano in [13] developed the boundedness of the Bessel–Riesz operator in generalized Morrey spaces and studied its relation with Schrödinger equation. Michael Ruzhansky and Durvudkhan Suragan in [14] studied its boundedness on Morrey spaces formulated over a homogeneous group. Nasir et al. [15, 16] studied Bessel–Riesz operators $I_{\alpha,\gamma}$ in variable exponent spaces and established their boundedness on variable Lebesgue spaces $L^{p(\cdot)}$ over Euclidean domains.

However, all existing studies on the Bessel–Riesz operator have been carried out for Bessel–Riesz operators of constant order. In contrast, this manuscript introduces a new class of integral operator with Bessel–Riesz kernels of variable-order and establishes their boundedness in Lebesgue spaces with variable exponents. We first establish the boundedness in the diagonal case on $L^{p(\cdot)}(\mathbb{R}_+)$ under suitable assumptions on the order function and the exponent function. For the nondiagonal setting, a pointwise estimate is derived based on the Hardy–Littlewood maximal operator and the norm of the underlying function, which allows us to obtain boundedness results from $L^{p(\cdot)}(\mathbb{R}_+)$ to $L^{q(\cdot)}(\mathbb{R}_+)$ under appropriate relations between the exponent functions. Furthermore, we develop results on the boundedness for the variable-order Bessel–Riesz operator in a form analogous to Young’s inequality for convolution. Because the classical Young inequality is not valid in $L^{p(\cdot)}$ spaces, we develop a suitable alternative formulation. To this end, we first determine sufficient conditions on the exponent function for the variable-order Bessel–Riesz kernel to belong to a variable Lebesgue space. These results lead to convolution-type estimates that provide boundedness of the operator between variable Lebesgue spaces under suitable assumptions on the exponents. A variable-order Bessel–Riesz operator allows the fractional order to vary with the spatial variable, capturing location-dependent singularities and nonlocal phenomena that constant-order operators cannot model.

This manuscript is organized as follows. Sections 1 and 2 introduce the fundamental concepts of variable $L^{p(\cdot)}$ spaces, the Hardy–Littlewood maximal operator, and the classical Bessel–Riesz operators, which provide the necessary background for the main results. In Section 3, we introduce the variable-order Bessel–Riesz operator. Section 4 is devoted to the statement of our main results on the boundedness of the variable-order Bessel–Riesz operator. The proofs of these results are presented in Section 5, where dyadic decomposition techniques and maximal operator estimates are employed. Finally, Section 6 concludes the paper with some remarks.

2. Preliminaries and definitions

An *exponent function* $p(\cdot)$ is a Lebesgue measurable function defined by

$$p(\cdot) : \Phi \rightarrow [1, \infty],$$

where $\Phi \subseteq \mathbb{R}^n$. The collection of all such exponent functions is denoted by $E(\Phi)$.

For any measurable set $A \subseteq \Phi$, we define

$$p_-(A) := \operatorname{ess\,inf}_{w \in A} p(w), \quad p_+(A) := \operatorname{ess\,sup}_{w \in A} p(w).$$

Simply, it is written as $p_- := p_-(\Phi)$ and $p_+ := p_+(\Phi)$.

The *conjugate exponent function* $p'(\cdot)$ is defined by

$$\frac{1}{p(w)} + \frac{1}{p'(w)} = 1.$$

It follows that

$$(p'(\cdot))_+ = (p_-)', \quad (p'(\cdot))_- = (p_+)'.$$

We summarize below some basic results for the variable Lebesgue spaces $L^{p(\cdot)}$ taken from [2].

Definition 2.1. [2, Definition 2.5] Assume that $p(\cdot) \in E(\Phi)$. We define the modular functional for a measurable function ψ as

$$\rho_{p(\cdot)}(\psi) = \int_{\Phi \setminus \Phi_\infty} |\psi(w)|^{p(w)} dw + \|\psi\|_{\Phi_\infty}, \quad (2.1)$$

where $\|\psi(\cdot)\|_{\Phi_\infty} = \operatorname{ess\,sup}_{w \in \Phi_\infty} |\psi(w)|$, and $\Phi_\infty := \{w \in \Phi : p(w) = \infty\}$. The modular functional can be represented as $\rho_{p(\cdot)}(\psi)$ or $\rho(\psi)$.

Definition 2.2. [2, Definition 2.8] Assume that $p(\cdot) \in E(\Phi)$; variable exponent Lebesgue space $L^{p(\cdot)}(\Phi)$ is defined as

$$L^{p(\cdot)}(\Phi) = \{\psi : \rho_{p(\cdot)}(\psi/j) < \infty, \text{ for some } j > 0\}. \quad (2.2)$$

Definition 2.3. [2, Definition 2.12] Suppose that ψ is a measurable function and $p(\cdot) \in E(\Phi)$; then,

$$\|\psi\|_{L^{p(\cdot)}} = \inf\{j > 0 : \rho(\psi/j) \leq 1\}. \quad (2.3)$$

It is easy to prove that the variable Lebesgue space $L^{p(\cdot)}(\Phi)$ is a normed space under this norm.

Proposition 2.1. [2, Proposition 2.9] Suppose that $p(\cdot) \in E(\Phi)$ and $p_+ < \infty$; then, so $\psi \in L^{p(\cdot)}(\Phi)$ if and only if

$$\rho_{p(\cdot)}(\psi) = \int_{\Phi \setminus \Phi_\infty} |\psi(w)|^{p(w)} dw < \infty. \quad (2.4)$$

Proposition 2.2. [2, Proposition 2.10] Suppose that $p(\cdot) \in E(\Phi)$ and $p_+ < \infty$, then $\forall j \geq 1$, we get $j^p \rho(\psi) \leq \rho(j\psi) \leq j^{p_+} \rho(\psi)$. Further, $0 < j < 1$, then we obtain reverse inequalities.

Proposition 2.3. [2, Proposition 2.15] Suppose that $p(\cdot) \in E(\Phi)$ and $\psi \in L^{p(\cdot)}(\Phi)$ with $\|\psi\|_{L^{p(\cdot)}} > 0$. Then, we get $\rho(\psi/\|\psi\|_{L^{p(\cdot)}}) \leq 1$. For the case when $p_+ < \infty$, we get $\rho(\psi/\|\psi\|_{L^{p(\cdot)}}) = 1, \forall$ nontrivial $\psi \in L^{p(\cdot)}(\Phi)$.

Corollary 2.1. Assume that $p_+ < \infty$ and $\|\psi\|_{L^{p(\cdot)}} > 1$; then, $\rho(\psi)^{1/p_+} \leq \|\psi\|_{L^{p(\cdot)}} \leq \rho(\psi)^{1/p_-}$. For the case when $0 < \|\psi\|_{L^{p(\cdot)}} \leq 1$, then we get $\rho(\psi)^{1/p_-} \leq \|\psi\|_{L^{p(\cdot)}} \leq \rho(\psi)^{1/p_+}$. Moreover, if $p(\cdot) = p$, that is, a constant, then $p_- = p_+ = p$, and we get the norm on classical Lebesgue spaces $L^p(\Phi)$. that is,

$$\|\psi\|_p = \left(\int_{\Phi} |\psi(w)|^p dw \right)^{1/p}.$$

Theorem 2.1. [Hölder's Inequality] Assume that $p(\cdot) \in E(\Phi)$. Then, $\forall \psi_1 \in L^{p(\cdot)}(\Phi)$, and $\psi_2 \in L^{p'(\cdot)}(\Phi)$; we get $\psi_1 \psi_2 \in L^1(\Phi)$ and

$$\int_{\Phi} |\psi_1(w)\psi_2(w)| dw \leq K_{p(\cdot)} \|\psi_1\|_{L^{p(\cdot)}} \|\psi_2\|_{L^{p'(\cdot)}},$$

where $K_{p(\cdot)}$ is a constant depending on $p(\cdot)$.

Definition 2.4. Assume that $\Phi \subseteq \mathbb{R}^n$ and $p(\cdot) : \Phi \rightarrow \mathbb{R}$, and $p(\cdot)$ is known as locally log-Hölder continuous, denoted by $p(\cdot) \in LH_0(\Phi)$. If one has a constant C_0 in a way that $\forall w_1, w_2 \in \Phi$ with $|w_1 - w_2| < 1/2$,

$$|p(w_1) - p(w_2)| \leq \frac{C_0}{-\log(|w_1 - w_2|)}.$$

$p(\cdot)$ is referred to as log-Hölder continuous at infinity represented by $p(\cdot) \in LH_{\infty}(\Phi)$, if one has constants C_{∞}, p_{∞} in a manner that $\forall w \in \Phi$,

$$|p(w) - p_{\infty}| \leq \frac{C_{\infty}}{\log(e + |w|)}.$$

Simply, $p(\cdot) \in LH(\Phi)$ denotes that $p(\cdot)$ is log-Hölder continuous both locally and at infinity.

Definition 2.5. The Hardy–Littlewood maximal operator $\mathcal{M}\psi$ of a function $\psi \in L^1_{loc}(\mathbb{R}^n)$ is given as

$$\mathcal{M}\psi(w) = \sup_{Q \ni w} \frac{1}{|Q|} \int_Q |\psi(z)| dz, \quad w \in \mathbb{R}^n; \quad (2.5)$$

the supremum is determined over the cubes $Q \subset \mathbb{R}^n$, which have sides parallel to coordinate axes, and $w \in Q$.

Theorem 2.2. [2, Theorem 3.16] Assume that $p(\cdot) \in E(\mathbb{R}^n)$ and $1/p(\cdot) \in LH(\mathbb{R}^n)$. Then,

$$\|\beta\chi_{\{w: \mathcal{M}\psi(w) > \beta\}}\|_{L^{p(\cdot)}} \leq C \|\psi\|_{L^{p(\cdot)}}.$$

For the case when $p_- > 1$, we get

$$\|\mathcal{M}\psi\|_{L^{p(\cdot)}} \leq C \|\psi\|_{L^{p(\cdot)}}. \quad (2.6)$$

Here, C denotes a constant depending on dimension n , p_- , and p_{∞} .

Remark 2.1. If one has $p_+ < \infty$, then $\frac{1}{p(\cdot)} \in LH(\Phi)$ is equivalent to $p(\cdot) \in LH(\Phi)$.

We now study the Bessel–Riesz operators that will be used in the sequel.

Definition 2.6. Assume that $\gamma \in (0, \infty)$ and $\alpha \in (0, n)$. Then the Bessel–Riesz operator $I_{\alpha, \gamma}$ is convolution of a function $\psi \in L_{loc}^{p(\cdot)}(\mathbb{R})$ with a Bessel–Riesz kernel;

$$I_{\alpha, \gamma} \psi = K_{\alpha, \gamma} * \psi(w) = \int_{\mathbb{R}} K_{\alpha, \gamma}(w - z) \psi(z) dz, \quad (2.7)$$

where $K_{\alpha, \gamma}(w) = \frac{|w|^{\alpha-1}}{(1+|w|)^\gamma}$, $0 \neq w \in \mathbb{R}$, and it is defined to be zero for $w = 0$. The Bessel–Riesz kernel $K_{\alpha, \gamma}$ is the product of the Bessel kernel $K_\gamma = \frac{1}{(1+|w|)^\gamma}$ and the Riesz kernel $K_\alpha = |w|^{\alpha-1}$.

Lemma 2.1. [20, Theorem 1.2.12] Assume that $q, s, t \in [1, \infty]$ and that $\frac{1}{t} = \frac{1}{q} + \frac{1}{s} - 1$. If $\psi_1 \in L^q(\mathbb{R})$, and $\psi_2 \in L^s(\mathbb{R})$, then $\psi_1 * \psi_2 \in L^t(\mathbb{R})$, and

$$\|\psi_1 * \psi_2\|_{L^t(\mathbb{R})} \leq \|\psi_1\|_{L^q(\mathbb{R})} \|\psi_2\|_{L^s(\mathbb{R})}. \quad (2.8)$$

Using the Young’s inequality, it is easy to develop the boundedness of the Bessel–Riesz in L^p spaces as given in the following corollary.

Corollary 2.2. Let $\alpha \in (0, 1)$, and $\gamma > 0$. Let $p, q, t \geq 1$, and define q by $\frac{1}{q} = \frac{1}{p} + \frac{1}{t} - 1$. If $K_{\alpha, \gamma} \in L^t(\mathbb{R})$, and $\psi \in L^p(\mathbb{R})$, then $I_{\alpha, \gamma} \psi \in L^q(\mathbb{R})$, and

$$\|I_{\alpha, \gamma} \psi\|_{L^q(\mathbb{R})} \leq \|K_{\alpha, \gamma}\|_{L^t(\mathbb{R})} \|\psi\|_{L^p(\mathbb{R})}. \quad (2.9)$$

However, the classical Young convolution inequality generally fails in variable Lebesgue spaces $L^{p(\cdot)}$. One of the main objectives of this paper is to develop an appropriate analog of Young’s inequality for the variable-order Bessel–Riesz operator.

3. Variable-order Bessel–Riesz operator

Let $\alpha(\cdot) : \mathbb{R}_+ \rightarrow [\alpha_-, \alpha_+] \subset (0, 1)$ be a measurable function. We define the *variable-order Bessel–Riesz kernel* by

$$K_{\alpha(\cdot), \gamma}(w) = \frac{w^{\alpha(w)-1}}{(1+w)^\gamma}, \quad w \in \mathbb{R}_+,$$

where $\gamma > 0$. The associated *variable-order Bessel–Riesz operator* is

$$I_{\alpha(\cdot), \gamma} \psi(w) = \int_{\mathbb{R}_+} \frac{|w-z|^{\alpha(w)-1}}{(1+|w-z|)^\gamma} \psi(z) dz, \quad w \in \mathbb{R}_+.$$

Basic pointwise domination: Because $0 < \alpha_- \leq \alpha(w) \leq \alpha_+ < 1$ for all $w \in \mathbb{R}_+$, we have $\alpha(w) - 1 < 0$ for every w . Hence, the map $a \mapsto |w|^a$ is *decreasing* in a when $|w| < 1$ and *increasing* in a when $|w| > 1$. Therefore:

- **When $|w| \geq 1$:** Because $a \mapsto |w|^a$ is increasing and $\alpha_- \leq \alpha(w) \leq \alpha_+$, we obtain

$$\frac{|w|^{\alpha_- - 1}}{(1+|w|)^\gamma} \leq \frac{|w|^{\alpha(w) - 1}}{(1+|w|)^\gamma} \leq \frac{|w|^{\alpha_+ - 1}}{(1+|w|)^\gamma}.$$

- **When $0 < |w| < 1$:** Because $a \mapsto |w|^a$ is decreasing (as $\ln|w| < 0$) and $\alpha_- \leq \alpha(w) \leq \alpha_+$, the inequalities *reverse*:

$$\frac{|w|^{\alpha_+-1}}{(1+|w|)^\gamma} \leq \frac{|w|^{\alpha(w)-1}}{(1+|w|)^\gamma} \leq \frac{|w|^{\alpha_- -1}}{(1+|w|)^\gamma}.$$

In each case, the kernel $K_{\alpha(\cdot),\gamma}$ is pointwise bounded between two standard Bessel–Riesz kernels.

4. Results and discussion

This section presents the main results concerning the boundedness of the variable-order Bessel–Riesz operator $I_{\alpha(\cdot),\gamma}$ on variable Lebesgue $L^{p(\cdot)}(\mathbb{R}_+)$ spaces. This section is subdivided in two subsections.

In the first subsection, we construct the results on the boundedness of variable-order Bessel–Riesz operator in both the diagonal and nondiagonal settings. These results describe the mapping properties of the operator between suitable variable Lebesgue spaces.

The second subsection is devoted to establishing boundedness results in a form analogous to Young’s inequality for convolution, which is not valid in Lebesgue spaces with variable exponents. As these spaces are not translation invariant, we derive an appropriate analog suitable for the present setting. Throughout the section, we assume that $1 < p_- \leq p_+ < \infty$, $q_+ < \infty$ and $t_+ < \infty$.

4.1. Diagonal and nondiagonal cases

We begin by establishing the boundedness of the variable-order Bessel–Riesz operator on the same variable Lebesgue space. This result corresponds to the diagonal case.

Theorem 4.1. *Let $\alpha(\cdot) : \mathbb{R}_+ \rightarrow [\alpha_-, \alpha_+]$ be a measurable function with $0 < \alpha_- \leq \alpha_+ < 1$. Let $\gamma \in (0, \infty)$ satisfy $\alpha_+ < \gamma$, and let $p(\cdot) \in LH(\mathbb{R}_+)$ with $1 < p_- \leq p_+ < \infty$. Then the variable-order Bessel–Riesz operator $I_{\alpha(\cdot),\gamma}$ is bounded on $L^{p(\cdot)}(\mathbb{R}_+)$, that is, there exists a constant $C > 0$ such that*

$$\|I_{\alpha(\cdot),\gamma}\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)} \leq C\|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)} \quad \text{for all } \psi \in L^{p(\cdot)}(\mathbb{R}_+).$$

In order to establish boundedness results in the nondiagonal setting, we first derive a pointwise estimate for the operator. The following lemma provides a useful decomposition estimate that will be employed in the proofs of subsequent results.

Lemma 4.1. *Let $\alpha(\cdot) : \mathbb{R}_+ \rightarrow [\alpha_-, \alpha_+] \subset (0, 1)$ be measurable, $\gamma > 0$ with $\alpha_+ - \frac{1}{p_+} < \gamma$, and $p(\cdot) \in LH(\mathbb{R}_+)$ with $1 < p_- \leq p_+ < \infty$. There exists a constant $C_4 > 0$, dependent on α_\pm , γ , and p_\pm (but not on ψ , w , or r), such that for every $r > 0$, there exists an integer G_r (depending on r only, satisfying $2^{G_r-1}r < 1 \leq 2^{G_r}r$), and for every $\psi \in L^{p(\cdot)}(\mathbb{R}_+)$, the following estimate holds for almost every $w \in \mathbb{R}_+$:*

$$|I_{\alpha(\cdot),\gamma}\psi(w)| \leq C_4 \left[M\psi(w) \sum_{n=-\infty}^{G_r-1} \frac{(2^n r)^{\alpha_-}}{(1+2^n r)^\gamma} + \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)} \sum_{n=G_r}^{\infty} \frac{(2^n r)^{\alpha_+-1/p_+}}{(1+2^n r)^\gamma} \right]. \quad (4.1)$$

The following theorem determines the conditions on the target space corresponding to a given domain space for which the variable-order Bessel–Riesz operator is bounded.

Theorem 4.2. Let $p(\cdot) \in E(\mathbb{R}_+)$ and $1 < p_- \leq p_+ < \infty$ with $p(\cdot) \in LH(\mathbb{R}_+)$. Let $\alpha(\cdot) : \mathbb{R} \rightarrow [\alpha_-, \alpha_+] \subset (0, 1)$ be measurable, and assume $\alpha_+ - \frac{1}{p_+} < \gamma$. Then, there exists $q(\cdot) \in E(\mathbb{R}_+)$ such that $\frac{\alpha_- p_+ p(w)}{(1 - (\alpha_+ - \gamma)p_+)q(w)} + \frac{p(w)}{q(w)} = 1$ a.e. Then, $I_{\alpha(\cdot), \gamma}$ is bounded from $L^{p(\cdot)}(\mathbb{R}_+)$ into $L^{q(\cdot)}(\mathbb{R}_+)$.

The next theorem provides the converse perspective, and it characterizes the domain space associated with a given range space that guarantees the boundedness of the variable-order operator.

Theorem 4.3. Suppose that $q(\cdot) \in E(\mathbb{R}_+)$ with $q_+ < \infty$, and $\frac{1}{q_-} + \frac{\alpha_+ q_+}{q_-} < 1$, where $\alpha(\cdot) : \mathbb{R} \rightarrow [\alpha_-, \alpha_+] \subset (0, 1)$ and $q(\cdot) \in LH(\mathbb{R}_+)$. Then $p(\cdot) \in E(\mathbb{R}_+)$ such that $\frac{1}{p(\cdot)} = \frac{1}{q(\cdot)} + \frac{\alpha_+ q_+}{q(\cdot)}$ almost everywhere (a.e) on \mathbb{R}_+ . Then, the operator $I_{\alpha(\cdot), \gamma}$ is bounded from $L^{p(\cdot)}(\mathbb{R}_+)$ into $L^{q(\cdot)}(\mathbb{R}_+)$.

4.2. Analog of Young's inequality

In this subsection, we develop the results on the boundedness of the variable-order Bessel–Riesz operator in a form analogous to Young's inequality for convolution. It is well known that the classical Young inequality does not generally hold in variable Lebesgue spaces, primarily because these spaces are not translation invariant. Consequently, a suitable alternative formulation is required.

To develop such an analog, we first determine the conditions under which the associated variable-order kernel belongs to $L^{t(\cdot)}$ space. The following lemma provides the necessary conditions on the exponent function.

Lemma 4.2. Let $\alpha(\cdot) : \mathbb{R}_+ \rightarrow [\alpha_-, \alpha_+] \subset (0, 1)$ be measurable, and let $\gamma > 0$. A sufficient condition for the variable-order Bessel–Riesz kernel

$$K_{\alpha(\cdot), \gamma}(w) = \frac{w^{\alpha(w)-1}}{(1+w)^\gamma}, \quad w \in \mathbb{R}_+,$$

to belong to $L^{t(\cdot)}(\mathbb{R}_+)$ is that $t(\cdot) \in E(\mathbb{R}_+)$ satisfies

$$\frac{1}{\gamma + 1 - \alpha_+} < t_- \leq t_+ < \frac{1}{1 - \alpha_-}. \quad (4.2)$$

Remark 4.1. When $\alpha(\cdot) \equiv \alpha$ is constant, Condition (4.2) reduces to $\frac{1}{\gamma+1-\alpha} < t_- \leq t_+ < \frac{1}{1-\alpha}$, recovering the membership criterion for the constant-order kernel $K_{\alpha, \gamma}$ established in [15]. The variable-order case is thus a strict extension, with the role of the single parameter α split between the essential infimum α_- (governing the near-field singularity at $w = 0$, which controls the right-hand bound on t_+) and the essential supremum α_+ (governing the far-field decay as $w \rightarrow \infty$, which controls the left-hand bound on t_-).

The next lemma provides lower bounds for the modular of the variable-order Bessel–Riesz kernel that will be required in subsequent estimates.

Lemma 4.3. Let $\alpha(\cdot) : \mathbb{R}_+ \rightarrow [\alpha_-, \alpha_+] \subset (0, 1)$ be measurable, $\gamma > 0$, and let $t(\cdot) \in E(\mathbb{R}_+)$ with $0 < t^- \leq t^+ < \infty$ satisfying $\frac{1}{\gamma+1-\alpha_+} < t^- \leq t^+ < \frac{1}{1-\alpha_-}$. For each $r > 0$, there exists an integer $G_r \in \mathbb{Z}$ (satisfying $2^{G_r-1}r < 1 \leq 2^{G_r}r$) such that for every integer $n < G_r$,

$$\rho_{t(\cdot)}(K_{\alpha(\cdot), \gamma}) \geq C_1 \frac{(2^n r)^{(\alpha_+ - 1)t_- + 1}}{(1 + 2^n r)^{\gamma t_+}}, \quad (4.3)$$

and for every integer $n \geq G_r$,

$$\rho_{t(\cdot)}(K_{\alpha(\cdot),\gamma}) \geq C_2 \frac{(2^n r)^{(\alpha_- - 1)t_+ + 1}}{(1 + 2^n r)^{\gamma t_+}}, \quad (4.4)$$

where the positive constants C_1, C_2 may be chosen as

$$C_1 = \frac{2^{(\beta_1 + 1)} - 1}{2^{\gamma t_+} (\beta_1 + 1)}, \quad C_2 = \frac{2^{(\beta_2 + 1)} - 1}{2^{\gamma t_+} (\beta_2 + 1)},$$

with $\beta_1 = (\alpha_+ - 1)t_-$ and $\beta_2 = (\alpha_- - 1)t_+$. Here, $\beta_i + 1 > 0$, so $C_i > 0$.

The following two lemmas provide useful estimates for the norm of the kernel depending on whether the norm is less than or greater than one.

Lemma 4.4. Suppose that $K_{\alpha(\cdot),\gamma} \in L^{t(\cdot)}(\mathbb{R}_+)$ and $0 < \|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}} < 1$. For each $r > 0$, there exists an integer $G_r \in \mathbb{Z}$ (satisfying $2^{G_r - 1} r < 1 \leq 2^{G_r} r$) such that for every integer $n < G_r$,

$$\|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}^{1/t_+} \geq D_1 \frac{(2^n r)^{(\alpha_+ - 1)t_+ + \frac{1}{t_+}}}{(1 + 2^n r)^\gamma}, \quad (4.5)$$

and for $n \geq G_r$,

$$\|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}^{1/t_+} \geq D_2 \frac{(2^n r)^{(\alpha_- - 1)t_+ + \frac{1}{t_+}}}{(1 + 2^n r)^\gamma}, \quad (4.6)$$

where one can take $D_1 = C_1^{1/t_+}$ and $D_2 = C_2^{1/t_+}$, and C_1, C_2 are the positive constants appearing in Lemma 4.3.

Lemma 4.5. Suppose that $K_{\alpha(\cdot),\gamma} \in L^{t(\cdot)}(\mathbb{R}_+)$ and $\|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}} \geq 1$. For each $r > 0$, there exists an integer $G_r \in \mathbb{Z}$ (satisfying $2^{G_r - 1} r < 1 \leq 2^{G_r} r$) such that for every integer $n < G_r$,

$$\|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}} \geq \tilde{C}_1 \frac{(2^n r)^{(\alpha_+ - 1)t_+ + \frac{1}{t_+}}}{(1 + 2^n r)^\gamma}, \quad (4.7)$$

and for $n \geq G_r$,

$$\|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}} \geq \tilde{C}_2 \frac{(2^n r)^{(\alpha_- - 1)t_+ + \frac{1}{t_+}}}{(1 + 2^n r)^\gamma}, \quad (4.8)$$

where $\tilde{C}_1 = C_1^{1/t_+}$ and $\tilde{C}_2 = C_2^{1/t_+}$, with $C_1 = \frac{2^{(\beta_1 + 1)} - 1}{2^{\gamma t_+} (\beta_1 + 1)}$, $C_2 = \frac{2^{(\beta_2 + 1)} - 1}{2^{\gamma t_+} (\beta_2 + 1)}$, and $\beta_1 = (\alpha_+ - 1)t_-$ and $\beta_2 = (\alpha_- - 1)t_+$. Here, $\beta_i + 1 > 0$, so $C_i > 0$.

Finally, we develop the boundedness of the variable-order Bessel–Riesz operator in a form analogous to Young’s inequality.

Theorem 4.4. Let $p(\cdot), q(\cdot), t(\cdot) \in E(\mathbb{R}_+)$, $\alpha(\cdot) : \mathbb{R}_+ \rightarrow [\alpha_-, \alpha_+] \subset (0, 1)$ be measurable, and let $0 < \gamma < \infty$ with $\text{Range}(t(\cdot)) \subset \left(\frac{1}{\gamma + 1 - \alpha_+}, \frac{1}{1 - \alpha_-}\right)$. Suppose that a constant $c > 0$ exists in a manner that $\frac{1}{q_+} \leq c \left(\frac{1}{p_+} + \frac{1}{t_+} - 1\right)$ and $\frac{q(\cdot)}{p(\cdot)} = 1 + \frac{1}{(\alpha_- - \alpha_+ + \frac{1}{c q_+})} \frac{\alpha_- t_+ + (1 - \alpha_+) t_- - 1}{t_+}$ a.e. with $p(\cdot) \in LH(\mathbb{R}_+)$; then, variable-order Bessel–Riesz operator $I_{\alpha(\cdot),\gamma}$ is bounded from $L^{p(\cdot)}(\mathbb{R}_+)$ to $L^{q(\cdot)}(\mathbb{R}_+)$. Moreover, suppose $\psi \in L^{p(\cdot)}$, then, we can find a positive constant C , in a way that, for $0 < \|K_{\alpha(\cdot),\gamma}\|_{L^{s(\cdot)}} < 1$, following inequality holds

$$\|I_{\alpha(\cdot),\gamma}\psi\|_{L^{q(\cdot)}} \leq C_5 \|K_{\alpha(\cdot),\gamma}\|_{L^{s(\cdot)}}^{1/t_+} \|\psi\|_{L^{p(\cdot)}}. \quad (4.9)$$

Further, if $\|K_{\alpha(\cdot),\gamma}\|_{L^{s(\cdot)}} \geq 1$, following inequality holds,

$$\|I_{\alpha(\cdot),\gamma}\psi\|_{L^{q(\cdot)}} \leq C_6 \|\psi\|_{L^{p(\cdot)}} \|K_{\alpha(\cdot),\gamma}\|_{L^{s(\cdot)}}. \quad (4.10)$$

Remark 4.2. If the measurable function $\alpha(\cdot)$ becomes constant such that $\alpha(\cdot) = \alpha_+ = \alpha_- = \alpha$, then Theorem 4.4 coincides with the boundedness of the constant order Bessel–Riesz operator in variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}_+)$, as shown in [15]. Furthermore, if all exponent functions become constants, such that $q(\cdot) = q_- = q_+ = q$, $p(\cdot) = p_- = p_+ = p$ and $t(\cdot) = t_- = t_+ = t$, with $c = 1$, the expression $\frac{1}{q_+} \leq c \left(\frac{1}{p_+} + \frac{1}{t_+} - 1 \right)$ becomes $\frac{1}{q} \leq \frac{1}{p} + \frac{1}{t} - 1$, further the relation $\frac{q(\cdot)}{p(\cdot)} = 1 + \frac{1}{\left(\alpha_- - \alpha_+ + \frac{1}{cq_+}\right) t_+} \frac{\alpha_- t_+ + (1 - \alpha_+) t_- - 1}{t_+}$ becomes $\frac{1}{q} = \frac{1}{p} + \frac{1}{t} - 1$. Thus, by combining both, we get $\frac{1}{q} = \frac{1}{p} + \frac{1}{t} - 1$. In such a particular situation, our results coincide with the results on the boundedness of the Bessel–Riesz operator in classical Lebesgue spaces, as in [18].

5. Proofs

5.1. Diagonal and nondiagonal cases

Proof of Theorem 4.1. For any $w \in \mathbb{R}_+$ and $\psi \in L^{p(\cdot)}(\mathbb{R}_+)$, we write

$$\begin{aligned} I_{\alpha(\cdot), \gamma} \psi(w) &= \int_{\mathbb{R}_+} \frac{|w-z|^{\alpha(w)-1}}{(1+|w-z|)^\gamma} \psi(z) dz \\ &= \int_{\{z \in \mathbb{R}_+ : |w-z| < 1\}} \frac{|w-z|^{\alpha(w)-1}}{(1+|w-z|)^\gamma} \psi(z) dz + \int_{\{z \in \mathbb{R}_+ : |w-z| \geq 1\}} \frac{|w-z|^{\alpha(w)-1}}{(1+|w-z|)^\gamma} \psi(z) dz \\ &= I_1(w) + I_2(w). \end{aligned}$$

Estimate of $I_1(w)$.

$$\begin{aligned} |I_1(w)| &\leq \int_{|w-z| < 1} \frac{|w-z|^{\alpha(w)-1} |\psi(z)|}{(1+|w-z|)^\gamma} dz \\ &\leq \sum_{n=-\infty}^{-1} \int_{2^n \leq |w-z| < 2^{n+1}} \frac{|w-z|^{\alpha(w)-1} |\psi(z)|}{(1+|w-z|)^\gamma} dz \\ &\leq \sum_{n=-\infty}^{-1} \frac{(2^n)^{\alpha(w)-1}}{(1+2^n)^\gamma} \int_{2^n \leq |w-z| < 2^{n+1}} |\psi(z)| dz \\ &\leq \mathcal{M}\psi(w) \sum_{n=-\infty}^{-1} \frac{(2^n)^{\alpha(w)}}{(1+2^n)^\gamma} \end{aligned} \tag{5.1}$$

$$\leq \mathcal{M}\psi(w) \sum_{n=-\infty}^{-1} 2^{\alpha-n}. \tag{5.2}$$

For $n \leq -1$, we have $2^n \in (0, 1/2]$. Because the function $a \mapsto (2^n)^a$ is decreasing in a (because $2^n < 1$), and $\alpha(w) \geq \alpha_-$, we obtain

$$(2^n)^{\alpha(w)} \leq (2^n)^{\alpha_-}.$$

Moreover, $(1+2^n)^\gamma \geq 1$, so

$$\frac{(2^n)^{\alpha(w)}}{(1+2^n)^\gamma} \leq (2^n)^{\alpha_-} = 2^{n\alpha_-}.$$

Therefore,

$$\sum_{n=-\infty}^{-1} \frac{(2^n)^{\alpha(w)}}{(1+2^n)^\gamma} \leq \sum_{n=-\infty}^{-1} 2^{n\alpha_-} = \sum_{m=1}^{\infty} 2^{-m\alpha_-} = \frac{2^{-\alpha_-}}{1-2^{-\alpha_-}} = C_1 < \infty.$$

Therefore, $|I_1(w)| \leq C_1 \mathcal{M}\psi(w)$, where the constant C_1 depends on α_- .

Estimate of $I_2(w)$.

$$\begin{aligned} |I_2(w)| &\leq \int_{|w-z| \geq 1} \frac{|w-z|^{\alpha(w)-1} |\psi(z)|}{(1+|w-z|)^\gamma} dz \\ &= \sum_{n=0}^{\infty} \int_{2^n \leq |w-z| < 2^{n+1}} \frac{|w-z|^{\alpha(w)-1} |\psi(z)|}{(1+|w-z|)^\gamma} dz \\ &\leq \sum_{n=0}^{\infty} \frac{(2^n)^{\alpha(w)-1}}{(1+2^n)^\gamma} \int_{2^n \leq |w-z| < 2^{n+1}} |\psi(z)| dz \\ &\leq \mathcal{M}\psi(w) \sum_{n=0}^{\infty} \frac{(2^n)^{\alpha(w)}}{(1+2^n)^\gamma} \end{aligned} \quad (5.3)$$

$$\leq \mathcal{M}\psi(w) \sum_{n=0}^{\infty} 2^{n(\alpha_+ - \gamma)}. \quad (5.4)$$

Because $\alpha_+ < \gamma$, so the geometric series converges. Hence,

$$|I_2(w)| \leq C_2 \mathcal{M}\psi(w).$$

Combining (5.1) and (5.3), we obtain the pointwise estimate

$$|I_{\alpha(\cdot), \gamma} \psi(w)| \leq C_3 \mathcal{M}\psi(w).$$

Let $j > 0$ be such that $\rho_{p(\cdot)}(\mathcal{M}\psi/j) \leq 1$. Then,

$$\left| \frac{I_{\alpha(\cdot), \gamma} \psi(w)/C_3}{j} \right| \leq \frac{|\mathcal{M}\psi(w)|}{j}.$$

By the definition of the modular,

$$\rho_{p(\cdot)}\left(\frac{I_{\alpha(\cdot), \gamma} \psi/C_3}{j}\right) \leq \rho_{p(\cdot)}\left(\frac{\mathcal{M}\psi}{j}\right) \leq 1.$$

Thus,

$$\left\| \frac{I_{\alpha(\cdot), \gamma} \psi}{C_3} \right\|_{L^{p(\cdot)}} \leq \|\mathcal{M}\psi\|_{L^{p(\cdot)}}.$$

Now, because $p_+ < \infty$, and $p(\cdot) \in LH(\mathbb{R}_+)$, the Hardy–Littlewood maximal operator \mathcal{M} is bounded on $L^{p(\cdot)}(\mathbb{R}_+)$. Hence, there exists a constant $c_0 > 0$ such that

$$\|\mathcal{M}\psi\|_{L^{p(\cdot)}} \leq c_0 \|\psi\|_{L^{p(\cdot)}}.$$

Therefore, we obtain

$$\|I_{\alpha(\cdot), \gamma} \psi\|_{L^{p(\cdot)}} \leq C_4 \|\psi\|_{L^{p(\cdot)}}.$$

Hence, the variable-order Bessel–Riesz operator is bounded on $L^{p(\cdot)}(\mathbb{R}_+)$. \square

Proof of Lemma 4.1. By definition of the variable-order Bessel–Riesz operator, we get following for any $w \in \mathbb{R}_+$ and $\psi \in L^{p(\cdot)}(\mathbb{R}_+)$:

$$I_{\alpha(\cdot), \gamma} \psi(w) = \int_{\mathbb{R}_+} \frac{|w-z|^{\alpha(w)-1} \psi(z)}{(1+|w-z|)^\gamma} dz = \sum_{n \in \mathbb{Z}} \int_{2^n r \leq |w-z| < 2^{n+1} r} \frac{|w-z|^{\alpha(w)-1} \psi(z)}{(1+|w-z|)^\gamma} dz.$$

Let $G_r \in \mathbb{Z}$ be the unique integer satisfying $2^{G_r-1} r < 1 \leq 2^{G_r} r$. Such an integer exists for any $r > 0$. This choice ensures that for all $n \leq G_r - 1$ we have $2^n r < 1$ (local regime, where the Bessel–Riesz kernel exhibits singularity), while for all $n \geq G_r$ we have $2^n r \geq 1$ (far regime, where the kernel decays).

$$\begin{aligned} I_{\alpha(\cdot), \gamma} \psi(w) &= \sum_{n=-\infty}^{G_r-1} \int_{2^n r \leq |w-z| < 2^{n+1} r} \frac{|w-z|^{\alpha(w)-1} \psi(z)}{(1+|w-z|)^\gamma} dz \\ &+ \sum_{n=G_r}^{\infty} \int_{2^n r \leq |w-z| < 2^{n+1} r} \frac{|w-z|^{\alpha(w)-1} \psi(z)}{(1+|w-z|)^\gamma} dz = I_1(w) + I_2(w). \end{aligned} \quad (5.5)$$

Because $\alpha(w) \geq \alpha_- > 0$ and $\alpha(w) \leq \alpha_+ < 1$, we get

$$\alpha(w) - 1 < 0, \quad \alpha_- - 1 < 0.$$

Because $\alpha(w) - 1 < 0$, the function $t \mapsto t_{\alpha(w)-1}$ is decreasing for $t > 0$. Therefore,

$$|w-z|^{\alpha(w)-1} \leq (2^n r)^{\alpha(w)-1}.$$

Now, if $2^n r < 1$, we have

$$\alpha(w) - 1 \geq \alpha_- - 1.$$

For $0 < t < 1$, if $a \geq b$, then $t_a \leq t_b$. Applying this with $t = 2^n r$, $a = \alpha(w) - 1$, and $b = \alpha_- - 1$, we obtain

$$(2^n r)^{\alpha(w)-1} \leq (2^n r)^{\alpha_- - 1}.$$

Combining the inequalities, we conclude that

$$|w-z|^{\alpha(w)-1} \leq (2^n r)^{\alpha_- - 1}, \quad \text{for } n \leq G_r - 1.$$

Similarly,

$$|w-z|^{\alpha(w)-1} \leq (2^n r)^{\alpha_+ - 1}, \quad \text{for } n \geq G_r.$$

Now, we estimate $I_1(w)$:

$$\begin{aligned} |I_1(w)| &\leq \sum_{n=-\infty}^{G_r-1} \int_{2^n r \leq |w-z| < 2^{n+1} r} \frac{|w-z|^{\alpha(w)-1} |\psi(z)|}{(1+|w-z|)^\gamma} dz \\ &\leq \sum_{n=-\infty}^{G_r-1} \frac{(2^n r)^{\alpha_- - 1}}{(1+2^n r)^\gamma} \int_{2^n r \leq |w-z| < 2^{n+1} r} |\psi(z)| dz \\ &= \sum_{n=-\infty}^{G_r-1} \frac{(2^n r)^{\alpha_-}}{(1+2^n r)^\gamma} \frac{1}{2^n r} \int_{2^n r \leq |w-z| < 2^{n+1} r} |\psi(z)| dz \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{n=-\infty}^{G_r-1} \frac{(2^n r)^{\alpha_-}}{(1+2^n r)^\gamma} \sup \frac{1}{2^n r} \int_{2^n r \leq |w-z| < 2^{n+1} r} |\psi(z)| dz \\
&= \mathcal{M}\psi(w) \sum_{n=-\infty}^{G_r-1} \frac{(2^n r)^{\alpha_-}}{(1+2^n r)^\gamma}.
\end{aligned} \tag{5.6}$$

To estimate $I_2(w)$, we proceed as follows:

$$\begin{aligned}
|I_2(w)| &= \sum_{n=G_r}^{\infty} \int_{2^n r \leq |w-z| < 2^{n+1} r} \frac{|w-z|^{\alpha(w)-1} |\psi(z)|}{(1+|w-z|)^\gamma} dz \\
&\leq \sum_{n=G_r}^{\infty} \frac{(2^n r)^{\alpha_+ - 1}}{(1+2^n r)^\gamma} \int_{2^n r \leq |w-z| < 2^{n+1} r} |\psi(z)| dz.
\end{aligned}$$

By Hölder's inequality, Theorem 2.1 for $|\psi|$, and 1, one obtains

$$\begin{aligned}
|I_2(w)| &\leq C_3 \sum_{n=G_r}^{\infty} \frac{(2^n r)^{\alpha_+ - 1}}{(1+2^n r)^\gamma} K_{p(\cdot)} \|\psi\|_{L^{p(\cdot)}[w+2^n r, w+2^{n+1} r]} \|\chi_{[w+2^n r, w+2^{n+1} r]}\|_{L^{p'(\cdot)}(w+2^n r, w+2^{n+1} r)} \\
&\leq C_4 \sum_{n=G_r}^{\infty} \frac{(2^n r)^{\alpha_+ - 1}}{(1+2^n r)^\gamma} \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)} \|\chi_{[w+2^n r, w+2^{n+1} r]}\|_{L^{p'(\cdot)}(w+2^n r, w+2^{n+1} r)},
\end{aligned} \tag{5.7}$$

where the constant $K_{p(\cdot)}$ of Hölder's inequality has been absorbed into the constant C_4 . Further, because $\|\psi\|_{L^{p(\cdot)}[w+2^n r, w+2^{n+1} r]} \leq \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)}$ as the norm is monotone with respect to set inclusion, we replace the local norm by the global norm. Now, we calculate the norm $\|\chi_{[w+2^n r, w+2^{n+1} r]}\|_{L^{p'(\cdot)}[w+2^n r, w+2^{n+1} r]}$. Now, by definition,

$$\|\chi_{[w+2^n r, w+2^{n+1} r]}\|_{L^{p'(\cdot)}[w+2^n r, w+2^{n+1} r]} = \inf \left\{ j > 0 : \rho_{L^{p'(\cdot)}} \left(\frac{\chi_{[w+2^n r, w+2^{n+1} r]}}{j} \right) \leq 1 \right\}. \tag{5.8}$$

Also, for $j \geq 1$, we have

$$\begin{aligned}
\rho_{L^{p'(\cdot)}} \left(\frac{\chi_{[w+2^n r, w+2^{n+1} r]}}{j} \right) &= \int_{2^n r \leq |w-z| < 2^{n+1} r} \left| \frac{\chi_{[w+2^n r, w+2^{n+1} r]}(z)}{j} \right|^{p'(z)} dz, \\
&= \int_{2^n r \leq |w-z| < 2^{n+1} r} \left| \frac{1}{j} \right|^{p'(z)} dz = \left(\frac{1}{j} \right)^{(p')_-} 2^n r.
\end{aligned}$$

Thus, $\rho_{L^{p'(\cdot)}} \left(\frac{\chi_{[2^n r, 2^{n+1} r]}}{j} \right) = \left(\frac{1}{j} \right)^{(p')_-} 2^n r \leq 1$, if $2^n r \leq j^{(p')_-}$; therefore, $\inf \left\{ j > 0 : \rho_{L^{p'(\cdot)}} \left(\frac{\chi_{[w+2^n r, w+2^{n+1} r]}}{j} \right) \leq 1 \right\} = (2^n r)^{\frac{1}{(p')_-}}$. Consequently, from (5.8), we get

$$\|\chi_{[w+2^n r, w+2^{n+1} r]}\|_{L^{p'(\cdot)}[w+2^n r, w+2^{n+1} r]} = (2^n r)^{\frac{1}{(p')_-}}. \tag{5.9}$$

Then, by following (5.7) and (5.9), and then by using $(p')_- = (p_+)'$, we get

$$|I_2(w)| \leq C_3 \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)} \sum_{n=G_r}^{\infty} \frac{(2^n r)^{\alpha_+ - 1} (2^n r)^{\frac{1}{(p')_-}}}{(1+2^n r)^\gamma}$$

$$\begin{aligned}
&= C_3 \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)} \sum_{n=G_r}^{\infty} \frac{(2^n r)^{\alpha_+ - 1} (2^n r)^{\frac{1}{(p_+)^{\gamma}}}}{(1 + 2^n r)^{\gamma}} \\
&= C_3 \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)} \sum_{n=G_r}^{\infty} \frac{(2^n r)^{\alpha_+ - \frac{1}{p_+}}}{(1 + 2^n r)^{\gamma}}. \tag{5.10}
\end{aligned}$$

Thus, by combining (5.6) and (5.9), we get

$$|I_{\alpha(\cdot), \gamma} \psi(w)| \leq C_4 \left(\mathcal{M}\psi(w) \sum_{n=-\infty}^{G_r-1} \frac{(2^n r)^{\alpha_-}}{(1 + 2^n r)^{\gamma}} + \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)} \sum_{n=G_r}^{\infty} \frac{(2^n r)^{\alpha_+ - \frac{1}{p_+}}}{(1 + 2^n r)^{\gamma}} \right).$$

□

Proof of Theorem 4.2. Let $w \in \mathbb{R}_+$ and $\psi \in L^{p(\cdot)}(\mathbb{R}_+)$ with $\psi \neq 0$. By Lemma 4.1, there exists an integer G_r depending on $r = r(w)$ such that

$$\begin{aligned}
|I_{\alpha(\cdot), \gamma} \psi(w)| &\leq C_4 \left(\mathcal{M}\psi(w) \sum_{n=-\infty}^{G_r-1} \frac{(2^n r)^{\alpha_-}}{(1 + 2^n r)^{\gamma}} \right. \\
&\quad \left. + \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)} \sum_{n=G_r}^{\infty} \frac{(2^n r)^{\alpha_+ - 1/p_+}}{(1 + 2^n r)^{\gamma}} \right).
\end{aligned}$$

Because $\alpha_+ - \frac{1}{p_+} - \gamma < 0$, both series converge, and we obtain

$$|I_{\alpha(\cdot), \gamma} \psi(w)| \leq C^{**} \left(r^{\alpha_-} \mathcal{M}\psi(w) + r^{\alpha_+ - \frac{1}{p_+} - \gamma} \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)} \right).$$

We choose $r = \left(\frac{\mathcal{M}\psi(w)}{C_0 \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)}} \right)^{\frac{p_+ p(w)}{((\alpha_+ - \gamma)p_+ - 1)q(w)}}$, where $(\alpha_+ - \gamma)p_+ - 1 < 0$. Hence, using $\frac{q(w)}{p(w)} = \frac{1 + \gamma p_+}{1 - (\alpha_+ - \gamma)p_+}$, a.e. for any $w \in \mathbb{R}_+$, we get

$$\begin{aligned}
r^{\alpha_-} \mathcal{M}\psi(w) + r^{\alpha_+ - \frac{1}{p_+} - \gamma} \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)} &= \left(\frac{\mathcal{M}\psi(w)}{C_0 \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)}} \right)^{\frac{\alpha_- p_+ p(w)}{((\alpha_+ - \gamma)p_+ - 1)q(w)}} \mathcal{M}\psi(w) + \left(\frac{\mathcal{M}\psi(w)}{C_0 \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)}} \right)^{\frac{p(w)}{q(w)}} \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)} \\
&= \frac{\left(C_0 \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)} \right)^{\frac{\alpha_- p_+ p(w)}{(1 - (\alpha_+ - \gamma)p_+)q(w)}}}{(\mathcal{M}\psi(w))^{\frac{\alpha_- p_+ p(w)}{(1 - (\alpha_+ - \gamma)p_+)q(w)} - 1}} + \frac{(\mathcal{M}\psi(w))^{\frac{p(w)}{q(w)}}}{C_0 \left(C_0 \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)} \right)^{\frac{p(w)}{q(w)} - 1}} \\
&= \frac{C_0 \left(C_0 \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)} \right)^{\frac{\alpha_- p_+ p(w)}{(1 - (\alpha_+ - \gamma)p_+)q(w)} - 1} + (\mathcal{M}\psi(w))^{\frac{\alpha_- p_+ p(w)}{(1 - (\alpha_+ - \gamma)p_+)q(w)} + \frac{p(w)}{q(w)} - 1}}{C_0 (\mathcal{M}\psi(w))^{\frac{\alpha_- p_+ p(w)}{(1 - (\alpha_+ - \gamma)p_+)q(w)} - 1} \left(C_0 \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)} \right)^{\frac{p(w)}{q(w)} - 1}} \\
&= (C_0 + 1) \left(\frac{\mathcal{M}\psi(w)}{C_0 \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)}} \right)^{\frac{p(w)}{q(w)}} \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)}. \tag{5.11}
\end{aligned}$$

Therefore,

$$|I_{\alpha(\cdot),\gamma}\psi(w)| \leq C_1 \left(\frac{\mathcal{M}\psi(w)}{C_0\|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)}} \right)^{\frac{p(w)}{q(w)}} \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)},$$

where $C_1 = (C_0 + 1)C^{**}$. Hence,

$$\left(\frac{|I_{\alpha(\cdot),\gamma}\psi(w)|}{C_1\|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)}} \right)^{q(w)} \leq \left(\frac{\mathcal{M}\psi(w)}{C_0\|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)}} \right)^{p(w)}.$$

Integrating over \mathbb{R}_+ and using the definition of the modular gives

$$\rho_{q(\cdot)} \left(\frac{I_{\alpha(\cdot),\gamma}\psi}{C_1\|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)}} \right) \leq \rho_{p(\cdot)} \left(\frac{\mathcal{M}\psi}{C_0\|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)}} \right). \quad (5.12)$$

Now, because $1 < p_- \leq p_+ < \infty$ and $p(\cdot) \in LH(\mathbb{R}_+)$, the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}_+)$. Hence $C_0 > 0$ exists in a way that

$$\|\mathcal{M}\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)} \leq C_0\|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)},$$

because,

$$\|\mathcal{M}\psi\|_{L^{p(\cdot)}} = \inf\{j > 0 : \rho_{p(\cdot)}(\mathcal{M}\psi/j) \leq 1\}.$$

Thus,

$$C_0\|\psi\|_{L^{p(\cdot)}} \in \left\{ j > 0 : \rho_{p(\cdot)} \left(\frac{\mathcal{M}\psi}{j} \right) \leq 1 \right\}.$$

Therefore, we obtain from (5.12)

$$\|I_{\alpha(\cdot),\gamma}\psi\|_{L^{q(\cdot)}(\mathbb{R}_+)} \leq C_1\|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)}.$$

Thus, the variable-order Bessel–Riesz operator is bounded from $L^{p(\cdot)}(\mathbb{R}_+)$ to $L^{q(\cdot)}(\mathbb{R}_+)$. \square

Proof of Theorem 4.3. Because $q_+ < \infty$, and

$$\frac{1}{q_-} + \frac{\alpha_+q_+}{q_-} < 1,$$

together with

$$\frac{1}{p(w)} = \frac{1}{q(w)} + \frac{\alpha_+q_+}{q(w)} \quad \text{a.e.},$$

it follows that

$$1 < p_- \leq p_+ < \infty.$$

Moreover, because $q(\cdot) \in LH(\mathbb{R}_+)$, and

$$p(\cdot) = \frac{q(\cdot)}{1 + \alpha_+q_+},$$

the log-Hölder continuity of $q(\cdot)$ implies that $p(\cdot)$ also satisfies the log-Hölder condition. Hence,

$$p(\cdot) \in LH(\mathbb{R}_+).$$

Therefore, by the boundedness of the maximal operator, there exists a constant $C_0 > 0$ such that for every $\psi \in L^{p(\cdot)}(\mathbb{R}_+)$,

$$\|\mathcal{M}\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)} \leq C_0 \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)}.$$

Let $w \in \mathbb{R}_+$ and $\psi \in L^{p(\cdot)}(\mathbb{R}_+)$ with $\psi \neq 0$. By Lemma 4.1, we obtain the estimate

$$|I_{\alpha(\cdot), \gamma} \psi(w)| \leq C_4 \left(\mathcal{M}\psi(w) \sum_{n=-\infty}^{G_r-1} \frac{(2^n r)^{\alpha_-}}{(1+2^n r)^\gamma} + \|\psi\|_{L^{p(\cdot)}} \sum_{n=G_r}^{\infty} \frac{(2^n r)^{\alpha_+ - \frac{1}{p_+}}}{(1+2^n r)^\gamma} \right).$$

Because $(1+2^n r)^{-\gamma} \leq 1$, we obtain

$$|I_{\alpha(\cdot), \gamma} \psi(w)| \leq C_4 \left(\mathcal{M}\psi(w) \sum_{n=-\infty}^{G_r-1} (2^n r)^{\alpha_-} + \|\psi\|_{L^{p(\cdot)}} \sum_{n=G_r}^{\infty} (2^n r)^{\alpha_+ - \frac{1}{p_+} - \gamma} \right).$$

Because

$$\alpha_+ - \frac{1}{p_+} - \gamma < 0,$$

both series are convergent. Consequently, there exists a constant $C_6 > 0$ such that

$$|I_{\alpha(\cdot), \gamma} \psi(w)| \leq C_6 \left(r^{\alpha_-} \mathcal{M}\psi(w) + r^{\alpha_+ - \frac{1}{p_+} - \gamma} \|\psi\|_{L^{p(\cdot)}} \right).$$

Now, choose

$$r = \left(\frac{\mathcal{M}\psi(w)}{C_0 \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)}} \right)^{-\frac{q_+ p(w)}{q(w)}}.$$

Substituting this value of r into the above estimate and using the relation

$$\frac{1}{p(\cdot)} = \frac{1}{q(\cdot)} + \frac{\alpha_+ q_+}{q(\cdot)},$$

we obtain

$$|I_{\alpha(\cdot), \gamma} \psi(w)| \leq C_7 \left(\frac{\mathcal{M}\psi(w)}{C_0 \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)}} \right)^{\frac{p(w)}{q(w)}} \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)}.$$

Therefore,

$$\left(\frac{|I_{\alpha(\cdot), \gamma} \psi(w)|}{C_7 \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)}} \right)^{q(w)} \leq \left(\frac{\mathcal{M}\psi(w)}{C_0 \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)}} \right)^{p(w)}.$$

By the definition of the modular function we obtain

$$\rho_{q(\cdot)} \left(\frac{I_{\alpha(\cdot), \gamma} \psi}{C_7 \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)}} \right) \leq \rho_{p(\cdot)} \left(\frac{\mathcal{M}\psi}{C_0 \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)}} \right).$$

Finally, by the boundedness of \mathcal{M} , one obtains

$$\|I_{\alpha(\cdot), \gamma} \psi\|_{L^{q(\cdot)}(\mathbb{R}_+)} \leq C_7 \|\psi\|_{L^{p(\cdot)}(\mathbb{R}_+)}.$$

Hence, the operator $I_{\alpha(\cdot), \gamma}$ is bounded from $L^{p(\cdot)}(\mathbb{R}_+)$ into $L^{q(\cdot)}(\mathbb{R}_+)$. \square

5.2. Analog of Young's inequality

Proof of Lemma 4.2. Recall that a measurable function ψ belongs to $L^{(\cdot)}(\mathbb{R}_+)$ if and only if the modular $\rho_{t(\cdot)}(\psi/j) = \int_{\mathbb{R}_+} |\psi(w)/j|^{t(w)} dw$ is finite for some $j > 0$. We show that under Condition (4.2), the integral

$$\mathcal{I} = \int_0^\infty (K_{\alpha(\cdot), \gamma(w)})^{t(w)} dw = \int_0^\infty \frac{w^{(\alpha(w)-1)t(w)}}{(1+w)^{\gamma t(w)}} dw \quad (5.13)$$

is finite, from which the conclusion follows immediately because $\rho_{t(\cdot)}(K_{\alpha(\cdot), \gamma}/j) \leq j^{-t_-} \mathcal{I}$ for $j \geq 1$ and $\leq j^{-t_+} \mathcal{I}$ for $0 < j < 1$.

Decompose \mathcal{I} at $w = 1$:

$$\mathcal{I} = \underbrace{\int_0^1 \frac{w^{(\alpha(w)-1)t(w)}}{(1+w)^{\gamma t(w)}} dw}_{=\mathcal{I}_1} + \underbrace{\int_1^\infty \frac{w^{(\alpha(w)-1)t(w)}}{(1+w)^{\gamma t(w)}} dw}_{=\mathcal{I}_2}. \quad (5.14)$$

Finiteness of \mathcal{I}_1 . For $w \in (0, 1]$, one has $1+w \leq 2$, so $(1+w)^{\gamma t(w)} \geq 1$, and therefore,

$$\mathcal{I}_1 \leq \int_0^1 w^{(\alpha(w)-1)t(w)} dw. \quad (5.15)$$

Because $w \in (0, 1)$, we have $\ln w < 0$, so the power function $a \mapsto w^a$ is *strictly decreasing* in a . The exponent $(\alpha(w)-1)t(w)$ is negative (as $\alpha(w) < 1$ and $t(w) > 0$) and attains its most negative value when $\alpha(w) - 1$ is most negative, and $t(w)$ is largest, namely

$$(\alpha(w) - 1)t(w) \geq (\alpha_- - 1)t_+.$$

Because $a \mapsto w^a$ is decreasing on $(0, 1)$, the largest value of the integrand corresponds to the *most negative* exponent:

$$w^{(\alpha(w)-1)t(w)} \leq w^{(\alpha_- - 1)t_+}, \quad w \in (0, 1).$$

Hence,

$$\mathcal{I}_1 \leq \int_0^1 w^{(\alpha_- - 1)t_+} dw = \frac{1}{1 + (\alpha_- - 1)t_+} = \frac{1}{1 - (1 - \alpha_-)t_+}. \quad (5.16)$$

This is finite if and only if $1 - (1 - \alpha_-)t_+ > 0$; that is, $t_+ < \frac{1}{1 - \alpha_-}$, which is guaranteed by the right-hand inequality in (4.2).

Finiteness of \mathcal{I}_2 . For $w \geq 1$, one has $1+w > w$, so

$$\frac{w^{(\alpha(w)-1)t(w)}}{(1+w)^{\gamma t(w)}} < \frac{w^{(\alpha(w)-1)t(w)}}{w^{\gamma t(w)}} = w^{(\alpha(w)-1-\gamma)t(w)}. \quad (5.17)$$

Because $\alpha(w) - 1 - \gamma < 0$ (as $\alpha(w) \leq \alpha_+ < 1$ and $\gamma > 0$), the exponent $(\alpha(w) - 1 - \gamma)t(w)$ is strictly negative. Now $w \geq 1$ gives $\ln w \geq 0$, so $a \mapsto w^a$ is *strictly increasing* in a . The largest value of the integrand therefore corresponds to the *least negative* exponent. The least negative value of $(\alpha(w) - 1 - \gamma)t(w)$ is obtained when $\alpha(w)$ is largest and $t(w)$ is smallest, namely

$$(\alpha(w) - 1 - \gamma)t(w) \leq (\alpha_+ - 1 - \gamma)t_-, \quad w \geq 1.$$

Consequently,

$$\mathcal{I}_2 < \int_1^\infty w^{(\alpha_+ - 1 - \gamma)t_-} dw. \quad (5.18)$$

This integral converges if and only if $(\alpha_+ - 1 - \gamma)t_- < -1$; that is

$$(\gamma + 1 - \alpha_+)t_- > 1, \quad \text{i.e., } t_- > \frac{1}{\gamma + 1 - \alpha_+},$$

which is exactly the left-hand inequality in (4.2). When this holds,

$$\mathcal{I}_2 < \left[\frac{w^{(\alpha_+ - 1 - \gamma)t_- + 1}}{(\alpha_+ - 1 - \gamma)t_- + 1} \right]_1^\infty = \frac{1}{(\gamma + 1 - \alpha_+)t_- - 1} < \infty. \quad (5.19)$$

Combining (5.16) and (5.19) with (5.14) gives

$$\mathcal{I} \leq \frac{1}{1 - (1 - \alpha_-)t_+} + \frac{1}{(\gamma + 1 - \alpha_+)t_- - 1} < \infty. \quad (5.20)$$

Hence, $\rho_{t(\cdot)}(K_{\alpha(\cdot), \gamma}/i) < \infty$ for some $i > 0$, which by definition gives $K_{\alpha(\cdot), \gamma} \in L^{t(\cdot)}(\mathbb{R}_+)$. \square

Proof of Lemma 4.3. By definition of the modular,

$$\rho_{t(\cdot)}(K_{\alpha(\cdot), \gamma}) = \int_0^\infty \left| \frac{w^{\alpha(w)-1}}{(1+w)^\gamma} \right|^{t(w)} dw = \int_0^\infty w^{(\alpha(w)-1)t(w)} (1+w)^{-\gamma t(w)} dw.$$

Fix $r > 0$, and decompose \mathbb{R}_+ dyadically:

$$\rho_{t(\cdot)}(K_{\alpha(\cdot), \gamma}) = \sum_{n \in \mathbb{Z}} \int_{2^n r}^{2^{n+1} r} w^{(\alpha(w)-1)t(w)} (1+w)^{-\gamma t(w)} dw.$$

For each $w \in [2^n r, 2^{n+1} r)$, we have $1+w \leq 1+2^{n+1} r$. Because $w \mapsto w^{-\gamma t}$ is decreasing for $w > 0$ and $t(w) \leq t_+$,

$$(1+w)^{-\gamma t(w)} \geq (1+2^{n+1} r)^{-\gamma t(w)} \geq (1+2^{n+1} r)^{-\gamma t_+}.$$

Hence,

$$\int_{2^n r}^{2^{n+1} r} w^{(\alpha(w)-1)t(w)} (1+w)^{-\gamma t(w)} dw \geq (1+2^{n+1} r)^{-\gamma t_+} \int_{2^n r}^{2^{n+1} r} w^{(\alpha(w)-1)t(w)} dw.$$

Using $1+2^{n+1} r \leq 2(1+2^n r)$ we may further bound

$$(1+2^{n+1} r)^{-\gamma t_+} \geq 2^{-\gamma t_+} (1+2^n r)^{-\gamma t_+}.$$

Therefore

$$\rho_{t(\cdot)}(K_{\alpha(\cdot), \gamma}) \geq 2^{-\gamma t_+} \sum_{n \in \mathbb{Z}} \frac{1}{(1+2^n r)^{\gamma t_+}} \int_{2^n r}^{2^{n+1} r} w^{(\alpha(w)-1)t(w)} dw.$$

Choose $G_r \in \mathbb{Z}$ so that $2^{G_r} r \geq 1$, and $2^{G_r-1} r < 1$. Thus, intervals with $n \leq G_r - 1$ lie in $(0, 1)$, and those with $n \geq G_r$ lie in $[1, \infty)$. We treat these two ranges separately.

(i) **Case** $n \leq G_r - 1$ (**small** w , $w \in (0, 1)$).

On $(0, 1)$ the map $a \mapsto w^a$ is decreasing (because $\ln w < 0$). Put

$$a(w) = (\alpha(w) - 1)t(w).$$

Because $a(w) \in [(\alpha_- - 1)t_+, (\alpha_+ - 1)t_-]$, the largest value of $w^{a(w)}$ on $(0, 1)$ occurs at the smallest exponent $(\alpha_- - 1)t_+$, and the smallest value occurs at the largest exponent $(\alpha_+ - 1)t_-$. Consequently, for every $w \in [2^n r, 2^{n+1} r) \subset (0, 1)$,

$$w^{(\alpha(w)-1)t(w)} \geq w^{(\alpha_+ - 1)t_-}.$$

Thus, for such n ,

$$\int_{2^n r}^{2^{n+1} r} w^{(\alpha(w)-1)t(w)} dw \geq \int_{2^n r}^{2^{n+1} r} w^{(\alpha_+ - 1)t_-} dw = \frac{2^{(\beta_1+1)} - 1}{\beta_1 + 1} (2^n r)^{\beta_1+1},$$

where $\beta_1 = (\alpha_+ - 1)t_-$, and $\beta_1 + 1 > 0$ by the standing assumptions. Therefore, for every $n \leq G_r - 1$,

$$\rho_{t(\cdot)}(K_{\alpha(\cdot), \gamma}) \geq 2^{-\gamma t_+} \frac{2^{(\beta_1+1)} - 1}{\beta_1 + 1} \cdot \frac{(2^n r)^{\beta_1+1}}{(1 + 2^n r)^{\gamma t_+}}.$$

This yields (4.3) with the choice $C_1 = \frac{2^{(\beta_1+1)} - 1}{2^{\gamma t_+} (\beta_1 + 1)} > 0$.

(ii) **Case** $n \geq G_r$ (**large** w , $w \geq 1$).

For $w \geq 1$, the map $a \mapsto w^a$ is increasing (because $\ln w \geq 0$). Again with $a(w) = (\alpha(w) - 1)t(w) \in [(\alpha_- - 1)t_+, (\alpha_+ - 1)t_-]$, the minimal value of $w^{a(w)}$ on $[1, \infty)$ occurs at the smallest exponent $(\alpha_- - 1)t_+$. Hence, for $w \in [2^n r, 2^{n+1} r) \subset [1, \infty)$,

$$w^{(\alpha(w)-1)t(w)} \geq w^{(\alpha_- - 1)t_+}.$$

Thus, for such n ,

$$\int_{2^n r}^{2^{n+1} r} w^{(\alpha(w)-1)t(w)} dw \geq \int_{2^n r}^{2^{n+1} r} w^{(\alpha_- - 1)t_+} dw = \frac{2^{(\beta_2+1)} - 1}{\beta_2 + 1} (2^n r)^{\beta_2+1},$$

where $\beta_2 = (\alpha_- - 1)t_+$ and $\beta_2 + 1 > 0$ by our assumptions. Therefore, for every $n \geq G_r$,

$$\rho_{t(\cdot)}(K_{\alpha(\cdot), \gamma}) \geq 2^{-\gamma t_+} \frac{2^{(\beta_2+1)} - 1}{\beta_2 + 1} \cdot \frac{(2^n r)^{\beta_2+1}}{(1 + 2^n r)^{\gamma t_+}}.$$

This yields (4.4) with the choice $C_2 = \frac{2^{(\beta_2+1)} - 1}{2^{\gamma t_+} (\beta_2 + 1)} > 0$.

Combining the two ranges of n completes the proof. \square

Proof of Lemma 4.4. From Proposition 2.3, we have

$$\rho_{t(\cdot)}\left(\frac{K_{\alpha(\cdot), \gamma}}{\|K_{\alpha(\cdot), \gamma}\|_{L^{t(\cdot)}}}\right) = 1.$$

Because $0 < \|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}} < 1$, and $t_- \geq 1$, Proposition 2.2 (applied with $j = 1/\|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}} > 1$) gives

$$1 = \rho_{t(\cdot)}\left(\frac{K_{\alpha(\cdot),\gamma}}{\|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}}\right) \geq \frac{1}{\|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}^{t_-}} \rho_{t(\cdot)}(K_{\alpha(\cdot),\gamma}),$$

so

$$\|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}^{t_-} \geq \rho_{t(\cdot)}(K_{\alpha(\cdot),\gamma}).$$

Because $0 < \|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}} < 1$, and $t_- \geq 1$, we also have $\|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}} \geq \|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}^{t_-}$; hence,

$$\|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}} \geq \rho_{t(\cdot)}(K_{\alpha(\cdot),\gamma}).$$

Raising both sides to the power $1/t_+$ (recall $t_+ < \infty$) yields

$$\|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}^{1/t_+} \geq \rho_{t(\cdot)}(K_{\alpha(\cdot),\gamma})^{1/t_+}.$$

Now, from Lemma 4.3, we have the lower bounds

$$\rho_{t(\cdot)}(K_{\alpha(\cdot),\gamma}) \geq C_1 \frac{(2^n r)^{\beta_1+1}}{(1+2^n r)^{\gamma t_+}} \quad (n < G_r), \quad \rho_{t(\cdot)}(K_{\alpha(\cdot),\gamma}) \geq C_2 \frac{(2^n r)^{\beta_2+1}}{(1+2^n r)^{\gamma t_+}} \quad (n \geq G_r),$$

with $\beta_1 = (\alpha_+ - 1)t_-$ and $\beta_2 = (\alpha_- - 1)t_+$. Taking the $1/t_+$ -power of these inequalities and using the preceding estimate $\|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}^{1/t_+} \geq \rho_{t(\cdot)}(K_{\alpha(\cdot),\gamma})^{1/t_+}$ gives

$$\|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}^{1/t_+} \geq C_1^{1/t_+} \frac{(2^n r)^{(\beta_1+1)/t_+}}{(1+2^n r)^\gamma} \quad (n < G_r),$$

and

$$\|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}^{1/t_+} \geq C_2^{1/t_+} \frac{(2^n r)^{(\beta_2+1)/t_+}}{(1+2^n r)^\gamma} \quad (n \geq G_r).$$

Replacing $(\beta_1 + 1)/t_+ = ((\alpha_+ - 1)t_- + 1)/t_+ = (\alpha_+ - 1)\frac{t_-}{t_+} + \frac{1}{t_+}$ and $(\beta_2 + 1)/t_+ = (\alpha_- - 1) + \frac{1}{t_+}$ yields (4.5) and (4.6) with $D_i = C_i^{1/t_+}$. \square

Proof of Lemma 4.5. As before, Proposition 2.3 gives

$$\rho_{t(\cdot)}\left(\frac{K_{\alpha(\cdot),\gamma}}{\|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}}\right) = 1.$$

Because now $\|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}} \geq 1$ and $t_+ < \infty$, Proposition 2.2 (with $j = 1/\|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}} \leq 1$) yields

$$1 = \rho_{t(\cdot)}\left(\frac{K_{\alpha(\cdot),\gamma}}{\|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}}\right) \geq \frac{1}{\|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}^{t_+}} \rho_{t(\cdot)}(K_{\alpha(\cdot),\gamma}),$$

so

$$\|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}^{t_+} \geq \rho_{t(\cdot)}(K_{\alpha(\cdot),\gamma}).$$

Taking t_+ -th roots gives

$$\|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}} \geq \rho_{t(\cdot)}(K_{\alpha(\cdot),\gamma})^{1/t_+}.$$

Applying Lemma 4.3 (the same dyadic lower bounds for $\rho_{t(\cdot)}(K_{\alpha(\cdot),\gamma})$) and taking $1/t_+$ -powers as in the previous lemma yields, the claimed inequalities (4.7) and (4.8) hold with the stated constants. \square

Proof of Theorem 4.4. For any $w \in \mathbb{R}_+$ and $\psi \in L^{p(\cdot)}(\mathbb{R}_+)$, we get from Lemma 4.1

$$|I_{\alpha(\cdot),\gamma}\psi(w)| \leq C_4 \left(\mathcal{M}\psi(w) \sum_{n=-\infty}^{G_r-1} \frac{(2^n r)^{\alpha_-}}{(1+2^n r)^\gamma} + \|\psi\|_{L^{p(\cdot)}} \sum_{n=G_r}^{\infty} \frac{(2^n r)^{\alpha_+ - \frac{1}{p_+}}}{(1+2^n r)^\gamma} \right). \quad (5.21)$$

We discuss two cases depending on $\|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}$.

Case 1. If $0 < \|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}} < 1$, following (4.5), we can write

$$\begin{aligned} \mathcal{M}\psi(w) \sum_{n=-\infty}^{G_r-1} \frac{(2^n r)^{\alpha_-}}{(1+2^n r)^\gamma} &\leq C_1 \mathcal{M}\psi(w) \|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}^{1/t_+} \sum_{n=-\infty}^{G_r-1} \frac{(2^n r)^{\alpha_-}}{(2^n r)^{(\alpha_+ - 1)\frac{t_-}{t_+} + \frac{1}{t_+}}} \\ &= C_1 r^{\frac{\alpha_- t_+ + (1 - \alpha_+) t_- - 1}{t_+}} \mathcal{M}\psi(w) \|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}^{1/t_+} \sum_{n=-\infty}^{G_r-1} 2^{\frac{k}{t_+}(\alpha_- t_+ + (1 - \alpha_+) t_- - 1)}. \end{aligned}$$

Because $t_+ > t_-$ and $\alpha_- t_+ \geq 1$ hence, the series is convergent; therefore,

$$\mathcal{M}\psi(w) \sum_{n=-\infty}^{G_r-1} \frac{(2^n r)^{\alpha_-}}{(1+2^n r)^\gamma} \leq C_2 r^{\frac{\alpha_- t_+ + (1 - \alpha_+) t_- - 1}{t_+}} \mathcal{M}\psi(w) \|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}^{1/t_+}. \quad (5.22)$$

Now, from (4.6) and from $1 - \frac{1}{p_+} - \frac{1}{t_+} \leq -\frac{1}{cq_+}$, we obtain

$$\begin{aligned} \|\psi\|_{L^{p(\cdot)}} \sum_{n=G_r}^{\infty} \frac{(2^n r)^{\alpha_+ - \frac{1}{p_+}}}{(1+2^n r)^\gamma} &\leq C_3 \|\psi\|_{L^{p(\cdot)}} \|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}^{1/t_+} \sum_{n=G_r}^{\infty} \frac{(2^n r)^{\alpha_+ - \frac{1}{p_+}}}{(2^n r)^{(\alpha_- - 1)\frac{t_-}{t_+} + \frac{1}{t_+}}} \\ &= C_3 \|\psi\|_{L^{p(\cdot)}} \|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}^{1/t_+} \sum_{n=G_r}^{\infty} (2^n r)^{\alpha_+ - \alpha_- + 1 - \frac{1}{p_+} - \frac{1}{t_+}} \\ &\leq C_3 \|\psi\|_{L^{p(\cdot)}} \|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}^{1/t_+} \sum_{n=G_r}^{\infty} (2^n r)^{\alpha_+ - \alpha_- - \frac{1}{cq_+}} \\ &= C_3 r^{\alpha_+ - \alpha_- - \frac{1}{cq_+}} \|\psi\|_{L^{p(\cdot)}} \|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}^{1/t_+} \sum_{n=G_r}^{\infty} (2^n)^{\alpha_+ - \alpha_- - \frac{1}{cq_+}}. \end{aligned}$$

One may determine a $c > 0$ in a manner that $\alpha_+ - \alpha_- - \frac{1}{cq_+} < 0$. Hence, the series is convergent, therefore,

$$\|\psi\|_{L^{p(\cdot)}} \sum_{n=G_r}^{\infty} \frac{(2^n r)^{\alpha_+ - \frac{1}{p_+}}}{(1+2^n r)^\gamma} \leq C_4 r^{\alpha_+ - \alpha_- - \frac{1}{cq_+}} \|\psi\|_{L^{p(\cdot)}} \|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}^{1/t_+}. \quad (5.23)$$

By (5.21)–(5.23), we have

$$|I_{\alpha(\cdot),\gamma}\psi(w)| \leq C_5 \left(r^{\frac{\alpha_- t_+ + (1 - \alpha_+) t_- - 1}{t_+}} \mathcal{M}\psi(w) + r^{\alpha_+ - \alpha_- - \frac{1}{cq_+}} \|\psi\|_{L^{p(\cdot)}} \right) \|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}^{1/t_+}. \quad (5.24)$$

We choose $r = \left(\frac{\mathcal{M}\psi(w)}{\|\psi\|_{L^{p(\cdot)}}} \right)^{\frac{p(w)}{q(w)} \frac{1}{\alpha_+ - \alpha_- - \frac{1}{cq_+}}}$, and then

$$r^{\frac{\alpha_- t_+ + (1 - \alpha_+) t_- - 1}{t_+}} \mathcal{M}\psi(w) + r^{\alpha_+ - \alpha_- - \frac{1}{cq_+}} \|\psi\|_{L^{p(\cdot)}} = \left(\frac{\mathcal{M}\psi(w)}{\|\psi\|_{L^{p(\cdot)}}} \right)^{\frac{p(w)}{q(w)} \frac{1}{\alpha_+ - \alpha_- - \frac{1}{cq_+}} \frac{\alpha_- t_+ + (1 - \alpha_+) t_- - 1}{t_+}} \mathcal{M}\psi(w) + \left(\frac{\mathcal{M}\psi(w)}{\|\psi\|_{L^{p(\cdot)}}} \right)^{\frac{p(w)}{q(w)}} \|\psi\|_{L^{p(\cdot)}}$$

$$\begin{aligned}
&= \frac{\|\psi\|_{L^{p(\cdot)}}^{\frac{bp(w)}{q(w)t_+} - \frac{1}{\alpha_- - \alpha_+ + \frac{1}{cq_+}}}}{(\mathcal{M}\psi(w))^{\frac{bp(w)}{q(w)t_+} - \frac{1}{\alpha_- - \alpha_+ + \frac{1}{cq_+}} - 1}} + \frac{(\mathcal{M}\psi(w))^{\frac{p(w)}{q(w)}}}{\|\psi\|_{L^{p(\cdot)}}^{\frac{p(w)}{q(w)} - 1}} \\
&= \frac{\|\psi\|_{L^{p(\cdot)}}^{\frac{bp(w)}{q(w)t_+} \cdot \frac{1}{\alpha_- - \alpha_+ + \frac{1}{cq_+}} + \frac{p(w)}{q(w)} - 1}}{(\mathcal{M}\psi(w))^{\frac{bp(w)}{q(w)t_+} - \frac{1}{\alpha_- - \alpha_+ + \frac{1}{cq_+}} - 1}} + \frac{(\mathcal{M}\psi(w))^{\frac{bp(w)}{q(w)t_+} - \frac{1}{\alpha_- - \alpha_+ + \frac{1}{cq_+}} + \frac{p(w)}{q(w)} - 1}}{\|\psi\|_{L^{p(\cdot)}}^{\frac{p(w)}{q(w)} - 1}},
\end{aligned}$$

where $b = \alpha_- t_+ + (1 - \alpha_+) t_- - 1$.

Because $\frac{q(w)}{p(w)} = 1 + \frac{1}{(\alpha_- - \alpha_+ + \frac{1}{cq_+})} \cdot \frac{\alpha_- t_+ + (1 - \alpha_+) t_- - 1}{t_+}$ a.e, it implies that $\frac{bp(w)}{q(w)t_+} \cdot \frac{1}{(\alpha_- - \alpha_+ + \frac{1}{cq_+})} + \frac{p(w)}{q(w)} - 1 = 0$, and we get

$$r^{\frac{\alpha(t_+ - t_-) + t_- - 1}{t_+}} \mathcal{M}\psi(w) + r^{\alpha_+ - \alpha_- - \frac{1}{cq_+}} \|\psi\|_{L^{p(\cdot)}} = \frac{2}{(\mathcal{M}\psi(w))^{-\frac{p(w)}{q(w)}} \|\psi\|_{L^{p(\cdot)}}^{\frac{p(w)}{q(w)} - 1}} = 2 \frac{(\mathcal{M}\psi(w))^{\frac{p(w)}{q(w)}}}{\|\psi\|_{L^{p(\cdot)}}^{\frac{p(w)}{q(w)} - 1}} = 2 \left(\frac{\mathcal{M}\psi(w)}{\|\psi\|_{L^{p(\cdot)}}} \right)^{\frac{p(w)}{q(w)}} \|\psi\|_{L^{p(\cdot)}}. \quad (5.25)$$

Therefore, following (5.24), we get

$$\frac{|I_{\alpha(\cdot), \gamma} \psi| / C_5 \|K_{\alpha(\cdot), \gamma}\|_{L^{p(\cdot)}}^{1/t_+}}{\|\psi\|_{L^{p(\cdot)}}} \leq \left(\frac{\mathcal{M}\psi}{\|\psi\|_{L^{p(\cdot)}}} \right)^{\frac{p(w)}{q(w)}},$$

or

$$\left(\frac{|I_{\alpha(\cdot), \gamma} \psi| / C_5 \|K_{\alpha(\cdot), \gamma}\|_{L^{p(\cdot)}}^{1/t_+}}{\|\psi\|_{L^{p(\cdot)}}} \right)^{q(w)} \leq \left(\frac{\mathcal{M}\psi}{\|\psi\|_{L^{p(\cdot)}}} \right)^{p(w)}.$$

Therefore,

$$\rho_{q(\cdot)} \left(\frac{|I_{\alpha(\cdot), \gamma} \psi| / C_5 \|K_{\alpha(\cdot), \gamma}\|_{L^{p(\cdot)}}^{1/t_+}}{\|\psi\|_{L^{p(\cdot)}}} \right) \leq \rho_{p(\cdot)} \left(\frac{\mathcal{M}\psi}{\|\psi\|_{L^{p(\cdot)}}} \right). \quad (5.26)$$

Now, because $p_+ < \infty$ and $p(\cdot) \in LH(\mathbb{R}_+)$, the Hardy-Littlewood maximal operator \mathcal{M} is bounded on $L^{p(\cdot)}$; that is, for some constant $C_0 > 0$; we get, $\|\mathcal{M}\psi\|_{L^{p(\cdot)}} \leq C_0 \|\psi\|_{L^{p(\cdot)}}$. It can be written as $\|\mathcal{M}\psi\|_{L^{p(\cdot)}} \leq \|C_0 \psi\|_{L^{p(\cdot)}}$, as $L^{p(\cdot)}$ is a vector space, therefore, $C_0 \psi \in L^{p(\cdot)}$ and hence, we can choose $\psi \in L^{p(\cdot)}$ in a manner that $\|\mathcal{M}\psi\|_{L^{p(\cdot)}} \leq \|\psi\|_{L^{p(\cdot)}}$. Therefore

$$\|\psi\|_{L^{p(\cdot)}} \in \{j > 0 : \rho_{p(\cdot)}(\mathcal{M}\psi/j) \leq 1\}. \quad (5.27)$$

Consequently, from (5.26), we get

$$\rho_{q(\cdot)} \left(\frac{|I_{\alpha(\cdot), \gamma} \psi| / C_5 \|K_{\alpha(\cdot), \gamma}\|_{L^{p(\cdot)}}^{1/t_+}}{\|\psi\|_{L^{p(\cdot)}}} \right) \leq \rho_{p(\cdot)} \left(\frac{\mathcal{M}\psi}{\|\psi\|_{L^{p(\cdot)}}} \right) \leq 1.$$

Thus,

$$\|\psi\|_{L^{p(\cdot)}} \in \{j > 0 : \rho_{q(\cdot)} \left(\frac{|I_{\alpha(\cdot), \gamma} \psi| / C_5 \|K_{\alpha(\cdot), \gamma}\|_{L^{p(\cdot)}}^{1/t_+}}{j} \right) \leq 1\}.$$

Hence,

$$\left\| \frac{I_{\alpha(\cdot), \gamma} \psi}{C_5 \|K_{\alpha(\cdot), \gamma}\|_{L^{p(\cdot)}}^{1/t_+}} \right\|_{L^{q(\cdot)}} \leq \|\psi\|_{L^{p(\cdot)}},$$

or

$$\|I_{\alpha(\cdot),\gamma}\psi\|_{L^{q(\cdot)}} \leq C_5 \|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}^{1/t_+} \|\psi\|_{L^{p(\cdot)}}.$$

Case 2. Suppose that $\|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}} \geq 1$.

From (4.7), and by same method as in previous case, we obtain

$$\mathcal{M}\psi(w) \sum_{n=-\infty}^{G_r-1} \frac{(2^n r)^{\alpha_-}}{(1+2^n r)^\gamma} \leq C_6 r^{\frac{\alpha_- t_+ + (1-\alpha_+)t_- - 1}{t_+}} \mathcal{M}\psi(w) \|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}. \quad (5.28)$$

From (4.8), we get

$$\|\psi\|_{L^{p(\cdot)}} \sum_{n=G_r}^{\infty} \frac{(2^n r)^{\alpha_+ - \frac{1}{p_+}}}{(1+2^n r)^\gamma} \leq C_7 r^{\alpha_+ - \alpha_- - \frac{1}{cq_+}} \|\psi\|_{L^{p(\cdot)}} \|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}. \quad (5.29)$$

By (5.21), (5.28), and (5.29), we have

$$|I_{\alpha(\cdot),\gamma}\psi(w)| \leq C_8 \left(r^{\frac{\alpha_- t_+ + (1-\alpha_+)t_- - 1}{t_+}} \mathcal{M}\psi(w) + r^{\alpha_+ - \alpha_- - \frac{1}{cq_+}} \|\psi\|_{L^{p(\cdot)}} \right) \|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}. \quad (5.30)$$

Now, from (5.25), we obtain

$$\frac{|I_{\alpha(\cdot),\gamma}\psi(w)|/C_8 \|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}}{\|\psi\|_{L^{p(\cdot)}}} \leq \left(\frac{\mathcal{M}\psi(w)}{\|\psi\|_{L^{p(\cdot)}}} \right)^{\frac{p(w)}{q(w)}},$$

and thus,

$$\rho_{q(\cdot)} \left(\frac{|I_{\alpha(\cdot),\gamma}\psi/C_8 \|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}}{\|\psi\|_{L^{p(\cdot)}}} \right) \leq \rho_{p(\cdot)} \left(\frac{\mathcal{M}\psi}{\|\psi\|_{L^{p(\cdot)}}} \right).$$

Finally, it follows from (5.27) that we get $\rho_{p(\cdot)}(\mathcal{M}\psi/\|\psi\|_{L^{p(\cdot)}}) \leq 1$. Therefore,

$$\|I_{\alpha(\cdot),\gamma}\psi\|_{L^{q(\cdot)}} \leq C_8 \|\psi\|_{L^{p(\cdot)}} \|K_{\alpha(\cdot),\gamma}\|_{L^{t(\cdot)}}.$$

□

6. Conclusions

In this manuscript, we introduced and studied a new class of integral operators with a Bessel–Riesz kernel of variable-order and established their boundedness in variable Lebesgue spaces. We established boundedness in the diagonal case under suitable assumptions on the exponent function. For the nondiagonal setting, a pointwise estimate of the operator was derived based on the maximal operator and norm of function, which we applied to obtain boundedness results from $L^{p(\cdot)}(\mathbb{R}_+)$ to $L^{q(\cdot)}(\mathbb{R}_+)$. Furthermore, we established boundedness of the variable-order Bessel–Riesz operator in a form analogous to Young’s inequality for convolution. For this, we first determined sufficient conditions on the exponent function for the variable-order Bessel–Riesz kernel to belong to a variable Lebesgue space. These results lead to convolution-type estimates that provide boundedness of the operator between variable Lebesgue spaces under suitable assumptions on the exponents. Our results provide a systematic approach for analyzing integral operators in the setting of variable exponent function spaces, and these results are more general than those already established for classical Bessel–Riesz cases.

7. Discussion and future work

In this work, the assumptions imposed on exponent functions are standard and natural in the setting of variable exponent analysis. They are required to ensure the boundedness of the Hardy–Littlewood maximal operator, to apply the variable-exponent Hölder inequality, and to control the behavior of the variable-order kernel. Some of these conditions, in particular $p_- > 1$, are essential; removing them leads to well-known failures even for classical operators. Establishing boundedness under weaker assumptions on exponent functions remains an interesting open problem. Possible extensions include the study of endpoint cases, weaker regularity conditions on the exponents, and alternative function spaces.

Author contributions

The authors equally contributed to the research and manuscript preparation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors confirm no conflicts of interest.

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