Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(6): 17951–17971.
DOI: 10.3934/math.2026732
Received: 10 April 2026
Revised: 07 June 2026
Accepted: 15 June 2026
Published: 18 June 2026

Research article

Controlled 4-dimensional metric type spaces with fixed point results

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Abstract: In this paper, we introduce the concept of controlled four–dimensional metric type spaces, which generalizes the structure of four-dimensional metric spaces by incorporating a control function into the generalized triangle inequality. We investigate several fundamental properties of this newly defined space and present illustrative examples to demonstrate its structure. Moreover, we introduce the concept of α_q–admissible mappings and generalize Wardowski's contraction principle by formulating $(\alpha_q$-$F)$–contractive mappings within the framework of controlled four–dimensional metric type spaces. We also propose a contraction condition defined via an iterative function P, thereby extending classical contraction principles to this setting. We prove the existence and uniqueness of fixed points in complete controlled four–dimensional metric type spaces, thereby extending and enhancing several related results in the literature. In addition, illustrative examples are provided for each main theorem to demonstrate the applicability and validity of the obtained results. Furthermore, a non-trivial application to a four-component boundary value problem is presented in Section 5.

Keywords: fixed point; controlled metric type spaces; $(\alpha_q$-$\mathcal{F})$ -contraction mapping;
4- dimensional-metric spaces; controlled functions
Mathematics Subject Classification: 47H10, 47J26, 54H25

1. Introduction

Metric spaces play a fundamental role in numerous branches of mathematics, particularly in analysis and fixed-point theory. The concept of a metric space was first introduced by Fréchet in 1906 and has since become one of the most important mathematical structures for studying convergence, continuity, and topological properties. Due to its wide applicability, many researchers have proposed various generalizations of metric spaces to extend the scope of fixed-point theory and to model more complex systems.

One important development in this direction was the introduction of S–metric spaces by Sedghi et al. in 2012 [1]. Unlike classical metric spaces, where the distance is defined between two elements, S–metric spaces define the distance using three variables. This generalization opened new directions for research in nonlinear analysis and fixed-point theory. Following this development, several related structures were introduced, including S_b–metric spaces [2], extended S_b–metric spaces [3], extended S–metric spaces of type (α,β) [4], and triple controlled S–metric type spaces [5]. These extensions significantly enriched the theory of generalized metric spaces and allowed researchers to obtain many new fixed-point results under weaker conditions.

Motivated by these developments, the concept of distance defined on three variables was further extended to four variables. In this context, Sarma et al. [6] introduced the notion of four–dimensional metric spaces as a natural extension of S–metric spaces. The introduction of an additional variable provides a more flexible framework for studying relationships among multiple elements simultaneously. Such higher–dimensional metric structures can be useful in modeling situations where the distance between objects depends on several parameters, such as spatial location, time, or other contextual variables. Therefore, four–dimensional metric spaces provide a promising mathematical framework for studying more complex systems and for developing new results in fixed-point theory.

Despite the significance of these structures, the study of four-dimensional metric spaces remains relatively limited, and many of their fundamental properties have yet to be fully explored. In particular, controlled versions of such spaces have received little attention in the literature, and to the best of our knowledge, no work has been conducted in this direction.

Motivated by these considerations, this paper introduces the notion of controlled 4–dimensional metric type spaces, extending the class of four–dimensional metric type spaces presented in [6]. We examine several basic properties of this newly defined structure and include examples to illustrate its characteristics.

Furthermore, we present the concept of α_q–admissible mappings and generalize Wardowski's contraction principle by introducing $(\alpha_q\text{-}F)$–contractive mappings tailored to controlled four–dimensional metric type spaces. In addition, we propose a contraction condition defined through an iterative function P, thereby extending classical contraction principles within this framework. Based on these notions, we derive existence and uniqueness results for fixed points in complete controlled four–dimensional metric type spaces. Moreover, illustrative examples are provided for each main theorem to demonstrate the applicability and validity of the obtained results. Furthermore, a non-trivial application to a four-component boundary value problem is presented in Section 5.

2. Preliminaries

We begin by recalling the notion of S-metric spaces, originally introduced by Sedghi et al. [1].

Definition 1. *[1] Let* $\Upsilon : \Omega^3 \to [0,\infty)$ *be a mapping, with* $\Omega \neq \emptyset$*, such that for all* $x, \wp, \xi,$ *and* $a \in \Omega$*, the following holds:*

1) $\Upsilon(x,\wp,\xi) = 0$ *if and only if* $x = \wp = \xi$*;*
2) $\Upsilon(x,\wp,\xi) \leq \Upsilon(x,x,a) + \Upsilon(\wp,\wp,a) + \Upsilon(\xi,\xi,a)$.

The pair (Ω,Υ) *is called an S - metric space.*

In 2023, Ekiz et al. [7] proposed the concept of controlled S-metric type spaces.

Definition 2. *[7] Let $\Upsilon : \Omega^3 \to [0, \infty)$ be a mapping, where Ω is a nonempty set, and let $\alpha : \Omega^2 \to [1, \infty)$ be a function such that for every $x, \wp, \xi, a \in \Omega$, the following is satisfied:*

1) $\Upsilon(x, \wp, \xi) = 0$ *if and only if* $x = \wp = \xi$;
2) $\Upsilon(x, \wp, \xi) \leq \alpha(x, a)\Upsilon(x, x, a) + \alpha(\wp, a)\Upsilon(\wp, \wp, a) + \alpha(\xi, a)\Upsilon(\xi, \xi, a)$.

The pair (Ω, Υ) is called a controlled S-metric type space.

Next, we state the definition of the four-dimensional metric space as initiated in [6].

Definition 3. *[6] Let $\Omega \neq \emptyset$, and let $B_4 : \Omega^4 \to [0, \infty)$ be a mapping such that, for every x, y, z, w and $a \in \Omega$, the following conditions hold:*

1) $B_4(x, y, z, w) = 0$ *if and only if* $x = y = z = w$;
2) $B_4(x, y, z, w) \leq B_4(x, x, x, , a) + B_4(y, y, y, a) + B_4(z, z, z, a) + B_4(w, w, w, a)$.

Then, the pair (Ω, B_4) is called a 4- dimensional metric space.

Remark 1. *Let (Ω, Υ) be an S- metric space. Then, one can define the mapping $B_4 : \Omega^4 \to [0, \infty)$ by*

$$B_4(x, y, z, w) = S(y, z, w) + S(x, z, w) + S(x, y, w) + S(x, y, z), \text{ for all } x, y, z, w \in \Omega.$$

It can be shown that (Ω, B_4) is a 4- dimensional metric space. For more details, see [6].

We proceed to present several examples of four-dimensional metric spaces.

Example 1. *[6] Let $\Omega = \mathbb{R}$, and define the mapping $B_4 : \Omega^4 \to [0, \infty)$, for all $x, y, z, w \in \mathbb{R}$, by*

$$B_4(x, y, z, w) = |x - z| + |y - z| + |x + y + z - 3w|.$$

Then, (Ω, B_4) is a 4- dimensional metric space.

Example 2. *[8] Let $\Omega = \mathbb{N} \cup 0$, and let $B_4 : \Omega^4 \to [0, \infty)$ be defined by*

$$B_4(x, y, z, w) = \begin{cases} 0 & \text{if } x = y = z = w, \\ x^2 + y^2 + z^2 + w^2 & \text{otherwise.} \end{cases}$$

Then, (Ω, B_4) is a 4- dimensional metric space.

3. Results

We introduce the notion of a controlled four-dimensional metric type space, which serves as a meaningful extension of the classical framework. First, we present its special case, namely the four-dimensional b-metric space, as a natural generalization of a four-dimensional metric space obtained by relaxing the classical triangle inequality through the introduction of a constant $s \geq 1$ in place of the unity. We then introduce the controlled four-dimensional metric type space, which further extends this structure by incorporating a control function that governs the generalized metric inequality.

Definition 4. *Let $\Omega \neq \emptyset$ and let $\Upsilon : \Omega^4 \to [0, \infty)$ be a mapping. Let $s \geq 1$ be a real constant, and for every $x, y, z, w, a \in \Omega$, the following holds:*

1) $\Upsilon(x, y, z, w) = 0$ *if and only if* $x = y = z = w$*;*
2) $\Upsilon(x, x, x, y) = \Upsilon(y, y, y, x)$*;*
3)

$$\Upsilon(x, y, z, w) \leq s[\Upsilon(x, x, x, a) + \Upsilon(y, y, y, a) + \Upsilon(z, z, z, a) + \Upsilon(w, w, w, a)].$$

Then the pair (Ω, Υ) is called a 4-dimensional b-metric space, abbreviated by b4D-MS.

Now, we introduce the concept of a controlled 4-dimensional metric type space.

Definition 5. *Let $\Omega \neq \emptyset$, and let $\Upsilon : \Omega^4 \to [0, \infty)$ be a mapping and consider the function $\mu : \Omega^2 \to [1, \infty)$, so that for every x, y, z, w, and $a \in \Omega$, the following is satisfied:*

1) $\Upsilon(x, y, z, w) = 0$ *iff* $x = y = z = w$*;*
2) $\Upsilon(x, x, x, y) = \Upsilon(y, y, y, x)$*;*
3) $\Upsilon(x, y, z, w) \leq \mu(x, a)\Upsilon(x, x, x, , a) + \mu(y, a)\Upsilon(y, y, y, a) + \mu(z, a)\Upsilon(z, z, z, a) + \mu(w, a)\Upsilon(w, w, w, a).$

The pair (Ω, Υ) is called a controlled 4-dimensional metric type space, and it will be abbreviated by $C4D - MS$.

Remark 2. *Note that by taking the function $\mu(x, a) = s \geq 1$, for all $x, a \in \Omega$, in Definition 5, we obtain Definition 4.*

In what follows, we present several examples of controlled four-dimensional metric type spaces.

Example 3. *Let $\Omega = \mathbb{N}$, and define the mapping $\Upsilon : \Omega^4 \to [0, \infty)$ and $\mu : \Omega^2 \to [1, \infty)$ by:*

$$\Upsilon(x, y, z, w) = \begin{cases} 0 & \textit{if } x = y = z = w, \\ \frac{1}{x} & \textit{if } x = y = z \textit{ is even, and } w \textit{ is odd}, \\ \frac{1}{w} & \textit{if } w \textit{ is even, and } x = y = z \textit{ is odd}, \\ 1 & \textit{otherwise}. \end{cases}$$

$$\mu(x, a) = \begin{cases} x & \textit{if } x \textit{ is even, and } a \textit{ is odd}, \\ a & \textit{if } a \textit{ is even, and } x \textit{ is odd}, \\ 1 & \textit{otherwise}. \end{cases}$$

Condition (1) of Definition 5 follows easily; we will show conditions (2) and (3). For this, we will consider several cases listed in Table 1.

Table 1. All possible cases.

cases	x	y	z	w
1	even	even	even	even
2	even	even	even	odd
3	even	even	odd	odd
4	even	odd	odd	odd
5	odd	odd	odd	odd
6	odd	even	even	even
7	odd	odd	even	even
8	odd	odd	odd	even

For example, consider case (2), where $x = y = z$ is even, and w is odd. Suppose $x = y = z = 2$ while $w = 3$. Then $\Upsilon(2,2,2,3) = 1/2$, and $\Upsilon(3,3,3,2) = 1/2$ by definition. Hence, $\Upsilon(2,2,2,3) = \Upsilon(3,3,3,2)$. Similarly, the remaining cases for condition (2) can be verified in the same manner.

Moving to condition (3), we use the same chosen values $x = y = z = 2$ and $w = 3$, and suppose a is odd, for example $a = 5$. Then,

$$\begin{aligned}\frac{1}{2} = \Upsilon(2,2,2,3) &\le \mu(2,5)\Upsilon(2,2,2,5) + \mu(2,5)\Upsilon(2,2,2,5) \\ &\quad +\mu(2,5)\Upsilon(2,2,2,5) + \mu(3,5)\Upsilon(3,3,3,5). \\ &= 2(\frac{1}{2}) + 2(\frac{1}{2}) + 2(\frac{1}{2}) + 1(1) = 4.\end{aligned}$$

Next, suppose we take $x = y = z = 2$ and $w = 3$, but a is even, for example $a = 6$. Then,

$$\begin{aligned}\frac{1}{2} = \Upsilon(2,2,2,3) &\le \mu(2,6)\Upsilon(2,2,2,6) + \mu(2,6)\Upsilon(2,2,2,6) \\ &\quad +\mu(2,6)\Upsilon(2,2,2,6) + \mu(3,6)\Upsilon(3,3,3,6). \\ &= 1(1) + 1(1) + 1(1) + 6(1/6) = 4.\end{aligned}$$

It is straightforward to verify that all cases satisfy condition (3). Thus, (Ω, Υ) is a controlled 4-dimensional metric type space.

Note that this example defines a controlled four-dimensional metric type space, which is not a four-dimensional b-metric space.

Example 4. *Let $\Omega = \{0,1,2,3\}$, and define $\Upsilon : \Omega^4 \to [0,\infty)$ by*

$$\Upsilon(x,y,z,w) = \begin{cases} 0 & \text{if } x = y = z = w, \\ 1 & \text{if } x \neq y \neq z \neq w, \\ \frac{3}{2} & \text{if } x = y = z, z \neq w. \end{cases}$$

Define $\mu : \Omega^2 \to [1,\infty)$ by , $\mu(x,y) = 1 + y + x$.

One can easily see that (Ω, Υ) is a $C4D - MS$.

We proceed to examine the topological properties of $C4D-MS$, focusing on the concepts of Cauchy and convergent sequences, completeness, and open balls.

Definition 6. *Let (Ω, Υ) be a $C4D-MS$, and let $\{x_n\}$ be any sequence in Ω:*

1) *We say that the sequence $\{x_n\}$ converges to w in Ω, if $\Upsilon(x_n, x_n, x_n, w) \to 0$, as $n \to \infty$. Furthermore, by symmetry, this also implies that $\Upsilon(w, w, w, x_n) \to 0$, as $n \to \infty$.*

2) *The sequence $\{x_n\}$ is referred to as a Cauchy sequence if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\Upsilon(x_n, x_n, x_n, x_m) < \varepsilon$ for all $m, n \geq N$.*

3) *The space (Ω, Υ) is called complete if every Cauchy sequence in Ω is convergent.*

4) *Let $x \in \Omega$ and $r > 0$. The open ball centered at x with radius r is defined by*

$$B(x, r) = \{y \in \Omega : \Upsilon(x, x, x, y) < r\}.$$

5) *Let $x \in \Omega$ and $r > 0$. The closed ball centered at x with radius r is defined by*

$$\overline{B}(x, r) = \{y \in \Omega : \Upsilon(x, x, x, y) \leq r\}.$$

6) *A mapping $f : (\Omega, \Upsilon) \to (\Omega, \Upsilon)$ is said to be continuous at $x \in \Omega$, if for every sequence $\{x_n\}$ in Ω such that $x_n \to x$, we have*

$$f(x_n) \to f(x).$$

The following lemma demonstrates the uniqueness of the limit of a convergent sequence in $C4D-MS$.

Lemma 5. *Let (Ω, Υ) be a $C4D-MS$ and let $\{x_n\}$ be a sequence in Ω. Suppose that $\{x_n\}$ converges to both x and y. Furthermore, assume that $\lim_{n\to\infty} \mu(x, x_n)$ and $\lim_{n\to\infty} \mu(y, x_n)$ exist and are finite. Then $x = y$.*

Proof.

$$\begin{aligned} \Upsilon(x, x, x, y) &\leq \mu(x, x_n)\Upsilon(x, x, x, x_n) + \mu(x, x_n)\Upsilon(x, x, x, x_n) \\ &+ \mu(x, x_n)\Upsilon(x, x, x, x_n) + \mu(y, x_n)\Upsilon(y, y, y, x_n). \end{aligned} \tag{3.1}$$

Passing to the limit as $n \to \infty$ in equation 3.1, we deduce that $x = y$. □

Samet et al. [9] originally introduced the class of α-admissible mappings in metric spaces. We now extend this notion by defining the concept of α_q-admissible mapping in the context of a controlled four-dimensional metric type space (Ω, Υ).

Definition 7. *Let $T : \Omega \longrightarrow \Omega$ be a mapping and consider the function $\alpha_q : \Omega^4 \to [0, \infty)$, where Ω is a nonempty set. The mapping T is called α_q-admissible if, for every $\hat{x}, \hat{y}, \hat{z}, \hat{w} \in \Omega$, we have*

$$\alpha_q(\hat{x}, \hat{y}, \hat{z}, \hat{w}) \geq 1 \Longrightarrow \alpha_q(T\hat{x}, T\hat{y}, T\hat{z}, T\hat{w}) \geq 1. \tag{3.2}$$

Example 6. *Assume that* $\Omega = [0, \infty)$. *Let the mapping* $T : \Omega \longrightarrow \Omega$ *and the function* $\alpha_q : \Omega^4 \rightarrow [0, \infty)$ *be defined by* $T(x) = ln(x + 2)$, *for all* $x \in \Omega$, *and* $\alpha_q(x, y, z, w) = e$ *if* $x \geq y \geq z \geq w$, *and* $\alpha_q(x, y, z, w) = 0$ *otherwise.*

Suppose that $x \geq y \geq z \geq w$. *Then* $\alpha_q(x, y, z, w) = e \geq 1$. *Since* $ln(x)$ *is a strictly increasing function, it follows that*

$$x \geq y \geq z \geq w \implies ln(x + 2) \geq ln(y + 2) \geq ln(z + 2) \geq ln(w + 2).$$

Therefore, we obtain $\alpha_q(Tx, Ty, Tz, Tw) = e \geq 1$. *Thus,* T *is* α_q*-admissible.*

Wardowski [10] proposed the notion of an $\mathcal{F}$–contraction and established several novel fixed point results within the framework of complete metric spaces. The corresponding definition is given below.

Definition 8. *[10] Let* $\mathcal{F}$ *denote the family of all functions* $F : (0, \infty) \rightarrow (-\infty, \infty)$ *meeting the following conditions:*
(W1) The function F *is strictly increasing.*
(W2) Let $\{s_n\}$ *be any sequence of positive real numbers; then*

$$\lim_{n\to\infty} s_n = 0 \text{ iff } \lim_{n\to\infty} F(s_n) = -\infty.$$

(W3) $\lim_{s\to 0^+} s^m F(s) = 0$, *for some* $m \in (0, 1)$.

Example 7. *Consider the functions* $G(t) = ln(t)$, *and* $K(t) = \frac{-1}{\sqrt{t}}$, *for* $t > 0$. *Then both functions* $G(t)$, *and* $K(t)$ *fulfill conditions* $(W1), (W2)$, *and* $(W3)$; *hence, they belong to* $\mathcal{F}$. *For additional details, see [10].*

Numerous authors have adapted the $\mathcal{F}$-contraction mapping introduced by Wardowski [10] to various metric space settings (see, e.g., [11–13]). Here, we propose a new modified $\mathcal{F}$-contraction mapping specifically tailored to the controlled four-dimensional metric space $C4D$-MS, highlighting its suitability for this framework.

Definition 9. *Let* (Ω, Υ) *be a* $C4D - MS$, *with* $\Omega \neq \emptyset$. *A mapping* $T : \Omega \rightarrow \Omega$ *is said to be a modified* $\mathcal{F}$*-contraction mapping if there exist a function* $F \in \mathcal{F}$ *and a constant* $\tau > 0$ *such that the following condition holds:*

$$\Upsilon(Tx, Ty, Tz, Tw) > 0 \implies \tau + F(\Upsilon(Tx, Ty, Tz, Tw)) \leq F(\Upsilon(x, y, z, w)), \tag{3.3}$$

for all $x, y, z, w \in \Omega$.

We now introduce a new notion of (α_q-$\mathcal{F}$)-contractive mappings in the setting of $C4D - MS$ space.

Definition 10. *Consider a* $C4D - MS$ (Ω, Υ), *where* $\Omega \neq \emptyset$. *A mapping* $T : \Omega \rightarrow \Omega$ *is said to be an* (α_q-$\mathcal{F}$)*-contraction mapping if there exists a function* $\alpha_q : \Omega^4 \rightarrow [0, \infty)$, *a function* $F \in \mathcal{F}$, *and a constant* $\tau > 0$ *such that the following holds;*

$$\tau + \alpha_q(x, y, z, w)F(\Upsilon(Tx, Ty, Tz, Tw)) \leq F(\Upsilon(x, y, z, w)), \tag{3.4}$$

for $x, y, z, w \in \Omega$, *with* $\Upsilon(Tx, Ty, Tz, Tw) > 0$.

4. Main results

In this section, we pursue two main objectives. We begin by proving the existence and uniqueness of fixed points in a complete four-dimensional metric type space (Ω, Υ) using $(\alpha_q\text{-}\mathcal{F})$-contractive mappings. Second, we introduce a contraction condition via an iterative function P, thereby extending classical contraction principles to controlled four-dimensional metric type spaces. In addition, we present illustrative examples corresponding to our main theorems.

Theorem 8. *Let (Ω, Υ) be a complete $C4D - MS$, where $\Omega \neq \emptyset$. Let $T : \Omega \to \Omega$ be an $(\alpha_q\text{-}\mathcal{F})$-contractive mapping. Suppose the following conditions are satisfied:*

(1) There exists $x_0 \in \Omega$, such that $\alpha_q(x_0, x_0, x_0, Tx_0) \geq 1$.
(2) The mapping T is α_q-admissible.
(3) For $x_0 \in \Omega$, the sequence $\{x_n\}$ is formed by $x_n = T^n x_0$; furthermore, the following holds:

$$\sup_{m\geq 1} \lim_{n\to\infty} \frac{\mu(x_{n+1}, x_m)\mu(x_{n+1}, x_{n+2})}{\mu(x_n, x_{n+1})} < 1. \tag{4.1}$$

Moreover,

$$\text{for every } x \in \Omega,\ \lim_{n\to\infty} \mu(x, x_n) \text{ and } \lim_{n\to\infty} \mu(x_n, x) \text{ exist and finite}. \tag{4.2}$$

Then T admits a fixed point. Moreover, if u and v are two fixed points of T in Ω such that $\alpha_q(u, u, u, v) \geq 1$, then the fixed point is unique in Ω.

Proof. Select $x_0 \in \Omega$, so $\alpha_q(x_0, x_0, x_0, Tx_0) \geq 1$. Consider the sequence $\{x_n\}$ defined by $x_1 = Tx_0, x_2 = Tx_1 = T^2 x_0$, and so on. Then, for any $n \in \mathbb{N}$, it holds that

$$x_n = Tx_{n-1} = T^n x_0.$$

Moreover, $T^n x_0 \neq T^{n+1} x_0$ true for all $n \geq 0$.

Since the mapping T is α_q-admissible, this implies $\alpha_q(x_n, x_n, x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$. The fact T is an $(\alpha_q\text{-}\mathcal{F})$-contraction mapping, thus we have

$$\begin{aligned}
\tau + F(\Upsilon(x_n, x_n, x_n, x_{n+1})) &= \tau + F(\Upsilon(Tx_{n-1}, Tx_{n-1}, Tx_{n-1}, Tx_n)). \\
&\leq \tau + \alpha_q(x_{n-1}, x_{n-1}, x_{n-1}, x_n) F(\Upsilon(Tx_{n-1}, Tx_{n-1}, Tx_{n-1}, Tx_n)). \\
&\leq F(\Upsilon(x_{n-1}, x_{n-1}, x_{n-1}, x_n)).
\end{aligned}$$

This yields,

$$\begin{aligned}
F(\Upsilon(x_n, x_n, x_n, x_{n+1})) &\leq F(\Upsilon(x_{n-1}, x_{n-1}, x_{n-1}, x_n)) - \tau. \\
&\leq F(\Upsilon(x_{n-2}, x_{n-2}, x_{n-2}, x_{n-1})) - 2\tau. \\
&\leq \cdots \leq F(\Upsilon(x_0, x_0, x_0, x_1)) - n\tau.
\end{aligned} \tag{4.3}$$

As $n \to \infty$ in equation 4.3, with the condition $\tau > 0$, we obtain

$$\lim_{n\to\infty} F(\Upsilon(x_n, x_n, x_n, x_{n+1})) = -\infty. \tag{4.4}$$

Since $F \in \mathcal{F}$, utilizing (W2) of Definition 8, we deduce that $\lim_{n\to\infty} \Upsilon(x_n, x_n, x_n, x_{n+1}) = 0$, and by (W3), there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty}(\Upsilon(x_n, x_n, x_n, x_{n+1}))^k F((\Upsilon(x_n, x_n, x_n, x_{n+1})) = 0. \tag{4.5}$$

From Eq (4.3), we obtain

$$F(\Upsilon(x_n, x_n, x_n, x_{n+1})) - F(\Upsilon(x_0, x_0, x_0, x_1)) \leq -n\tau.$$

Thus for any n, we have

$$\begin{aligned}(\Upsilon(x_n, x_n, x_n, x_{n+1}))^k F(\Upsilon(x_n, x_n, x_n, x_{n+1}) - (\Upsilon(x_n, x_n, x_n, x_{n+1})^k F(\Upsilon(x_0, x_0, x_0, x_1))\\ \leq -n\tau(\Upsilon(x_n, x_n, x_n, x_{n+1}))^k \leq 0.\end{aligned} \tag{4.6}$$

Letting $n \to \infty$ in Eq (4.6), we obtain

$$\lim_{n\to\infty} n(\Upsilon(x_n, x_n, x_n, x_{n+1}))^k = 0. \tag{4.7}$$

This yields, $\lim_{n\to\infty} n^{1/k}(\Upsilon(x_n, x_n, x_n, x_{n+1})) = 0$. Thus, there exists an $n_0 \in \mathbb{N}$, such that

$$\Upsilon(x_n, x_n, x_n, x_{n+1}) \leq \frac{1}{n^{1/k}}, \text{ for every } n \geq n_0. \tag{4.8}$$

To show $\{x_n\}$ is a Cauchy sequence, let $m, n \in \mathbb{N}$ with $n < m$. Then we obtain:

$$\begin{aligned}\Upsilon(x_n, x_n, x_n, x_m) &\leq \mu(x_n, x_{n+1})\Upsilon(x_n, x_n, x_n, x_{n+1}) + \mu(x_n, x_{n+1})\Upsilon(x_n, x_n, x_n, x_{n+1})\\ &+ \mu(x_n, x_{n+1})\Upsilon(x_n, x_n, x_n, x_{n+1}) + \mu(x_{n+1}, x_m)\Upsilon(x_{n+1}, x_{n+1}, x_{n+1}, x_m).\\ &\leq 3\mu(x_n, x_{n+1})\Upsilon(x_n, x_n, x_n, x_{n+1})\\ &+ \mu(x_{n+1}, x_m)\Big[3\mu(x_{n+1}, x_{n+2})\Upsilon(x_{n+1}, x_{n+1}, x_{n+1}, x_{n+2})\\ &+ \mu(x_{n+2}, x_m)\Upsilon(x_{n+2}, x_{n+2}, x_{n+2}, x_m)\Big].\\ &\leq 3\mu(x_n, x_{n+1})\Upsilon(x_n, x_n, x_n, x_{n+1})\\ &+ 3\mu(x_{n+1}, x_m)\mu(x_{n+1}, x_{n+2})\Upsilon(x_{n+1}, x_{n+1}, x_{n+1}, x_{n+2})\\ &+ \mu(x_{n+1}, x_m)\mu(x_{n+2}, x_m)\Big[3\mu(x_{n+2}, x_{n+3})\Upsilon(x_{n+2}, x_{n+2}, x_{n+2}, x_{n+3})\\ &+ \mu(x_{n+3}, x_m)\Upsilon(x_{n+3}, x_{n+3}, x_{n+3}, x_m)\Big].\\ &\vdots\end{aligned}$$

$$\begin{aligned}\Upsilon(x_n, x_n, x_n, x_m) \leq\ & 3\,\mu(x_n, x_{n+1})\Upsilon(x_n, x_n, x_n, x_{n+1})\\ &+ 3\sum_{i=n+1}^{m-2} \mu(x_i, x_{i+1})\Upsilon(x_i, x_i, x_i, x_{i+1})\left(\prod_{j=n+1}^{i} \mu(x_j, x_m)\right)\\ &+ \prod_{i=n+1}^{m-1} \mu(x_i, x_m)\Upsilon(x_{m-1}, x_{m-1}, x_{m-1}, x_m).\end{aligned}$$

$$\Upsilon(x_n, x_n, x_n, x_m) \leq 3\,\mu(x_n, x_{n+1})\Upsilon(x_n, x_n, x_n, x_{n+1}) + 3\sum_{i=n+1}^{m-1}[\,\mu(x_i, x_{i+1})]\Upsilon(x_i, x_i, x_i, x_{i+1})\left(\prod_{j=n+1}^{i}\mu(\,x_j, x_m)\right). \tag{4.9}$$

Substituting Eq (4.8) into inequality (4.9) yields

$$\begin{aligned}\Upsilon(\,x_n,\ x_n, x_n,\ x_m) &\leq 3\,\mu(x_n, x_{n+1})(\frac{1}{n^{1/k}})\\ &+ \sum_{i=n+1}^{m-1}[3\mu(x_i, x_{i+1})](\frac{1}{i^{1/k}})\left(\prod_{j=n+1}^{i}\mu(x_j, x_m)\right).\\ &\leq [3\,\mu(x_n, x_{n+1})(\frac{1}{n^{1/k}})\\ &+ \sum_{i=1}^{m-1}[\,3\mu(\,x_i,\ x_{i+1})](\frac{1}{i^{1/k}})\left(\prod_{j=1}^{i}\mu(x_j, x_m)\right).\\ &\leq [3\,\mu(x_n, x_{n+1})(\frac{1}{n^{1/k}})\\ &+ \sum_{i=1}^{\infty}[\,3\mu(\,x_i,\ x_{i+1})](\frac{1}{i^{1/k}})\left(\prod_{j=1}^{i}\mu(x_j, x_m)\right).\end{aligned} \tag{4.10}$$

We aim to show

$$\lim_{n,m\to\infty}\Upsilon(\,x_n,\ x_n, x_n,\ x_m) = 0.$$

For the first part, by Eq (4.2), we obtain

$$\lim_{n\to\infty} 3\,\mu(x_n, x_{n+1})(\frac{1}{n^{1/k}}) = 0. \tag{4.11}$$

Next, we show that the series $\sum_{i=1}^{\infty}[\,3\mu(\,x_i,\ x_{i+1})](\frac{1}{i^{1/k}})\left(\prod_{j=1}^{i}\mu(x_j, x_m)\right)$ is absolutely convergent using the ratio test.

Let

$$a_i = 3\mu(\,x_i,\ x_{i+1})](\frac{1}{i^{1/k}})\left(\prod_{j=1}^{i}\mu(x_j, x_m)\right).$$

Then

$$a_{i+1} = 3\mu(\,x_{i+1},\ x_{i+2})](\frac{1}{(i+1)^{1/k}})\left(\prod_{j=1}^{i+1}\mu(x_j, x_m)\right).$$

Thus,

$$\frac{a_{i+1}}{a_i} = \left(\frac{\mu(\,x_{i+1},\ x_{i+2})\mu(x_{i+1}, x_m)}{\mu(\,x_i,\ x_{i+1})}\right)(\frac{i}{i+1})^{1/k}.$$

Hence, by the ratio test along with Eq (4.1), we deduce

$$sup_m \lim_{i\to\infty}\Big(\frac{\mu(x_{i+1},\ x_{i+2})\mu(x_{i+1}, x_m)}{\mu(x_i,\ x_{i+1})}\Big)\big(\frac{i}{i+1}\big)^{1/k} < 1. \tag{4.12}$$

Therefore, by Eqs (4.11) and (4.12), we deduce

$$\lim_{n,m\to\infty} \Upsilon(x_n, x_n, x_n, x_m) = 0.$$

We conclude that $\{x_n\}$ is a Cauchy sequence. Since $(\Omega,\ \Upsilon)$ is complete, the sequence converges to some $u \in \Omega$, that is,

$$\lim_{n\to\infty} \Upsilon(x_n, x_n, x_n, u) = 0. \tag{4.13}$$

In the following paragraph, we will illustrate that u is a fixed point of T, i.e., $u = Tu$. Initially, we illustrate that $\lim_{n\to\infty} \Upsilon(Tx_n, Tx_n, Tx_n, Tu) = 0$.

Assume $\Upsilon(T\,x_n,\ T\,x_n, Tx_n,\ Tu) > 0$, for any n. Thus, Definition 9 gives

$$\tau + F(\Upsilon(Tx_n, Tx_n, Tx_n, Tu)) \le F(\Upsilon(x_n, x_n, x_n, u)). \tag{4.14}$$

Letting $n \to \infty$ in Eq (4.14), and applying Eq (4.13) along with conditions and (W2) from Definition 8, we get

$$\lim_{n\to\infty} F(\Upsilon(Tx_n, Tx_n, Tx_n, Tu)) = -\infty.$$

Consequently, by Definition 8, it follows that $\lim_{n\to\infty} \Upsilon(Tx_n, Tx_n, Tx_n, Tu) = 0$.
To verify that u is a fixed point, note that;

$$\begin{aligned}\Upsilon(\,Tu,\ Tu, Tu,\ u) = \Upsilon(\,u,\ u, u,\ Tu) &\le 3\mu(\,u,\ x_{n+1})\Upsilon(\,u,\ u, u,\ x_{n+1})\\ &\quad + \mu(Tu, x_{n+1})\Upsilon(Tu, Tu, Tu, Tx_n).\\ &\le 3\mu(u,\ x_{n+1})\Upsilon(u,\ u, u,\ x_{n+1})\\ &\quad + \mu(Tu, x_{n+1})\Upsilon(\,Tx_n,\ Tx_n, Tx_n,\ Tu).\end{aligned}$$

Letting $n \to \infty$, in the preceding inequality, we conclude that $\Upsilon(\,Tu,\ Tu, Tu,\ u) = 0$, which shows that $Tu\ =\ u$. Next, we address the uniqueness of the fixed point. Assume there exist two fixed points, u and v, such that u and v are not equal, and suppose $\alpha(\,u,\ u, u,\ v) \ge 1$. Since $u = Tu \neq Tv = v$, it follows that $\Upsilon(\,Tu,\ Tu,\ Tu, Tv\,) > 0$.

Given that T is a $(\alpha_q\text{-}\mathcal{F})$-contractive mapping, utilizing Eq (3.4), we obtain

$$\begin{aligned}\tau + F(\Upsilon(\,Tu,\ Tu, Tu,\ Tv)) &\le \tau + \alpha_q(\,u,\ u, u, v)F(\Upsilon(\,Tu,\ Tu, Tu, Tv)).\\ &\le F(\Upsilon(u, u, u, v)) = F(\Upsilon(\,Tu,\ Tu, Tu,\ Tv))\end{aligned}$$

This leads to $\tau\ \le\ 0$, which is a contradiction. Hence, $u\ =\ v$, establishing the uniqueness of the fixed point. □

Example 9. *Let* $\Omega = [0, \infty)$*, and consider the mapping* $\Upsilon : \Omega^4 \to [0, \infty)$*, defined by*

$$\Upsilon(x, y, z, w) = |x - z| + |y - z| + |x + y + z - 3w|.$$

Define

$$\mu(x,y) = \lambda(1+x+y), \qquad 0<\lambda<1.$$

Then, (Ω,Υ) *is a complete* $C4D-MS$*. Define the mapping* $T:\Omega\to\Omega$ *by* $T(x)=\frac{x}{3}$*. Furthermore, let* $\alpha_q:\Omega^4\to(-\infty,\infty)$*, and* $F:(0,\infty)\to(-\infty,\infty)$ *be defined by* $F(t)=Ln(t)$*, and*

$$\alpha_q(x,y,z,w) = \begin{cases} 1 & \text{if } x,y,z,w\in[0,1], \\ 0 & \text{otherwise}. \end{cases}$$

First, we show that T *is an* (α_q-$\mathcal{F}$)*-contractive mapping. Let* x, y, z*, and* $w\in\Omega$ *be arbitrary,*

$$\begin{aligned} \Upsilon(Tx,Ty,Tz,Tw) &= |Tx-Tz|+|Ty-Tz|+|Tx+Ty+Tz-T(3w)|. \\ &= \frac{1}{3}|x-z|+\frac{1}{3}|y-z|+\frac{1}{3}|x+y+z-3w|. \\ &= \frac{1}{3}\Upsilon(x,y,z,w)\le\frac{2}{3}\Upsilon(x,y,z,w). \end{aligned} \tag{4.15}$$

Thus,

$$\begin{aligned} \alpha_q(x,y,z,w)(ln(\frac{3}{2})+ln(\Upsilon(Tx,Ty,Tz,Tw))) &\le ln(\frac{3}{2})+\alpha_q(x,y,z,w)ln(\Upsilon(Tx,Ty,Tz,Tw)). \\ &\le ln(\Upsilon(x,y,z,w)), \text{ by Eq (4.15)}. \end{aligned}$$

We have shown that

$$\tau+\alpha_q(x,y,z,w)F(\Upsilon(Tx,Ty,Tz,Tw))\le F(\Upsilon(x,y,z,w)),$$

where $\tau=ln(\frac{3}{2})>0$*. Therefore,* T *is an* (α_q-$\mathcal{F}$)*-contractive mapping.*

Now, choose $x_0\in\Omega$*. By the assumptions of Theorem 8, we require* $\alpha_q(x_0,x_0,x_0,Tx_0)\ge1$*, which implies that* $x_0\in[0,1]$*. Moreover,*

$$x_n = T^n x_0 = \frac{x_0}{3^n}.$$

Then

$$x_n\to0 \qquad \text{as } n\to\infty.$$

Now,

$$\mu(x_n,x_{n+1}) = \lambda(1+\frac{x_0}{3^n}+\frac{x_0}{3^{n+1}}), \text{ and } \mu(x_{n+1},x_m)=\lambda(1+\frac{x_0}{3^{n+1}}+\frac{x_0}{3^m}).$$

we obtain

$$\sup_{m\ge1}\lim_{n\to\infty}\frac{\mu(x_{n+1},x_m)\mu(x_{n+1},x_{n+2})}{\mu(x_n,x_{n+1})}=\lambda<1.$$

Thus, condition (4.1) *is satisfied.*

Moreover, for every $x\in\Omega$*,*

$$\lim_{n\to\infty}\mu(x,x_n) \quad \text{and} \quad \lim_{n\to\infty}\mu(x_n,x),$$

exist and are finite. Hence, condition (4.2) *holds.*

Therefore, all hypotheses of Theorem 8 are satisfied. Consequently, T *has a fixed point in* Ω*.*

Next, we introduce a contraction-type condition expressed through an iterative function P. This condition allows us to extend classical contraction principles to the framework of controlled four-dimensional metric spaces. The following theorem establishes the existence and uniqueness of fixed points for mappings satisfying this contractive condition.

Theorem 10. *Let* (Ω, Υ) *be a complete controlled* 4*–dimensional metric type space and let* $g : \Omega \to \Omega$ *be a continuous mapping satisfying*

$$\Upsilon(g(x), g(y), g(z), g(w)) \leq P(\Upsilon(x, y, z, w)), \tag{4.16}$$

for every $x, y, z, w \in \Omega$*, where* $P : [0, +\infty) \to [0, +\infty)$ *is a strictly increasing function such that*

$$\lim_{n\to\infty} P^n(s) = 0,$$

for every $s \geq 0$*.*

For each $x \in \Omega$*, define*

$$M(\Upsilon, g, x) = \sup\{\Upsilon(x, x, x, g^j x) : j \in \mathbb{N} \cup \{0\}\}.$$

If there exists $x_0 \in \Omega$ *for which* $M(\Upsilon, g, x_0)$ *is finite, then g possesses a unique fixed point in* Ω*.*

Proof. Let $x_0 \in \Omega$ be such that $M(\Upsilon, g, x_0) < \infty$, and define the Picard sequence $\{x_n\}$ by

$$x_n = g^n(x_0), \qquad n \in \mathbb{N} \cup \{0\}.$$

We first prove that $\{x_n\}$ is a Cauchy sequence.

For every $n \geq 1$ and every $j \in \mathbb{N} \cup \{0\}$, by (4.16), we have

$$\Upsilon(x_n, x_n, x_n, x_{n+j}) = \Upsilon(g(x_{n-1}), g(x_{n-1}), g(x_{n-1}), g(x_{n+j-1})) \leq P\Big(\Upsilon(x_{n-1}, x_{n-1}, x_{n-1}, x_{n+j-1})\Big).$$

Repeating this argument n times, we obtain

$$\Upsilon(x_n, x_n, x_n, x_{n+j}) \leq P^n\Big(\Upsilon(x_0, x_0, x_0, x_j)\Big).$$

Since $x_j = g^j(x_0)$, from the definition of $M(\Upsilon, g, x_0)$, it follows that

$$\Upsilon(x_0, x_0, x_0, x_j) \leq M(\Upsilon, g, x_0).$$

Hence,

$$\Upsilon(x_n, x_n, x_n, x_{n+j}) \leq P^n(M(\Upsilon, g, x_0)).$$

Because $\lim_{n\to\infty} P^n(s) = 0$ for every $s \geq 0$, we deduce that

$$\lim_{n\to\infty} \Upsilon(x_n, x_n, x_n, x_{n+j}) = 0$$

for every fixed $j \in \mathbb{N} \cup \{0\}$.

Now let $m > n$. We use the controlled 4-dimensional metric condition (3) in Definition 5 with

$$x = x_n, \qquad y = x_n, \qquad z = x_n, \qquad w = x_m, \qquad a = x_{n+p},$$

where $p \in \mathbb{N}$ is fixed. Then,

$$\Upsilon(x_n, x_n, x_n, x_m) \leq \mu(x_n, x_{n+p})\Upsilon(x_n, x_n, x_n, x_{n+p}) + \mu(x_n, x_{n+p})\Upsilon(x_n, x_n, x_n, x_{n+p}) + \mu(x_n, x_{n+p})\Upsilon(x_n, x_n, x_n, x_{n+p}) + \mu(x_m, x_{n+p})\Upsilon(x_m, x_m, x_m, x_{n+p}).$$

That is,

$$\Upsilon(x_n, x_n, x_n, x_m) \leq 3\mu(x_n, x_{n+p})\Upsilon(x_n, x_n, x_n, x_{n+p}) + \mu(x_m, x_{n+p})\Upsilon(x_m, x_m, x_m, x_{n+p}).$$

Using condition (2) of Definition 5, we have

$$\Upsilon(x_m, x_m, x_m, x_{n+p}) = \Upsilon(x_{n+p}, x_{n+p}, x_{n+p}, x_m).$$

Therefore,

$$\Upsilon(x_n, x_n, x_n, x_m) \leq 3\mu(x_n, x_{n+p})\Upsilon(x_n, x_n, x_n, x_{n+p}) + \mu(x_m, x_{n+p})\Upsilon(x_{n+p}, x_{n+p}, x_{n+p}, x_m).$$

Now, from the first part already proved,

$$\Upsilon(x_n, x_n, x_n, x_{n+p}) \leq P^n(M(\Upsilon, g, x_0)) \to 0 \qquad \text{as } n \to \infty.$$

Similarly,

$$\Upsilon(x_{n+p}, x_{n+p}, x_{n+p}, x_m) \leq P^{n+p}(M(\Upsilon, g, x_0)) \to 0 \qquad \text{as } n \to \infty.$$

Hence, assuming that the control functions $\mu(x_n, x_{n+p})$ and $\mu(x_m, x_{n+p})$ remain finite, we obtain

$$\Upsilon(x_n, x_n, x_n, x_m) \to 0 \qquad \text{as } n, m \to \infty.$$

Thus $\{x_n\}$ is a Cauchy sequence in Ω.

As (Ω, Υ) is complete, there exists $u \in \Omega$ such that $x_n \to u$. By the continuity of g, it follows that

$$g(x_n) \to g(u).$$

But $g(x_n) = x_{n+1}$ for all n, and since $\{x_n\}$ converges to u, its tail $\{x_{n+1}\}$ also converges to u. Therefore,

$$g(u) = u.$$

So u is a fixed point of g.

To prove uniqueness, let $u, v \in \Omega$ be fixed points of g. Then,

$$u = g(u), \qquad v = g(v).$$

Using (4.16), we have

$$\Upsilon(u, u, u, v) = \Upsilon(g(u), g(u), g(u), g(v)) \leq P(\Upsilon(u, u, u, v)).$$

Iterating,

$$\Upsilon(u, u, u, v) \leq P^n(\Upsilon(u, u, u, v))$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$, we obtain

$$\Upsilon(u, u, u, v) = 0.$$

By condition (1) of Definition 5, it follows that

$$u = v.$$

Hence, g has a unique fixed point in Ω. □

Example 11. *Let $\Omega = \mathbb{R}$, and consider the mapping $\Upsilon : \Omega^4 \to [0, \infty)$, defined by*

$$\Upsilon(x, y, z, w) = |x - w| + |y - w| + |z - w|.$$

Define the control function

$$\mu : \Omega^2 \to [1, \infty)$$

by

$$\mu(x, a) = 1 + x^2 + a^2.$$

Then, (Ω, Υ) is a complete controlled 4-dimensional metric type space.
Indeed,

1)

$$\Upsilon(x, y, z, w) = 0 \text{ if and only if } x = y = z = w.$$

2)

$$\Upsilon(x, x, x, y) = 3|x - y| = 3|y - x| = \Upsilon(y, y, y, x).$$

3) For every $x, y, z, w, a \in \Omega$,

$$\begin{aligned}\Upsilon(x, y, z, w) &= |x - w| + |y - w| + |z - w| \\ &\le (|x - a| + |a - w|) + (|y - a| + |a - w|) \\ &\quad + (|z - a| + |a - w|) \\ &\le \mu(x, a)\Upsilon(x, x, x, a) + \mu(y, a)\Upsilon(y, y, y, a) \\ &\quad + \mu(z, a)\Upsilon(z, z, z, a) + \mu(w, a)\Upsilon(w, w, w, a).\end{aligned}$$

Hence, Definition 3.2 is satisfied, and (Ω, Υ) is a complete controlled 4-dimensional metric type space.
Now define the function $g : \Omega \to \Omega$ by $g(x) = \frac{x}{3}$.
Choose

$$P : [0, \infty) \to [0, \infty), \qquad P(t) = \frac{t}{3}.$$

Clearly, P is strictly increasing and for each t,

$$P^n(t) = \frac{t}{3^n} \to 0 \qquad (n \to \infty).$$

Next,

$$\begin{aligned}\Upsilon(g(x), g(y), g(z), g(w)) &= \left|\frac{x}{3} - \frac{w}{3}\right| + \left|\frac{y}{3} - \frac{w}{3}\right| + \left|\frac{z}{3} - \frac{w}{3}\right| \\ &= \frac{1}{3}\Big(|x-w| + |y-w| + |z-w|\Big) \\ &= \frac{1}{3}\,\Upsilon(x, y, z, w) \\ &= P(\Upsilon(x, y, z, w)).\end{aligned}$$

Thus, condition 4.16 in Theorem 10 holds.

Finally, for $x_0 = 1$, observe that

$$g^j(1) = \frac{1}{3^j},$$

hence,

$$\Upsilon(1, 1, 1, g^j(1)) = 3\left|1 - \frac{1}{3^j}\right| \leq 3.$$

Therefore,

$$M(\Upsilon, g, 1) = \sup_{j\in\mathbb{N}\cup\{0\}} \Upsilon(1, 1, 1, g^j(1)) < \infty.$$

All assumptions of Theorem 10 are satisfied. Hence, g possesses a unique fixed point in Ω. Indeed,

$$g(u) = u \quad\Longrightarrow\quad \frac{u}{3} = u \quad\Longrightarrow\quad u = 0.$$

Therefore, the unique fixed point of g is $u = 0$.

5. Application to a nontrivial four-component boundary value problem

In this application, we use Theorem 10 to establish the existence and uniqueness of the solution for a coupled four-component boundary value problem. Consider the system

$$\begin{cases} u_1''(t) = \dfrac{1}{10}\left(\sin t + u_2(t) + u_3(t)\right), \\ u_2''(t) = \dfrac{1}{12}\left(e^{-t} + u_1(t) + u_4(t)\right), \\ u_3''(t) = \dfrac{1}{15}\left(\cos t + u_1(t) + u_2(t)\right), \\ u_4''(t) = \dfrac{1}{18}\left(1 + t^2 + u_3(t)\right), \end{cases} \qquad t \in [0, 1],$$

subject to the boundary conditions

$$u_i(0) = 0, \qquad u_i(1) = 0, \qquad i = 1, 2, 3, 4.$$

This system may be interpreted as a model for four interacting thermal or chemical components in a distributed engineering process. The unknown functions

$$u_1, u_2, u_3, u_4$$

represent four coupled state variables. Each component is affected by external nonhomogeneous sources such as

$$\sin t, \qquad e^{-t}, \qquad \cos t, \qquad 1+t^2,$$

and by the other components of the system. Hence, the problem is genuinely coupled and naturally motivates the use of a four-dimensional metric-type framework.

Let

$$\Omega = C([0,1],\mathbb{R})^4.$$

For

$$U=(u_1,u_2,u_3,u_4), \quad V=(v_1,v_2,v_3,v_4), \quad W=(w_1,w_2,w_3,w_4), \quad Z=(z_1,z_2,z_3,z_4),$$

define

$$\|U-Z\| = \sum_{i=1}^{4} \|u_i - z_i\|_\infty,$$

where

$$\|u_i - z_i\|_\infty = \sup_{t\in[0,1]} |u_i(t) - z_i(t)|.$$

Define

$$\Upsilon(U,V,W,Z) = \|U-Z\| + \|V-Z\| + \|W-Z\|.$$

Also define the control function

$$\mu(U,Z) = 1 + \|U\| + \|Z\|.$$

Then, (Ω, Υ) is a complete controlled four-dimensional metric type space.

The Green function corresponding to the boundary value problem

$$u''(t) = f(t), \qquad u(0) = u(1) = 0,$$

is given by

$$G(t,s) = \begin{cases} s(t-1), & 0 \le s \le t \le 1, \\ t(s-1), & 0 \le t \le s \le 1. \end{cases}$$

Moreover,

$$\int_0^1 |G(t,s)|\, ds = \frac{t(1-t)}{2} \le \frac{1}{8}, \qquad t \in [0,1].$$

Therefore, the given boundary value problem is equivalent to the integral system

$$u_i(t) = \int_0^1 G(t,s) F_i(s, U(s))\, ds, \qquad i = 1,2,3,4,$$

where

$$F_1(t, U(t)) = \frac{1}{10}\left(\sin t + u_2(t) + u_3(t)\right),$$

$$F_2(t, U(t)) = \frac{1}{12}\left(e^{-t} + u_1(t) + u_4(t)\right),$$

$$F_3(t, U(t)) = \frac{1}{15}\left(\cos t + u_1(t) + u_2(t)\right),$$

$$F_4(t, U(t)) = \frac{1}{18}\left(1 + t^2 + u_3(t)\right).$$

Define the operator

$$T : \Omega \to \Omega$$

by

$$T(U) = (T_1(U), T_2(U), T_3(U), T_4(U)),$$

where

$$T_i(U)(t) = \int_0^1 G(t, s)F_i(s, U(s))\,ds, \qquad i = 1, 2, 3, 4.$$

Now, for any $U, Z \in \Omega$, we have

$$\|T_1(U) - T_1(Z)\|_\infty \le \frac{1}{8} \cdot \frac{1}{10}\left(\|u_2 - z_2\|_\infty + \|u_3 - z_3\|_\infty\right),$$

$$\|T_2(U) - T_2(Z)\|_\infty \le \frac{1}{8} \cdot \frac{1}{12}\left(\|u_1 - z_1\|_\infty + \|u_4 - z_4\|_\infty\right),$$

$$\|T_3(U) - T_3(Z)\|_\infty \le \frac{1}{8} \cdot \frac{1}{15}\left(\|u_1 - z_1\|_\infty + \|u_2 - z_2\|_\infty\right),$$

$$\|T_4(U) - T_4(Z)\|_\infty \le \frac{1}{8} \cdot \frac{1}{18}\|u_3 - z_3\|_\infty.$$

Adding these inequalities gives

$$\|T(U) - T(Z)\| \le \frac{1}{48}\|U - Z\|.$$

Hence, for all $U, V, W, Z \in \Omega$,

$$\begin{aligned}\Upsilon(TU, TV, TW, TZ) &= \|TU - TZ\| + \|TV - TZ\| + \|TW - TZ\| \\ &\le \frac{1}{48}\left(\|U - Z\| + \|V - Z\| + \|W - Z\|\right) \\ &= \frac{1}{48}\Upsilon(U, V, W, Z).\end{aligned}$$

Define

$$P(t) = \frac{t}{48}, \qquad t \ge 0.$$

Clearly, P is strictly increasing and

$$P^n(t) = \frac{t}{48^n} \to 0 \qquad \text{as } n \to \infty.$$

Therefore,

$$\Upsilon(TU, TV, TW, TZ) \le P(\Upsilon(U, V, W, Z)).$$

Observe that (Ω, Υ) is a complete controlled four-dimensional metric type space. Moreover, T maps Ω into itself and satisfies

$$\Upsilon(TU, TV, TW, TZ) \le P(\Upsilon(U, V, W, Z)),$$

where $P(t) = t/48$ is strictly increasing and satisfies

$$P^n(t) = \frac{t}{48^n} \to 0 \qquad (n \to \infty).$$

Furthermore, choose any $U_0 \in \Omega$. Since T is a contraction, its iterates $\{T^n(U_0)\}_{n\ge 0}$ remains bounded. Hence,

$$M(\Upsilon, T, U_0) = \sup_{n\in\mathbb{N}\cup\{0\}} \Upsilon(T^n(U_0), U_0, U_0, U_0) < \infty.$$

Therefore, all assumptions of Theorem 10 are satisfied. Hence, the operator T has a unique fixed point in Ω. Consequently, the coupled four-component boundary value problem has a unique solution on $[0, 1]$.

Finally, the solution is nontrivial. Indeed, the zero function cannot satisfy the system because, for example,

$$u_4''(t) = \frac{1}{18}(1 + t^2 + u_3(t)),$$

and if $u_3(t) = u_4(t) = 0$, then the right-hand side becomes

$$\frac{1}{18}(1 + t^2) \ne 0.$$

Therefore, the unique solution obtained above is nontrivial.

The present framework provides a natural setting for studying coupled four-component systems. Although existence and uniqueness could also be established by reformulating the problem in a classical product Banach space and applying a standard contraction principle, such an approach does not explicitly reflect the intrinsic four-component structure of the system. In contrast, the controlled four-dimensional metric-type framework developed in this paper is specifically designed to handle interactions among multiple coupled components through the generalized distance function and the associated control function. As a result, the present boundary value problem can be studied directly within the same abstract setting established in Section 4, without reducing it to a conventional one-dimensional framework. Consequently, this example demonstrates the practical applicability of the proposed theory and illustrates how the abstract fixed-point results can be employed to obtain the existence and uniqueness of solutions for a nontrivial coupled four-component boundary value problem.

6. Conclusions

In this paper, we introduced the concept of controlled four–dimensional metric type spaces as a natural extension of the recently developed four–dimensional metric structures. These spaces generalize classical metric spaces by extending the distance function to four variables and incorporating a control function into the generalized triangle inequality. Such an extension provides a more flexible framework for studying relationships among multiple elements simultaneously. It allows modeling systems in which several parameters, such as spatial location, time, or other contextual variables, influence the notion of distance. Motivated by developments in generalized metric spaces, including S, S_b, extended S, and controlled S–metric type spaces, this work advances higher-dimensional metric structures.

We investigated fundamental properties of controlled four-dimensional metric type spaces, provided illustrative examples, and introduced new contraction conditions, including iterative-function-based contractions and (α_q-F)–contractive mappings. Using these, we established the existence and uniqueness of fixed points, extending several known fixed point theorems and laying a foundation for further studies in generalized higher-dimensional metric spaces and nonlinear analysis. Furthermore, a nontrivial application to four-component boundary value problem was presented in the final section. Future work will extend this framework to quadruple controlled metric type spaces.

Author contributions

All authors fully contributed to writing this article.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article. AI tools used.

Acknowledgments

The author, Fatima M. Azmi, would like to thank Prince Sultan University for their support and for covering the article publication fees through the TAS Research Lab.

Conflict of interest

The authors declare no conflicts of interest in this paper.

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