AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(6): 17937–17950.
DOI: 10.3934/math.2026731
Received: 19 January 2026
Revised: 10 April 2026
Accepted: 22 April 2026
Published: 18 June 2026

Research article

Some Hermite-Hadamard-Mercer type inequalities via Atangana-Baleanu-conformable fractional operator

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Abstract: This paper investigated new Hermite-Hadamard-Mercer type inequalities associated with an Atangana-Baleanu-conformable fractional integral operator. By combining the structural features of Atangana-Baleanu fractional integrals and conformable kernels, we derived a fractional framework that contained both local and nonlocal effects. A fundamental identity was first established, transforming a symmetric combination of endpoint values and fractional integral terms into a weighted integral involving the first derivative. Based on this identity and Jensen-Mercer's inequality, several new bounds were obtained under convexity assumptions on $|f'|$ and $|f'|^q$, where $q > 1$. The results extended known Hermite-Hadamard-Mercer inequalities and reduced to classical or fractional special cases under suitable parameter choices. The proposed approach provided a flexible tool for fractional integral inequalities and related estimates in convex analysis.

Keywords: Hermite-Hadamard-Mercer-type inequalities; Atangana-Baleanu fractional operator; conformable fractional operator; integral inequalities; fractional calculus; convex functions
Mathematics Subject Classification: 26A51, 26D10, 26D15

1. Introduction

In recent years, fractional calculus has become an active research area for extending classical integral inequalities due to its ability to incorporate nonlocal memory effects and generalized kernel structures. In particular, fractional integral operators with non-singular kernels have attracted considerable attention because they overcome several limitations associated with the classical Riemann-Liouville framework.

Among these developments, Atangana-Baleanu (AB) fractional operators have emerged as an important class of fractional operators with non-singular kernels. Their applications to integral inequalities have been extensively investigated in the literature. For example, Set et al. [1]

established several new Hermite-Hadamard-type inequalities via AB fractional integral operators and demonstrated that the AB framework provides effective extensions of classical convexity inequalities. Subsequently, Set et al. [2] derived new integral inequalities for differentiable convex functions involving AB fractional integrals, thereby obtaining refined estimates for a broad class of convex mappings.

The interaction between Mercer-type inequalities and AB fractional operators has also received increasing attention. Jiu et al. [3] established Jensen-Mercer variants of Hermite-Hadamard inequalities via AB fractional operators and extended several classical Mercer-type results to the fractional setting. More recently, Tariq et al. [4] proposed new modifications of integral inequalities in the framework of AB fractional operators, showing that the AB structure can provide greater flexibility for deriving generalized integral estimates. Furthermore, Karim et al. [5] investigated Ostrowski-type inequalities for convex functions through AB fractional operators and obtained new error estimates involving fractional integral representations. Long et al. [6] developed Simpson-like inequalities associated with AB fractional integrals for functions whose third derivatives satisfy suitable convexity assumptions, further illustrating the applicability of AB operators in fractional approximation theory.

On the other hand, conformable fractional operators constitute another important branch of modern fractional calculus. Jarad et al. [7] introduced a new class of conformable fractional operators and established their fundamental analytical properties. Owing to their simple kernel structure and additional scaling parameter, conformable fractional operators have been widely applied in the study of fractional differential equations, generalized convexity, and fractional integral inequalities.

Although significant progress has been achieved in both AB fractional inequalities and conformable fractional operators, most existing studies treat these two frameworks independently. The literature cited above mainly focuses either on AB fractional operators with non-singular kernels [1–3] or on conformable fractional operators separately [7]. To the best of our knowledge, Hermite-Hadamard-Mercer type inequalities associated with a unified AB-conformable (ABConf) fractional integral operator have not yet been systematically investigated. In particular, there remains a lack of results that simultaneously incorporate the nonlocal memory effect of AB operators and the flexible scaling structure of conformable fractional kernels within a Mercer-type inequality framework.

Motivated by this observation, we introduce a new ABConf fractional integral operator by embedding a conformable kernel into the AB fractional integral framework. Based on this operator, we establish a fundamental fractional identity and derive several new Hermite-Hadamard-Mercer type inequalities under suitable convexity assumptions. The obtained results unify and extend a number of existing inequalities associated with AB fractional operators and conformable fractional integrals. Moreover, several previously known results can be recovered as special cases through appropriate choices of the involved parameters. Therefore, the present work contributes to the further development of fractional integral inequalities and provides a new framework for studying Mercer-type inequalities involving non-singular kernel fractional operators.

2. Background on classical integral inequalities

Definition 2.1. *A function $f : [a, b] \to \mathbb{R}$ is called convex on $[a, b]$ for all $(x, y) \in [a, b]$ and $t \in [0, 1]$ if it satisfies the following inequality:*

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y).$$

In 2003, Mercer [8] introduced a refined version of the classical Jensen inequality, now known as the Jensen-Mercer inequality, stated as follows:

For a convex mapping $f:[a,b]\to\mathbb{R}$, the following inequality holds for each $x_j\in[a,b]$:

$$f\left(a+b-\sum_{j=1}^{n}u_jx_j\right)\leq f(a)+f(b)-\sum_{j=1}^{n}u_jf\left(x_j\right),$$

where $u_j\in[0,1]$ and $\sum_{j=1}^{n}u_j=1$.

First, we recall the notion of the Caputo-Fabrizio (CF) derivative operator.

Definition 2.2. *[9] Let $f\in H^1(a,b)$, $\eta\in[0,1]$ and $b>a$. Then the new CF derivative is defined as*

$${}^{CF}D^{\eta}f(t)=\frac{B(\eta)}{1-\eta}\int_a^t f'(s)\exp\left[\frac{-\eta}{1-\eta}(t-s)\right]ds,$$

where $B(\eta)$ denotes normalization function.

Moreover, the corresponding CF fractional integral operator is given as the following:

Definition 2.3. *[9] Let $f\in H^1(a,b)$, $\eta\in[0,1]$, and $b>a$,*

$${}^{CF}I_{a^+}^{\eta}f(t)=\frac{1-\eta}{B(\eta)}f(t)+\frac{\eta}{B(\eta)}\int_a^t f(s)\,ds,$$

and

$${}^{CF}I_{b^-}^{\eta}f(t)=\frac{1-\eta}{B(\eta)}f(t)+\frac{\eta}{B(\eta)}\int_t^b f(s)\,ds,$$

with normalization function $B(\eta)>0$ satisfying the property $B(0)=B(1)=1$.

Recently, Atangana and Baleanu [10] proposed a new fractional derivative operator which also overcomes this deficiency. Atangana and Baleanu introduced the derivative operator both in CF and Riemann-Liouville senses.

Definition 2.4. *[10] Let $f\in H^1(a,b)$, $\eta\in[0,1]$, and $b>a$. The Atangana-Baleanu-Caputo (ABC) fractional derivative is defined as*

$${}^{ABC}{}_aD^{\eta}[f(t)]=\frac{B(\eta)}{1-\eta}\int_a^t f'(s)E_{\eta}\left[\frac{-\eta}{1-\eta}(t-s)^{\eta}\right]ds.$$

Definition 2.5. *[10] Let $f\in H^1(a,b)$, $\eta\in[0,1]$, and $b>a$. The Atangana-Baleanu-Riemann (ABR) fractional derivative is defined as*

$${}^{ABR}{}_aD^{\eta}[f(t)]=\frac{B(\eta)}{1-\eta}\frac{d}{dt}\int_a^t f(s)E_{\eta}\left[\frac{-\eta}{1-\eta}(t-s)^{\eta}\right]ds.$$

Following Atangana and Baleanu [10], the AB fractional integral operator with a nonlocal kernel is defined as follows.

Definition 2.6. *(AB fractional integral operator [10]) Let $f \in H^1(a,b)$, $\eta \in [0,1]$, and $b > a$. The AB fractional integral operator with a nonlocal kernel is defined by*

$$^{ABR}{}_{a^+}I^{\eta}\{f(t)\} = \frac{1-\eta}{B(\eta)}f(t) + \frac{\eta}{B(\eta)\Gamma(\eta)}\int_a^t f(s)(t-s)^{\eta-1}ds,$$

where $\Gamma(\eta)$ is the Gamma function.

Definition 2.7. *(Right-sided AB fractional integral operator [10]) Let $f \in H^1(a,b)$, $\eta \in [0,1]$, and $b > a$. The right-sided AB fractional integral operator is defined by*

$$^{AB}{}_{b^-}I^{\eta}\{f(t)\} = \frac{1-\eta}{B(\eta)}f(t) + \frac{\eta}{B(\eta)\Gamma(\eta)}\int_t^b f(s)(s-t)^{\eta-1}ds.$$

In [11], Tariq et al. established the Simpson-Mercer-type inequalities including AB fractional operators as follows:

Theorem 2.1. *[11] Suppose the mapping $\Phi : I = [d,D] \to \mathbb{R}$ is differentiable on (d,D) with $D > d$. If Φ' is s-convex function on $[d,D]$, for $s > 0$. Then for all $\omega_1, \omega_{2\in} [d,D]$ and $\omega_1 < \omega_2$ and $\delta > 0$, the following Simpson-Mercer type inequality for the AB fractional integral holds for $k \in [0,1]$:*

$$\begin{aligned}
&\left|\frac{1}{6}\{\Phi(d+D+\omega_1)+\Phi(d+D-\omega_2)\}+\frac{2}{3}\Phi\left(d+D-\frac{\omega_1+\omega_2}{2}\right)\right.\\
&+\frac{2^{\delta}(1-\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^{\delta}}\Phi\left(d+D-\frac{\omega_1+\omega_2}{2}\right)\\
&-\frac{2^{\delta}M(\delta)\Gamma(\delta)}{(\omega_2-\omega_1)^{\delta}}\left\{{}^{AB}_{(d+D-\omega_1)}I^{\delta}_{\left(d+D-\frac{\omega_1+\omega_2}{2}\right)}\Phi\left(d+D-\frac{\omega_1+\omega_2}{2}\right)\right.\\
&\left.\left.+{}^{AB}_{\left(d+D-\frac{\omega_1+\omega_2}{2}\right)}I^{\delta}_{(d+D-\omega_1)}\Phi\left(d+D-\frac{\omega_1+\omega_2}{2}\right)\right\}\right|\\
\leq\ &\frac{\omega_2-\omega_1}{2}\left\{2\left[\left(\frac{2}{3}\right)^{1+\frac{1}{\delta}}\left(\frac{\delta}{\delta+1}\right)+\frac{1-2\delta}{6(\delta+1)}\right][|\Phi'(d)|+|\Phi'(D)|]\right.\\
&\left.-\frac{1}{2^s}[|\Phi'(\omega_1)|+|\Phi'(\omega_2)|][J_3(\delta)+J_4(\delta)]\right\},
\end{aligned}$$

where

$$J_3(\delta) = \int_0^{\left(\frac{2}{3}\right)^{\frac{1}{\delta}}}\left(\frac{1}{3}-\frac{k^{\delta}}{2}\right)[(1+k)^s+(1-k)^s]dk,$$

and

$$J_4(\delta) = \int_{\left(\frac{2}{3}\right)^{\frac{1}{\delta}}}^1\left(\frac{k^{\delta}}{2}-\frac{1}{3}\right)[(1+k)^s+(1-k)^s]dk.$$

In [7], Jarad et al. established a new fractional operator called conformable fractional integrals.

Definition 2.8. *The fractional conformable integral operator $I^{\beta}_{a^+}f(x)$ and $I^{\beta}_{b^-}f(x)$ of order $\beta \in \mathbb{C}$, $Re(\beta) > 0$, and $\alpha \in (0,1]$ are given by*

$$\left({}^{C}I^{\beta,\alpha}_{a^+}f\right)(t) = \frac{1}{\Gamma(\beta)}\int_a^t \frac{1}{(s-a)^{1-\alpha}}\left[\frac{(t-a)^{\alpha}-(s-a)^{\alpha}}{\alpha}\right]^{\beta-1}f(s)ds,$$

$$\left({}^{C}I_{b^-}^{\beta,\alpha}f\right)(t)=\frac{1}{\Gamma(\beta)}\int_t^b\frac{1}{(b-s)^{1-\alpha}}\left[\frac{(b-t)^{\alpha}-(b-s)^{\alpha}}{\alpha}\right]^{\beta-1}f(s)\,ds.$$

Throughout the paper, we use the following parameters:

Let $0\le a<b$ and $f:[a,b]\to\mathbb{R}$ be a function such that the involved integrals exist. $\eta\in[0,1]$ denotes the AB fractional order, $\alpha\in(0,1]$ denotes the conformable parameter, and $B(\eta)>0$ is the AB normalization function satisfying $B(0)=B(1)=1$.

We denote by $\Gamma(\cdot)$ the Gamma function and by $E_\eta(\cdot)$ the one-parameter Mittag-Leffler function.

Next, we denote the new operator with ABConf fractional integrals.

Definition 2.9. *Let $f:[a,b]\to\mathbb{R}$ be a function and $\eta\in[0,1]$, $\alpha\in(0,1]$. We denote by*

$${}^{ABConf}{}_{a^+}I^{\eta,\alpha}\{f(t)\}=\frac{1-\eta}{B(\eta)}f(t)+\frac{\eta}{B(\eta)\Gamma(\eta)}\int_a^t\frac{1}{(s-a)^{1-\alpha}}\left[\frac{(t-a)^{\alpha}-(s-a)^{\alpha}}{\alpha}\right]^{\eta-1}f(s)\,ds,$$

and

$${}^{ABConf}{}_{b^-}I^{\eta,\alpha}\{f(t)\}=\frac{1-\eta}{B(\eta)}f(t)+\frac{\eta}{B(\eta)\Gamma(\eta)}\int_t^b\frac{1}{(b-s)^{1-\alpha}}\left[\frac{(b-t)^{\alpha}-(b-s)^{\alpha}}{\alpha}\right]^{\eta-1}f(s)\,ds.$$

Remark 2.1. *Let $\eta\in[0,1]$ and set $\alpha=1$. Then for all $s\in[a,b]$,*

$$\left({}^{ABConf}{}_{a^+}I^{\eta,1}\right)\{f(t)\}=\left({}^{AB}{}_{a^+}I^{\eta}\right)\{f(t)\},\quad\left({}^{ABConf}{}_{b^-}I^{\eta,1}\right)\{f(t)\}=\left({}^{AB}{}_{b^-}I^{\eta}\right)\{f(t)\}.$$

Remark 2.2. *Let $\alpha\in(0,1]$ and set $\eta=1$. Using $B(1)=1$ and $\Gamma(1)=1$, we obtain*

$${}^{ABConf}{}_{a^+}I^{1,\alpha}\{f(t)\}=\int_a^t\frac{1}{(s-a)^{1-\alpha}}f(s)\,ds,$$

and

$${}^{ABConf}{}_{b^-}I^{1,\alpha}\{f(t)\}=\int_t^b\frac{1}{(b-s)^{1-\alpha}}f(s)\,ds.$$

That is, ${}^{ABConf}I^{1,\alpha}\{f(t)\}$ coincides with the conformable integral of order 1 with parameter α.

Remark 2.3. *If $\alpha=1$ and $\eta=1$, then ${}^{ABConf}I^{1,\alpha}\{f(t)\}$ integral reduces the classical (Riemann) integral over the same endpoints.*

3. Results

Theorem 3.1. *For a positive convex function $f:[a,b]\to\mathbb{R}$ with $0\le a<b$ and $\eta\in[0,1]$, $\alpha\in(0,1]$. Then for $a\le m<n\le b$, the inequalities for ABConf fractional integral operators hold:*

$$\begin{aligned}&\frac{2}{B(\eta)\Gamma(\eta)}f\left(a+b-\frac{m+n}{2}\right)\\ \le\;&\frac{\alpha^{\eta}}{(n-m)^{\alpha\eta-\alpha+1}}\left\{{}^{ABConf}{}_{a+b-n^+}I^{\eta,\alpha}\{f(a+b-m)\}+{}^{ABConf}{}_{a+b-m^-}I^{\eta,\alpha}\{f(a+b-n)\}\right.\\&\left.-\frac{1-\eta}{B(\eta)}\left[f(a+b-m)+f(a+b-n)\right]\right\}\end{aligned}$$

$$\leq \frac{1}{B(\eta)\Gamma(\eta)}\left[2f(a)+2f(b)-f(m)-f(n)\right], \tag{3.1}$$

and

$$\begin{aligned}
&\frac{1}{B(\eta)\Gamma(\eta)}f\left(a+b-\frac{m+n}{2}\right)\\
\leq\;& \frac{1}{B(\eta)\Gamma(\eta)}\left[f(a)+f(b)\right]-\frac{\alpha^{\eta}}{2(n-m)^{\alpha\eta-\alpha+1}}\left\{{}^{ABConf}{}_{m^+}I^{\eta,\alpha}\{f(n)\}+{}^{ABConf}{}_{n^-}I^{\eta,\alpha}\{f(m)\}\right.\\
&\left.-\frac{1-\eta}{B(\eta)}\left[f(m)+f(n)\right]\right\}\\
\leq\;& \frac{1}{B(\eta)\Gamma(\eta)}\left[f(a)+f(b)-f\left(\frac{m+n}{2}\right)\right].
\end{aligned} \tag{3.2}$$

Proof. Since f is convex on $[a,b]$, we obtain

$$2f\left(a+b-\frac{x+y}{2}\right)\leq f(a+b-x)+f(a+b-y)$$

for all $x,y\in[a,b]$.

Replacing $x=tm+(1-t)n$ and $y=(1-t)m+tn$, for all $m,n\in[a,b]$ and $t\in[0,1]$, we have

$$2f\left(a+b-\frac{m+n}{2}\right)\leq f(a+b-[tm+(1-t)n])+f(a+b-[(1-t)m+tn]).$$

Multiplying $\frac{\alpha\eta}{B(\eta)\Gamma(\eta)}[1-t^{\alpha}]^{\eta-1}t^{\alpha-1}$ on both sides and integrating the inequality w.r.t $t\in[0,1]$, we have

$$\begin{aligned}
&\frac{2}{B(\eta)\Gamma(\eta)}f\left(a+b-\frac{m+n}{2}\right)\\
\leq\;& \frac{\alpha\eta}{B(\eta)\Gamma(\eta)}\int_0^1[1-t^{\alpha}]^{\eta-1}t^{\alpha-1}f(a+b-[tm+(1-t)n])\,dt\\
&+\frac{\alpha\eta}{B(\eta)\Gamma(\eta)}\int_0^1[1-t^{\alpha}]^{\eta-1}t^{\alpha-1}f(a+b-[(1-t)m+tn])\,dt\\
=\;& \frac{\alpha\eta}{B(\eta)\Gamma(\eta)}\int_0^1[1-t^{\alpha}]^{\eta-1}t^{\alpha-1}f(a+b-[tm+(1-t)n])\,dt\\
&+\frac{\alpha\eta}{B(\eta)\Gamma(\eta)}\int_0^1[1-(1-s)^{\alpha}]^{\eta-1}(1-s)^{\alpha-1}f(a+b-[sm+(1-s)n])\,ds\\
=\;& \frac{\alpha^{\eta}}{(n-m)^{\alpha\eta-\alpha+1}}\left\{{}^{ABConf}{}_{a+b-n^+}I^{\eta,\alpha}\{f(a+b-m)\}+{}^{ABConf}{}_{a+b-m^-}I^{\eta,\alpha}\{f(a+b-n)\}\right.\\
&\left.-\frac{1-\eta}{B(\eta)}\left[f(a+b-m)+f(a+b-n)\right]\right\},
\end{aligned}$$

and we obtain the first inequality of (2.1).

Now, we show the other side inequality of (2.1). Since f is convex on $[a,b]$, for $t\in[0,1]$, it gives

$$f(a+b-[tm+(1-t)n])\leq f(a)+f(b)-tf(m)-(1-t)f(n)$$

and

$$f(a+b-[(1-t)m+tn]) \leq f(a)+f(b)-(1-t)f(m)-tf(n),$$

so we get

$$f(a+b-[tm+(1-t)n])+f(a+b-[(1-t)m+tn]) \leq 2f(a)+2f(b)-f(m)-f(n).$$

Multiplying $\frac{\alpha\eta}{B(\eta)\Gamma(\eta)}[1-t^{\alpha}]^{\eta-1}t^{\alpha-1}$ on both sides and integrating the inequality w.r.t $t \in [0,1]$, we get the required inequality (2.1).

Next, we prove the inequality of (2.2). Since f is convex, we have

$$f\left(a+b-\frac{x+y}{2}\right) \leq f(a)+f(b)-\frac{f(x)+f(y)}{2}.$$

By replacing $x = tm+(1-t)n$ and $y = (1-t)m+tn$, for all $m, n \in [a,b]$ and $t \in [0,1]$, we have

$$f\left(a+b-\frac{m+n}{2}\right) \leq f(a)+f(b)-\frac{f(tm+(1-t)n)+f((1-t)m+tn)}{2}.$$

Multiplying $\frac{\alpha\eta}{B(\eta)\Gamma(\eta)}[1-t^{\alpha}]^{\eta-1}t^{\alpha-1}$ on both sides and integrating the inequality w.r.t $t \in [0,1]$, we get

$$\begin{aligned}
&\frac{1}{B(\eta)\Gamma(\eta)}f\left(a+b-\frac{m+n}{2}\right)\\
\leq\ &\frac{1}{B(\eta)\Gamma(\eta)}[f(a)+f(b)]\\
&-\frac{\alpha^{\eta}}{2(n-m)^{\alpha\eta-\alpha+1}}\left\{{}^{ABConf}_{m^+}I^{\eta,\alpha}\{f(n)\}+{}^{ABConf}{}_{n^-}I^{\eta,\alpha}\{f(m)\}-\frac{1-\eta}{B(\eta)}[f(m)+f(n)]\right\},
\end{aligned}$$

and we obtain the first inequality of (2.2).

To prove the other inequality in (2.2), since f is convex on $[a,b]$, for $t \in [0,1]$, it gives

$$f\left(\frac{m+n}{2}\right) \leq \frac{1}{2}[f(tm+(1-t)n)+f((1-t)m+tn)].$$

Multiplying $\frac{\alpha\eta}{B(\eta)\Gamma(\eta)}[1-t^{\alpha}]^{\eta-1}t^{\alpha-1}$ on both sides and integrating the inequality w.r.t $t \in [0,1]$, we get

$$\begin{aligned}
\frac{1}{B(\eta)\Gamma(\eta)}f\left(\frac{m+n}{2}\right) \leq\ &\frac{\alpha^{\eta}}{2(n-m)^{\alpha\eta-\alpha+1}}\left\{{}^{ABConf}{}_{m^+}I^{\eta,\alpha}\{f(n)\}+{}^{ABConf}{}_{n^-}I^{\eta,\alpha}\{f(m)\}\right.\\
&\left.-\frac{1-\eta}{B(\eta)}[f(m)+f(n)]\right\}.
\end{aligned}$$

Multiplying (-1) in the above inequality,

$$\begin{aligned}
-\frac{1}{B(\eta)\Gamma(\eta)}f\left(\frac{m+n}{2}\right) \geq\ &\frac{-\alpha^{\eta}}{2(n-m)^{\alpha\eta-\alpha+1}}\left\{{}^{ABConf}{}_{m^+}I^{\eta,\alpha}\{f(n)\}+{}^{ABConf}{}_{n^-}I^{\eta,\alpha}\{f(m)\}\right.\\
&\left.-\frac{1-\eta}{B(\eta)}[f(m)+f(n)]\right\}.
\end{aligned}$$

By adding $\frac{1}{B(\eta)\Gamma(\eta)}[f(a)+f(b)]$ in above, we get

$$\begin{aligned}
&\frac{1}{B(\eta)\Gamma(\eta)}[f(a)+f(b)] \\
&-\frac{\alpha^{\eta}}{2(n-m)^{\alpha\eta-\alpha+1}}\left\{{}^{ABConf}{}_{m^+}I^{\eta,\alpha}\{f(n)\}+{}^{ABConf}{}_{n^-}I^{\eta,\alpha}\{f(m)\}-\frac{1-\eta}{B(\eta)}[f(m)+f(n)]\right\} \\
\leq\ &\frac{1}{B(\eta)\Gamma(\eta)}\left[f(a)+f(b)-f\left(\frac{m+n}{2}\right)\right].
\end{aligned}$$

Corollary 3.1. *In Theorem 3.1, set $\alpha = 1$. Then, the inequalities for the AB fractional integral operators hold:*

$$\begin{aligned}
&\frac{2}{B(\eta)\Gamma(\eta)}f\left(a+b-\frac{m+n}{2}\right) \\
\leq\ &\frac{\alpha^{\eta}}{(n-m)^{\alpha\eta-\alpha+1}}\left\{{}^{AB}_{a+b-n^+}I^{\eta}\{f(a+b-m)\}+{}^{AB}_{a+b-m^-}I^{\eta}\{f(a+b-n)\}\right. \\
&\left.-\frac{1-\eta}{B(\eta)}[f(a+b-m)+f(a+b-n)]\right\} \\
\leq\ &\frac{1}{B(\eta)\Gamma(\eta)}[2f(a)+2f(b)-f(m)-f(n)],
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{B(\eta)\Gamma(\eta)}f\left(a+b-\frac{m+n}{2}\right) \\
\leq\ &\frac{1}{B(\eta)\Gamma(\eta)}[f(a)+f(b)]-\frac{1}{2(n-m)^{\eta}}\left\{{}^{AB}{}_{m^+}I^{\eta}\{f(n)\}+{}^{AB}{}_{n^-}I^{\eta}\{f(m)\}-\frac{1-\eta}{B(\eta)}[f(m)+f(n)]\right\} \\
\leq\ &\frac{1}{B(\eta)\Gamma(\eta)}\left[f(a)+f(b)-f\left(\frac{m+n}{2}\right)\right].
\end{aligned}$$

Corollary 3.2. *In Theorem 3.1, set $\eta = 1$. Then, the inequalities for conformable fractional integral operators hold:*

$$2f\left(a+b-\frac{m+n}{2}\right) \leq \frac{2\alpha}{(n-m)}\int_{a+b-n}^{a+b-m}\frac{1}{(s-a)^{1-\alpha}}f(s)\,ds \leq [2f(a)+2f(b)-f(m)-f(n)],$$

and

$$\begin{aligned}
f\left(a+b-\frac{m+n}{2}\right) &\leq [f(a)+f(b)]-\frac{\alpha}{(n-m)}\int_{a+b-n}^{a+b-m}\frac{1}{(s-a)^{1-\alpha}}f(s)\,ds \\
&\leq \left[f(a)+f(b)-f\left(\frac{m+n}{2}\right)\right].
\end{aligned}$$

Corollary 3.3. *In Theorem 3.1, set $\alpha = \eta = 1$. Then, the inequalities for classical fractional integral operators hold:*

$$\begin{aligned}
2f\left(a+b-\frac{m+n}{2}\right) &\leq \frac{2}{(n-m)}\int_{a+b-n}^{a+b-m}f(s)\,ds \\
&\leq [2f(a)+2f(b)-f(m)-f(n)],
\end{aligned}$$

and

$$\begin{aligned} f\left(a+b-\frac{m+n}{2}\right) &\leq [f(a)+f(b)]-\frac{1}{(n-m)}\int_{a+b-n}^{a+b-m}f(s)\,ds \\ &\leq \left[f(a)+f(b)-f\left(\frac{m+n}{2}\right)\right]. \end{aligned}$$

Lemma 3.1. *Let $f : [a,b] \to \mathbb{R}$ with $0 \leq a < b$, $\eta \in [0,1]$ and $\alpha \in (0,1]$, f is differentiable, and $f' \in L^1$. Then, the following identity is valid for ABConf fractional integral operators:*

$$\begin{aligned} &\left(\frac{(n-m)^{\alpha\eta}}{B(\eta)\Gamma(\eta)}+\frac{\alpha^{\eta}(1-\eta)}{B(\eta)}\right)[f(a+b-m)+f(a+b-n)] \\ &-\alpha^{\eta}\left[{}^{ABConf}{}_{a+b-n^{+}}I^{\eta,\alpha}\{f(a+b-m)\}+{}^{ABConf}{}_{a+b-m^{-}}I^{\eta,\alpha}\{f(a+b-n)\}\right] \\ =\; &\frac{(n-m)^{\alpha\eta+1}}{B(\eta)\Gamma(\eta)}\int_0^1\{[1-(1-t)^{\alpha}]^{\eta}-[1-t^{\alpha}]^{\eta}\}f'(a+b-[tm+(1-t)n])\,dt, \end{aligned}$$

for all $a \leq m < n \leq b$.

Proof. Let

$$\begin{aligned} I_1 &= \int_0^1[1-(1-t)^{\alpha}]^{\eta}f'(a+b-[tm+(1-t)n])\,dt \\ &= \frac{f(a+b-m)}{n-m}-\frac{1}{n-m}\int_0^1 f(a+b-[tm+(1-t)n])\,\eta\,[1-(1-t)^{\alpha}]^{\eta-1}\left(\alpha(1-t)^{\alpha-1}\right)dt \\ &= \frac{f(a+b-m)}{n-m}-\frac{\alpha\eta}{n-m}\int_{a+b-n}^{a+b-m}\left(\frac{n-m}{(a+b-m)-x}\right)^{1-\alpha} \\ &\quad\times\left[\left(\frac{(a+b-m)-(a+b-n)}{n-m}\right)^{\alpha}-\left(\frac{(a+b-m)-x}{n-m}\right)^{\alpha}\right]^{\eta-1}f(x)\,\frac{1}{n-m}dx \\ &= \frac{f(a+b-m)}{n-m}-\frac{\eta\alpha^{\eta}}{(n-m)^{\alpha\eta+1}}\int_{a+b-n}^{a+b-m}\frac{1}{[(a+b-m)-x]^{1-\alpha}} \\ &\quad\times\left[\frac{[(a+b-m)-(a+b-n)]^{\alpha}-[(a+b-m)-x]^{\alpha}}{\alpha}\right]^{\eta-1}f(x)\,dx, \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_0^1[1-t^{\alpha}]^{\eta}f'(a+b-[tm+(1-t)n])\,dt \\ &= -\frac{f(a+b-n)}{n-m}-\frac{1}{n-m}\int_0^1 f(a+b-[tm+(1-t)n])\,\eta\,[1-t^{\alpha}]^{\eta-1}\left(-\alpha t^{\alpha-1}\right)dt \\ &= -\frac{f(a+b-n)}{n-m}+\frac{\alpha\eta}{n-m}\int_{a+b-n}^{a+b-m}\left(\frac{n-m}{x-(a+b-n)}\right)^{1-\alpha} \\ &\quad\times\left[\left(\frac{(a+b-m)-(a+b-n)}{n-m}\right)^{\alpha}-\left(\frac{x-(a+b-n)}{n-m}\right)^{\alpha}\right]^{\eta-1}f(x)\,\frac{1}{n-m}dx \end{aligned}$$

$$= -\frac{f(a+b-n)}{n-m} + \frac{\eta\alpha^{\eta}}{(n-m)^{\alpha\eta+1}} \int_{a+b-n}^{a+b-m} \frac{1}{[x-(a+b-n)]^{1-\alpha}}$$
$$\times \left[\frac{[(a+b-m)-(a+b-n)]^{\alpha} - [x-(a+b-n)]^{\alpha}}{\alpha} \right]^{\eta-1} f(x)\,dx.$$

Now, $I_1 - I_2$ and multiplying by $\frac{(n-m)^{\alpha\eta+1}}{B(\eta)\Gamma(\eta)}$, using the definition of ABConf fractional intergals, we get

$$\left(\frac{(n-m)^{\alpha\eta}}{B(\eta)\Gamma(\eta)} + \frac{\alpha^{\eta}(1-\eta)}{B(\eta)} \right) [f(a+b-m) + f(a+b-n)]$$
$$-\alpha^{\eta} \left[{}^{ABConf}{}_{a+b-n^{+}}I^{\eta,\alpha} \{f(a+b-m)\} + {}^{ABConf}{}_{a+b-m^{-}}I^{\eta,\alpha} \{f(a+b-n)\} \right]$$
$$= \frac{(n-m)^{\alpha\eta+1}}{B(\eta)\Gamma(\eta)} \int_0^1 \{[1-(1-t)^{\alpha}]^{\eta} - [1-t^{\alpha}]^{\eta}\} f'(a+b-[tm+(1-t)n])\,dt.$$

Remark 3.1. *If we choose $\alpha = 1$ in Lemma 3.1, we recover Lemma 3.1 of [12].*

Theorem 3.2. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function with $0 \le a < b$ and $f' \in L^1[a,b]$ such that $|f'|$ is a convex function on $[a,b]$. Then for $a \le m < n \le b$, the following ineqaulity for the ABConf fractional integral holds:*

$$\left| \left(\frac{(n-m)^{\alpha\eta}}{B(\eta)\Gamma(\eta)} + \frac{\alpha^{\eta}(1-\eta)}{B(\eta)} \right) [f(a+b-m) + f(a+b-n)] \right.$$
$$\left. -\alpha^{\eta} \left[{}^{ABConf}{}_{a+b-n^{+}}I^{\eta} \{f(a+b-m)\} + {}^{ABConf}{}_{a+b-m^{-}}I^{\eta} \{f(a+b-n)\} \right] \right|$$
$$\le \frac{(n-m)^{\alpha\eta+1}}{B(\eta)\Gamma(\eta)} \int_0^1 \left| [1-(1-t)^{\alpha}]^{\eta} - [1-t^{\alpha}]^{\eta} \right| |f'(a+b-[tm+(1-t)n])|\,dt$$
$$\le \frac{(n-m)^{\alpha\eta+1}}{B(\eta)\Gamma(\eta)} \int_0^1 \left| [1-(1-t)^{\alpha}]^{\eta} - [1-t^{\alpha}]^{\eta} \right|$$
$$\times \{|f'(a)| + |f'(b)| - t|f'(m)| - (1-t)|f'(n)|\}\,dt$$
$$= \frac{(n-m)^{\alpha\eta+1}}{B(\eta)\Gamma(\eta)} \{[|f'(a)| + |f'(b)|] A_1 - |f'(m)| A_2 - |f'(n)| A_3\},$$

where

$$A_1 = \int_{\frac{1}{2}}^{1} [1-(1-t)^{\alpha}]^{\eta} - [1-t^{\alpha}]^{\eta}\,dt + \int_0^{\frac{1}{2}} [1-t^{\alpha}]^{\eta} - [1-(1-t)^{\alpha}]^{\eta}\,dt,$$
$$A_2 = \int_{\frac{1}{2}}^{1} t[1-(1-t)^{\alpha}]^{\eta} - t[1-t^{\alpha}]^{\eta}\,dt + \int_0^{\frac{1}{2}} t[1-t^{\alpha}]^{\eta} - t[1-(1-t)^{\alpha}]^{\eta}\,dt,$$

and

$$A_3 = \int_{\frac{1}{2}}^{1} (1-t)[1-(1-t)^{\alpha}]^{\eta} - (1-t)[1-t^{\alpha}]^{\eta}\,dt$$
$$+ \int_0^{\frac{1}{2}} (1-t)[1-t^{\alpha}]^{\eta} - (1-t)[1-(1-t)^{\alpha}]^{\eta}\,dt,$$

for $\eta \in [0,1]$ and $\alpha \in (0,1]$.

Proof. By Lemma 2.5 and using the Jensen-Mercer inequality, we have

$$\begin{aligned}
&\left|\left(\frac{(n-m)^{\alpha\eta}}{B(\eta)\Gamma(\eta)}+\frac{\alpha^{\eta}(1-\eta)}{B(\eta)}\right)[f(a+b-m)+f(a+b-n)]\right.\\
&\left.-\alpha^{\eta}\left[{}^{AB-C}{}_{a+b-n^{+}}I^{\eta}\{f(a+b-m)\}+{}^{AB-C}{}_{a+b-m^{-}}I^{\eta}\{f(a+b-n)\}\right]\right|\\
\leq\ & \frac{(n-m)^{\alpha\eta+1}}{B(\eta)\Gamma(\eta)}\int_0^1\left|[1-(1-t)^{\alpha}]^{\eta}-[1-t^{\alpha}]^{\eta}\right|\left|f'(a+b-[tm+(1-t)n])\right|dt\\
\leq\ & \frac{(n-m)^{\alpha\eta+1}}{B(\eta)\Gamma(\eta)}\int_0^1\left|[1-(1-t)^{\alpha}]^{\eta}-[1-t^{\alpha}]^{\eta}\right|\\
&\times\{|f'(a)|+|f'(b)|-t|f'(m)|-(1-t)|f'(n)|\}\,dt\\
=\ & \frac{(n-m)^{\alpha\eta+1}}{B(\eta)\Gamma(\eta)}\{[|f'(a)|+|f'(b)|]A_1-|f'(m)|A_2-|f'(n)|A_3\}.
\end{aligned}$$

Theorem 3.3. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function with $0\leq a<b$ and $f'\in L^1[a,b]$ such that $|f'|^q$ is a convex function on $[a,b]$. Then, the following ineqaulity for the ABConf fractional integral holds:*

$$\begin{aligned}
&\left|\left(\frac{(n-m)^{\alpha\eta}}{B(\eta)\Gamma(\eta)}+\frac{\alpha^{\eta}(1-\eta)}{B(\eta)}\right)[f(a+b-m)+f(a+b-n)]\right.\\
&\left.-\alpha^{\eta}\left[{}^{ABConf}{}_{a+b-n^{+}}I^{\eta}\{f(a+b-m)\}+{}^{ABConf}{}_{a+b-m^{-}}I^{\eta}\{f(a+b-n)\}\right]\right|\\
\leq\ & \frac{(n-m)^{\alpha\eta+1}}{B(\eta)\Gamma(\eta)}\left(\int_0^1\left|[1-(1-t)^{\alpha}]^{\eta}-[1-t^{\alpha}]^{\eta}\right|^{p}dt\right)^{\frac{1}{p}}\\
&\times\left[|f'(a)|^q+|f'(b)|^q-\frac{|f'(m)|^q+|f'(n)|^q}{2}\right]^{\frac{1}{q}},
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, for $\eta\in[0,1]$, $\alpha\in(0,1]$, and $a\leq m<n\leq b$.

Proof. By Lemma 2.5 and using Hölder's inequality with Jensen-Mercer's inequality and the convexity of $|f'|^q$, we have

$$\begin{aligned}
&\left|\left(\frac{(n-m)^{\alpha\eta}}{B(\eta)\Gamma(\eta)}+\frac{\alpha^{\eta}(1-\eta)}{B(\eta)}\right)[f(a+b-m)+f(a+b-n)]\right.\\
&\left.-\alpha^{\eta}\left[{}^{ABConf}{}_{a+b-n^{+}}I^{\eta}\{f(a+b-m)\}+{}^{ABConf}{}_{a+b-m^{-}}I^{\eta}\{f(a+b-n)\}\right]\right|\\
\leq\ & \frac{(n-m)^{\alpha\eta+1}}{B(\eta)\Gamma(\eta)}\left(\int_0^1\left|[1-(1-t)^{\alpha}]^{\eta}-[1-t^{\alpha}]^{\eta}\right|^{p}dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1|f'(a+b-[tm+(1-t)n])|^q\,dt\right)^{\frac{1}{q}}\\
=\ & \frac{(n-m)^{\alpha\eta+1}}{B(\eta)\Gamma(\eta)}\left(\int_0^1\left|[1-(1-t)^{\alpha}]^{\eta}-[1-t^{\alpha}]^{\eta}\right|^{p}dt\right)^{\frac{1}{p}}\\
&\times\left[|f'(a)|^q+|f'(b)|^q-\frac{|f'(m)|^q+|f'(n)|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

Theorem 3.4. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function with $0 \le a < b$ and $f' \in L^1[a,b]$ such that $|f'|^q$ is a convex function on $[a,b]$. Then, the following ineqaulity for ABConf fractional integral holds:*

$$\begin{aligned}
&\left|\left(\frac{(n-m)^{\alpha\eta}}{B(\eta)\Gamma(\eta)}+\frac{\alpha^{\eta}(1-\eta)}{B(\eta)}\right)[f(a+b-m)+f(a+b-n)]\right.\\
&\left.-\alpha^{\eta}\left[{}^{ABConf}{}_{a+b-n^{+}}I^{\eta}\{f(a+b-m)\}+{}^{ABConf}{}_{a+b-m^{-}}I^{\eta}\{f(a+b-n)\}\right]\right|\\
\le\;&\frac{(n-m)^{\alpha\eta+1}}{B(\eta)\Gamma(\eta)}\left(\frac{1}{p}\int_0^1\left|[1-(1-t)^{\alpha}]^{\eta}-[1-t^{\alpha}]^{\eta}\right|^p dt\right)\\
&\times\left(\frac{1}{q}\int_0^1|f'(a+b-[tm+(1-t)n])|^q dt\right)\\
=\;&\frac{(n-m)^{\alpha\eta+1}}{B(\eta)\Gamma(\eta)}\left(\frac{1}{p}\int_0^1\left|[1-(1-t)^{\alpha}]^{\eta}-[1-t^{\alpha}]^{\eta}\right|^p dt\right)\\
&\times\frac{1}{q}\left[|f'(a)|^q+|f'(b)|^q-\frac{|f'(m)|^q+|f'(n)|^q}{2}\right],
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, for $\eta\in[0,1]$, $\alpha\in(0,1]$, and $a\le m<n\le b$.

Proof. By Lemma 2.5 and using Hölder's inequality with Young's inequality $xy \le \frac{1}{p}x^p + \frac{1}{q}y^q$ and the convexity of $|f'|^q$, we have

$$\begin{aligned}
&\left|\left(\frac{(n-m)^{\alpha\eta}}{B(\eta)\Gamma(\eta)}+\frac{\alpha^{\eta}(1-\eta)}{B(\eta)}\right)[f(a+b-m)+f(a+b-n)]\right.\\
&\left.-\alpha^{\eta}\left[{}^{AB-C}{}_{a+b-n^{+}}I^{\eta}\{f(a+b-m)\}+{}^{AB-C}{}_{a+b-m^{-}}I^{\eta}\{f(a+b-n)\}\right]\right|\\
\le\;&\frac{(n-m)^{\alpha\eta+1}}{B(\eta)\Gamma(\eta)}\left(\frac{1}{p}\int_0^1\left|[1-(1-t)^{\alpha}]^{\eta}-[1-t^{\alpha}]^{\eta}\right|^p dt\right)\\
&\times\left(\frac{1}{q}\int_0^1|f'(a+b-[tm+(1-t)n])|^q dt\right)\\
=\;&\frac{(n-m)^{\alpha\eta+1}}{B(\eta)\Gamma(\eta)}\left(\frac{1}{p}\int_0^1\left|[1-(1-t)^{\alpha}]^{\eta}-[1-t^{\alpha}]^{\eta}\right|^p dt\right)\\
&\times\frac{1}{q}\left[|f'(a)|^q+|f'(b)|^q-\frac{|f'(m)|^q+|f'(n)|^q}{2}\right].
\end{aligned}$$

4. Conclusions

This paper proposes fractional integral operators and establishes new Hermite-Hadamard-Mercer type inequalities in this setting. A fundamental integral identity is proved, expressing a symmetric endpoint-operator combination in terms of a weighted integral involving f. This identity provides a robust framework for obtaining Hermite-Hadamard-Mercer type bounds under standard smoothness and integrability assumptions.

Under convexity assumptions on $|f'|$ and $|f'|^q$, three families of inequalities are derived via Jensen-Mercer's inequality combined with Hölder's and Young's inequalities. The obtained bounds explicitly reflect the underlying kernel and parameters, and they admit reductions to known special cases through appropriate parameter selections, highlighting both generality and compatibility with existing fractional operators.

Potential future work includes extending the approach to broader convexity classes and higher-order derivatives, developing multi-variable analogues (e.g., co-ordinated convexity on rectangles), and optimizing kernel-dependent constants to sharpen the estimates and enhance applicability to fractional numerical quadrature and related error bounds.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

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