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Research Article

A five-step approximation method for a nonlinear delay integral equation in hyperbolic spaces: A qualitative study

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Abstract: In this manuscript, we design and present a Jungck-type iterative method to find the common fixed point of a pair of mappings with weak compatibility in hyperbolic metric spaces. Strong convergence is perfomed to estimate the common fixed point, and stability of the designed iterative scheme is established under suitable assumptions. Further, Δ-convergence is established, and a theoretical result is outlined to exhibit the coherence and effectiveness of our scheme with some of its counterparts. A numerical case study is set forth to evidence the convergence and effectiveness of the proposed method. Finally, the efficacy and applicability of our proposed method is illustrated by implementing it to explore a nonlinear delay integral equation (NDIE).

Keywords: hyperbolic metric spaces; coincidence point; Jungck-type iterative scheme; convergence and stability; delayed integral equation
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1. Introduction

Over the last few years, the fixed point theory has shown a multifaceted development and become one of the most applicable and significant research fields. Since a number of diverse problems that we frequently deal with, including ordinary differential equations (ODEs), partial differential equations (PDEs), integral equations, variational inequalities, inclusion problems, optimization problems, etc., can be examined by transforming into a fixed point problem (FPP). Meanwhile, the availability of a vast range of techniques for solving FPPs has made this field admirable among researchers. Among these techniques, iterative methods are the most significant, convenient, and effective procedures for approximating fixed points. Iterative approximations are useful for exploring the convergence behavior, and simultaneously they are useful for stability and data independence. The novelty of iterative methods fascinated researchers for its constantly innovating accelerated and efficient convergence; see, [2, 6, 16, 17].

The conception of multi-step iterative methods was initially developed in the context of linear problems. The transition of linear structures into nonlinear structures is not so trivial because of the construction procedure of the iteration schemes. However, nonlinear iterative procedures are more efficient for dealing with complex, nonlinear functions. They impart a faster rate of convergence as well as widen the impact of fixed point approximation in various interdisciplinary fields such as dynamical systems, chaos theory, machine learning, artificial intelligence, optimization, game theory, operations research, etc.

Takahashi introduced the concept of metric space with convex structure, investigated the properties of the space, and named it the convex metric space. A Banach space and its convex subsets are convex metric spaces. However, a Frechet space need not be a convex metric space. The author examined the fixed point results for nonexpansive mappings, which generalized the fixed point theorems due to Browder [10] and Kirk [24]. Kohlenbach [27] coined a new class of spaces called hyperbolic spaces by imposing a convex structure to the metric space. These spaces allow analyzation of the convergence behavior of iterative methods. Hyperbolic spaces defined in [27] are more general spaces, which include the hyperbolic space due to Reich and Shafrir [36], normed spaces, CAT(0) spaces, and the Poincaré disk, the Poincaré half-plane, or the Beltrami-Klein model; see [18, 26, 39] as well as the Hilbert ball with hyperbolic metric; for details, we refer to [37, 38].

The conception of Δ-convergence was planted by Lim [31] in metric spaces. Further, Kirk and Panyanak [25] specialized the conception of Lim to CAT(0) spaces and reported some properties such as the Opial property, the Kadec–Klee property, and the demiclosedness principle for LANE mappings in this setting. An analogous concept of convergence, namely, almost convergence, was given by Kuczumow [28] in Banach spaces.

Some of the most common and frequently used iterative schemes for exploring fixed points include Mann [32], Ishikawa [22], Noor [33], S-iteration [1], M-iteration [43], and their hybrid forms. For the sake of efficient convergence rate and horizontal growth of fixed point theory, numerous schemes have been designed by researchers. Recently, Alam and Rohen [4] proposed a refined iterative method for approximating the fixed point of a contraction mapping as under:

$$\begin{cases} \omega_1 \in \mathcal{B}, \\ \omega_{m+1} = \mathcal{G}^2 \varrho_m, \\ \varrho_m = \mathcal{G}[(1-p_m)\sigma_m + p_m \mathcal{G}\sigma_m], \\ \sigma_m = \mathcal{G}[(1-r_m)\omega_m + r_m \mathcal{G}\varpi_m], \\ \varpi_m = \mathcal{G}[(1-s_m)\omega_m + s_m \mathcal{G}\omega_m], m \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\{p_m\}, \{r_m\}, \{s_m\} \subset (0, 1)$. They proved the stability of the proposed scheme and substantiated the convergence analysis to compare the convergence rate with some well-known schemes. Further, the authors demonstrated the data dependence and finally implemented their theoretical outcomes to explore a fractional Volterra-Fredholm integro-differential problem. Okeke et al. [34] designed an efficient four step-iterative scheme called the AG iterative described as under:

$$\begin{cases} \omega_0 \in \mathcal{B}, \\ \omega_{m+1} = \mathcal{G}\varrho_m, \\ \varrho_m = \mathcal{G}[(1-p_m)\sigma_m + p_m\mathcal{G}\sigma_m], \\ \sigma_m = (1-r_m)\mathcal{G}\omega_m + r_m\mathcal{G}\varpi_m, \\ \varpi_m = (1-s_m)\omega_m + s_m\mathcal{G}\omega_m, \ m \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\{p_m\}, \{r_m\}, \{s_m\} \subset [0,1]$. The authors studied the convergence theorem by approximating the fixed point of a contraction mapping. They established the stability of the proposed scheme and summarized the weak convergence for Suzuki's generalized non-expansive mapping.

Let $\mathcal{G}, \mathcal{S} : C \to \mathcal{B}$ be a pair of non-self mappings such that $\mathcal{G}(C) \subseteq \mathcal{S}(C)$. For an arbitrary point $\omega_0 \in C$, the Jungck-iterative scheme [23] is described as

$$\mathcal{S}\omega_{m+1} = \mathcal{G}\omega_m. \tag{1.3}$$

Jungck [23] utilized the scheme (1.3) to estimate the common fixed point of the mappings $\mathcal{G}$ and $\mathcal{S}$, obeying the contraction condition

$$d(\mathcal{G}\omega, \mathcal{G}\varrho) \leq \tau d(\mathcal{S}\omega, \mathcal{S}\varrho), \tau \in [0,1). \tag{1.4}$$

In 2005, Singh et al. [40] proposed the Jungck–Mann iterative method as given below:

$$\mathcal{S}\omega_{m+1} = (1-p_m)\mathcal{S}\omega_m + p_m\mathcal{G}\omega_m, \tag{1.5}$$

where $\{p_m\}$ lies in $[0,1]$. Authors reported the stability of scheme (1.5) for a pair of Jungck–Osilike-type mappings on an arbitrary set. In this sequel, Sen and Karapinar [14] investigated a cyclic Jungck modified TS-iterative scheme. They analyzed the convergence behavior of quasi-cyclic and cyclic Jungck modified TS-iterative schemes in complete metric spaces and Banach spaces. They also reported uniqueness of the best proximity points. Later, Hussain et al. [21] designed a Jungck-CR iterative scheme as under:

$$\begin{cases} \mathcal{S}\omega_{m+1} = p_m\mathcal{G}\zeta_m + (1-p_m)\mathcal{S}\zeta_m, \\ \mathcal{S}\zeta_m = r_m\mathcal{G}\varrho_m + (1-r_m)\mathcal{G}\omega_m, \\ \mathcal{S}\varrho_m = s_m\mathcal{G}\omega_m + (1-s_m)\mathcal{S}\omega_m, \ m \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\{p_m\}, \{r_m\}, \{s_m\} \subset [0,1]$. The authors analyzed their scheme to set up the strong convergence in arbitrary Banach spaces. They compared the efficiency of its convergence with some Jungck-type iterative methods and tackled a recurrent neural networks analysis by implementing their scheme. Recently, a Jungck-DK iterative scheme was constructed by Guran et al. [19] as follows:

$$\begin{cases} \mathcal{S}\omega_{m+1} = p_m\mathcal{G}\zeta_m + (1-p_m)\mathcal{G}\varrho_m, \\ \mathcal{S}\zeta_m = p_m\mathcal{G}\varrho_m + (1-p_m)\mathcal{G}\omega_m, \\ \mathcal{S}\varrho_m = \mathcal{S}\omega_m, \ m \in \mathbb{N}, \end{cases} \tag{1.7}$$

where $\{p_m\}, \{r_m\} \subset [0, 1]$. The authors demonstrated that their scheme converges faster than some key Jungck-type schemes and visualizes Mandelbrot and Julia sets for polynomial functions generated by their iteration procedure.

Recently, Copur et al. [13] proposed the Jungck-type iteration algorithm for a pair of quasi-contractive operators and investigated strong convergence. Authors established stability of their proposed scheme and noted data dependence results. In continuation of the hybridization of Jungck-type iterative methods, very recently, Alam and Rohen [3] developed a Jungck-AI iterative scheme as follows:

$$\begin{cases} \mathcal{S}\omega_{m+1} = \mathcal{G}\zeta_m, \\ \mathcal{S}\zeta_m = \mathcal{G}\varrho_m, \\ \mathcal{S}\varrho_m = \mathcal{G}\varpi_m, \\ \mathcal{S}\varpi_m = p_m\mathcal{G}\omega_m + (1 - p_m)\mathcal{S}\omega_m, m \in \mathbb{N}, \end{cases} \tag{1.8}$$

where $\{p_m\} \subseteq (0, 1)$. The authors presented a strong convergence result and established the stability of the proposed scheme. They also manifested that their scheme is more efficient than (1.6) and (1.7).

Driven and encouraged by the earlier revealed results and discussions, we construct a Jungck-AG iterative scheme in hyperbolic metric spaces as under:

$$\begin{cases} \mathcal{S}\omega_{m+1} = \mathcal{G}\varrho_m, \\ \mathcal{S}\varrho_m = \mathcal{G}\zeta_m, \\ \mathcal{S}\zeta_m = (1 - p_m)\mathcal{S}\sigma_m + p_m\mathcal{G}\sigma_m, \\ \mathcal{S}\sigma_m = (1 - r_m)\mathcal{G}\omega_m + r_m\mathcal{G}\varpi_m, \\ \mathcal{S}\varpi_m = (1 - s_m)\mathcal{S}\omega_m + s_m\mathcal{G}\omega_m, \quad m \in \mathbb{N}, \end{cases} \tag{1.9}$$

where $\{p_m\}, \{r_m\}, \{s_m\} \subset (0, 1)$. The goal of our work is to present strong convergence and stability of our proposed scheme under generalized contractive conditions (2.2) and (2.3) in hyperbolic spaces. We also establish Δ-convergence of the Jungk-AG iterative method. A theoretical result is outlined to exhibit the effectiveness of our scheme, and a numerical case study is provided to illustrate the convergence and efficacy of the proposed method. Finally, the significance of our designed scheme is ensured by implementing it to explore a NDIE.

2. Preliminaries

Herein, we shall collect some basic definitions and properties and present useful results related to hyperbolic spaces, which will guide us to accomplish the goals of this paper.

Definition 2.1. *[27] A hyperbolic space $(\mathcal{B}, d, \Phi)$ is a metric space $(\mathcal{B}, d)$ with a convex mapping $\Phi : \mathcal{B} \times \mathcal{B} \times [0, 1] \to \mathcal{B}$ if for all $\omega, \varrho, \varpi, \varsigma \in \mathcal{B}$ and $a, b \in [0, 1]$,*

(i) $d(\omega, \Phi(\varrho, \varpi, a)) \leq ad(\omega, \varpi) + (1 - a)d(\omega, \varrho)$,
(ii) $d(\Phi(\omega, \varrho, a), \Phi(\omega, \varrho, b)) \leq |a - b|d(\omega, \varrho)$,
(iii) $\Phi(\omega, \varrho, a) = \Phi(\omega, \varrho, 1 - a)$,
(iv) $d(\Phi(\omega, \varrho, a), \Phi(\varpi, \varsigma, a)) \leq ad(\varrho, \varsigma) + (1 - a)d(\omega, \varpi)$.

Note that hyperbolic spaces are unified frameworks which contain CAT(0) spaces, Banach spaces, and linear spaces. A hyperbolic space $(\mathcal{B}, d, \Phi)$ satisfies condition (i), then $(\mathcal{B}, d, \Phi)$ turns into a convex metric space; see [42]. More precisely, every hyperbolic space is a convex metric space.

Definition 2.2. *[12] A hyperbolic space $(\mathcal{B}, d, \Phi)$ is referred to as uniformly convex if for any $\epsilon \in (0, 2]$ and $\theta > 0$, there exists $\tau \in (0, 1]$ complying with $d(\omega, \varrho) \leq \theta, d(\varpi, \omega) \leq \theta$, and $d(\varrho, \varpi) \leq \theta\epsilon$ such that $d(\Phi(\varrho, \varpi, \frac{1}{2}), \omega) \leq (1 - \tau)\theta, \forall \omega, \varrho, \varpi \in \mathcal{B}$.*

Definition 2.3. *[47] A mapping $\zeta : (0, \infty) \times (0, 2] \to (0, 1)$ is referred to as a modulus of uniform convexity of hyperbolic space $(\mathcal{B}, d, \Phi)$ if for each $\epsilon \in (0, 2]$, there exists $r > 0$ such that $\tau = \zeta(r, \epsilon)$. For a fixed ϵ, if ζ decreases with r, then ζ is as monotone.*

In a hyperbolic space $(\mathcal{B}, d, \Phi)$, if a sequence $\{\omega_m\}$ is bounded, then the radius function $R(\cdot, \{\omega_m\}) : \mathcal{B} \to [0, \infty)$ is defined as by $R(\omega, \{\omega_m\}) = \limsup_{m\to\infty} d(\omega, \omega_m), \forall \omega \in \mathcal{B}$. If $\emptyset \neq C \subseteq \mathcal{B}$, then the asymptotic radius and asymptotic center of C are expressed as $\mathcal{B}r_C(\{\omega_m\}) = \limsup_{m\to\infty} \inf\{R(\omega, \{\omega_m\}) : \omega \in C\}$ and $\mathcal{B}c_C(\{\omega_m\}) = \{\omega \in \mathcal{B} : R(\omega, \{\omega_m\}) \leq R(\varrho, \{\omega_m\}), \forall \varrho \in C\}$.

Definition 2.4. *[24] A sequence $\{\omega_m\}$ in $(\mathcal{B}, d, \Phi)$ is referred to as Δ-convergent to $\omega \in \mathcal{B}$ if for every subsequence $\{\omega_{m_k}\}$ of $\{\omega_m\}$,*

$$\mathcal{B}c_{\mathcal{B}}(\{\omega_{m_k}\}) = \{\omega\}.$$

Then ω is known as the Δ-limit of $\{\omega_m\}$, and we express it as $\Delta\text{-}\lim_{m\to\infty} \omega_m = \omega$.

Lemma 2.1. *[30] Let $\emptyset \neq C \subseteq \mathcal{B}$ be a closed convex set. Then each bounded sequence $\{\omega_m\}$ in a complete uniformly hyperbolic space $(\mathcal{B}, d, \Phi)$ with a monotone modulus of convexity ξ has a unique asymptotic center in C.*

The following definition is crucial for comparing the efficiency of the iterative processes.

Definition 2.5. *[5] Suppose that $\{\omega_m\}$ and $\{\varrho_m\}$ are two real sequences such that $\omega_m \to \omega$ and $\varrho_m \to \omega$ as $m \to \infty$. Then, convergence of $\{\omega_m\}$ is faster than $\{\varrho_m\}$ if*

$$\lim_{m\to\infty} \frac{|\omega_m - \omega|}{|\varrho_m - \varrho|} = 0. \tag{2.1}$$

Definition 2.6. *[41] Suppose that $\mathcal{G}$ and $\mathcal{S}$ are two self-mappings on $\mathcal{B}$. An element $\omega \in \mathcal{B}$ is known as the common fixed point of $\mathcal{G}$ and $\mathcal{S}$ if $\mathcal{G}\omega = \mathcal{S}\omega = \omega$. It is the coincidence point of the pair $(\mathcal{G}, \mathcal{S})$ if $\mathcal{G}\omega = \mathcal{S}\omega$. If for some $\omega \in \mathcal{B}$, $\varrho = \mathcal{G}\omega = \mathcal{S}\omega$, then ϱ is referred to as the point of coincidence of $\mathcal{G}$ and $\mathcal{S}$. A pair $(\mathcal{G}, \mathcal{S})$ is called weakly compatible if $\mathcal{G}$ and $\mathcal{S}$ commute at the coincidence point.*

Definition 2.7. *[35] Let $\mathcal{G}, \mathcal{S} : C \to \mathcal{B}$ be non-self mappings such that $\mathcal{G}(C) \subseteq \mathcal{S}(C)$ and $\psi : (0, \infty) \to (0, \infty)$ be a monotonic function with $\psi(0) = 0$. Then $\mathcal{G}, \mathcal{S}$ are said to obey the general contractive condition if, for all $\omega, \varrho \in C$, there exists $\tau \in [0, 1)$ such that*

$$d(\mathcal{G}\omega, \mathcal{G}\varrho) \leq \psi(d(\mathcal{S}\omega, \mathcal{G}\omega)) + \tau d(\mathcal{S}\omega, \mathcal{S}\varrho). \tag{2.2}$$

Remark 2.1. *The contractive condition expressed in (2.2) is a generalized form of contractive condition (1.4). It is worth to mention that the pair $(\mathcal{G}, \mathcal{S})$ satisfying (2.2) may not hold the coincidence point.*

The following example is illustrated to support Remark 2.1.

Example 2.1. *[29] Let $\mathcal{B} = [0,\infty)$ with usual the metric and define $\mathcal{G}, \mathcal{S} : \mathcal{B} \to \mathcal{B}$ such that*

$$\mathcal{G}\omega = \begin{cases} 3, & 0 \le \omega \le 1, \\ 2, & 1 < \omega, \end{cases} \qquad \textit{and} \qquad \mathcal{S}\omega = \omega, \forall \omega \in [0,\infty).$$

Then $\mathcal{S}([0,\infty))$ is a complete subspace of $[0,\infty)$ and $\mathcal{G}([0,\infty)) \subset \mathcal{S}([0,\infty))$. Define the increasing function $\psi : [0,\infty) \to [0,\infty)$ by $\psi(\omega) = l\sqrt{\omega}, l \ge 1$. Clearly, $\psi(0) = 0$. Assume that $\omega < \varrho$ and take the following cases into consideration.
Case 1. *If $0 \le \omega < \varrho \le 1$ or $1 < \omega < \varrho$, then $d(\mathcal{G}\omega, \mathcal{G}\varrho) = |\mathcal{G}\omega - \mathcal{G}\varrho| = 0$. Then there is nothing to prove.*
Case 2. *If $0 \le \omega \le 1 < \varrho$, then $d(\mathcal{G}\omega, \mathcal{G}\varrho) = |\mathcal{G}\omega - \mathcal{G}\varrho| = 1$ and $d(\mathcal{S}\omega, \mathcal{G}\omega) = |\mathcal{S}\omega - \mathcal{G}\omega| = 3 - \omega$. Now, $\psi(d(\mathcal{S}\omega, \mathcal{G}\omega))) = l\sqrt{3-\omega} \ge l \ge 1$. Thus, for any $0 \le \tau < 1$, we can write*

$$1 = d(\mathcal{G}\omega, \mathcal{G}\varrho) \le \psi(d(\mathcal{S}\omega, \mathcal{G}\omega)) \le \psi(d(\mathcal{S}\omega, \mathcal{G}\omega)) + \tau d(\mathcal{S}\omega, \mathcal{S}\varrho).$$

Thus, the inequality (2.2) holds; however, the pair $(\mathcal{S}, \mathcal{G})$ does not have a coincidence point.

The following crucial result guarantees the existence of the coincidence point of the pair $(\mathcal{S}, \mathcal{G})$ in which one more assumption in addition to condition (2.2) is considered.

Proposition 2.1. *[8] Let $\mathcal{B}$ be a complete metric space and let $\mathcal{S}, \mathcal{G} : \mathcal{B} \to \mathcal{B}$ be non-self mappings such that $\mathcal{G}(\mathcal{B}) \subseteq \mathcal{S}(\mathcal{B})$ and $\mathcal{S}(\mathcal{B})$ is a complete subspace of $\mathcal{B}$. If there exists a $\tau \in [0,1)$ and $l \ge 0$ such that*

$$d(\mathcal{G}\omega, \mathcal{G}\varrho) \le \tau d(\mathcal{S}\omega, \mathcal{S}\varrho) + l d(\mathcal{S}\varrho, \mathcal{G}\omega), \forall \omega, \varrho \in \mathcal{B}, \tag{2.3}$$

then $\mathcal{G}$ and $\mathcal{S}$ have a point of coincidence in $\mathcal{B}$.

Definition 2.8. *[40] Let $\mathcal{B} \neq \emptyset$ be a convex Banach space. For some function Λ and sequence $\{\mathcal{S}\gamma_m\}$ in $\mathcal{B}$, there is a pair of non-self mapping $(\mathcal{G}, \mathcal{S})$ such that $\mathcal{G}(\mathcal{B}) \subseteq \mathcal{S}(\mathcal{B})$. Then a converging iterative scheme $\mathcal{S}\omega_{m+1} = \Lambda(\omega_m, \mathcal{G})$ converging to a point of coincidence ω is said to be $(\mathcal{S}, \mathcal{G})$-stable if*

$$\lim_{m\to\infty} d(\mathcal{S}\gamma_m, \Lambda(\gamma_m, \mathcal{G})) = 0 \Leftrightarrow \lim_{m\to\infty} \mathcal{S}\gamma_m = \omega. \tag{2.4}$$

Lemma 2.2. *[45] Suppose the nonnegative real sequences $\{\omega_m\}_{m=1}^{\infty}$ and $\{\varrho_m\}_{m=1}^{\infty}$ satisfy*

$$\omega_{m+1} \le (1 - \nu_m)\omega_m + \varrho_m,$$

where $\nu_m \in (0,1)$, $\sum\limits_{m=0}^{\infty} \nu_m = \infty$, and $\lim\limits_{m\to\infty} \dfrac{\varrho_m}{\nu_m} = 0$. Then $\lim\limits_{m\to\infty} \omega_m = 0$.

Now onward, we consider $C \neq \emptyset$ a closed convex subset of $\mathcal{B}$, and $(\mathcal{B}, d, \Phi)$ is a complete uniformly convex hyperbolic space with a monotone modulus of convexity ξ.

3. Convergence and stability results

Next, we shall present convergence analysis and stability of our Jungck-AG iterative scheme (1.9). We shall also establish an analytical proof to showcase the better convergence rate of our newly proposed scheme. First of all, we shall express Jungck-AG scheme (1.9) in hyperbolic space as under:

$$\begin{cases} \mathcal{S}\omega_{m+1} = \mathcal{G}\varrho_m, \\ \mathcal{S}\varrho_m = \mathcal{G}\zeta_m, \\ \mathcal{S}\zeta_m = \Phi(\mathcal{S}\sigma_m, \mathcal{G}\sigma_m, p_m), \\ \mathcal{S}\sigma_m = \Phi(\mathcal{G}\omega_m, \mathcal{G}\varpi_m, r_m), \\ \mathcal{S}\varpi_m = \Phi(\mathcal{S}\omega_m, \mathcal{G}\omega_m, s_m), \forall m \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{p_m\}, \{r_m\}, \{s_m\} \subset (0, 1)$.

Theorem 3.1. *Let $(\mathcal{B}, d, \Phi)$ be a hyperbolic space and let the non-self mappings $\mathcal{S}, \mathcal{G} : C \to \mathcal{B}$ obey contractive conditions (2.2) and (2.3) such that $\mathcal{G}(C) \subseteq \mathcal{S}(C)$, $\mathcal{S}(C)$ is a complete subspace of $\mathcal{B}$ and $\mathcal{S}\varrho = \mathcal{G}\varrho = \omega$ (say). Then the sequence $\{\mathcal{S}\omega_m\}$ produced by the Jungck-AG iterative scheme (3.1) converges strongly to the coincidence point ω. Also, ω is the unique common fixed point of $\mathcal{S}$ and $\mathcal{G}$, provided $C = \mathcal{B}$ and $\mathcal{S}$ and $\mathcal{G}$ are weakly compatible.*

Proof. Now, first we show that $\{\mathcal{S}\omega_m\}$ produced by the Jungck-AG iterative scheme (3.1) converges to ω. From (3.1) and (2.2), we obtain the following inequalities:

$$\begin{aligned} d(\mathcal{S}\omega_{m+1}, \omega) &= d(\mathcal{G}\varrho_m, \omega) \\ &= d(\mathcal{G}\varrho_m, \mathcal{G}\varrho) \\ &\leq \psi(d(\mathcal{S}\varrho, \mathcal{G}\varrho)) + \tau d(\mathcal{S}\varrho_m, \mathcal{S}\varrho) \\ &\leq \tau d(\mathcal{S}\varrho_m, \mathcal{S}\varrho) = \tau d(\mathcal{S}\varrho_m, \omega). \end{aligned} \tag{3.2}$$

Likewise, other relations in (3.2) yield

$$\begin{aligned} d(\mathcal{S}\zeta_m, \omega) &= d(\Phi(\mathcal{S}\sigma_m, \mathcal{G}\sigma_m, p_m), \omega) \\ &\leq p_m d(\mathcal{G}\sigma_m, \omega) + (1 - p_m) d(\mathcal{S}\sigma_m, \omega) \\ &= p_m d(\mathcal{G}\sigma_m, \mathcal{G}\varrho) + (1 - p_m) d(\mathcal{S}\sigma_m, \omega) \\ &\leq p_m[\psi(d(\mathcal{S}\varrho, \mathcal{G}\varrho)) + \tau d(\mathcal{S}\sigma_m, \mathcal{S}\varrho)] + (1 - p_m) d(\mathcal{S}\sigma_m, \omega) \\ &\leq p_m \tau d(\mathcal{S}\sigma_m, \omega) + (1 - p_m) d(\mathcal{S}\sigma_m, \omega) \\ &= [1 - p_m(1 - \tau)] d(\mathcal{S}\sigma_m, \omega), \end{aligned} \tag{3.3}$$

$$\begin{aligned} d(\mathcal{S}\varrho_m, \omega) &= d(\mathcal{G}\zeta_m, \omega) \\ &= d(\mathcal{G}\zeta_m, \mathcal{G}\varrho) \\ &\leq \psi(d(\mathcal{S}\varrho, \mathcal{G}\varrho)) + \tau d(\mathcal{S}\zeta_m, \mathcal{S}\varrho) \\ &\leq \tau d(\mathcal{S}\zeta_m, \mathcal{S}\varrho) = \tau d(\mathcal{S}\zeta_m, \omega), \end{aligned}$$

which together with (3.3) turns into

$$d(\mathcal{S}\varrho_m, \omega) \leq \tau[1 - p_m(1 - \tau)] d(\mathcal{S}\sigma_m, \omega), \tag{3.4}$$

$$\begin{aligned} d(\mathcal{S}\sigma_m, \omega) &= d(\Phi(\mathcal{G}\omega_m, \mathcal{G}\varpi_m, r_m), \omega) \\ &\leq (1 - r_m)d(\mathcal{G}\omega_m, \omega) + r_m d(\mathcal{G}\varpi_m, \omega) \\ &= (1 - r_m)d(\mathcal{G}\omega_m, \mathcal{G}\varrho) + r_m d(\mathcal{G}\varpi_m, \mathcal{G}\varrho) \\ &\leq (1 - r_m)[\psi(d(\mathcal{S}\varrho, \mathcal{G}\varrho)) + \tau d(\mathcal{S}\omega_m, \mathcal{S}\varrho)] \\ &\quad + r_m[\psi(d(\mathcal{S}\varrho, \mathcal{G}\varrho)) + \tau d(\mathcal{S}\varpi_m, \mathcal{S}\varrho)] \\ &\leq (1 - r_m)\tau d(\mathcal{S}\omega_m, \omega) + r_m\tau d(\mathcal{S}\varpi_m, \omega), \end{aligned} \tag{3.5}$$

$$\begin{aligned} d(\mathcal{S}\varpi_m, \omega) &= d(\Phi(\mathcal{S}\omega_m, \mathcal{G}\omega_m, s_m), \omega) \\ &\leq s_m d(\mathcal{G}\omega_m, \omega) + (1 - s_m)d(\mathcal{S}\omega_m, \omega) \\ &= s_m d(\mathcal{G}\omega_m, \mathcal{G}\varrho) + (1 - s_m)d(\mathcal{S}\omega_m, \omega) \\ &\leq s_m[\psi(d(\mathcal{S}\varrho, \mathcal{G}\varrho)) + \tau d(\mathcal{S}\omega_m, \mathcal{S}\varrho)] \\ &\quad + (1 - s_m)d(\mathcal{S}\omega_m, \omega) \\ &\leq s_m\tau d(\mathcal{S}\omega_m, \omega) + (1 - s_m)d(\mathcal{S}\omega_m, \omega) \\ &= [1 - s_m(1 - \tau)]d(\mathcal{S}\omega_m, \omega). \end{aligned} \tag{3.6}$$

Thus, the back substitution from (3.2)–(3.6) yields

$$d(\mathcal{S}\omega_{m+1}, \omega) \leq \tau^3[1 - p_m(1 - \tau)][1 - r_m[1 - (1 - s_m(1 - \tau))]]d(\mathcal{S}\omega_m, \omega). \tag{3.7}$$

Taking the advantage of the hypotheses $\{p_m\}, \{r_m\}, \{s_m\} \subseteq (0, 1)$ and $\tau \in [0, 1)$, we acquire $1 - p_m(1 - \tau) < 1, 1 - r_m[1 - (1 - s_m(1 - \tau))] < 1$, and thus (3.7) turns into

$$\begin{aligned} d(\mathcal{S}\omega_{m+1}, \omega) &\leq \tau^3 d(\mathcal{S}\omega_m, \omega) \\ &\leq \tau^6 d(\mathcal{S}\omega_{m-1}, \omega) \\ &\leq \tau^{3m} d(\mathcal{S}\omega_1, \omega). \end{aligned} \tag{3.8}$$

Again, $0 \leq \tau < 1$ leads to the conclusion $\lim_{m\to\infty} d(\mathcal{S}\omega_{m+1}, \omega) = 0$, and hence $\{\mathcal{S}\omega_m\}$ converges strongly to ω. Next, it remains to substantiate ω is the common fixed point of $\mathcal{S}$ and $\mathcal{G}$. To substantiate the uniqueness, suppose ω^* is another point of coincidence. Then there exists $\varrho^* \in C$ such that $\mathcal{S}\varrho^* = \mathcal{G}\varrho^* = \omega^*$. From (2.2), we can express

$$\begin{aligned} 0 \leq d(\omega, \omega^*) &= d(\mathcal{G}\varrho, \mathcal{G}\varrho^*) \\ &\leq \psi(d(\mathcal{S}\varrho, \mathcal{G}\varrho)) + \tau d(\mathcal{S}\varrho, \mathcal{S}\varrho^*) \\ &\leq \tau d(\mathcal{S}\varrho, \mathcal{S}\varrho^*) \\ &= \tau d(\omega, \omega^*). \end{aligned} \tag{3.9}$$

Since $0 \leq \tau < 1$, (3.9) yields $d(\omega, \omega^*) = 0$, and thus $\omega = \omega^*$. Also, the weak compatibility of $\mathcal{S}$ and $\mathcal{G}$ yields $\mathcal{G}\omega = \mathcal{G}\mathcal{S}\varrho = \mathcal{G}\mathcal{G}\varrho$, i.e., $\mathcal{G}\omega$ is a point of coincidence of $\mathcal{S}$, and $\mathcal{G}$, and the uniqueness gives $\omega = \mathcal{G}\omega$. Accordingly, $\mathcal{S}\omega = \omega = \mathcal{G}\omega$, i.e., ω is the unique common fixed point of $\mathcal{S}$ and $\mathcal{G}$. □

Next, we shall present a significant property called the stability of our scheme (3.1). In fact, the stability of an iteration scheme is the property that does not affect the convergence regardless of numerical error that occurred during approximation or small changes in the initial values. This important characterization was brought for the first time by Harder and Hicks [20]. Later, Berinde [9] and Olatinwo and Postolache [35] implemented this conception in various iterative schemes.

Theorem 3.2. *Suppose $\mathcal{S}$ and $\mathcal{G}$ are the same and obey the same assumptions as in Theorem 3.1. Then the sequence $\{\mathcal{S}\omega_m\}$ produced by the Jungck-AG iterative scheme (3.1) is $(\mathcal{S},\mathcal{G})$-stable, provided $\{s_m\}$ is bounded away from 0.*

Proof. Let Λ be an arbitrary function, and the sequence $\{\mathcal{S}\omega_m\}$ initiated by (3.1) is expressed as $\mathcal{S}\omega_{m+1} = \Lambda(\omega_m,\mathcal{G})$ so that for some $\varrho \in \mathcal{B}$, $\omega_m \to \omega = \mathcal{S}\varrho = \mathcal{G}\varrho$. Suppose that $\{\mathcal{S}\gamma_m\}$ is an arbitrary sequence, then recalling the triangle inequality, we obtain

$$d(\mathcal{S}\gamma_{m+1},\omega) \le d(\mathcal{S}\gamma_{m+1},\Lambda(\gamma_m,\mathcal{G})) + d(\Lambda(\gamma_m,\mathcal{G}),\omega), \tag{3.10}$$

where

$$\begin{cases} \Lambda(\gamma_m,\mathcal{G}) = \mathcal{G}\varrho_m, \\ \mathcal{S}\varrho_m = \mathcal{G}\zeta_m, \\ \mathcal{S}\zeta_m = \Phi(\mathcal{S}\sigma_m,\mathcal{G}\sigma_m,p_m), \\ \mathcal{S}\sigma_m = \Phi(\mathcal{G}\omega_m,\mathcal{G}\varpi_m,r_m), \\ \mathcal{S}\varpi_m = \Phi(\mathcal{S}\gamma_m,\mathcal{G}\gamma_m,s_m), \forall m \in \mathbb{N}. \end{cases} \tag{3.11}$$

Proceeding in the same manner as from (3.2)–(3.7), inequality (3.10) turns into

$$d(\mathcal{S}\gamma_{m+1},\omega) \le d(\mathcal{S}\gamma_{m+1},\Lambda(\gamma_m,\mathcal{G})) + \tau^3[1-p_m(1-\tau)][1-r_m[1-(1-s_m(1-\tau))]]d(\mathcal{S}\gamma_m,\omega). \tag{3.12}$$

Since $1-p_m(1-\tau) < 1$, take $\vartheta_m = d(\mathcal{S}\gamma_m,\omega)$, $\nu_m = r_m[1-(1-s_m(1-\tau))]$ and $\delta_m = d(\mathcal{S}\gamma_{m+1},\Lambda(\gamma_m,\mathcal{G}))$. If $\lim_{m\to\infty} \delta_m = 0$, then $\lim_{m\to\infty} \frac{\delta_m}{\nu_m} = 0$. Hence by Lemma 2.2, $\lim_{m\to\infty} d(\mathcal{S}\gamma_m,\omega) = 0$, that is, $\lim_{m\to\infty} \mathcal{S}\gamma_m = \omega$.

On the other hand, suppose that $\mathcal{S}\gamma_m \to \omega$ as $m \to \infty$, which amounts to, say, $\lim_{m\to\infty} d(\mathcal{S}\gamma_m,\omega) = 0$, and so $\lim_{m\to\infty} d(\mathcal{S}\gamma_{m+1},\omega) = 0$. Then, one can express

$$\begin{aligned} &d(\mathcal{S}\gamma_{m+1},\Lambda(\gamma_m,\mathcal{G})) \\ &\le d(\mathcal{S}\gamma_{m+1},\omega) + d(\Lambda(\gamma_m,\mathcal{G}),\omega) \\ &\le d(\mathcal{S}\gamma_{m+1},\omega) + \tau^3[1-p_m(1-\tau)][1-r_m[1-(1-s_m(1-\tau))]]d(\mathcal{S}\gamma_m,\omega), \end{aligned} \tag{3.13}$$

which yields $\lim_{m\to\infty} d(\mathcal{S}\gamma_{m+1},\Lambda(\gamma_m,\mathcal{G})) = 0$, and hence the Jungck-AG scheme (3.1) is $(\mathcal{S},\mathcal{G})$-stable. □

In the next theorem, we shall put forward the Δ-convergence of the Jungck-AG iterative scheme (3.1).

Theorem 3.3. *Let $(\mathcal{B},d,\Phi)$ be a hyperbolic space and let the mappings $\mathcal{S},\mathcal{G} : C \to \mathcal{B}$ obey contractive condition (2.2) such that $\mathcal{G}(C) \subseteq \mathcal{S}(C)$, $\mathcal{S}(C)$ is a complete subspace of $\mathcal{B}$ and $\mathcal{S}\varrho = \mathcal{G}\varrho = \omega$ (say). Then the sequence $\{\mathcal{S}\omega_m\}$ produced by the Jungck-AG iterative scheme (3.1) is Δ-convergent to ω. Also, ω is the unique common fixed point of $\mathcal{S}$ and $\mathcal{G}$, provided $C = \mathcal{B}$ and $\mathcal{S}$ and $\mathcal{G}$ are weakly compatible.*

Proof. It is evident from Theorem 3.1 that the sequence $\{\mathcal{S}\omega_m\}$ produced by the Jungck-AG iterative scheme (3.1) is bounded and converges to the unique common fixed point $\omega = \mathcal{S}\varrho = \mathcal{G}\varrho$. Consequently, $\{\mathcal{S}\omega_m\}$ owns a Δ-convergent subsequence. It remains to validate that each Δ-convergent subsequence of $\{\mathcal{S}\omega_m\}$ converges to the unique Δ-limit. Assume that the subsequences $\{\mathcal{S}\omega_{m_u}\}$ and $\{\mathcal{S}\omega_{m_v}\}$ of $\{\mathcal{S}\omega_m\}$ own distinct Δ-limits ω_u and ω_v, respectively. Thus, by virtue of Lemma 2.1, we obtain $\mathcal{B}c_{\mathcal{B}}(\{\omega_{m_u}\}) = \{\omega_u\}$ and $\mathcal{B}c_{\mathcal{B}}(\{\omega_{m_v}\}) = \{\omega_v\}$. It follows from Theorem 3.1 that $\lim_{u\to\infty} d(\mathcal{S}\omega_{m_u}, \omega_u) = 0, \mathcal{S}\varrho_u = \mathcal{G}\varrho_u = \omega_u$ and $\lim_{v\to\infty} d(\mathcal{S}\omega_{m_v}, \omega_v) = 0, \mathcal{S}\varrho_v = \mathcal{G}\varrho_v = \omega_v$ for some $\omega_u, \omega_v \in C$.

Further, we shall establish the uniqueness. Recalling the general contractive condition for $\mathcal{S}$ and, $\mathcal{G}$, we express

$$\begin{aligned} 0 \le d(\omega_u, \omega_v) &= d(\mathcal{G}\varrho_u, \mathcal{G}\varrho_v) \\ &\le \psi(d(\mathcal{S}\varrho_u, \mathcal{G}\varrho_v)) + \tau d(\mathcal{S}\varrho_u, \mathcal{S}\varrho_v) \\ &\le \tau d(\mathcal{S}\varrho_u, \mathcal{S}\varrho_v) \\ &= \tau d(\omega_u, \omega_v). \end{aligned} \tag{3.14}$$

Since $0 \le \tau < 1$, consequently, we obtain from (3.14) that $d(\omega_u, \omega_v) = 0$ or equivalently, $\omega_u = \omega_v$. Thus, $\{\mathcal{S}\omega_m\}$ is Δ-convergent to a unique common fixed point. □

Next, we shall present the following analytical result to manifest that our scheme (3.1) is more efficient than Jungck-AI scheme (1.8) and Jungck-DK scheme (1.7).

Theorem 3.4. *Suppose $\mathcal{S}$ and $\mathcal{G}$ are the same and obey the same assumptions as in Theorem 3.1. If the sequences $\{p_m\}, \{r_m\}$, and $\{s_m\}$ are bounded away inside $(0,1)$, then the sequence $\{\mathcal{S}\omega_m\}$ produced by the Jungck-AG iterative scheme (3.1) converges to ω faster than Jungck-AI scheme (1.8).*

Proof. Using the hypothesis that $\{p_m\}, \{r_m\}$, and $\{s_m\}$ are bounded away inside $(0,1)$, $\exists p^{'}, p^{''}, r^{'}, r^{''}, s^{'}, s^{''} \in (0,1)$ satisfies $0 < p^{'} \le p_m \le p^{''} < 1$, $0 < r^{'} \le r_m \le r^{''} < 1$, and $0 < s^{'} \le s_m \le s^{''} < 1$. Suppose that $\{\mathcal{S}\omega_m\}$ is generated by (3.1), then from (3.7), we acquire

$$\begin{aligned} d(\mathcal{S}\omega_{m+1}, \omega) &\le \tau^3[1 - p_m(1-\tau)][1 - r_m[1 - (1 - s_m(1-\tau))]]d(\mathcal{S}\omega_m, \omega) \\ &\le \tau^3[1 - p^{'}(1-\tau)][1 - r^{'}[1 - (1 - s^{'}(1-\tau))]]d(\mathcal{S}\omega_m, \omega) \\ &\le \tau^{3m}[1 - p^{'}(1-\tau)]^m[1 - r^{'}[1 - (1 - s^{'}(1-\tau))]]^m d(\mathcal{S}\omega_1, \omega) = \mathbb{A}_m. \end{aligned} \tag{3.15}$$

Accordingly, from (1.8), we obtain

$$\begin{aligned} d(\mathcal{S}\omega_{m+1}, \omega) = d(\mathcal{G}\zeta_m, \omega) &= d(\mathcal{G}\zeta_m, \mathcal{G}\varrho) \\ &\le \psi(d(\mathcal{S}\varrho, \mathcal{G}\varrho)) + \tau d(\mathcal{S}\zeta_m, \mathcal{S}\varrho) \\ &\le \tau d(\mathcal{S}\zeta_m, \mathcal{S}\varrho) = \tau d(\mathcal{S}\zeta_m, \omega), \end{aligned} \tag{3.16}$$

$$\begin{aligned} d(\mathcal{S}\zeta_m, \omega) &= d(\mathcal{G}\varrho_m, \omega) \\ &= d(\mathcal{G}\varrho_m, \mathcal{G}\varrho) \\ &\le \psi(d(\mathcal{S}\varrho, \mathcal{G}\varrho)) + \tau d(\mathcal{S}\varrho_m, \mathcal{S}\varrho) \\ &\le \tau d(\mathcal{S}\varrho_m, \mathcal{S}\varrho) \\ &= \tau d(\mathcal{S}\varrho_m, \omega), \end{aligned} \tag{3.17}$$

$$\begin{aligned} d(\mathcal{S}\varrho_m, \omega) &= d(\mathcal{G}\varpi_m, \omega) \\ &= d(\mathcal{G}\varpi_m, \mathcal{G}\varrho) \\ &\le \psi(d(\mathcal{S}\varrho, \mathcal{G}\varrho)) + \tau d(\mathcal{S}\varpi_m, \mathcal{S}\varrho) \\ &\le \tau d(\mathcal{S}\varpi_m, \mathcal{S}\varrho) \\ &= \tau d(\mathcal{S}\varpi_m, \omega), \end{aligned} \tag{3.18}$$

$$
\begin{aligned}
d(\mathcal{S}\varpi_m,\omega) &= d(\Phi(\mathcal{S}\omega_m,\mathcal{G}\omega_m,p_m),\omega) \\
&\leq p_m d(\mathcal{G}\omega_m,\omega) + (1-p_m)d(\mathcal{S}\omega_m,\omega) \\
&\leq p_m[\psi(d(\mathcal{S}\varrho,\mathcal{G}\varrho) + \tau d(\mathcal{S}\omega_m,\mathcal{S}\varrho)] + (1-p_m)d(\mathcal{S}\omega_m,\omega) \\
&\leq p_m\tau d(\mathcal{S}\omega_m,\mathcal{S}\varrho) + (1-p_m)d(\mathcal{S}\omega_m,\omega) \\
&\leq p_m\tau d(\mathcal{S}\omega_m,\omega) + (1-p_m)d(\mathcal{S}\omega_m,\omega) \\
&= (1-p_m(1-\tau))d(\mathcal{S}\omega_m,\omega).
\end{aligned} \tag{3.19}
$$

Thus, by substituting (3.17)–(3.19) in (3.16), we acquire

$$
\begin{aligned}
d(\mathcal{S}\omega_{m+1},\omega) &\leq \tau^3(1-p_m(1-\tau))d(\mathcal{S}\omega_m,\omega) \\
&\leq \tau^3(1-p^{'}(1-\tau))d(\mathcal{S}\omega_m,\omega) \\
&\leq \tau^{3m}[1-p^{'}(1-\tau)]^m d(\mathcal{S}\omega_1,\omega) = \mathbb{B}_m.
\end{aligned} \tag{3.20}
$$

Hypotheses of the theorem yield $1 - r^{'}[1-(1-s^{'}(1-\tau))] < 1$, then

$$
\lim_{m\to\infty}\frac{\mathbb{A}_m}{\mathbb{B}_m} = \lim_{m\to\infty}\frac{\tau^{3m}[1-p^{'}(1-\tau)]^m[1-(1-r^{'})s^{'}(1-\tau)]^m d(\mathcal{S}\omega_1,\omega)}{\tau^{3m}[1-p^{'}(1-\tau)]^m d(\mathcal{S}\omega_1,\omega)} = 0.
$$

Thus, the Jungck-AG iterative scheme (3.1) converges faster than the Jungck-AI scheme (1.8). □

It is demonstrated in [3] that Jungck-AI iterative scheme (1.8) converges faster than Jungck-DK scheme (1.7). Thus, our scheme is trivially faster than (1.7). So, we omit the proof.

Before heading to the application of the proposed scheme, we shall illustrate a numerical example to evidence the reliability and potential of the considered scheme.

In the following example, we validate Theorem 3.1. It is shown that our iterative scheme (3.1) converges to the common fixed point $\omega = 8$ for distinct initial values in Figure 1, and the comparison of our scheme (3.1) with its counterpart (1.6)–(1.8) is shown in Figures 2–5 for distinct initial values.

Example 3.1. *Let $([5,\infty), d, \Phi)$ be a hyperbolic space, where convex mapping $\Phi : [5,\infty)\times[5,\infty)\times[0,1] \to [5,\infty)$ is described as $\Phi(\omega,\varrho,a) = (1-a)\omega + a\varrho, \forall a \in [0,1]$, and d is a usual metric on $\mathcal{R}$. Define $\mathcal{S},\mathcal{G} : [5,\infty) \to [5,\infty)$ as $\mathcal{S}\omega = \dfrac{\omega^4}{64} - 56$ and $\mathcal{G}\omega = \omega^2 - 8\omega + 8, \forall\omega \in [5,\infty)$. Define $\psi : (0,\infty) \to (0,\infty)$ by $\psi(\omega) = 2\omega$, then ψ is monotonic and $\psi(0) = 0$. Also, $\mathcal{S}$ and $\mathcal{G}$ satisfy (2.2) and (2.3) for $\tau = \frac{1}{5}, l \geq 1$ and for all $\omega,\varrho \in [5,\infty)$.*

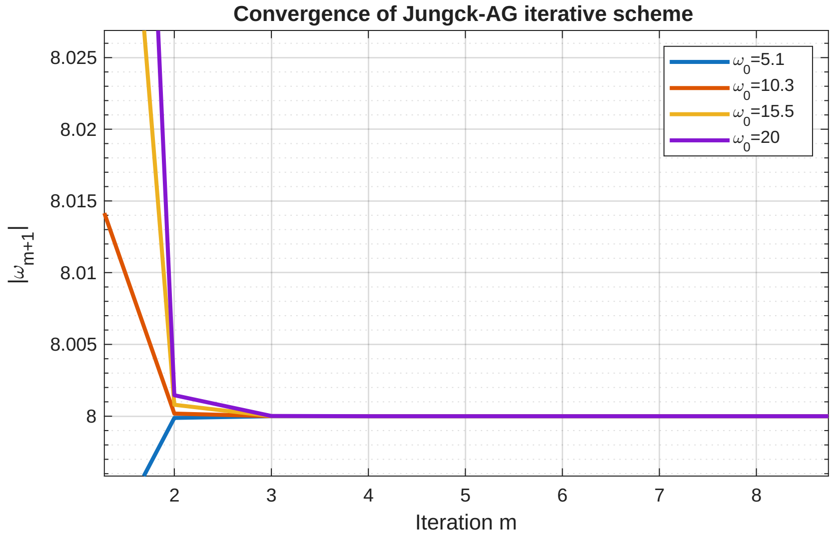

Figure 1. Convergence behavior of the Jungck-AG iterative scheme (3.1) with distinct initial guess $\omega_0 = 5.1, 10.3, 15.5$, and 20.

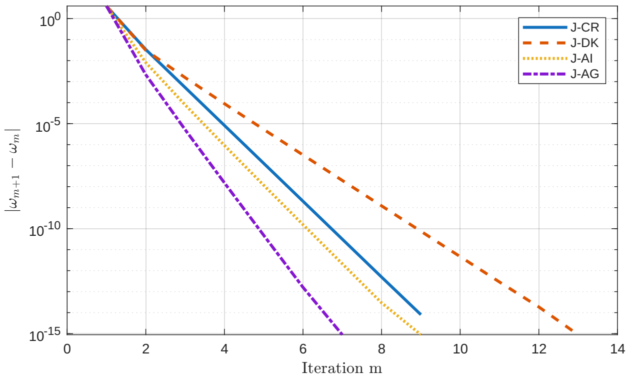

Figure 2. Comparison of convergence analysis with initial value $\omega_0 = 4$.

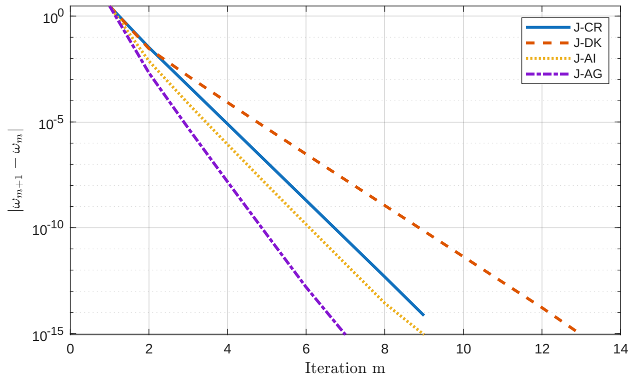

Figure 3. Comparison of convergence analysis with initial value $\omega_0 = 5$.

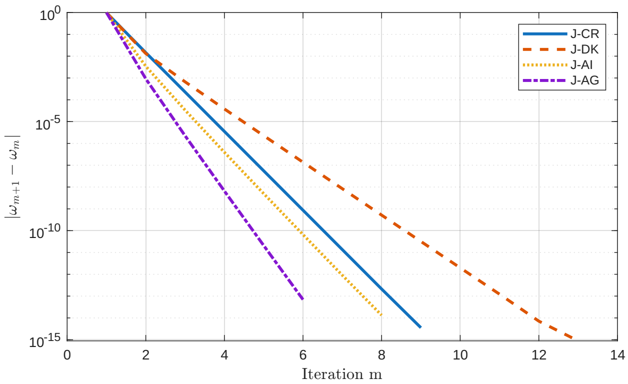

Figure 4. Comparison of convergence analysis with initial value $\omega_0 = 7$.

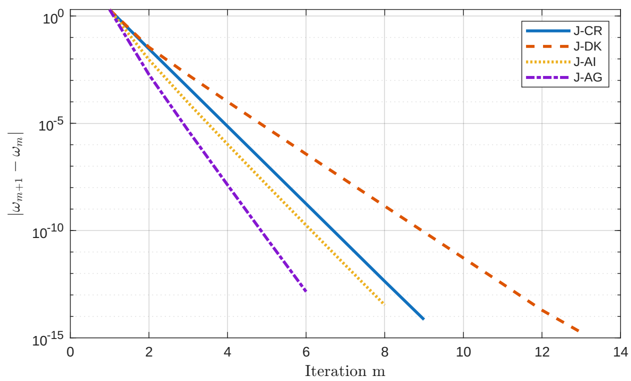

Figure 5. Comparison of convergence analysis with initial value $\omega_0 = 10$.

We shall furnish the following example to validate Theorem 3.1 and the computational reliability of our scheme. We demonstrated the convergence of our scheme for distinct initial values and showcased the rate of convergence with the schemes (1.6)–(1.8) in Figures 6–9.

Example 3.2. *Let* $([0,2], d, \Phi)$ *be a hyperbolic space with* $d(\omega, \varrho) = |\omega - \varrho|, \forall \omega, \varrho \in \mathcal{R}$, *and the convex mapping* $\Phi : [0,2] \times [0,2] \times [0,1] \to [0,2]$ *is described as* $\Phi(\omega, \varrho, a) = (1-a)\omega + a\varrho, \forall a \in [0,1]$. *Define* $\mathcal{S}, \mathcal{G} : [0,2] \to [0,15]$ *as* $\mathcal{S}\omega = 3\omega^2$ *and* $\mathcal{G}\omega = e^{\omega}, \forall \omega \in [0,2]$. *Then, one can easily verify that* $\mathcal{G}([0,2]) \subseteq \mathcal{S}([0,2])$ *and* $\mathcal{S}([0,2])$ *is complete. Define* $\psi : (0,\infty) \to (0,\infty)$ *by* $\psi(\omega) = 2\omega$, *then* ψ *is monotonic and* $\psi(0) = 0$. *Also,* $\mathcal{S}$ *and* $\mathcal{G}$ *satisfy (2.2) and (2.3) for* $\tau = \frac{1}{2}, l \geq 1$ *and for all* $\omega, \varrho \in [0,2]$ *with coincidence point* 0.91008.

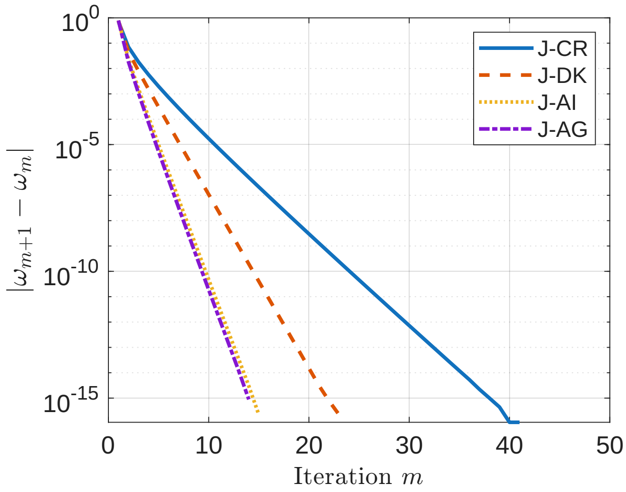

Figure 6. Comparison of convergence analysis with initial value $\omega_0 = 0.1$.

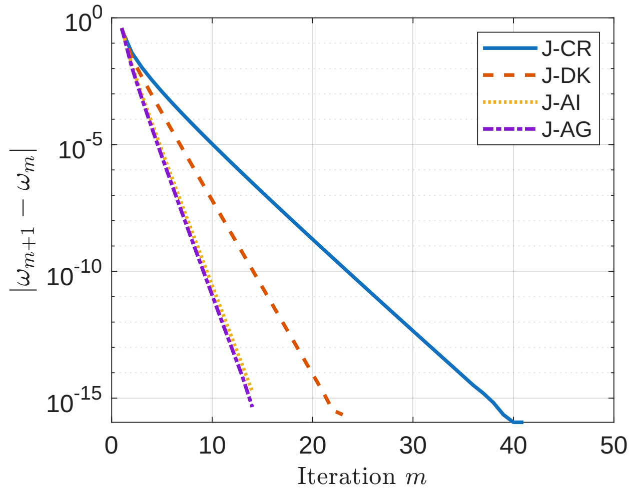

Figure 7. Comparison of convergence analysis with initial value $\omega_0 = 0.5$.

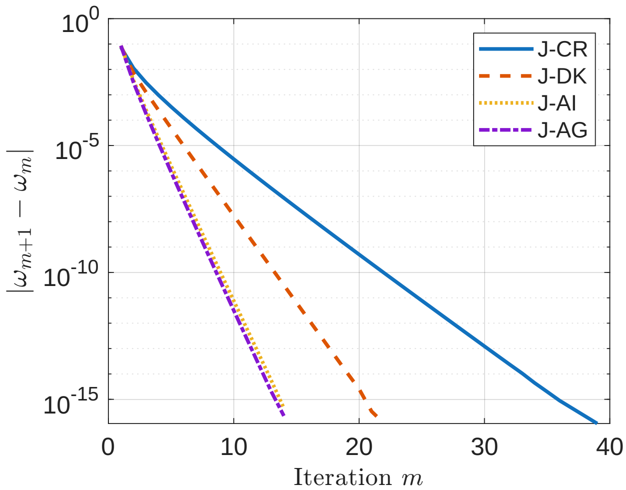

Figure 8. Comparison of convergence analysis with initial value $\omega_0 = 1.0$.

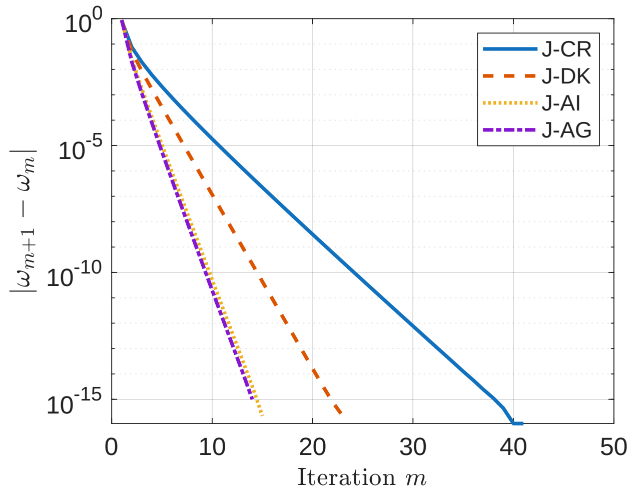

Figure 9. Comparison of convergence analysis with initial value $\omega_0 = 1.5$.

4. Application

The equations in which the unknown function is to be estimated appears under one or more integral signs are referred to as integral equations [44]. NDIEs are characterized by the time variable called delay, which a plays key role in determining the solution. Brunner's equation is the general form of NDIEs, which was used to examine nonlinear Volterra integral equations of the second kind. These equations are used to deal with wide-ranging phenomena including electric circuits, quantum mechanics, mechanical systems, models of economic systems with time delay, etc. [7, 15, 46].

Herein, our goal is to examine the following NDIE with two delays by implementing our Jungck-AG scheme (3.1).

$$\gamma(\omega^*) = \zeta\left(\omega^*, \gamma(\omega^*), \gamma(\varphi(\omega^*)), \int_a^b \xi(\omega^*, s, \gamma(s), \gamma(\eta(s)))ds\right), \forall \omega^* \in [a,b], a, b \in \mathbb{R}, \tag{4.1}$$

where $\zeta : [a,b] \times \mathbb{R}^3 \to \mathbb{R}, \xi : [a,b]^2 \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, and the continuous functions $\varphi, \eta : [a,b] \to [a,b]$ represent delays such that $\varphi(\varrho) \leq \varrho, \eta(\varrho) \leq \varrho, \forall \varrho \in [a,b]$. Now, we consider the hyperbolic space $(C[a,b], d, \Phi)$ of all real-valued continuous functions equipped with metric d expressed as follows:

$$d(\gamma_1, \gamma_2) = \max_{\omega^* \in [a,b]} \frac{\gamma_1(\omega^*) - \gamma_2(\omega^*)}{u(\omega^*)}, \tag{4.2}$$

where $u : [a,b] \to \mathbb{R}^+$ is a non-decreasing function. Suppose that the assertions given below are fulfilled:

(a_1) For some $\mathbb{Q} \in \mathbb{R}, \int_a^b u(s)ds \leq \mathbb{Q}u(\omega^*)d\omega^*, \forall \omega^* \in [a,b]$;

(a_2) for some $\mathbb{M} > 0, |\zeta(\varrho, \gamma_1(\omega^*), \gamma_1(\varphi(\omega^*)), \xi_1(\omega^*)) - \zeta(\omega^*, \gamma_2(\omega^*), \gamma_2(\varphi(\omega^*)), \xi_2(\omega^*))| \leq \mathbb{M}|\gamma_1(\omega^*) - \gamma_2(\omega^*)| + |\gamma_1(\varphi(\omega^*)) - \gamma_2(\varphi(\omega^*))| + |\xi_1(\omega^*) - \xi_2(\omega^*)|, \forall \omega^* \in [a,b]$;

(a_3) for some $\mathbb{P} > 0, |\xi(\omega^*, s, \gamma_1(s), \gamma_1(\eta(s)) - \xi(\omega^*, s, \gamma_2(s), \gamma_2(\eta(s))| \leq \mathbb{P}|\gamma_1(\eta(s)) - \gamma_2(\eta(s))|$, $\forall s, \varrho \in [a,b]$;

(a_4) $\mathbb{M}\mathbb{Q}\mathbb{P} < 1 - 2\mathbb{M}$.

It is documented in Castro and Simoes [11] that NDIE (4.1) possesses a unique solution, provided the assertions (a_1)–(a_4) are fulfilled. Now, we shall put forward our scheme (3.1) to estimate the unique solution of NDIE (4.1).

Theorem 4.1. *Let* $(C([a,b]), d, \Phi)$ *be a hyperbolic space and* $\mathcal{S}, \mathcal{G} : C[a,b] \to C[a,b]$ *be defined as* $\mathcal{S}\gamma(\varrho) = \gamma(\varrho)$ *and*

$$\mathcal{G}\gamma(\omega^*) = \zeta\left(\omega^*, \gamma(\omega^*), \gamma(\varphi(\omega^*)), \int_a^b \xi(\omega^*, s, \gamma(s), \gamma(\eta(s)))ds\right), \forall \omega^* \in [a,b], \gamma \in C[a,b]. \tag{4.3}$$

If all the assertions (a_1)–(a_4) *are satisfied and the sequence* $\{\mathcal{S}\omega_m\}$ *is produced by (3.1), then* $\mathcal{S}\omega_m \to \omega^* \in C[a,b]$*, the unique solution of NDIE (4.1).*

Proof. Since all the assertions (a_1)–(a_4) are fulfilled, we assume that $\omega^* \in C([a,b])$, the unique solution of NDIE (4.1). Next, we exhibit $\mathcal{S}\omega_m \to \omega^* \in C([a,b])$ as $m \to \infty$. Employing the scheme (3.1) and making use of assumptions (a_1)–(a_4) and (4.3) along with equipped metric d defined in (4.2), we obtain

$$\begin{aligned}
& d(\mathcal{S}\omega_{m+1},\omega^*) = d(\mathcal{G}\varrho_m,\mathcal{G}\omega^*) \\
& = \max_{\varrho\in[a,b]} \tfrac{|\mathcal{G}\varrho_m(t)-\mathcal{G}\omega^*(t)|}{u(\varrho)} \\
& = \max_{\varrho\in[a,b]} \tfrac{1}{u(\varrho)}\Big|\zeta\big(\varrho,\varrho_m(\varrho),\varrho_m(\varphi(\varrho)),\int_a^b \xi(\varrho,s,\varrho_m(s),\varrho_m(\eta(s)))ds\big) \\
& \quad - \zeta\big(\varrho,\omega^*(\varrho),\omega^*(\varphi(\varrho)),\int_a^b \xi(\varrho,s,\omega^*(s),\omega^*(\eta(s)))ds\big)\Big| \\
& \le \max_{\varrho\in[a,b]} \tfrac{\mathbb{M}}{u(\varrho)}\big[|\varrho_m(\varrho)-\omega^*(\varrho)| + |\varrho_m(\varphi(\varrho))-\omega^*(\varphi(\varrho))| \\
& \quad + \Big|\int_a^b \xi(\varrho,s,\varrho_m(s),\varrho_m(\eta(s)))ds - \int_a^b \xi(\varrho,s,\omega^*(s),\omega^*(\eta(s)))ds\Big|\big] \\
& \le \max_{\varrho\in[a,b]} \tfrac{\mathbb{M}}{u(\varrho)}\big[|\varrho_m(\varrho)-\omega^*(\varrho)| + |\varrho_m(\varphi(\varrho))-\omega^*(\varphi(\varrho))| \\
& \quad + \int_a^b |\xi(\varrho,s,\varrho_m(s),\varrho_m(\eta(s)) - \xi(\varrho,s,\omega^*(s),\omega^*(\eta(s))|ds\big] \\
& \le \max_{\varrho\in[a,b]} \tfrac{\mathbb{M}}{u(\varrho)}\big[|\varrho_m(\varrho)-\omega^*(\varrho)| + |\varrho_m(\varphi(\varrho))-\omega^*(\varphi(\varrho))| \\
& \quad + \mathbb{P}\int_a^b |\varrho_m(\eta(s))-\omega^*(\eta(s))|ds\big] \\
& \le \mathbb{M}[2\max_{\varrho\in[a,b]} \tfrac{|\varrho_m(\varrho)-\omega^*(\varrho)|}{u(\varrho)} + \max_{\varrho\in[a,b]} \tfrac{\mathbb{P}}{u(\varrho)}\int_a^b |\varrho_m(\eta(s))-\omega^*(\eta(s))|ds] \\
& \le \mathbb{M}[2d(\varrho_m,\omega^*) + \mathbb{P}\max_{\varrho\in[a,b]} \tfrac{|\varrho_m(\eta(s))-\omega^*(\eta(s))|}{u(\varrho)} \max_{\varrho\in[a,b]} \tfrac{1}{u(\varrho)}\int_a^b u(s)ds] \\
& \le \mathbb{M}\Big[2d(\varrho_m,\omega^*) + \mathbb{P}d(\varrho_m,\omega^*)\max_{\varrho\in[a,b]} \tfrac{\mathbb{Q}u(\varrho)}{u(\varrho)}\Big] \\
& = \mathbb{M}[2+\mathbb{P}\mathbb{Q}]d(\mathcal{S}\varrho_m,\omega^*).
\end{aligned} \tag{4.4}$$

Again duplicating the process as in (4.4), one can achieve

$$\begin{aligned}
d(\mathcal{S}\zeta_m,\omega^*) &= d(\Phi(\mathcal{S}\sigma_m,\mathcal{G}\sigma_m,p_m),\omega^*) \\
&\le p_m d(\mathcal{G}\sigma_m,\omega^*) + (1-p_m)d(\mathcal{S}\sigma_m,\omega^*) \\
&\le p_m d(\mathcal{G}\sigma_m,\mathcal{G}\omega^*) + (1-p_m)d(\mathcal{S}\sigma_m,\omega^*) \\
&\le p_m\mathbb{M}[2+\mathbb{P}\mathbb{Q}]d(\mathcal{S}\sigma_m,\omega^*) + (1-p_m)d(\mathcal{S}\sigma_m,\omega^*) \\
&= \left(1-p_m(1-\mathbb{M}[2+\mathbb{P}\mathbb{Q}])\right)d(\mathcal{S}\sigma_m,\omega^*),
\end{aligned} \tag{4.5}$$

$$d(\mathcal{S}\varrho_m,\omega^*) = d(\mathcal{G}\zeta_m,\omega^*) \le \mathbb{M}[2+\mathbb{P}\mathbb{Q}]d(\mathcal{S}\zeta_m,\omega^*), \tag{4.6}$$

$$\begin{aligned}
d(\mathcal{S}\sigma_m,\omega^*) &= d(\Phi(\mathcal{G}\omega_m,\mathcal{G}\varpi_m,r_m),\omega^*) \\
&\le (1-r_m)d(\mathcal{G}\omega_m,\omega^*) + r_m d(\mathcal{G}\varpi_m,\omega^*) \\
&\le (1-r_m)d(\mathcal{G}\omega_m,\mathcal{G}\omega^*) + r_m d(\mathcal{G}\varpi_m,\mathcal{G}\omega^*) \\
&\le (1-r_m)\mathbb{M}[2+\mathbb{P}\mathbb{Q}]d(\mathcal{S}\omega_m,\omega^*) + r_m\mathbb{M}[2+\mathbb{P}\mathbb{Q}]d(\mathcal{S}\varpi_m,\omega^*),
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
d(\mathcal{S}\varpi_m,\omega^*) &= d(\Phi(\mathcal{S}\omega_m,\mathcal{G}\omega_m,s_m),\omega^*) \\
&\le s_m d(\mathcal{G}\omega_m,\omega^*) + (1-s_m)d(\mathcal{S}\omega_m,\omega^*) \\
&\le s_m d(\mathcal{G}\omega_m,\mathcal{G}\omega^*) + (1-s_m)d(\mathcal{S}\omega_m,\mathcal{G}\omega^*) \\
&\le s_m\mathbb{M}[2+\mathbb{P}\mathbb{Q}]d(\mathcal{S}\omega_m,\omega^*) + (1-s_m)d(\mathcal{S}\omega_m,\omega^*) \\
&= (1-s_m(1-\mathbb{M}[2+\mathbb{P}\mathbb{Q}]))d(\mathcal{S}\omega_m,\omega^*).
\end{aligned} \tag{4.8}$$

Implementing back substitution from (4.5)–(4.8), (4.4) turns into

$$\begin{aligned} d(\mathcal{S}\omega_{m+1},\omega^*) &\le [\mathbb{M}(2+\mathbb{P}\mathbb{Q})]^3[1-p_m(1-\mathbb{M}(2+\mathbb{P}\mathbb{Q}))]^3 \\ &\quad \times[1-r_m[1-(1-s_m(1-\mathbb{M}(2+\mathbb{P}\mathbb{Q})))]]^3|d(\mathcal{S}\omega_m,\omega^*) \\ &\quad \cdot \\ &\quad \cdot \\ &\quad \cdot \\ &\le [\mathbb{M}(2+\mathbb{P}\mathbb{Q})]^{3m}[1-p_m(1-\mathbb{M}(2+\mathbb{P}\mathbb{Q}))]^{3m} \\ &\quad \times[1-r_m[1-(1-s_m(1-\mathbb{M}(2+\mathbb{P}\mathbb{Q})))]]^{3m}d(\mathcal{S}\omega_1,\omega^*). \end{aligned} \tag{4.9}$$

Assumption (a_4) guarantees $\mathbb{M}(2+\mathbb{P}\mathbb{Q})<1$, and $\{p_m\},\{r_m\},\{s_m\}\subset(0,1)$ yields $1-p_m(1-\mathbb{M}(2+\mathbb{P}\mathbb{Q}))<1$ and $1-r_m[1-(1-s_m(1-\mathbb{M}(2+\mathbb{P}\mathbb{Q})))]<1$. Consequently, $\lim_{m\to\infty} d(\mathcal{S}\omega_m,\omega^*)=0$, i.e., $\mathcal{S}\omega_m\to\omega^*$ as $m\to\infty$. □

Example 4.1. *Let $(C([0,1]),d,\Phi)$ be a hyperbolic space equipped with metric d expressed as*

$$d(\gamma_1,\gamma_2)=\max_{\varrho\in[a,b]}\frac{\gamma_1(\varrho)-\gamma_2(\varrho)}{u(\varrho)}, \tag{4.10}$$

where the non-decreasing function $u:[0,1]\to\mathbb{R}^+$ is expressed as $u(\varrho)=0.0065\varrho+0.0007$, which satisfies

$$\int_0^{\varrho}(0.0065s+0.0007)ds\le(e^{b-a}-1)(0.0065\varrho+0.0007)=1.7183u(\varrho),\forall\varrho\in[0,1].$$

Let $\gamma:[0,1]\to\mathbb{R}$ be a nonlinear integral equation with two delays given as

$$\gamma(\varrho)=\frac{\varrho^3}{60}-\frac{\varrho^2}{5}+\varrho+\frac{1}{3}\gamma(\varphi(\varrho))+\frac{1}{6}\int_0^{\varrho}(s-\varrho)\gamma(\eta(s))ds,\varrho\in[0,1], \tag{4.11}$$

where the continuous delay functions $\varphi,\eta:[0,1]\to[0,1]$ are expressed as $\varphi(\varrho)=\varrho^3$ and $\eta(\varrho)=\varrho^2,\forall\varrho\in[0,1]$. Clearly, φ and η are continuous functions satisfying $\varphi(\varrho)\le\varrho$ and $\varphi(\varrho)\le\varrho$.

Also, the continuous function $\zeta:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ is defined as

$$\zeta(\varrho,\gamma(\varrho),\gamma(\varphi(\varrho)),\xi(\varrho))=\frac{\varrho^3}{60}-\frac{\varrho^2}{5}+\varrho+\frac{1}{3}\gamma(\varphi(\varrho))+\frac{1}{6}\xi(\varrho),$$

which satisfies

$$\begin{aligned}&|\zeta(\varrho,\gamma_1(\varrho),\gamma_1(\varphi(\varrho)),\xi_1(\varrho))-\zeta(\varrho,\gamma_2(\varrho),\gamma_2(\varphi(\varrho)),\xi_2(\varrho))|\\ &\le\tfrac{1}{6}(|\gamma_1(\varrho)-\gamma_2(\varrho)|+|\gamma_1(\varphi(\varrho))-\gamma_2(\varphi(\varrho))|+|\xi_1(\varrho))-\xi_2(\varrho)|,\forall\varrho\in[0,1].\end{aligned}$$

For $\mathbb{P}=1$, the kernel $\xi:[0,1]^2\times\mathbb{R}^2\to\mathbb{R}$ is defined as $\xi(\varrho,s,\gamma(s),\gamma(\eta(s))=(s-\varrho)\gamma(\eta(s))$, which satisfies

$$|\xi(\varrho,s,\gamma_1(s),\gamma_1(\eta(s))-\xi(\varrho,s,\gamma_1(s),\gamma_1(\eta(s))|\le\mathbb{P}|\gamma_1(\eta(s))-\gamma_2(\eta(s))|,\forall s\in[0,\varrho],\varrho\in[0,1].$$

Thus, the constants $\mathbb{Q}=1.7183,\mathbb{M}=\frac{1}{6}$, and $\mathbb{P}=1$ estimated from the hypotheses satisfy the assumption (a_4), i.e., $\mathbb{M}\mathbb{Q}\mathbb{P}<1-2\mathbb{M}$. Thus, for the given parameters and estimated constants, Theorem 4.1 is validated.

5. Conclusions

A novel Jungck-type iterative scheme involving a pair of mappings with weak compatibility is designed and presented in hyperbolic metric spaces. In this accomplishment, we investigated a common fixed point of the pair of non-self mappings by analyzing the strong convergence. The stability of the proposed iterative method is established under generalized contractive conditions (2.2) and (2.3). Δ-convergence is also shown, thereby reinforcing the robustness of the scheme. A theoretical result is provided to exhibit the efficiency of our scheme with some known counterpart, and numerical experiments further substantiate the rate of convergence and computational reliability of the approach. Finally, the significance and applicability of our designed scheme is illustrated by implementing it to explore an NDIE, underscoring its utility in solving complex functional problems.

Beyond these contributions, the work presented here enriches the theory of fixed point approximation in hyperbolic spaces and furnishes a flexible multifaceted tool for handling nonlinear equations. The methodological framework constructed here can be extended to broader classes of mappings such as multi-valued and hybrid mappings. Additionally, the stability and convergence properties suggest potential relevance in applied fields, including dynamical systems, control theory, and mathematical models in physics and engineering. In the future, the results discussed here can be refined under weaker contractive conditions. Thus, the proposed Jungck-type iterative scheme advances the theoretical landscape of fixed point theory and simultaneously opens new directions for exploring problems appearing in pure and applied sciences.

Author contributions

Doaa Filali: Funding, writing review and editing; Mohammad Dilshad: Conceptualization, writing review and editing; Mohammad Akram: Conceptualization, writing original draft preparation, writing review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

Authors declare no conflicts of interest in this paper.

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