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Research article

Optimized properties for noncanonical neutral differential equations with distributed deviating arguments and their oscillation analysis

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Abstract: This work develops new oscillation criteria for second-order neutral differential equations with distributed deviating arguments. The analysis covers both the noncanonical case and the situation in which the coefficient of the neutral delay term exceeds one, settings that remain insufficiently addressed in the existing literature. The main contribution of this study is the derivation of several refined inequalities that strengthen the relation between the solution and its associated function. These improved relations allow us to establish sharper oscillation criteria, including a Kneser-type criterion and additional forms that extend and refine earlier approaches. Several examples are provided to apply the theoretical results and to compare them with related findings. The findings indicate that the developed framework offers a more sensitive and robust tool for detecting oscillatory behavior in this class of neutral equations.

Keywords: differential equations; second order neutral equation; distributed deviating arguments; oscillatory behavior; noncanonical case
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1. Introduction

Distributed delay differential equations arise in the modeling of systems whose dynamics are shaped not only by their current state but also by a temporally distributed memory of past behavior. These equations naturally appear in settings where inertia, memory effects, and gradual response are integral to the underlying process. In mechanical and structural engineering, they are used to describe viscoelastic materials and vibration systems with distributed damping, where the system's response reflects a continuum of previous deformations. They also characterize heat transfer and deformation in materials that exhibit thermal memory. In control theory, second-order models with distributed delays emerge in feedback systems where delayed or distributed information influences acceleration or force generation. Additionally, these equations are employed in biological and neural models to represent propagation phenomena and population dynamics involving non-instantaneous interactions; see [1–3].

Neutral differential equations (NDE) are regarded as a principal category of functional differential equations (FDE), distinguished by the fact that the highest-order derivative of the solution depends on delayed as well as non-delayed terms. Their theoretical prominence, coupled with the notable analytical challenges encountered in analyzing their dynamical properties, has ensured that they remain a prominent subject of scholarly inquiry; see [4–6].

This work utilizes an enhanced analytical approach to explore the oscillatory characteristics of solutions to the NDE

$$\left(r(u)\left[\left(x(u)+\frac{1}{p(u)}x(\delta(u))\right)'\right]^{\kappa}\right)'+\int_a^b \varphi(u,\mu)\left[x(\sigma(u,\mu))\right]^{\kappa}\mathrm{d}\mu=0, \tag{1.1}$$

where $u \in \mathbb{J}_0 := [u_0,\infty)$. Our investigation proceeds under the following assumptions:

(C1) a, b are nonnegative real numbers, $a < b$, and $\kappa \geq 1$ is a ratio of odd integers;
(C2) $r, p, \delta \in \mathbf{C}([u_0,\infty),(0,\infty))$, $p(u) < 1$, $\delta(u)$ strictly increasing on $[u_0,\infty)$, and $\lim_{u\to\infty}\delta(u)=\infty$;
(C3) $\varphi, \sigma \in \mathbf{C}([u_0,\infty)\times[a,b],[0,\infty))$, $\sigma(u,\mu) \leq u$, $\lim_{u\to\infty}\sigma(u,\mu)=\infty$, $\sigma(u,\mu)$ is nondecreasing in u and monotonic in μ, and φ does not vanish identically on any half line $[u_*,\infty)\times[a,b]$ for $u_* \in \mathbb{J}_0$.

Let

$$y(u)=x(u)+\frac{x(\delta(u))}{p(u)}. \tag{1.2}$$

We define a function $x \in \mathbf{C}([u_x,\infty),\mathbb{R})$, for $u_x \in \mathbb{J}_0$, as a solution to (1.1) when y and $r[y']^{\kappa}$ both belong to $\mathbf{C}^1([u_x,\infty),\mathbb{R})$, and x satisfies (1.1) for all $u \in \mathbb{J}_x := [u_x,\infty)$. Our focus is limited to solutions of (1.1) for which $\sup\{|x(u)| : u \geq u_*\} > 0$ holds for all $u_* \in \mathbb{J}_x$. A solution is classified as oscillatory if it is not eventually positive or eventually negative; otherwise, it is considered nonoscillatory. Equation (1.1) is referred to as oscillatory whenever all of its solutions exhibit oscillatory behavior.

Over the past decade, there has undoubtedly been a remarkable surge of research activity concerning the oscillatory behavior of differential equations (see, e.g., [7–9]), particularly those of second order (see, e.g., [10–12]). Numerous studies have focused on various aspects aimed at refining oscillation criteria, including iterative improvements of the monotonic properties of positive solutions, enhancements of well-known methodologies, and the optimization of fundamental inequalities used in oscillation theory, among other approaches.

One of the intriguing research directions related to neutral equations is the relationship between a solution x and its associated function y. Earlier works relied on the classical form of this relationship until it was further developed by Moaaz et al. [13] and Hassan et al. [14]. This was followed by substantial and impactful progress in exploiting this relationship to obtain sharp oscillation criteria for second-order neutral equations, as demonstrated in [5]. The development was later extended to third-order equations (see [15]) and to higher-order cases (see [16]).

The primary motivation behind improving the relationship between the solution and its associated function lies in its direct and significant influence on strengthening oscillation criteria. Despite the extensive literature addressing this topic, there are, to the best of our knowledge, no works that have considered neutral equations with distributed delays in the form of Eq (1.1). Addressing this gap constitutes the main objective of the present study.

Recently, there exist numerous studies that have examined the oscillatory behavior of Eq (1.1) in the case where $p(u) > 1$, see, e.g., [17–19]. In what follows, we outline several previous works that dealt with Eq (1.1) when $p(u) < 1$, and analyze their results in preparation for a later comparison.

Li et al. [20] studied the oscillatory properties of the NDE

$$\left(r(u)\,\psi(w'(u))\right)' + \int_a^b \varphi(u,\mu)\,\psi(x(\sigma(u,\mu)))\,\mathrm{d}\sigma(\mu) = 0, \tag{1.3}$$

where

$$w(u) = x(u) + h(u)\,x(\delta(u)),\ \psi(v) = |v|^{\kappa-1} v,\ h \in \mathbf{C}([u_0,\infty),[0,h_0]).$$

They employed the approach used in [21] together with the Riccati technique, combining two distinct formulations of Eq (1.3) in order to avoid deriving a relation between x and w. Moreover, they considered both the canonical and non-canonical cases. In the following theorem, we present their results for the non-canonical case.

Theorem 1.1. *[20, Theorem 2.5] Suppose that $\delta'(u) \geq \delta_0 > 0$, $\sigma(\delta(u),\mu) = \delta(\sigma(u,\mu))$, $\sigma(u,a) \leq \delta(u) \leq u$, and*

$$\sigma(u,a) \leq \sigma(u,\mu) \leq \sigma(u,b) \text{ for } \mu \in [a,b]. \tag{1.4}$$

Equation (1.3) is oscillatory if there are $\rho \in \mathbf{C}^1([u_0,\infty),(0,\infty))$ and $\theta \in \mathbf{C}^1([u_0,\infty),\mathbb{R})$ such that $\theta(u) \geq u$, $\theta'(u) \geq 0$,

$$\limsup_{u\to\infty} \int_{u_0}^u \left(\frac{1}{2^{\kappa-1}}\rho(v)\,G(v) - \frac{h_*}{(\kappa+1)^{1+\kappa}} \frac{r(\sigma(v,a))\,[\rho'(v)]^{\kappa+1}}{[\rho(v)\,\sigma'(v,a)]^{\kappa}} \right) \mathrm{d}v = \infty,$$

and

$$\limsup_{u\to\infty} \int_{u_0}^u \left(\left(\int_{\theta(v)}^\infty [r(t)]^{-1/\kappa}\,\mathrm{d}t \right)^{\kappa} \frac{G(v)}{2^{\kappa-1}} - \frac{\frac{h_*\kappa^{1+\kappa}}{(\kappa+1)^{1+\kappa}}\theta'(v)}{r^{1/\kappa}(\theta(v)) \int_{\theta(v)}^\infty [r(t)]^{-1/\kappa}\,\mathrm{d}t} \right) \mathrm{d}v = \infty,$$

where $h_* = 1 + h_0^{\kappa}/\delta_0$ and

$$G(u) := \int_a^b \min\{\varphi(u,\mu),\varphi(\delta(u),\mu)\}\,\mathrm{d}\sigma(\mu).$$

Very recently, Tunç et al. [22] investigated the oscillatory behavior of the NDE

$$\left(r(u)\,w'(u)\right)' + \int_a^b \varphi(u,\mu)\,x\left[(\sigma(u,\mu))\right]^{\kappa}\,\mathrm{d}\mu = 0, \tag{1.5}$$

where $\kappa \in (0,1]$, $h(t) \geq h_0 > 1$, and $\sigma(u,\mu)$ is decreasing in μ. They considered the non-canonical case, that is, $\varrho(u_0) < \infty$, where

$$\varrho(u) = \int_u^\infty [r(v)]^{-1}\,\mathrm{d}v.$$

The approach adopted in their study was grounded in extending the methodology employed in [23,24].

Theorem 1.2. *[22, Theorem 2.1] Suppose that $\sigma(u,a) \leq \delta(u)$ and*

$$\int_{u_0}^\infty \varrho(u) \int_a^b \varphi(u,\mu) \left[h\left(\delta^{-1}(\sigma(u,\mu))\right)\right]^{-\kappa} \mathrm{d}\mu \mathrm{d}u = \infty. \tag{1.6}$$

Equation (1.5) is oscillatory if

$$\limsup_{u\to\infty} W_1(u) > \begin{cases} 1, & \text{if } \kappa = 1, \\ 0, & \text{if } \kappa < 1, \end{cases}$$

where $g(u) = \delta^{-1}(\sigma(u,a))$,

$$W_1(u) = \varrho(u) \int_{u_0}^u Q(v)\,\mathrm{d}v + [\varrho(g(u))]^{-\kappa} \int_u^\infty \varrho(v) [\varrho(g(v))]^\kappa Q(v)\,\mathrm{d}v,$$

and

$$Q(u) = \int_a^b \frac{\varphi(u,\mu)}{h(\delta^{-1}(\sigma(u,\mu)))} \left[1 - \frac{1}{h(\delta^{-1}(\delta^{-1}(\sigma(u,\mu))))}\right]^\kappa \mathrm{d}\mu.$$

Theorem 1.3. *[22, Theorem 2.5] Suppose that $\sigma(u,a) \geq \delta(u)$ and (1.6) holds. Equation (1.5) is oscillatory if*

$$\limsup_{u\to\infty} W_2(u) > \begin{cases} 1, & \text{if } \kappa = 1, \\ 0, & \text{if } \kappa < 1, \end{cases}$$

where g, Q are defined as in Theorem 1.2 and

$$\begin{aligned} W_2(u) &= \varrho(g(u)) \int_{u_0}^u Q(v)\,\mathrm{d}v + [\varrho(g(u))]^{1-\kappa} \int_u^{g_2(u)} [\varrho(g(s))]^\kappa Q(v)\,\mathrm{d}v \\ &\quad + [\varrho(g(u))]^{-\kappa} \int_{g_2(u)}^\infty \varrho(v) [\varrho(g(v))]^\kappa Q(v)\,\mathrm{d}v. \end{aligned}$$

This paper is devoted to examining the oscillatory characteristics of solutions to NDEs of the form (1.1). Despite the abundance of results concerning the oscillatory behavior of differential equations, only limited attention has been given to the case $h(u) > 1$. In the present study, we establish refined and novel relationships between x and y, taking into account both scenarios $\delta(u) \leq u$ and $\delta(u) > u$. These refined relationships are subsequently utilized to develop new oscillation criteria. In the examples section, we demonstrate the applicability of our results to several particular instances of the considered equation. Furthermore, a comparative analysis with previously known results reveals that our criteria constitute a significant enhancement over the existing ones.

2. Main results

In this section, our results are organized into three parts. We begin by establishing some properties of the function y associated with the positive solutions of Eq (1.1). The subsequent two parts address oscillation criteria for the cases $\delta(u) \leq u$ and $\delta(u) > u$, respectively.

2.1. Preliminary lemmas and notations

Throughout our results, we denote by $\mathbb{S}$ the class of solutions to Eq (1.1) that are eventually positive. We further use the symbols "↑" and "↓" to represent, respectively, increasing and decreasing functions. In addition, we require the definition of the following functions:

$$f^{\circ(0)}(u) = u,\ f^{\circ(-i)}(u) = f^{-1}\left(f^{\circ(-i+1)}(u)\right), \text{ for } i \in \mathbb{Z}^{+},$$

$$\sigma_i(u) := \begin{cases} \sigma(u,a), & \text{if } (-1)^i \frac{\partial}{\partial\mu}\sigma(u,\mu) \geq 0, \\ \sigma(u,b), & \text{if } (-1)^i \frac{\partial}{\partial\mu}\sigma(u,\mu) \leq 0, \end{cases}$$

and

$$\mathcal{A}(u) := \int_{u_1}^{u} [r(v)]^{-1/\kappa}\,\mathrm{d}v,\ \eta(u) := \int_{u}^{\infty} [r(v)]^{-1/\kappa}\,\mathrm{d}v,$$

for $u \geq u_1$, where $u_1 \in \mathbb{J}$.

Lemma 2.1. *Suppose that $x \in \mathbb{S}$. Hence, $y(u) > 0$, $\left[r(u)(y'(u))^{\kappa}\right]\downarrow$, and y has a constant sign.*

Proof. If $x \in \mathbb{S}$, then $(x \circ \sigma)$ and $(x \circ \delta)$ are positive for $u \geq u_1$, where $u_1 \in \mathbb{J}_0$. It follows from (1.1) and (1.2) that $y > 0$ and $\left[r\ (y')^{\kappa}\right]\downarrow$. Moreover, y must be monotonic.

This concludes the proof. □

Lemma 2.2. *Suppose that $x \in \mathbb{S}$. Hence, for $m \in \mathbb{Z}^{+}$,*

$$x(u) \geq \sum_{i=1}^{m}\left(\prod_{j=1}^{2i-1} p\left(\delta^{\circ(-j)}(u)\right)\right)\left[y\left(\delta^{\circ(-2i+1)}(u)\right) - p\left(\delta^{\circ(-2i)}(u)\right)y\left(\delta^{\circ(-2i)}(u)\right)\right].$$

Proof. It follows from (1.2) that

$$\begin{aligned} x(u) &= p\left(\delta^{-1}(u)\right)\left[y\left(\delta^{-1}(u)\right) - x\left(\delta^{-1}(u)\right)\right] \\ &= p\left(\delta^{-1}(u)\right)y\left(\delta^{-1}(u)\right) - p\left(\delta^{-1}(u)\right)p\left(\delta^{\circ(-2)}(u)\right)y\left(\delta^{\circ(-2)}(u)\right) \\ &\quad + p\left(\delta^{-1}(u)\right)p\left(\delta^{\circ(-2)}(u)\right)x\left(\delta^{\circ(-2)}(u)\right). \end{aligned}$$

By repeating this procedure m times, we obtain

$$\begin{aligned} x(u) &= \sum_{i=1}^{m}\left(\prod_{j=1}^{2i-1} p\left(\delta^{\circ(-j)}(u)\right)\right)\left[y\left(\delta^{\circ(-2i+1)}(u)\right) - p\left(\delta^{\circ(-2i)}(u)\right)y\left(\delta^{\circ(-2i)}(u)\right)\right] \\ &\quad + \prod_{i=1}^{2m}\left(p\left(\delta^{\circ(-i)}(u)\right)\right)x\left(\delta^{\circ(-2i)}(u)\right) \\ &\geq \sum_{i=1}^{m}\left(\prod_{j=1}^{2i-1} p\left(\delta^{\circ(-j)}(u)\right)\right)\left[y\left(\delta^{\circ(-2i+1)}(u)\right) - p\left(\delta^{\circ(-2i)}(u)\right)y\left(\delta^{\circ(-2i)}(u)\right)\right]. \end{aligned}$$

This concludes the proof. □

2.2. Canonical status of Eq (1.1)

This section presents a Kneser-type criterion for testing the oscillation of Eq (1.1) when $\mathcal{A}(u) \to \infty$ as $u \to \infty$. The following assumption will be needed here:

(L) There is a $p_* \in (0,1)$ such that

$$p(u) \leq \begin{cases} \frac{\mathcal{A}(\delta(u))}{\mathcal{A}(u)} p_*, & \text{if } \delta(u) \leq u, \\ p_*, & \text{if } \delta(u) \geq u. \end{cases}$$

Next, we define the following notations:

$$\mathcal{K}_m(u) := \begin{cases} \sum_{i=1}^{m} \prod_{j=1}^{2i-1} p\left(\delta^{\circ(-j)}(u)\right), & \text{if } \delta(u) \leq u, \\ \sum_{i=1}^{m} \frac{\mathcal{A}\left(\delta^{\circ(-2i+1)}(u)\right)}{\mathcal{A}(u)} \prod_{j=1}^{2i-1} p\left(\delta^{\circ(-j)}(u)\right), & \text{if } \delta(u) \geq u, \end{cases}$$

and

$$\varphi_*(u) := [1-p_*]^{\kappa} \int_a^b \varphi(u,\mu) \mathcal{K}_m^{\kappa}(\sigma(u,\mu))\, d\mu.$$

Theorem 2.1. *Suppose that $\mathcal{A}(u) \to \infty$ as $u \to \infty$, and that (L) holds. If the delay equation*

$$\left[r^{1/\kappa}(u) y'(u)\right]' + \frac{1}{\kappa}\varphi_*(u) \mathcal{A}^{\kappa-1}(\sigma_0(u)) y(\sigma_0(u)) = 0 \tag{2.1}$$

oscillates, then so does Eq (1.1).

Proof. Assume, contrary to our claim, that $x \in \mathbb{S}$. It follows from Lemma 2.1 that $y(u) > 0$ and $\left[r(u)(y'(u))^{\kappa}\right]\downarrow$. Since $\mathcal{A}(u) \to \infty$ as $u \to \infty$, we have that $y'(u) > 0$, eventually (see, e.g., Lemma 1 in [13]). Furthermore, we find

$$y(u) \geq \int_{u_1}^{u} [r(v)]^{1/\kappa} y'(v) [r(v)]^{-1/\kappa} dv \geq \mathcal{A}(u) [r(u)]^{1/\kappa} y'(u), \tag{2.2}$$

which directly leads to $[y/\mathcal{A}]\downarrow$ for $u \geq u_1$.

First, let $\delta(u) \leq u$. So, we obtain

$$y\left(\delta^{\circ(-2i)}(u)\right) \leq \frac{\mathcal{A}\left(\delta^{\circ(-2i)}(u)\right)}{\mathcal{A}\left(\delta^{\circ(-2i+1)}(u)\right)} y\left(\delta^{\circ(-2i+1)}(u)\right)$$

and

$$y\left(\delta^{\circ(-2i+1)}(u)\right) \geq y(u).$$

Using Lemma 2.2, we get

$$\begin{aligned} x(u) &\geq \sum_{i=1}^{m} \left(\prod_{j=1}^{2i-1} p\left(\delta^{\circ(-j)}(u)\right)\right) \left[1 - p\left(\delta^{\circ(-2i)}(u)\right) \frac{\mathcal{A}\left(\delta^{\circ(-2i)}(u)\right)}{\mathcal{A}\left(\delta^{\circ(-2i+1)}(u)\right)}\right] y\left(\delta^{\circ(-2i+1)}(u)\right) \\ &\geq [1-p_*] y(u) \sum_{i=1}^{m} \prod_{j=1}^{2i-1} p\left(\delta^{\circ(-j)}(u)\right). \end{aligned} \tag{2.3}$$

On the other hand, if $\delta(u) \geq u$, then we find

$$y\left(\delta^{\circ(-2i)}(u)\right) \leq y\left(\delta^{\circ(-2i+1)}(u)\right)$$

and

$$y\left(\delta^{\circ(-2i+1)}(u)\right) \geq \frac{\mathcal{A}\left(\delta^{\circ(-2i+1)}(u)\right)}{\mathcal{A}(u)} y(u),$$

and thus

$$\begin{aligned} x(u) &\geq y(u) \sum_{i=1}^{m} \left(\prod_{j=1}^{2i-1} p\left(\delta^{\circ(-j)}(u)\right) \right) \left[1 - p\left(\delta^{\circ(-2i)}(u)\right)\right] \frac{\mathcal{A}\left(\delta^{\circ(-2i+1)}(u)\right)}{\mathcal{A}(u)} \\ &\geq [1 - p_*] y(u) \sum_{i=1}^{m} \frac{\mathcal{A}\left(\delta^{\circ(-2i+1)}(u)\right)}{\mathcal{A}(u)} \prod_{j=1}^{2i-1} p\left(\delta^{\circ(-j)}(u)\right). \end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4), we arrive at

$$x(u) \geq [1 - p_*] \mathcal{K}_m(u) y(u),$$

which with (1.1) gives

$$\begin{aligned} \left(r(u)[y'(u)]^{\kappa}\right)' &\leq -\int_a^b \varphi(u,\mu) [1 - p_*]^{\kappa} \mathcal{K}_m^{\kappa}(\sigma(u,\mu)) y^{\kappa}(\sigma(u,\mu)) \mathrm{d}\mu \\ &\leq -\varphi_*(u) y^{\kappa}(\sigma_0(u)). \end{aligned} \tag{2.5}$$

From (2.2), we have

$$\left(\frac{y(\sigma_0(u))}{\mathcal{A}(\sigma_0(u))}\right)^{\kappa} \geq \left(\frac{y(u)}{\mathcal{A}(u)}\right)^{\kappa} \geq r(u)[y'(u)]^{\kappa}.$$

Thus, from (2.5), we obtain

$$\begin{aligned} \left[r^{1/\kappa}(u) y'(u)\right]' &= \frac{1}{\kappa} \left(r(u)[y'(u)]^{\kappa}\right)^{-1+1/\kappa} \left(r(u)[y'(u)]^{\kappa}\right)' \\ &\leq -\frac{1}{\kappa} \varphi_*(u) y^{\kappa}(\sigma_0(u)) \left[\frac{y(\sigma_0(u))}{\mathcal{A}(\sigma_0(u))}\right]^{-\kappa+1} \\ &= -\frac{1}{\kappa} \varphi_*(u) \mathcal{A}^{\kappa-1}(\sigma_0(u)) y(\sigma_0(u)). \end{aligned} \tag{2.6}$$

This means that inequality (2.6) admits a positive solution $y(u)$ that does not converge to zero (since $y'(u) > 0$). Therefore, Corollary 1 in [25] implies that the corresponding equation (2.1) must also admit a positive solution, which is a contradiction.

This concludes the proof. □

Corollary 2.1. *Suppose that $\mathcal{A}(u) \to \infty$ as $u \to \infty$, and that (L) holds. If*

$$\lambda_0 := \liminf_{u\to\infty} \frac{\mathcal{A}(u)}{\mathcal{A}(\sigma_0(u))} < \infty$$

and

$$\liminf_{u\to\infty} \left[r^{1/\kappa}(u) \mathcal{A}(u) \mathcal{A}^{\kappa}(\sigma_0(u)) \varphi_*(u)\right] > \kappa \max\left\{l(1-l)\lambda_0^{-l} : l \in (0,1)\right\}, \tag{2.7}$$

then Eq (1.1) oscillates.

Proof. Assume, contrary to our claim, that $x \in \mathbb{S}$. Proceeding further exactly as in the proof of Theorem 2.1, we obtain that Eq (2.1) must admit a positive solution. Nevertheless, Theorem 2 in [26] asserts that all solutions of this equation are oscillatory under condition (2.7), and this is a contradiction. This concludes the proof. □

2.3. Noncanonical status of Eq (1.1)

In our analysis, we consider two cases for the function δ, and we require the following conditions:

(P1) $\delta(u) \leq u$ and there is a $p_0 \in (0,1)$ such that $p(u) \leq p_0$;
(P2) $\delta(u) > u$ and there is a $p_1 \in (0,1)$ such that

$$p(u)\frac{\eta(u)}{\eta(\delta(u))} \leq p_1.$$

It is readily seen that η is decreasing. Therefore, in Case (P2), we deduce that $p(u) < p(u)[\eta(u)/\eta(\delta(u))] \leq p_1$. Next, we define the following notations:

$$\widehat{\varphi}(u) := \int_a^b \varphi(u,\mu)\left[p\left(\delta^{-1}(\sigma(u,\mu))\right)\right]^{\kappa} \mathrm{d}\mu,$$

$$\mathcal{H}_m(u) := \begin{cases} \sum_{i=1}^m \frac{\eta\left(\delta^{\circ(-2i+1)}(u)\right)}{\eta(u)} \prod_{j=1}^{2i-1} p\left(\delta^{\circ(-j)}(u)\right), & \text{if } \delta(u) \leq u, \\ \sum_{i=1}^m \prod_{j=1}^{2i-1} p\left(\delta^{\circ(-j)}(u)\right), & \text{if } \delta(u) > u. \end{cases}$$

$$\widetilde{\varphi}(u) := [1-p^*]^{\kappa} \int_a^b \varphi(u,\mu)\mathcal{H}_m^{\kappa}(\sigma(u,\mu))\,\mathrm{d}\mu$$

and

$$p^* = \begin{cases} p_0 & \text{if } \delta(u) \leq u, \\ p_1 & \text{if } \delta(u) > u. \end{cases}$$

2.3.1. The delay case of $\delta(u)$

Lemma 2.3. *Suppose that $x \in \mathbb{S}$, $\eta(u_0) < \infty$, and that (P1) holds. If*

$$\int_{u_0}^{\infty}\left(\frac{1}{r(\upsilon)}\int_{u_0}^{\upsilon}\widehat{\varphi}(v)\,\mathrm{d}v\right)^{1/\kappa}\mathrm{d}\upsilon = \infty, \tag{2.8}$$

then $y\downarrow$ and $[y/\eta]\uparrow$.

Proof. Let $x \in \mathbb{S}$. It follows from Lemma 2.1 that y is monotonic.

Assume first that $y(u)\uparrow$ for $u \geq u_1$. Thus, there is a constant $y_0 > 0$ such that

$$y(u) \geq y_0 \ \text{ for } u \geq u_1. \tag{2.9}$$

As in the proof of Theorem 2.1, we obtain that (2.2) holds, and so $[y/\mathcal{A}]\downarrow$ for $u \geq u_1$. Using Lemma 2.2 with $m = 1$, we obtain

$$x \geq p\left(\delta^{-1}\right)\left[y\left(\delta^{-1}\right) - p\left(\delta^{\circ(-2)}\right)y\left(\delta^{\circ(-2)}\right)\right], \tag{2.10}$$

and this, with the facts that $[y/\mathcal{A}]\downarrow$ and $\delta^{-1} \leq \delta^{\circ(-2)}$, results in

$$x \geq p\left(\delta^{-1}\right) y\left(\delta^{-1}\right)\left[1 - p\left(\delta^{\circ(-2)}\right)\frac{\mathcal{A}\left(\delta^{\circ(-2)}\right)}{\mathcal{A}\left(\delta^{-1}\right)}\right].$$

Since $\lim_{u\to\infty} \mathcal{A}(u) < \infty$, $\mathcal{A}'(u) > 0$, and $\delta^{-1}(u) \to \infty$ as $u \to \infty$, we obtain that

$$\frac{\mathcal{A}\left(\delta^{\circ(-2)}\right)}{\mathcal{A}\left(\delta^{-1}\right)} \to 1 \text{ as } u \to \infty,$$

and so, eventually,

$$x \geq \left[1 - p_0(1+\varepsilon)\right] p\left(\delta^{-1}\right) y\left(\delta^{-1}\right), \tag{2.11}$$

for all $0 < \varepsilon < \frac{1-p_0}{p_0}$. Combining (1.1), (2.9), and (2.11), we arrive at

$$\left(r(u)\left[y'(u)\right]^{\kappa}\right)' \leq -\left[1 - p_0(1+\varepsilon)\right]^{\kappa} y_0^{\kappa}\widehat{\varphi}(u). \tag{2.12}$$

Integrating (2.12), we get

$$r(u)\left[y'(u)\right]^{\kappa} \leq r(u_1)\left[y'(u_1)\right]^{\kappa} - \left[1 - p_0(1+\varepsilon)\right]^{\kappa} y_0^{\kappa}\int_{u_1}^{u}\widehat{\varphi}(v)\,\mathrm{d}v.$$

By taking the lim of both sides as $u \to \infty$, we find, based on condition (2.8) and the fact that $\eta(u_0) < \infty$, that $r\ \left[y'\right]^{\kappa} \to -\infty$, which contradicts the fact that it is positive.

Now, we deduce that $y\downarrow$. Then,

$$y(u) \geq -\int_{u}^{\infty}\left[r(v)\right]^{1/\kappa} y'(v)\left[r(v)\right]^{-1/\kappa}\mathrm{d}v \geq -\eta(u)\left[r(u)\right]^{1/\kappa} y'(u),$$

which directly leads to $[y/\eta]\uparrow$.

This concludes the proof. □

Through the following theorem, we will be able to obtain Kneser-type oscillation criteria for Eq (1.1).

Theorem 2.2. *Suppose that $\eta(u_0) < \infty$, and that (2.8) and (P1) hold. If the delay equation*

$$\left[r^{1/\kappa}(u)\,\eta^2(u)\,\omega'(u)\right]' + \frac{1}{\kappa}\eta^{\kappa}(u)\,\eta(\sigma_1(u))\,\widetilde{\varphi}(u)\,\omega(\sigma_1(u)) = 0 \tag{2.13}$$

oscillates, then so does Eq (1.1).

Proof. Assume, contrary to our claim, that $x \in \mathbb{S}$. It follows from Lemma 2.3 that $y\downarrow$ and $[y/\eta]\uparrow$. Since $u \leq \delta^{\circ(-2i+1)} \leq \delta^{\circ(-2i)}$ for $i \in \mathbb{Z}^+$, we obtain

$$y\left(\delta^{\circ(-2i+1)}(u)\right) \geq y\left(\delta^{\circ(-2i)}(u)\right)$$

and

$$y\left(\delta^{\circ(-2i+1)}(u)\right) \geq \frac{\eta\left(\delta^{\circ(-2i+1)}(u)\right)}{\eta(u)} y(u).$$

Consequently, from Lemma 2.2, we conclude that

$$x(u) \geq \frac{y(u)}{\eta(u)} \sum_{i=1}^{m} \left(\prod_{j=1}^{2i-1} p\left(\delta^{\circ(-j)}(u)\right) \right) \left[1 - p\left(\delta^{\circ(-2i)}(u)\right)\right] \eta\left(\delta^{\circ(-2i+1)}(u)\right),$$

which with (P1) yields

$$\begin{aligned} x(u) &\geq [1 - p_0] \frac{y(u)}{\eta(u)} \sum_{i=1}^{m} \left(\prod_{j=1}^{2i-1} p\left(\delta^{\circ(-j)}(u)\right) \right) \eta\left(\delta^{\circ(-2i+1)}(u)\right) \\ &\geq [1 - p_0] \mathcal{H}_m(u) y(u). \end{aligned}$$

Now, Eq (1.1) reduces to

$$\begin{aligned} (r(u)[y'(u)]^{\kappa})' &\leq -[1 - p_0]^{\kappa} \int_a^b \varphi(u,\mu) \mathcal{H}_m^{\kappa}(\sigma(u,\mu)) [y(\sigma(u,\mu))]^{\kappa} \mathrm{d}\mu \\ &\leq -\widetilde{\varphi}(u) [y(\sigma_1(u))]^{\kappa}. \end{aligned} \tag{2.14}$$

Since $y\downarrow$ and $[y/\eta]\uparrow$, we find that

$$-r(u)[y'(u)]^{\kappa} \leq \frac{[y(u)]^{\kappa}}{\eta^{\kappa}(u)} \leq \frac{[y(\sigma_1(u))]^{\kappa}}{\eta^{\kappa}(u)}. \tag{2.15}$$

Therefore, we have

$$(-r(u)[y'(u)]^{\kappa})^{-1+1/\kappa} \geq \left(\frac{[y(\sigma_1(u))]^{\kappa}}{\eta^{\kappa}(u)} \right)^{-1+1/\kappa}.$$

Using (2.14) and (2.15), we deduce that

$$\begin{aligned} \left[r^{1/\kappa}(u) y'(u)\right]' &= \frac{1}{\kappa} (r(u)[y'(u)]^{\kappa})^{-1+1/\kappa} (r(u)[y'(u)]^{\kappa})' \\ &\leq -\frac{1}{\kappa} \frac{[y(\sigma_1(u))]^{1-\kappa}}{\eta^{1-\kappa}(u)} \widetilde{\varphi}(u) [y(\sigma_1(u))]^{\kappa} \\ &= -\frac{1}{\kappa} \eta^{\kappa-1}(u) \widetilde{\varphi}(u) y(\sigma_1(u)). \end{aligned} \tag{2.16}$$

In addition, we obtain

$$\begin{aligned} \left[r^{1/\kappa}(u) \eta^2(u) \left(\frac{y(u)}{\eta(u)} \right)' \right]' &= \left[\eta(u) r^{1/\kappa}(u) y'(u) + y(u)\right]' \\ &= \eta(u) \left[r^{1/\kappa}(u) y'(u)\right]' \\ &\leq -\frac{1}{\kappa} \eta^{\kappa}(u) \widetilde{\varphi}(u) y(\sigma_1(u)). \end{aligned} \tag{2.17}$$

Using the oscillation-preserving transformation $\omega = y/\eta$, Eq (2.17) takes the form

$$\left[r^{1/\kappa}(u) \eta^2(u) \omega'(u)\right]' + \frac{1}{\kappa} \eta^{\kappa}(u) \eta(\sigma_1(u)) \widetilde{\varphi}(u) \omega(\sigma_1(u)) \leq 0. \tag{2.18}$$

Since $\lim_{u\to\infty}\sigma_1(u)=\infty$, $[y/\eta]\uparrow$, and

$$\int_{u_1}^{u}\left[r^{1/\kappa}(v)\,\eta^2(v)\right]\mathrm{d}v=\frac{1}{\eta(u)}-\frac{1}{\eta(u_1)}\to\infty \text{ as } t\to\infty,$$

this means that inequality (2.18) admits a positive solution that does not converge to zero, and moreover this inequality is in the canonical case. Therefore, Corollary 1 in [25] implies that the corresponding equation (2.13) must also admit a positive solution, which is a contradiction.

This concludes the proof. □

Corollary 2.2. *Suppose that $\eta(u_0)<\infty$, and that (2.8) and (P1) hold. If*

$$\lambda:=\liminf_{u\to\infty}\frac{\psi(u)}{\psi(\sigma_1(u))}<\infty \tag{2.19}$$

and

$$\liminf_{u\to\infty}\left[r^{1/\kappa}(u)\,\eta^{\kappa+2}(u)\,\eta(\sigma_1(u))\,\psi(u)\,\psi(\sigma_1(u))\,\widetilde{\varphi}(u)\right]>\kappa\max\left\{l(1-l)\,\lambda^{-l}:l\in(0,1)\right\}, \tag{2.20}$$

then Eq (1.1) oscillates, where

$$\psi(u):=\int_{u_1}^{u}r^{-1/\kappa}(v)\,[\eta(v)]^{-2}\,\mathrm{d}v. \tag{2.21}$$

Proof. Assume, contrary to our claim, that $x\in\mathbb{S}$. Proceeding further exactly as in the proof of Theorem 2.2, we obtain that the equation

$$\left[r^{1/\kappa}(u)\,\eta^2(u)\,\omega'(u)\right]'+\frac{1}{\kappa}\eta^{\kappa}(u)\,\eta(\sigma_1(u))\,\widetilde{\varphi}(u)\,\omega(\sigma_1(u))=0$$

must admit a positive solution. Nevertheless, Theorem 2 in [26] asserts that all solutions of this equation are oscillatory under condition (2.20), and this is a contradiction.

This concludes the proof. □

As a refinement and extension of the method employed in Theorem 2.2 in [28], the next theorem offers an improved framework for deriving criteria of the form $\limsup_{u\to\infty}(\cdot)>1$.

Theorem 2.3. *Suppose that $\eta(u_0)<\infty$, and that (2.8) and (P1) hold. If*

$$\limsup_{u\to\infty}\left([\widetilde{\eta}(u)]^{\kappa}\int_{u_1}^{u}\widetilde{\varphi}(v)\,\mathrm{d}v\right)>1, \tag{2.22}$$

then Eq (1.1) oscillates, where

$$\widetilde{\eta}(u)=\eta(u)+\frac{1}{\kappa}\frac{\eta(u)}{\eta(\sigma_1(u))}\int_{u}^{\infty}\eta^{\kappa}(v)\,\eta(\sigma_1(v))\,\widetilde{\varphi}(v)\,\mathrm{d}v.$$

Proof. Assume, contrary to our claim, that $x \in \mathbb{S}$. It follows from Lemma 2.3 that $y\downarrow$ and $[y/\eta]\uparrow$. Proceeding as in the proof of Theorem 2.2, we obtain (2.14) and (2.17). Upon integrating (2.14), we arrive at

$$\begin{aligned} -r(u)[y'(u)]^{\kappa} &\geq -r(u_1)[y'(u_1)]^{\kappa} + \int_{u_1}^{u} \widetilde{\varphi}(v)[y(\sigma_1(v))]^{\kappa} \mathrm{d}v \\ &\geq [y(u)]^{\kappa} \int_{u_1}^{u} \widetilde{\varphi}(v) \mathrm{d}v. \end{aligned} \tag{2.23}$$

Next, integrating (2.17), we obtain

$$y(u) \geq -\eta(u) r^{1/\kappa}(u) y'(u) + \frac{1}{\kappa} \int_{u}^{\infty} \eta^{\kappa}(v) \widetilde{\varphi}(v) y(\sigma_1(v)) \mathrm{d}v. \tag{2.24}$$

Since $y\downarrow$, $\sigma_1(u) \leq u$, and $[y/\eta]\uparrow$, we find that

$$y(\sigma_1(v)) \geq \frac{\eta(\sigma_1(v))}{\eta(\sigma_1(u))} y(\sigma_1(u)) \geq \frac{\eta(\sigma_1(v))}{\eta(\sigma_1(u))} y(u) \geq -\frac{\eta(\sigma_1(v))}{\eta(\sigma_1(u))} \eta(u)[r(u)]^{1/\kappa} y'(u),$$

for all $v \geq u$. Hence, (2.24) becomes

$$\begin{aligned} y(u) &\geq -\eta(u) r^{1/\kappa}(u) y'(u) \left[1 + \frac{1}{\kappa} \frac{1}{\eta(\sigma_1(u))} \int_{u}^{\infty} \eta^{\kappa}(v) \eta(\sigma_1(v)) \widetilde{\varphi}(v) \mathrm{d}v\right] \\ &= -\widetilde{\eta}(u) r^{1/\kappa}(u) y'(u). \end{aligned} \tag{2.25}$$

Combining (2.23) and (2.25), we get

$$-r(u)[y'(u)]^{\kappa} \geq -r(u)[y'(u)]^{\kappa} [\widetilde{\eta}(u)]^{\kappa} \int_{u_1}^{u} \widetilde{\varphi}(v) \mathrm{d}v,$$

which contradicts (2.22).

This concludes the proof. □

By applying the same approach used in Theorem 1.3, the following theorem investigates the sensitivity of the criteria to improvements in the relationship between x and y.

Theorem 2.4. *Suppose that $\eta(u_0) < \infty$, (2.8), and (P1) hold, and that $\rho(u) := \delta^{-1}(\sigma_1(u)) \geq u$, and $\rho'(u) \geq 0$. If*

$$\limsup_{u \to \infty} \mathcal{F}(u) > \kappa, \tag{2.26}$$

then Eq (1.1) oscillates, where

$$\begin{aligned} \mathcal{F}(u) &= \eta(\rho(u)) \int_{u_1}^{u} \eta^{\kappa-1}(v) \widetilde{\varphi}(v) \mathrm{d}v + \int_{u}^{\rho(u)} \eta^{\kappa-1}(v) \eta(\rho(v)) \widetilde{\varphi}(v) \mathrm{d}v \\ &\quad + \frac{1}{\eta(\rho(u))} \int_{\rho(u)}^{\infty} \eta^{\kappa}(v) \eta(\rho(v)) \widetilde{\varphi}(v) \mathrm{d}v. \end{aligned}$$

Proof. Assume, contrary to our claim, that $x \in \mathbb{S}$. It follows from Lemma 2.3 that $y\downarrow$ and $[y/\eta]\uparrow$. Proceeding as in the proof of Theorem 2.2, we obtain (2.16). Integrating (2.16), we conclude that

$$-r^{1/\kappa}(u)\,y'(u) \geq \frac{1}{\kappa}\int_{u_1}^{u} \eta^{\kappa-1}(v)\,\widetilde{\varphi}(v)\,y(\sigma_1(v))\,\mathrm{d}v. \tag{2.27}$$

By following the same steps used in the proof of Theorem 2.3, we get (2.24). From (2.24) and (2.27), we arrive at

$$\begin{aligned}\kappa y(u) &\geq \eta(u)\int_{u_1}^{u}\eta^{\kappa-1}(v)\,\widetilde{\varphi}(v)\,y(\sigma_1(v))\,\mathrm{d}v + \int_{u}^{\infty}\eta^{\kappa}(v)\,\widetilde{\varphi}(v)\,y(\sigma_1(v))\,\mathrm{d}v\\ &\geq \eta(u)\int_{u_1}^{u}\eta^{\kappa-1}(v)\,\widetilde{\varphi}(v)\,y(\rho(v))\,\mathrm{d}v + \int_{u}^{\infty}\eta^{\kappa}(v)\,\widetilde{\varphi}(v)\,y(\rho(v))\,\mathrm{d}v,\end{aligned}$$

or

$$\begin{aligned}\kappa y(\rho(u)) &\geq \eta(\rho(u))\int_{u_1}^{u}\eta^{\kappa-1}(v)\,\widetilde{\varphi}(v)\,y(\rho(v))\,\mathrm{d}v + \eta(\rho(u))\int_{u}^{\rho(u)}\eta^{\kappa-1}(v)\,\widetilde{\varphi}(v)\,y(\rho(v))\,\mathrm{d}v\\ &\quad+\int_{\rho(u)}^{\infty}\eta^{\kappa}(v)\,\widetilde{\varphi}(v)\,y(\rho(v))\,\mathrm{d}v.\end{aligned}$$

Using the facts that $y\downarrow$, $\delta(u) \leq \sigma_1(u) \leq u$, and $[y/\eta]\uparrow$, we arrive at

$$\begin{aligned}\kappa &\geq \eta(\rho(u))\int_{u_1}^{u}\eta^{\kappa-1}(v)\,\widetilde{\varphi}(v)\,\mathrm{d}v + \int_{u}^{\rho(u)}\eta^{\kappa-1}(v)\,\eta(\rho(v))\,\widetilde{\varphi}(v)\,\mathrm{d}v\\ &\quad+\frac{1}{\eta(\rho(u))}\int_{\rho(u)}^{\infty}\eta^{\kappa}(v)\,\eta(\rho(v))\,\widetilde{\varphi}(v)\,\mathrm{d}v.\end{aligned}$$

This contradicts (2.26).

This concludes the proof. □

2.3.2. The advanced case of $\delta(u)$

Lemma 2.4. *Suppose that $x \in \mathbb{S}$, $\eta(u_0) < \infty$, and that (P2) and (2.8) hold. Then, $y\downarrow$ and $[y/\eta]\uparrow$.*

Proof. Let $x \in \mathbb{S}$. It follows from Lemma 2.1 that y is monotonic.

Assume first that $y(u)\uparrow$ for $u \geq u_1$. Proceeding as in the proof of Lemma 2.3, we obtain that (2.9) and (2.10) hold. Since $y(u)\uparrow$ and $\delta^{-1} > \delta^{\circ(-2)}$, it follows from (2.10) that

$$\begin{aligned}x &\geq p\left(\delta^{-1}\right)\left[1 - p\left(\delta^{\circ(-2)}\right)\right]y\left(\delta^{-1}\right)\\ &\geq [1-p_1]\,p\left(\delta^{-1}\right)y\left(\delta^{-1}\right).\end{aligned} \tag{2.28}$$

Combining (1.1), (2.9), and (2.28), we arrive at

$$(r(u)\,[y'(u)]^{\kappa})' \leq -[1-p_1]^{\kappa}\,y_0^{\kappa}\widehat{\varphi}(u).$$

The rest of the proof follows the same steps as in the proof of Lemma 2.3, and is thus omitted. □

Theorem 2.5. *Suppose that $\eta(u_0) < \infty$, and that (2.8) and (P2) hold. If the delay equation (2.13) oscillates, then so does Eq (1.1).*

Proof. Assume, contrary to our claim, that $x \in \mathbb{S}$. It follows from Lemma 2.4 that $y\downarrow$ and $[y/\eta]\uparrow$. Since $u > \delta^{\circ(-2i+1)} > \delta^{\circ(-2i)}$ for $i \in \mathbb{Z}^+$, we obtain

$$y\left(\delta^{\circ(-2i+1)}(u)\right) > y(u)$$

and

$$y\left(\delta^{\circ(-2i)}(u)\right) < \frac{\eta\left(\delta^{\circ(-2i)}(u)\right)}{\eta\left(\delta^{\circ(-2i+1)}(u)\right)} y\left(\delta^{\circ(-2i+1)}(u)\right).$$

Consequently, from Lemma 2.2, we conclude that

$$\begin{aligned} x(u) &\geq \sum_{i=1}^{m}\left(\prod_{j=1}^{2i-1} p\left(\delta^{\circ(-j)}(u)\right)\right)\left[1 - p\left(\delta^{\circ(-2i)}(u)\right)\frac{\eta\left(\delta^{\circ(-2i)}(u)\right)}{\eta\left(\delta^{\circ(-2i+1)}(u)\right)}\right] y\left(\delta^{\circ(-2i+1)}(u)\right) \\ &\geq [1 - p_1]\, y(u) \sum_{i=1}^{m}\left(\prod_{j=1}^{2i-1} p\left(\delta^{\circ(-j)}(u)\right)\right) \\ &= [1 - p_1]\, \mathcal{H}_m(u)\, y(u). \end{aligned}$$

The rest of the proof follows the same steps as in the proof of Theorem 2.2, and is thus omitted. □

The following results are obtained directly by applying the same approach used in the case (P1); therefore, the proofs are omitted.

Corollary 2.3. *Suppose that $\eta(u_0) < \infty$, (P2), and (2.19) hold. If (2.20) is satisfied, then Eq (1.1) oscillates, where ψ is defined as in (2.21).*

Theorem 2.6. *Suppose that $\eta(u_0) < \infty$, (P2), and (2.22) hold. Then, Eq (1.1) oscillates, where $\widetilde{\eta}(u)$ is defined as in Theorem 2.3.*

3. Examples and comparison

Example 3.1. *Consider the NDE*

$$\left(u^{\kappa+1}\left[(x(u) + \ell x(\alpha u))'\right]^{\kappa}\right)' + \varphi_0 \int_{1/2}^{1} [x(\mu u)]^{\kappa}\, \mathrm{d}\mu = 0, \tag{3.1}$$

where $\alpha > 0$, $\ell > \max\left\{\sqrt[\kappa]{\alpha}, 1\right\}$, and $\varphi_0 > 0$. It can be readily verified that $p(u) = 1/\ell$, $\eta(u) = \kappa/\sqrt[\kappa]{u}$, $\sigma_1(u) = u$, and

$$p^* = \begin{cases} p_0 = 1/\ell, & \text{if } \alpha \leq 1, \\ p_1 = \sqrt[\kappa]{\alpha}/\ell, & \text{if } \alpha > 1. \end{cases}$$

Furthermore, we have

$$\widehat{\varphi}(u) = \frac{1}{2\ell^{\kappa}}\varphi_0,$$

$$\mathcal{H}_m(u) = \begin{cases} \sum_{i=1}^{m} \left(\frac{\sqrt[\kappa]{\alpha}}{\ell}\right)^{2i-1}, & \text{if } \alpha \leq 1, \\ \sum_{i=1}^{m} \left(\frac{1}{\ell}\right)^{2i-1}, & \text{if } \alpha > 1, \end{cases}$$

and

$$\widetilde{\varphi}(u) = \frac{1}{2}\left[1 - p^*\right]^{\kappa} \mathcal{H}_m^{\kappa} \varphi_0.$$

Then, condition (2.8) is satisfied. To apply our results, we find that

$$\psi(u) = \frac{1}{\kappa}\left(u^{1/\kappa} - u_0^{1/\kappa}\right)$$

and $\lambda = 1$. *Now, according to Corollaries 2.2 and 2.3, Eq (3.1) is oscillatory if*

$$\kappa^{\kappa}\left[1 - p^*\right]^{\kappa} \mathcal{H}_m^{\kappa} \varphi_0 > \frac{1}{2}. \tag{3.2}$$

Remark 3.1. *For Eq (3.1), the results in [20] are constrained by the condition* $\sigma(u, a) \leq \delta(u) \leq u$, *which means that they apply only when* $\alpha \in [0.5, 1]$. *Moreover, we observe that* $\sigma(u, \mu)$ *is increasing in* μ; *therefore, the results in [22] cannot be applied to Eq (3.1). Consequently, our results are more efficient in testing the oscillation of Eq (3.1) than those in [20, 22].*

Example 3.2. *Consider the NDE*

$$\left(u^2\left[x(u) + \ell x(\alpha u)\right]'\right)' + \varphi_0 \int_1^2 \left[1 + \mathbf{e}^{-u} \sin(2\pi\mu)\right] x\left(\frac{u}{\mu}\right) \mathrm{d}\mu, \tag{3.3}$$

where $\alpha \in (0, 1]$, $\ell > 1$, *and* $\varphi_0 > 0$. *It can be readily verified that* $p(u) = 1/\ell$, $\eta(u) = 1/u$, $\sigma_1(u) = u$, *and* $p^* = p_0 = 1/\ell$. *Hence, we obtain* $\delta^{\circ(-2i+1)}(u) = \alpha^{-2i+1}u$,

$$\widehat{\varphi}(u) = \frac{1}{\ell}\varphi_0 \int_1^2 \left(1 + \mathbf{e}^{-u} \sin(2\pi\mu)\right) \mathrm{d}\mu = \frac{\varphi_0}{\ell},$$

$$\mathcal{H}_m(u) = \sum_{i=1}^{m} \left(\frac{\alpha}{\ell}\right)^{2i-1},$$

and

$$\widetilde{\varphi}(u) = \left[1 - \frac{1}{\ell}\right] \varphi_0 \sum_{i=1}^{m} \left(\frac{\alpha}{\ell}\right)^{2i-1}.$$

Then, condition (2.8) is satisfied. Moreover, we have

$$\varphi_0^* = \lim_{m\to\infty} \widetilde{\varphi}(u) = \frac{\alpha\left[\ell - 1\right]\varphi_0}{\ell^2 - \alpha^2},$$

$$\widetilde{\eta}(u) = \left(1 + \varphi_0^*\right)\frac{1}{u},$$

and

$$\mathcal{F}(u) = \alpha\varphi_0^* \left[2 + \ln\frac{1}{\alpha}\right].$$

Using Corollary 2.2, Eq (3.3) is oscillatory if

$$\varphi_0 > \frac{\ell^2 - \alpha^2}{4\alpha\,[\ell - 1]}. \tag{3.4}$$

Next, from Theorems 2.3 and 2.4, Eq (3.3) is oscillatory if

$$\varphi_0^*\,(1 + \varphi_0^*) > 1 \tag{3.5}$$

or

$$\alpha\varphi_0^*\left[2 + \ln\frac{1}{\alpha}\right] > 1, \tag{3.6}$$

respectively.

Remark 3.2. *Regarding Eq (3.3), we note that condition (1.4) is violated because σ is decreasing in μ. Therefore, the results in [20] cannot be applied to Eq (3.3). On the other hand, applying Theorem 1.2 requires that $\sigma(u,a) \leq \delta(u)$, which means that it is applicable only when $\alpha = 1$ (non-neutral case). In this case, using Theorem 1.2, the resulting oscillation criterion takes the form*

$$\varphi_0 > \frac{\ell^2}{2\,[\ell - 1]}.$$

Whereas applying Theorem 1.3 leads to the criterion

$$\varphi_0 > \frac{\ell^2}{\alpha\,[\ell - 1]\,[2 + \ln(1/\alpha)]}. \tag{3.7}$$

Now, by comparing our results with those, we conclude the following:

- *It is straightforward to see that, for $\alpha = 1$, condition (3.4) reduces to*

  $$\varphi_0 > \frac{\ell + 1}{4}.$$

  Since

  $$\frac{\ell + 1}{4} < 2\frac{\ell + 1}{4} + \frac{1}{2\,[\ell - 1]} = \frac{\ell^2}{2\,[\ell - 1]},$$

  our results provide a sharper oscillation criterion in this case.
- *Figure 1 illustrates a comparison between the lower bounds of φ_0 that satisfy conditions (3.5)–(3.7) over $\alpha \in (0, 1]$. From this figure, two important observations can be made. First, the criterion obtained from Theorem 2.4 (3.6) provides a complete improvement over that derived from Theorem 1.3 (3.7), with the effect of this improvement becoming more pronounced as α approaches* 1. *Second, the criterion obtained from Theorem 2.3 yields a sharper bound over the subinterval* $[0.133, 1]$.

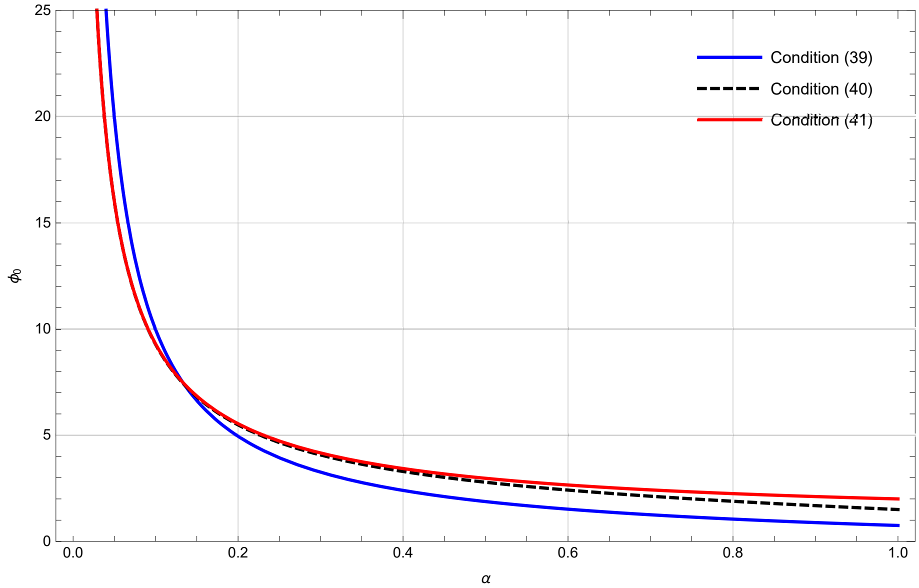

Figure 1. Minimal values of φ_0 obtained under criteria (3.5)–(3.7) for $\alpha \in (0, 1]$.

Example 3.3. *Consider the NDE*

$$(\mathbf{e}^u [x(u) + \ell x(u+c)]')' + \varphi_0 \int_0^1 \mathbf{e}^{u+\mu} x(u+\mu)\,\mathrm{d}\mu, \tag{3.8}$$

where $c \geq 0$, $\ell > \mathbf{e}^c$, *and* $\varphi_0 > 0$. *It can be readily verified that* $\eta(u) = \mathbf{e}^{-u}$, $\delta(u) = u + c \geq u$, $\sigma_1(u) = u + 1$, *and*

$$p^* = p_1 = \frac{\mathbf{e}^c}{\ell} < 1.$$

Hence, we obtain

$$\widehat{\varphi}(u) = \varphi_0 \frac{1}{\ell} \int_0^1 \mathbf{e}^{u+\mu} \mathrm{d}\mu = \frac{\varphi_0 (\mathbf{e} - 1)}{\ell} \mathbf{e}^u,$$

$$\mathcal{H}_m(u) = \sum_{i=1}^{m} \left(\frac{1}{\ell}\right)^{2i-1} \to \frac{\ell}{\ell^2 - 1} \text{ as } m \to \infty,$$

and

$$\widetilde{\varphi}(u) = \varphi_0 \frac{(\ell - \mathbf{e}^c)(\mathbf{e} - 1)}{\ell^2 - 1} \mathbf{e}^u.$$

Now, we see that

$$\frac{\varphi_0 (\mathbf{e} - 1)}{\ell} \int_{u_0}^{\infty} \frac{1}{\mathbf{e}^{\upsilon}} \int_{u_0}^{\upsilon} \mathbf{e}^{v} \mathrm{d}v \mathrm{d}\upsilon = \infty,$$

which implies that condition (2.8) is satisfied. Applying Corollary 2.3, we observe that $\psi(u) = \mathbf{e}^u - \mathbf{e}^{u_1}$ *and* $\lambda = 1/\mathbf{e}$. *Therefore, Eq (3.8) is oscillatory if*

$$\varphi_0 > \frac{\left(\ell^2 - 1\right)\left(-2 + \sqrt{5}\right)}{(\ell - \mathbf{e}^c)(\mathbf{e} - 1)} \mathbf{e}^{(-1+\sqrt{5})/2}. \tag{3.9}$$

On the other hand, we have

$$\widetilde{\eta}(u) = \left[1 + \varphi_0 \frac{(\ell - \mathbf{e}^c)(\mathbf{e} - 1)}{\ell^2 - 1}\right] e^{-u}.$$

It follows from Theorem 2.6 that Eq (3.8) is oscillatory if

$$\varphi_0 \frac{(\ell - \mathbf{e}^c)(\mathbf{e} - 1)}{\ell^2 - 1}\left[1 + \varphi_0 \frac{(\ell - \mathbf{e}^c)(\mathbf{e} - 1)}{\ell^2 - 1}\right] > 1. \tag{3.10}$$

Remark 3.3. *In the previous example, we observe that $\delta(u) \geq u$, which means that the results in [20, 22] cannot be applied to Eq (3.8). On the other hand, Figure 2 presents a comparison between the efficiency of conditions (3.9) and (3.10) in testing the oscillation of Eq (3.8). It is evident that (3.9) provides sharper results.*

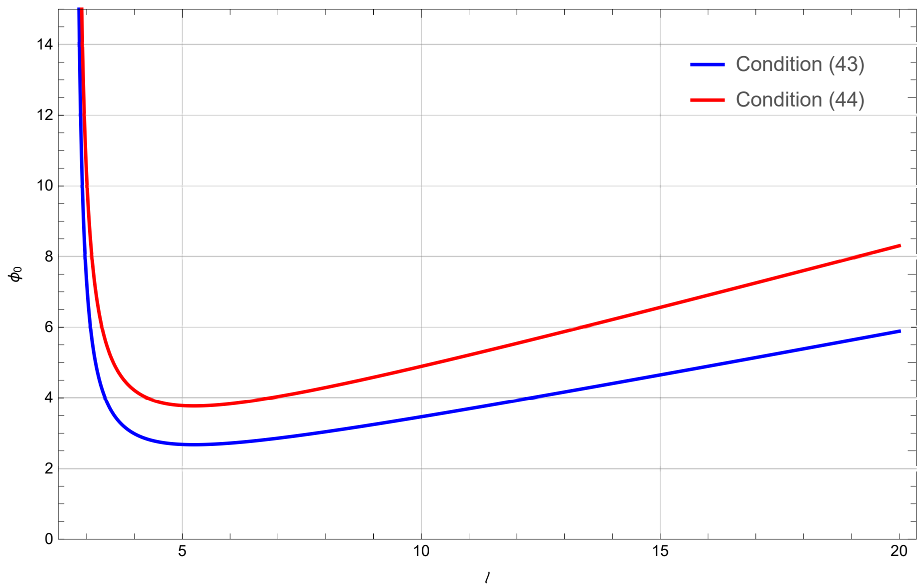

Figure 2. Minimal values of φ_0 obtained under criteria (3.9) and (3.10) for $\ell > 1$.

Remark 3.4. *It is further observed that the results obtained in this study are applicable to the equations*

$$\left(r(u)\left[y'(t)\ \operatorname{sign}(y'(t))\right]^{\kappa}\right)' + \int_a^b \varphi(u,\mu)\left[x(\sigma(u,\mu))\ \operatorname{sign}(x(\sigma(u,\mu)))\right]^{\kappa} \mathrm{d}\mu = 0$$

and

$$\left(r(u)\,|y'(t)|^{\kappa-1}\, y'(t)\right)' + \int_a^b \varphi(u,\mu)\,|x(\sigma(u,\mu))|^{\kappa-1}\, x(\sigma(u,\mu))\,\mathrm{d}\mu = 0$$

where $\kappa \geq 1$ is a real number.

4. Conclusions

This study examined the oscillatory properties of solutions to the NDE (1.1), which incorporates distributed deviating arguments. We considered the noncanonical case, as well as the two situations $\delta(u) \leq u$ and $\delta(u) > u$. Several refined inequalities were established and subsequently employed to derive new oscillation criteria. These include a Kneser-type criterion and another that refines and extends the methodology introduced in [22]. The obtained results were compared with existing ones in the literature through a set of illustrative examples. These examples demonstrate that our findings extend the theoretical framework to handle advanced arguments $\delta(u) > u$ and decreasing delays, successfully relaxing the structural constraints required by earlier works such as [22]. It would be of further interest to extend our analysis to higher-order equations and to equations involving damping terms.

Author contributions

O. Moaaz and A. Al-Jaser was responsible for writing the original draft; M. Anis contributed to the study's conceptualization and methodology; M. F. Abouelenein conducted the comparative analysis and software. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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