Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(6): 17820–17837.
DOI: 10.3934/math.2026726
Received: 18 April 2026
Revised: 26 May 2026
Accepted: 01 June 2026
Published: 18 June 2026

Research article

Dynamics of a modified Leslie–Gower model with dual Allee effects under unilateral and bilateral control

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Abstract: This paper establishes unilateral and bilateral impulsive control frameworks based on a modified Leslie–Gower model with dual Allee effects. The existence and orbital stability of order-1 and order-2 periodic solutions are proved by differential equation geometric theory. Numerical simulations are performed to validate the theoretical results, and bifurcation diagrams reveal parameter-dependent dynamical properties. Unilateral control can prevent prey extinction caused by the Allee effect or suppress excessive prey growth, thereby avoiding predator outbreaks and the resulting prey loss. In contrast, bilateral control precisely keeps prey and predator populations within reasonable ecological thresholds, and achieves stable population persistence as well as sustainable utilization of ecological resources.

Keywords: Allee effect; unilateral and bilateral control; successor function; periodic solution; stability
Mathematics Subject Classification: 92D25, 92D40

1. Introduction

Over the past few decades, overexploitation has been the primary driver of species decline and extinction [1, 2]. Due to global warming caused by human activities, this threat has intensified sharply [3]. Therefore, species that survived early overexploitation are now particularly vulnerable to the impacts of climate change. For example, the population of Pangolins in China has significantly declined, mainly due to overhunting. Moreover, current climate change is causing their extinction in certain regions [4]. Similarly, deep-sea fish species that have suffered from overfishing are also threatened by rising temperatures and sustained fishing pressure [5]. Meanwhile, biodiversity is increasingly threatened by habitat loss, overexploitation, and climate change, which often interact and amplify their impacts [6–8].

Given the severity of threats posed by climate change and overexploitation, how can we formulate effective strategies to mitigate these risks? Sustainable management of renewable biological resources represents a systematic framework for the scientific utilization and conservation of diverse biological resources [9]. It emphasizes balancing exploitation and protection and serves as a critical pathway toward effective biodiversity conservation. Traditional management strategies centered on maximum sustainable yield may fail to adequately address the ecological challenges [10]. Hence, there is an urgent demand for more advanced management strategies to mitigate the ongoing loss of biodiversity.

Predator-prey models are vital in population dynamics, as they characterize how interspecific interactions regulate population abundance. They enable resource managers to forecast population trends, identify rational utilization thresholds, and prevent overexploitation, offering key theoretical support for the scientific management of renewable biological resources. As a typical example, the Leslie–Gower predator-prey model describes the interaction between generalist predators and one or more prey species, reflecting the predator's dependence on prey density [11, 12].

However, the studies in [11, 12] assume logistic population growth and neglect the Allee effect, which causes reduced per capita growth in sparse populations due to mate limitation, deficient feeding, and reduced antipredator defense [13]. For species already threatened by overexploitation and climate change, the Allee effect can reshape predator-prey dynamics. For instance, an overfished prey population may drop below the Allee threshold and collapse abruptly. Thus, the study of Allee effect is important to conservation biology [14]. Most studies focus on prey Allee effects in predator-prey models, but the Allee effect also occurs in predator populations. A typical example is seabirds: Recent studies have confirmed that they exhibit the Allee effect. Most of them are generalist; when the primary prey is scarce, they will seek alternative prey [15, 16].

Therefore, it is necessary to take into account that both the prey and the predator are subject to the Allee effect. Specially, Mandal et al. [17] proposed the following modified Leslie–Gower model with dual Allee effects and cooperative hunting:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{K}\right)\left(\frac{x}{K} - \frac{a}{K}\right) - \frac{\left(p_0 + \frac{a_0 y}{1+b_0 y}\right)xy}{1+h_0\left(p_0 + \frac{a_0 y}{1+b_0 y}\right)x}, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = sy\left(\frac{y}{y+b_1} - \frac{a_1 y}{x+c_1}\right), \end{cases} \tag{1.1}$$

where $x(t)$ and $y(t)$ denote the population densities of the prey and predator, respectively. The specific meanings of each parameter and their representations are detailed in Table 1.

The dynamical behaviors of system (1.1) are very complex; for further details, readers are referred to the literature [17]. Ignoring this dual Allee effect may lead to misjudgment in resource management, as it fails to reflect the true complexity of the ecosystem. The success of renewable resource management depends on effective control strategies; some scholars have explored unilateral and bilateral control schemes in ecological systems [18, 19]. Early studies focused on the unilateral control and confirmed that targeted threshold control could curb overexploitation [20, 21].

Existing studies on population dynamical systems have largely focused on the single Allee effect or merely adopted unilateral control strategies. Few investigations simultaneously incorporate dual Allee effects for two interacting species, and theoretical comparisons of bilateral control schemes in such complex dual-Allee dynamical systems remain insufficient. Addressing this research gap, this paper establishes a two-species dynamical model with dual Allee effects and further introduces bilateral control strategies.

Table 1. Parameters and descriptions.

Parameter	Description
r	Intrinsic growth rate of the prey
K	Environmental carrying capacity of the prey
a	Strength of the Allee effect in the prey
p_0	Predation rate in absence of hunting cooperation
a_0	Cooperation coefficient
b_0	Saturated constant
h_0	Handling time
s	Infestation rate of susceptible trees
b_1	Allee effect coefficient for the predator
a_1	System constant
c_1	Protective effects of the environment on the predator species

Considering the need for the protection of renewable resources and the theoretical limitation, this study based on the model (1.1) explores unilateral and bilateral state feedback control strategies, aiming to reveal the dynamic mechanism of the system, provide scientific guidance for rational harvesting, and ensure the long-term stability of the renewable resource ecosystem.

The rest of the paper is organized as follows. In Section 2, based on a modified Leslie–Gower model incorporating dual Allee effects, we formulate mathematical models with unilateral and bilateral control strategies for renewable resource management. In Section 3, we mainly establish the existence and stability of the order-1 periodic solution for the unilateral control system and the order-2 periodic solution for the bilateral control system, respectively. In Section 4, we carry out some numerical simulations to illustrate our theoretical results. Section 5 gives a conclusion for the investigation.

2. Model formulation

Assume that the prey and predator are two types of biological resources with economic value, and the quantity of the prey population can be easily monitored using modern devices. Given the significant influence of dual Allee effects on population dynamics, feasible population regulation strategies need to set two critical density thresholds. One threshold restrains excessive population density to avoid ecological instability, and the other prevents population collapse induced by the Allee effect under low-density conditions. Accordingly, the population density $x(t)$ should be maintained within a reasonable interval $[h_1, h_2]$. Based on this principle, we improve the original dynamical system by introducing two pulse control thresholds $x = h_1$ and $x = h_2$. When the prey density declines to the lower threshold h_1, we implement supplementary regulation: Increase the prey population by proportion p_1 and reduce the predator population by quantity τ_1. When the prey density rises to the upper threshold h_2, we carry out harvesting and regulation: harvest prey individuals at proportion p_2 and supplement predator individuals by quantity τ_2. Thus, we establish a modified Leslie–Gower model under dual Allee effects with bilateral control as follows:

$$
\left\{
\begin{array}{l}
\left.\begin{array}{l}
\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{K}\right)\left(\frac{x}{K} + \frac{a}{K}\right) - \frac{\left(p_0 + \frac{a_0 y}{1+b_0 y}\right)xy}{1+h_0\left(p_0 + \frac{a_0 y}{1+b_0 y}\right)x},\\
\frac{\mathrm{d}y}{\mathrm{d}t} = sy\left(\frac{y}{y+b_1} - \frac{a_1 y}{x+c_1}\right),
\end{array}\right\} \quad h_1 < x < h_2,\\
\left.\Delta x = p_1 x,\ \Delta y = -\tau_1,\right\} \quad x = h_1,\\
\left.\Delta x = -p_2 x,\ \Delta y = \tau_2,\right\} \quad x = h_2,\\
y(0) > 0,\ h_1 < x(0) < h_2,
\end{array}
\right. \tag{2.1}
$$

where h_1 is the lower density bound of prey population, h_2 is the upper density bound of prey population, p_1 means the proportional release rate of prey, τ_1 is the harvesting quantity of predators, p_2 represents the proportional harvesting rate of prey, and τ_2 is the release quantity of juvenile predators.

This bilateral threshold control strategy embedded in the modified model perfectly integrates ecological theories with resource management practices. It ensures population survival and restrains overpopulation, balances natural evolution rules, and artificial regulation requirements, thereby preventing species extinction and ecological imbalance and achieving stable population persistence as well as sustainable utilization of ecological resources.

For system (2.1), based on the geometric theory of differential equations, we show that when only one of the two impulsive functions is active, the system reduces to two unilateral control systems listed as follows:

$$
\left\{
\begin{array}{l}
\left.\begin{array}{l}
\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{K}\right)\left(\frac{x}{K} + \frac{a}{K}\right) - \frac{\left(p_0 + \frac{a_0 y}{1+b_0 y}\right)xy}{1+h_0\left(p_0 + \frac{a_0 y}{1+b_0 y}\right)x},\\
\frac{\mathrm{d}y}{\mathrm{d}t} = sy\left(\frac{y}{y+b_1} - \frac{a_1 y}{x+c_1}\right),
\end{array}\right\} \quad x > h_1,\\
\Delta x = p_1 x,\ \Delta y = -\tau_1,\ \ x = h_1,\\
x(0) > h_1,\ y(0) > 0,\ p_1 > 0,
\end{array}
\right. \tag{2.2}
$$

and

$$
\left\{
\begin{array}{l}
\left.\begin{array}{l}
\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1 - \frac{x}{K}\right)\left(\frac{x}{K} + \frac{a}{K}\right) - \frac{\left(p_0 + \frac{a_0 y}{1+b_0 y}\right)xy}{1+h_0\left(p_0 + \frac{a_0 y}{1+b_0 y}\right)x},\\
\frac{\mathrm{d}y}{\mathrm{d}t} = sy\left(\frac{y}{y+b_1} - \frac{a_1 y}{x+c_1}\right),
\end{array}\right\} \quad x < h_2,\\
\Delta x = -p_2 x,\ \Delta y = \tau_2,\ \ x = h_2,\\
x(0) < h_2,\ y(0) > 0.
\end{array}
\right. \tag{2.3}
$$

3. Existence and stability of order-1 and order-2 periodic solutions

In the following analysis, we concentrate on the dynamical behaviors of the three systems.

3.1. Dynamics of the two unilateral control systems

Suppose that l_1 and l_2 are two isoclines of system (1.1), with their intersection points being $E_1^*(x_1^*, y_1^*)$ and $E_2^*(x_2^*, y_2^*)$. This means that system (1.1) has two positive equilibrium points, as illustrated in Figure 1. Specially, E_1^* is an unstable saddle point, whereas E_2^* may be stable, unstable, or enclosed by a stable limit cycle. In addition, E_{01} is locally stable, and E_{10} is unstable. Assume that l_3 and l_4 are two stable manifolds, which divide the first quadrant into two attractive domains. In this paper, we assume that E_2^* is locally stable, and we investigate the existence of order-1 periodic solutions for the unilateral control system and order-2 periodic solutions for the bilateral control system. Specifically, we take the weak Allee effect of the prey as an example, and the derivation process of the strong Allee effect is similar, so it is omitted here. The proof then follows for the existence of the order-1 periodic solution of systems (2.2) and (2.3). Therefore, we obtain the following two theorems.

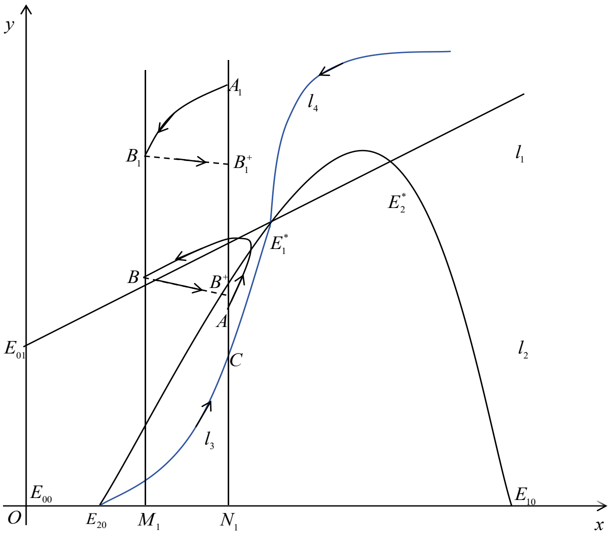


Figure 1. Existence of the order-1 periodic solution of system (2.2) for $h_1 < (1 + p_1)h_1 < x_1^*$.

Theorem 3.1. *If* $h_1 < (1 + p_1)h_1 < x_1^*$, *and* $y_B - \tau_1 \geq y_A$, *then system* (2.2) *emits an order-1 periodic solution.*

Proof. Assume the impulsive set

$$M_1 = \{(x, y) \in R_+^2 | x = h_1\}$$

and phase set

$$N_1 = \{(x, y) \in R_+^2 | x = (1 + p_1)h_1\}$$

with $(1 + p_1)h_1 < x_1^*$. We further assume that the stable manifold l_3 intersects the phase set N_1 at point $C(x_C, y_C)$ with $x_C = (1 + p_1)h_1$. Now, we choose a point A with $x_A = (1 + p_1)h_1 < x_1^*$ and $y_A > y_C$; the trajectory passing through point $A(x_A, y_A)$ intersects the impulsive set M_1 at point B. Due to the impulsive effect, it jumps to point $B^+ \in N_1$. According to the magnitude relationship between y_A and y_{B^+}, the discussion can be divided into following three cases.

(1) If $y_{B^+} < y_A$, then after finite pulses, the trajectory enters the basin of attraction E_2^* and will approach E_2^* asymptotically. This indicates that there is no order-1 periodic solution for this case.

(2) If $y_{B^+} = y_A$, then the curve $\widehat{AB} \cup \overline{BA}$ forms an order-1 periodic solution.

(3) If $y_{B^+} > y_A$, then the successor function of point A satisfies that

$$f(A) = y_{B^+} - y_A = y_B(1 - p_1) - y_A > 0.$$

Now, we only need to find another point $A_1 \in N_1$ such that the successor function $f(A_1) < 0$ holds. Choose a point A_1 such that A_1 is far away from the x-axis, and the trajectory passing through point A_1 intersects the impulsive set M_1 at point B_1 with $y_{B_1} > y_B$. It then jumps to $B_1^+ \in N_1$. Then the successor function

$$f(A_1) = y_{B_1^+} - y_{A_1} = (y_{B_1} + \tau_1) - (y_B + \tau_1) = y_{B_1} - y_B > 0.$$

Because the successor function$f$ is continuous [19], according to the intermediate value theorem, there must exist at least one zero point within this interval; that is, there exists a point $G \in N_1$ between point A and A_1 satisfying $f(G) = 0$. Therefore, system (2.2) has an order-1 periodic solution.

This completes the proof. □

In the following, we prove the existence of an order-1 periodic solution for the system (2.3) if $x_1^* < (1 - p_2)h_2 < h_2 < x_2^*$. The case of $x_1^* < (1 - p_2)h_2 < x_2^* < h_2$ can be established analogously and is omitted.

Then, we have the following Theorem 3.2.

Theorem 3.2. *If one of following three conditions holds, system* (2.3) *exists an order-1 periodic solution: (1)* $y_{C_2^+} = y_{A_2}$*; (2)* $y_{A_2} - y_{C_1} > 0$*,* $y_{A_2} > y_{C_2^+}$*; (3)* $y_{A_2} < y_{C_2^+} < y_{C'}$.

Proof. Assume that curve l_4 intersects phase set N_2 at point $C'(x_{C'}, y_{C'})$. Choose a point $A_2(x_{A_2}, y_{A_2})$ with $y_{A_2} < y_{C'}$; then the trajectory passing through point $A_2 \in N_2$ intersects impulsive set M_2 at point C_2 and then jumps to $C_2^+ \in N_2$, as is shown in Figure 2.

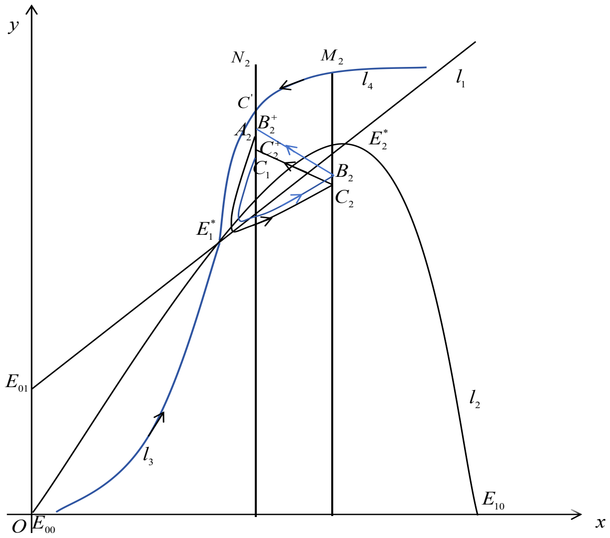


Figure 2. Existence of the order-1 periodic solution for system (2.2) for $x_1^* < (1 - p_2)h_2 < h_2 < x_2^*$.

The existence of order-1 periodic solutions can be proved by dividing into the following three cases:

(1) If $y_{C_2^+} = y_{A_2}$, the the curve $\widehat{A_2C_2} \cup \overline{C_2A_2}$ forms an order-1 periodic solution.

(2) If $y_{C_2^+} < y_{A_2}$, then the successor function $f(A_2) = y_{A_2} - y_{B_2^+} < 0$ holds. Now, we only need to find a point such that the corresponding successor function is bigger than 0. Choose a point C_1 with $y_{C_1} < y_{C_2^+}$ so that the trajectory passing through C_1 intersects impulsive set M_2 at point B_2 with $y_{B_2} > y_{C_2}$. Then $y_{B_2^+} > y_{C_2^+}$, and the successor function $f(C_1) = y_{B_2^+} - y_{C_1} > 0$ holds. This indicates that system (2.3) has an order-1 periodic solution.

(3) If $y_{A_2} < y_{C_2^+} < y_{C'}$, similar to the proof process in [22, 23], there also exists an order-1 periodic solution.

This completes the proof. □

Now, we prove the stability of the order-1 periodic solution of system (2.2). Assume that $\widehat{AB} \cup \overline{BA}$ forms an order-1 periodic solution of system (2.2) with period T. Let $(\xi(t), \eta(t))$ be such an order-1 periodic solution with period T satisfying $\xi(0) = x_A, \eta(0) = y_A$, $\xi(T) = x_B = h_1, \eta(T) = y_B$, and the impulsive conditions $\xi_1(T_1 + 0) = (1 + p_1)h_1 = x_A, \eta_1(T_1 + 0) = y_B - \tau_1 = y_A$.

Theorem 3.3. *The order-1 periodic solution of system* (2.2) *is orbitally asymptotically stable if* $\left|\Theta_1\right| < 1$ *holds, where*

$$\begin{aligned}
\Theta_1 =& \frac{G_1 L_1}{H_1} \frac{y_B}{(1 + p_1)(y_B - \tau_1)},\\
G_1 =& r(1 + p_1)h_1 \left(1 - \frac{(1 + p_1)h_1}{K}\right)\left(\frac{(1 + p_1)h_1 + a}{K}\right) - \frac{\left(p_0 + \frac{a_0(y_B-\tau_1)}{1+b_0(y_B-\tau_1)}\right)(1 + p_1)h_1(y_B - \tau_1)}{1 + h_0\left(p_0 + \frac{a_0(y_B-\tau_1)}{1+b_0(y_B-\tau_1)}\right)(1 + p_1)h_1},\\
H_1 =& rh_1 \left(1 - \frac{h_1}{K}\right)\left(\frac{h_1 + a}{K}\right) - \frac{\left(p_0 + \frac{a_0 y_B}{1+b_0 y_B}\right)h_1 y_B}{1 + h_0\left(p_0 + \frac{a_0 y_B}{1+b_0 y_B}\right)h_1}
\end{aligned}$$

and

$$L_1 = \exp\left\{\int_{0^+}^{T_1} sy\left(\frac{b_1}{(y + b_1)^2} - \frac{a_1}{x + c_1}\right)\mathrm{d}t\right\} \times \exp\left\{\int_{0^+}^{T_1} \left(\frac{rx}{K} - \frac{2rx^2}{K^2} + \frac{h_0(p_0 + \frac{a_0 y}{1+b_0 y})xy}{(1 + h_0(p_0 + \frac{a_0 y}{1+b_0 y})x)^2}\right)\mathrm{d}t\right\}.$$

Proof. Define

$$\begin{aligned}
P(x, y) &= rx\left(1 - \frac{x}{K}\right)\left(\frac{x}{K} - \frac{a}{K}\right) - \frac{\left(p_0 + \frac{a_0 y}{1+b_0 y}\right)xy}{1 + h_0\left(p_0 + \frac{a_0 y}{1+b_0 y}\right)x},\\
Q(x, y) &= sy\left(\frac{y}{y + b_1} - \frac{a_1 y}{x + c_1}\right),\\
E_1(x, y) &= -p_1 x,\quad F_1(x, y) = \tau_1,\quad \varphi_1(x, y) = x - h_1;
\end{aligned}$$

then, we obtain that

$$\begin{aligned}
\frac{\partial P}{\partial x} =& r\left(1 - \frac{x}{K}\right)\left(\frac{x}{K} - \frac{a}{K}\right) - \frac{(p_0 + \frac{a_0 y}{1+b_0 y})y}{1 + h_0(p_0 + \frac{a_0 y}{1+b_0 y})x} + \frac{rx}{K} - \frac{2rx^2}{K^2} + \frac{h_0(p_0 + \frac{a_0 y}{1+b_0 y})xy}{(1 + h_0(p_0 + \frac{a_0 y}{1+b_0 y})x)^2},\\
\frac{\partial Q}{\partial y} =& s\left(\frac{y}{y + b_1} - \frac{a_1 y}{x + c_1}\right) + sy\left(\frac{b_1}{(y + b_1)^2} - \frac{a_1}{x + c_1}\right),\\
\frac{\partial E_1}{\partial x} =& - p_1,\ \frac{\partial E_1}{\partial y} = 0,\ \frac{\partial F_1}{\partial x} = 0,\ \frac{\partial F_1}{\partial y} = 0,\ \frac{\partial \varphi_1}{\partial x} = 1,\ \frac{\partial \varphi_1}{\partial y} = 0.
\end{aligned}$$

By the stability criterion [24], we can get

$$
\begin{aligned}
\Delta_1 =& \frac{P_+(\frac{\partial F_1}{\partial y}\frac{\partial \varphi_1}{\partial x} - \frac{\partial F_1}{\partial x}\frac{\partial \varphi_1}{\partial y} + \frac{\partial \varphi_1}{\partial x}) + Q_+(\frac{\partial E_1}{\partial x}\frac{\partial \varphi_1}{\partial y} - \frac{\partial E_1}{\partial y}\frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_1}{\partial y})}{P(\frac{\partial \varphi_1}{\partial x}) + Q(\frac{\partial \varphi_1}{\partial y})} \\
=& \frac{P_+(\xi_1(T+0), \eta_1(T+0))}{P(\xi_1(T), \eta_1(T))} \\
=& \frac{G_1}{H_1}
\end{aligned}
$$

and

$$
\begin{aligned}
\mu_2 =& \Delta_1 \exp\left\{ \int_{0^+}^{T_1} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \mathrm{d}t \right\} \\
=& \frac{G_1 L_1}{H_1} \frac{y_B}{(1+p_1)(y_B - \tau_1)},
\end{aligned}
$$

where G_1, H_1, and L_1 have been defined in Theorem 3.3.

Combining the above results, we obtain $\mu_2 = \Theta_1$, where Θ_1 is given in Theorem 3.3. Therefore, if $|\Theta_1| < 1$, then $|\mu_2| < 1$; thus, the order-1 periodic solution is orbitally asymptotically stable. □

3.2. *Existence and stability of order-2 periodic solution*

In this subsection, we prove the existence and stability of the order-2 periodic solution for system (2.1). Then, we have the following Theorem 3.4.

Theorem 3.4. *If* $h_1 < (1 - p_2)h_2 < x_2^* < (1 + p_1)h_1 < h_2$, $y_{D_3^+} - y_{A_3} > 0$ *and* $y_{A_4} \geqslant y_{D_4^+}$ *hold, then system* (2.1) *emits an order-2 periodic solution.*

Proof. Assume that $A_3(x_{A_3}, y_{A_3}) \in N_1$ with $x_{A_3} > 0$ and $y_{A_3} \to 0$; then, the trajectory passing through point A_3 intersects impulsive set M_2 at point $B_3 \in M_2$, and then jumps to point $C_3 \in N_2$. Then, the trajectory passing through from C_3 intersects impulsive set M_1 at point $D_3(x_{D_3}, y_{D_3})$, and then jumps to $D_3^+(x_{D_3^+}, y_{D_3^+}) \in N_1$ with $x_{D_3^+} = (1 + p_1)x_{D_3}$ and $y_{D_3^+} = y_{D_3} - \tau_1 > 0$. Then the successor function $f(A_3) = y_{D_3^+} - y_{A_3} > 0$ holds, as is shown in Figure 3.

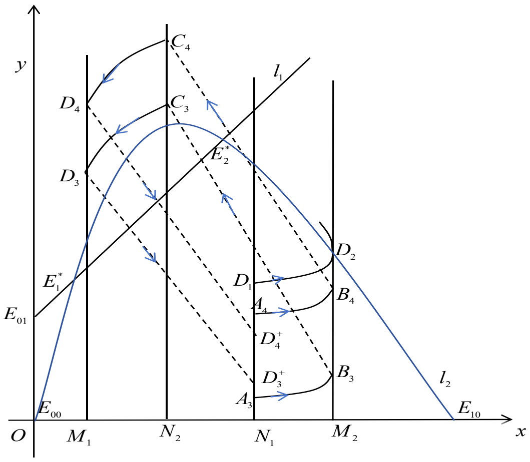


Figure 3. Existence of the order-2 periodic solution for $h_1 < (1 - p_1)h_2 < x_2^* < h_1 + \tau_1 < h_2$.

In the following, we only need to find a point such that the corresponding successor function is less than 0. Assume that $D_1 \in N_1$ and that the trajectory passing through point D_1 is tangent to the impulse set M_2 at point D_2. Choose a point $A_4 \in N_1$, which is very close to point D_1 satisfying $y_{A_4} < y_{D_1}$. The trajectory passing through A_4 intersects impulsive set at point $B_4 \in M_2$, then jumps to point $C_4 \in N_2$ due to impulsive effect, and the trajectory starting from C_4 intersects impulsive set M_1 at point D_4 and then jumps to point D_4^+, as is shown in Figure 3. There are three cases for the location of A_4 and D_4^+.

(1) If $y_{A_4} < y_{D_4^+}$, then the successor function $f(A_4) = y_{D_4^+} - y_{A_4} < 0$ holds, which means that there is no order-2 periodic solution. The trajectory tends to be the positive equilibrium (x_2^*, y_2^*).

(2) If $y_{A_4} = y_{D_4^+}$, then $\widehat{A_4B_4} \cup \overline{B_4C_4} \cup \widehat{C_4D_4} \cup \overline{D_4A_4}$ forms an order-2 periodic solution.

(3) If $y_{A_4} > y_{D_4^+}$, then the successor function $f(A_4) = y_{D_4^+} - y_{A_4} > 0$ holds. By the continuity of the successor function, this indicates that there is a point $S \in \overline{A_3A_4}$ such that $f(S) = 0$, which means that the system exists an order-2 periodic solution.

This completes the proof. □

Assume that $\widehat{A_4B_4} \cup \overline{B_4C_4} \cup \widehat{C_4D_4} \cup \overline{D_4A_4}$ forms an order-2 periodic solution. Let $(\varphi(t), \psi(t))$ be the order-2 periodic solution of system (2.1) with period $T' = T_1 + T_2$. Denote $\varphi(0) = x_{A_4}, \psi(0) = y_{A_4}$, $\varphi(T_1) = x_{B_4} = h_2, \psi(T_1) = y_{B_4}$, $\varphi(T_1 + 0) = x_{C_4} = (1 - p_2)h_2, \psi(T_1 + 0) = y_{C_4}$, $\varphi(T_1 + T_2) = x_{D_4} = h_1, \psi(T_1 + T_2) = y_{D_4}$, and $\varphi(T_1 + T_2 + 0) = x_{A_4} = (1 + p_1)h_1, \psi(T_1 + T_2 + 0) = y_{A_4}$. In what follows, the stability of order-2 periodic solution of system (2.1) is proved. We then have the following Theorem 3.5.

Theorem 3.5. *The order-2 periodic solution of system* (2.1) *is orbitally asymptotically stable if* $\left|\Phi_2\right| < 1$ *holds, where*

$$\Theta_2 = \Delta_1 \Delta_2 L_2 \frac{(1 - p_2)}{(1 + p_1)} \frac{(y_{A_4} + \tau_1)(y_{B_4} + \tau_2)}{y_{A_4} y_{B_4}}$$

and

$$\Delta_1 = \frac{P_+(x_{C_4}, y_{C_4})}{P(x_{B_4}, y_{B_4})}, \quad \Delta_2 = \frac{P_+(x_{A_4}, y_{A_4})}{P(x_{D_4}, y_{D_4})},$$

$$\begin{aligned} P_+(x_{C_4}, y_{C_4}) =& rh_2(1 - p_2)\left(1 - \frac{h_2(1 - p_2)}{K}\right)\left(\frac{h_2(1 - p_2)}{K} + \frac{a}{K}\right) \\ & - \frac{\left(p_0 + \frac{a_0(y_{B_4}+\tau_2)}{1+b_0(y_{B_4}+\tau_2)}\right) h_2(1 - p_2)(y_{B_4} + \tau_2)}{1 + h_0\left(p_0 + \frac{a_0(y_{B_4}+\tau_2)}{1+b_0(y_B+\tau_2)}\right) h_2(1 - p_2)}, \\ P(x_{B_4}, y_{B_4}) =& rh_2\left(1 - \frac{h_2}{K}\right)\left(\frac{h_2}{K} + \frac{a}{K}\right) - \frac{\left(p_0 + \frac{a_0 y_{B_4}}{1+b_0 y_{B_4}}\right) h_2 y_{B_4}}{1 + h_0\left(p_0 + \frac{a_0 y_{B_4}}{1+b_0 y_{B_4}}\right) h_2}, \\ P_+(x_{A_4}, y_{A_4}) =& r(1 + p_1)h_1\left(1 - \frac{(1 + p_1)h_1}{K}\right)\left(\frac{(1 + p_1)h_1}{K} + \frac{a}{K}\right) \\ & - \frac{\left(p_0 + \frac{a_0(y_{D_3}-\tau_2)}{1+b_0(y_{D_4}-\tau_2)}\right)(1 + p_1)h_1(y_{D_4} - \tau_2)}{1 + h_0\left(p_0 + \frac{a_0(y_{D_3}-\tau_2)}{1+b_0(y_{D_4}-\tau_2)}\right)(1 + p_1)h_1}, \end{aligned}$$

$$P(x_{D_4}, y_{D_4}) = rh_1\left(1 - \frac{h_1}{K}\right)\left(\frac{h_1}{K} + \frac{a}{K}\right) - \frac{\left(p_0 + \frac{a_0 y_{D_4}}{1+b_0 y_{D_4}}\right) h_1 y_{D_4}}{1 + h_0\left(p_0 + \frac{a_0 y_{D_4}}{1+b_0 y_{D_4}}\right) h_1},$$

and

$$\begin{aligned} L_2 = &\exp\left\{\int_{0^+}^{T_1+T_2} sy\left(\frac{b_1}{(y+b_1)^2} - \frac{a_1}{x+c_1}\right)\mathrm{d}t\right\} \\ &\times \exp\left\{\int_{0^+}^{T_1+T_2}\left(\frac{rx}{K} - \frac{2rx^2}{K^2} + \frac{h_0(p_0 + \frac{a_0 y}{1+b_0 y})xy}{(1 + h_0(p_0 + \frac{a_0 y}{1+b_0 y})x)^2}\right)\mathrm{d}t\right\}. \end{aligned}$$

Proof. Define

$$\begin{gathered} E_2(x,y) = -p_2 x, \;\; F_2(x,y) = \tau_2, \;\; \varphi_2(x,y) = x - h_2, \\ \frac{\partial E_2}{\partial x} = -p_1, \;\; \frac{\partial E_2}{\partial y} = 0, \;\; \frac{\partial F_2}{\partial x} = 0, \;\; \frac{\partial F_2}{\partial y} = 0, \;\; \frac{\partial \varphi_2}{\partial x} = 1, \;\; \frac{\partial \varphi}{\partial y} = 0. \end{gathered}$$

By simple calculation, we immediately obtain the expressions of Δ_1 and Δ_2.

Construct the stability criterion

$$\Theta_2 = \Delta_1 \Delta_2 \cdot \exp\left\{\int_{0^+}^{T_1+T_2}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\mathrm{d}t\right\}.$$

If $|\Theta_2| < 1$, then the corresponding Floquet multiplier satisfies $|\mu_2| < 1$, which implies the order-2 periodic solution is orbitally asymptotically stable. □

4. Numerical simulation

In this section, numerical simulations are conducted to verify the theoretical results derived in the previous sections, with different types of periodic solutions obtained by varying the control parameters. This study focuses on the dynamical behaviors of system (2.1), where the prey population is subject to a weak Allee effect. We choose the specific parameter values as follows:

$$\begin{gathered} r = 0.85, \;\; K = 100, \;\; a = 4, \;\; p_0 = 0.26, \;\; a_0 = 1, \\ b_0 = 2, \;\; h_0 = 5, \;\; s = 0.25, \;\; b_1 = 1, \;\; c_1 = 4, \;\; a_1 = 1. \end{gathered}$$

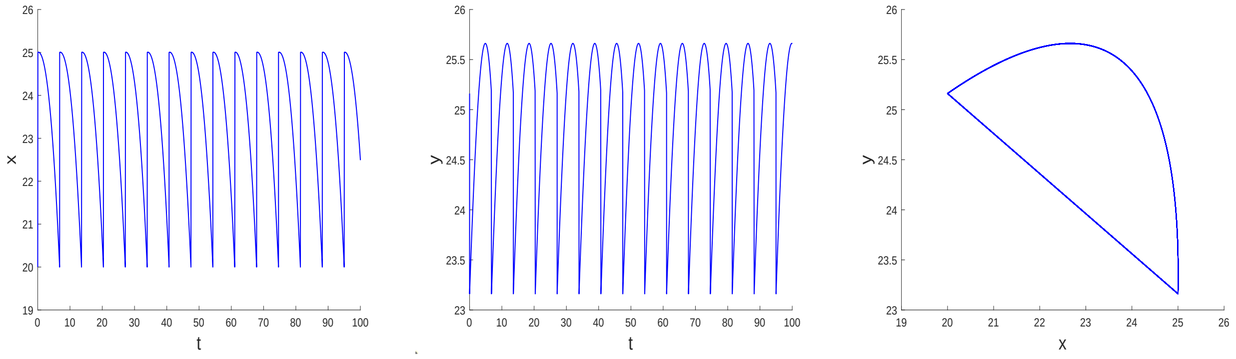
For these parameters, $E_1^*(35.01, 38.01)$ is an unstable positive equilibrium, and $E_2^*(63.83, 66.83)$ is a locally asymptotically stable positive equilibrium.

Example 4.1. *Under the weak Allee effect, the prey population maintains a positive per capita growth rate even at low density. Although low abundance impairs mating efficiency, group defense, and social cooperation, thereby reducing the net population growth rate, the prey can still persist in the absence of predators. However, extra mortality caused by predation may continuously suppress prey abundance and eventually drive the population to extinction. Therefore, targeted management measures should be implemented when the prey density falls below a certain threshold to ensure the persistence of the population.*

To illustrate the existence of order-1 periodic solution of system (2.2)*, the specific system is given by:*

$$\begin{cases} \left.\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= 0.85x\left(1-\frac{x}{100}\right)\left(\frac{x}{K}+\frac{4}{K}\right)-\frac{\left(0.26+\frac{y}{1+2y}\right)xy}{1+5\left(0.26+\frac{y}{1+2y}\right)x},\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 0.25y\left(\frac{y}{y+1}-\frac{y}{x+4}\right), \end{aligned}\right\} \quad x>20,\\ \Delta x = p_1 x, \quad \Delta y = -\tau_1, \quad x = 20,\\ x(0) > h_1, \quad y(0) > 0. \end{cases}$$

For the case of $h_1 < (1+p_1)h_1 < x_1^*$*, we choose the control parameters:* $h_1 = 20, \tau_1 = 2$*, and* $p_1 = 0.25$*. System* (2.2) *emits an order-1 periodic solution (see Figure 4), which corresponds to the case of Theorem 3.1.*



(a) (b) (c)

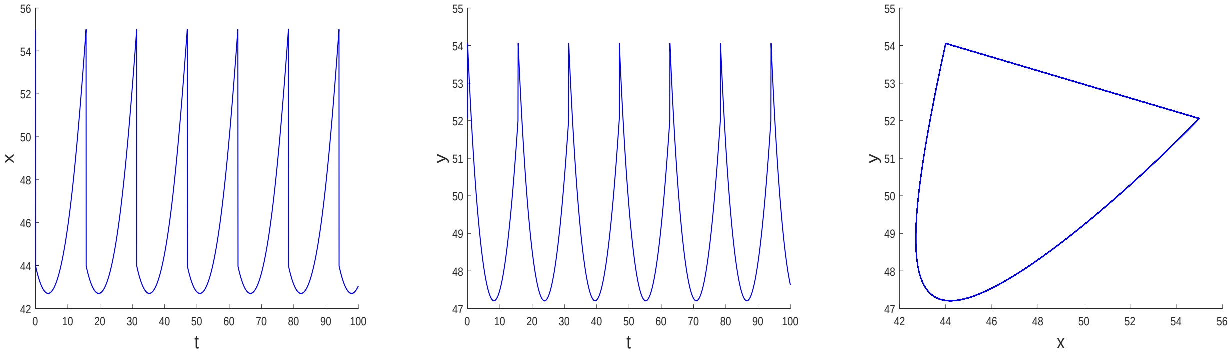
Figure 4. Existence of the order-1 periodic solution of system (2.2) for the case of $h_1 < (1+p_1)h_1 < x_1^*$.

Example 4.2. *When prey is highly abundant, the large population promotes rapid predator growth and enhances their cooperative hunting efficiency. Although this imposes intense predation pressure, it cannot suppress the large prey population in the short term, thus maintaining a stable high density coexistence equilibrium. However, this stability is fragile: Once external disturbances reduce the prey population sharply below a critical threshold, the weak Allee effect will immediately lower its growth rate and recovery ability. Combined with sustained predation, this may eventually drive the prey to extinction. Therefore, to maintain stable coexistence, the prey population should not exceed a critical upper threshold.*

To illustrate the existence of the order-1 periodic solution of system (2.3)*, the specific system is given by*

$$\begin{cases} \left.\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= 0.85x\left(1-\frac{x}{100}\right)\left(\frac{x}{K}+\frac{4}{K}\right)-\frac{\left(0.26+\frac{y}{1+2y}\right)xy}{1+5\left(0.26+\frac{y}{1+2y}\right)x},\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 0.25y\left(\frac{y}{y+1}-\frac{y}{x+4}\right), \end{aligned}\right\} \quad x<55,\\ \Delta x = -p_2 x, \quad \Delta y = \tau_2, \quad x = 55. \end{cases}$$

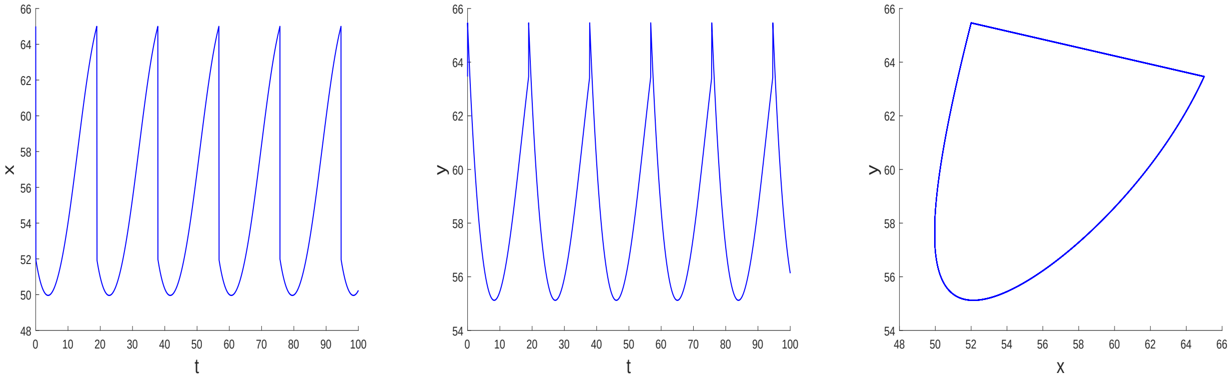
For the case of $(1-p_2)h_2 < h_2 < x_2^*$*, we choose the control parameters:* $h_2 = 55, \tau_1 = 2$*, and* $p_1 = 0.2$*. System* (2.3) *emits an order-1 periodic solution (see Figure 5), which corresponds to the case* *of Theorem 3.2.*



(a) (b) (c)

Figure 5. Order-1 periodic solution of system (2.2) with $(1-p_2)h_2 < h_2 < x_2^*$.

For the case of $(1-p_2)h_2 < x_2^* < h_2$, *we choose the following control parameters:* $h_2 = 65, \tau_1 = 2$ *and* $p_1 = 0.2$. *System* (2.3) *emits an order-1 periodic solution (see Figure 6), which corresponds to the case of Theorem 3.3.*



(a) (b) (c)

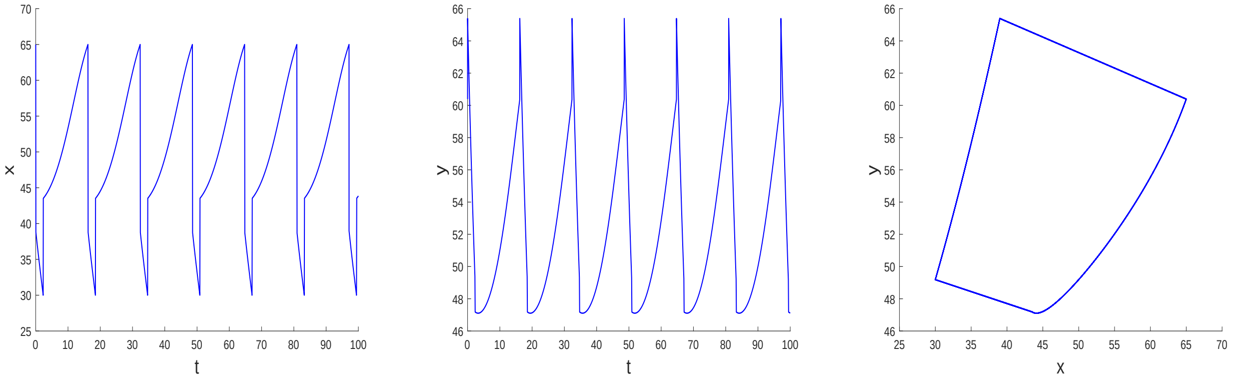
Figure 6. Order-1 periodic solution of system (2.2) with $(1-p_2)h_2 < x_2^* < h_2$.

Example 4.3. *As described above, under a weak Allee effect, the prey population can persist at low densities without predators, but predation may suppress its abundance and lead to extinction if it falls below a threshold. At high abundances, a stable coexistence equilibrium is maintained temporarily, but this stability is fragile, as external disturbances can push the prey below the critical threshold, and combined with sustained predation, drive it to extinction. Thus, the prey population must be kept within a reasonable range to ensure stable coexistence with the predator.*

To illustrate the existence of the order-2 periodic solution, the specific system is given by

$$\begin{cases} \left.\begin{aligned} &\frac{\mathrm{d}x}{\mathrm{d}t} = 0.85x\left(1-\frac{x}{100}\right)\left(\frac{x}{K}+\frac{4}{K}\right)-\frac{\left(0.26+\frac{y}{1+2y}\right)xy}{1+5\left(0.26+\frac{y}{1+2y}\right)x},\\ &\frac{\mathrm{d}y}{\mathrm{d}t} = 0.25y\left(\frac{y}{y+1}-\frac{a_1 y}{x+4}\right), \end{aligned}\right\} \quad 30 < x < 65,\\ \left.\Delta x = 0.45x, \quad \Delta y = -2, \right\} \quad x = 30,\\ \left.\Delta x = -0.4x, \quad \Delta y = 5, \right\} \quad x = 65. \end{cases}$$

For the case of $h_1 < (1-p_2)h_2 < (1+p_1)h_1 < x_2^* < h_2$*, we choose the following control parameters:* $h_1 = 30$*,* $h_2 = 65, \tau_1 = 2, p_1 = 0.45, p_2 = 0.4, \tau_2 = 5$*. Then, system* (2.1) *emits an order-2 periodic solution (see Figure 7), which corresponds to the case of Theorem 3.4.*



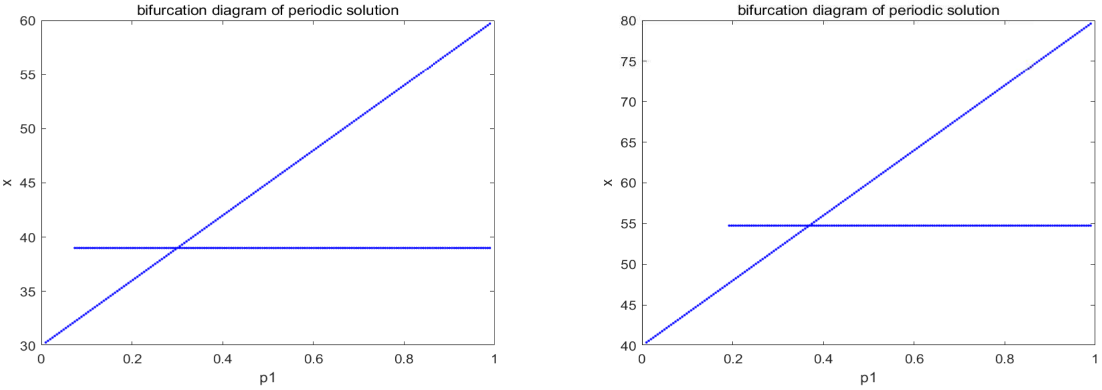
(a) (b) (c)

Figure 7. Existence of the order-2 periodic solution of system (2.1).

Example 4.4. *Inspired by the research ideas in [18, 25, 26], this paper explores the influences of control parameters on the dynamical behaviors of system* (2.1)*. To plot the bifurcation diagram, we take* $p_1 \in (0.01, 0.99)$ *as the bifurcation parameter, and set control parameters as* $h_1 = 30$*,* $h_2 = 65$*,* $p_2 = 0.4$*,* $\tau_1 = 2$*, and* $\tau_2 = 5$*. As illustrated in Figure 8a, the system experiences successive state transitions with the growing value of the control parameter. An order-1 periodic solution emerges initially, followed by the generation of an order-2 periodic solution. The two bifurcation branches intersect at a single point when the condition*

$$(1+p_1)h_1 = (1-p_2)h_2$$

holds. Once p_2 *exceeds this critical threshold, the crossed branches gradually split apart. To further discuss the dynamical evolution under a strong Allee effect, we fix other parameters and set* $r = 1.55$*,* $a = -25$*. Meanwhile, the control parameters are adjusted to* $h_1 = 40$*,* $h_2 = 73$*,* $p_2 = 0.25$*,* $\tau_1 = 5$*, and* $\tau_2 = 2.5$*. Numerical results in Figure 8b demonstrate that the system still admits order-1 and order-2 periodic orbits, and Figure 8b possesses analogous dynamical features to Figure 8a.*



(a) (b)

Figure 8. Bifurcation diagram of system (2.1): (a) Weak Allee effect in the prey population; (b) Strong Allee effect in the prey population.

5. Conclusions

In this paper, based on a modified Leslie–Gower model incorporating dual Allee effects, we formulate and analyze mathematical models with unilateral and bilateral control strategies for renewable resource management. Taking the weak Allee effect as an example, we investigate the dynamical behaviors of the system under impulsive control, including the existence and stability of the order-1 periodic solution for the unilateral control system and the order-2 periodic solution for the bilateral control system, respectively. To verify the effectiveness of the theoretical results, numerical simulations are performed; meanwhile, bifurcation diagrams are constructed to reveal the impact of control parameters on the transition of system dynamics. Moreover, we demonstrate that when prey populations are subject to a strong Allee effect, the lower control threshold of prey populations needs to be configured above the strong Allee threshold.

The comparative results reveal that unilateral control can effectively prevent population extinction triggered by Allee effects and can curb excessive prey growth which induces predator surges and consequent prey loss. Nevertheless, its regulation capacity is relatively constrained, which may easily lead to unbalanced population distribution and fail to achieve stable long-term population regulation from an ecological perspective. By contrast, bilateral control is capable of precisely keeping both prey and predator populations within reasonable ecological thresholds. From the biological viewpoint, this control mode can well coordinate the interspecific interaction relationship, balance the survival and reproduction rhythm of the two species, and ultimately realize stable and sustainable symbiosis of predator and prey populations in real ecological environments.

This study enriches the dynamical theory of Leslie–Gower predator-prey models with Allee effects and provides a theoretical basis and decision-making for the practical application in biological resource management and ecological restoration. By comparing the two control strategies, we suggest that bilateral control is more favorable than unilateral control in practical resource management.

Author contributions

Jing Xu: supervision, writing-review and editing; Haoyang Shen: software, methodology and visualization; Xinyi Cao: software, methodology and visualization. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this paper.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Nos. 12401639 and 62473134), Hubei Province University's outstanding young and middle-aged scientific and technological innovation team project (T2023020), and the Henan Provincial Natural Science Foundation (No. 252300421492).

Conflict of interest

The authors declare that there are no conflicts of interest.

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Appendix

Some definitions and lemmas

In this section, we introduce some definitions and lemmas about the geometric theory of the semi-continuous dynamical system.

Consider the following semi-continuous dynamical system [24]:

$$\begin{cases} \left.\begin{aligned} \frac{\mathrm{d}x}{\mathrm{dt}} &= P(x,y), \\ \frac{\mathrm{d}y}{\mathrm{dt}} &= Q(x,y), \end{aligned}\right\} \quad if \;\; M(x,y) \neq 0, \\ \left.\begin{aligned} \Delta x &= E(x,y), \\ \Delta y &= F(x,y), \end{aligned}\right\} \quad if \;\; M(x,y) = 0, \end{cases}$$

where $M(x,y)$ is called the impulsive set. Denote the impulsive map ϕ: $(x,y) \to (x+\Delta x, y+\Delta y)$, that is, $N(x,y) = \phi(M(x,y))$. Further, if $(x,y) \notin M(x,y)$, the system develops under the regulation of

$$f(x,y) = \left(\frac{\mathrm{d}x}{\mathrm{dt}} = P(x,y), \frac{\mathrm{d}y}{\mathrm{dt}} = Q(x,y)\right);$$

this part is similar to a continuous system.

Lemma A.1. *(Stability criterion [24]) The T_L period-1 solution $X(t) = (\xi(t), \eta(t))$ of the proposed model is orbitally asymptotically stable if the convergency ratio ρ_{γ_L} is less than one, where*

$$\rho_{\gamma_L} \triangleq \left| \frac{P_+^I[(1+\beta_y)\Phi_x - \beta_x\Phi_y] + Q_+^I[(1+\alpha_x)\Phi_y - \alpha_y\Phi_x]}{P^I\Phi_x + Q^I\Phi_y} \right| \exp\left(\int_{0^+}^{T} \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right]_{(\xi(t),\eta(t))} \right),$$

$P^I(Q^I)$ represents the value of $P(Q)$ at $L^-(\xi(T_L), \eta(T_L)) \in M_{imp}$, and $P_+^I(Q_+^I)$ represents the value of $P(Q)$ at $L(\xi(0^+), \eta(0^+)) \in N_{pha}$.

Next, the definition of the successor function is introduced.

Definition A.1. *(Successor function [18, 24]) Let* M_{IMP} *and* N_{PHA} *be two disjoint lines. Denote* O' *as the intersection point between* N_{PHA} *and the x-axis (or y-axis if* $N_{PHA} \cap$ *x-axis=*$\emptyset$*). For a given* $Q_1 \in N_{PHA}$*, the trajectory from* Q_1*, passing through the point* A_2*, intersecting the impulsive set at the point* Q_3*, then jumping to point* $Q_4 \in N_{PHA}$*, and* $Q_4 = \phi(Q_3)$*. Then, a type-I successor function* f^I_{SOR} *is defined by*

$$f^I_{SOR} = d(Q_4, O') - d(Q_2, O'),$$

and a type-II function f^{II}_{SOR} *is defined by*

$$f^{II}_{SOR} = d(Q_4, O') - d(Q_1, O'),$$

as is shown in Figure A.1.

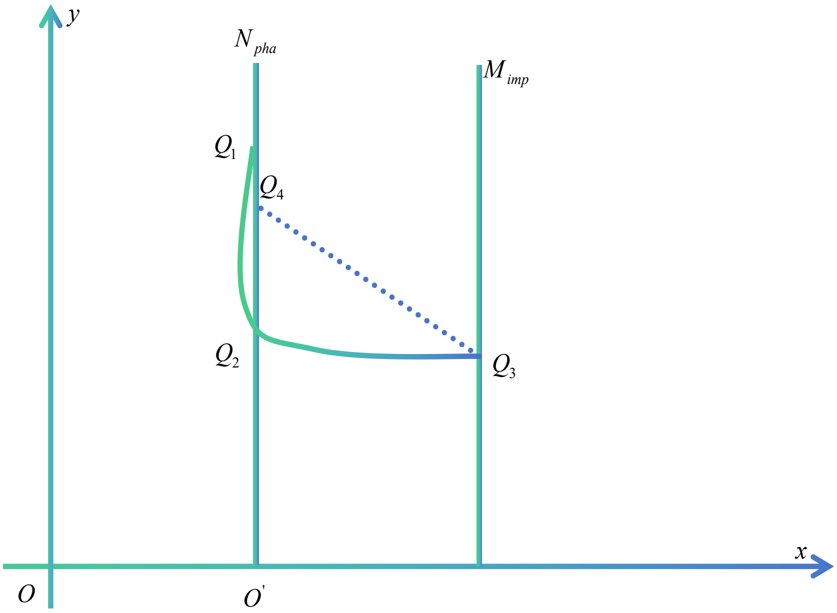

Figure A.1. Schematic diagram of the successor function.