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Research article

Fixed point theorems for nonlinear contractive mappings in cone-valued θ-type multiplicative metric spaces

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Abstract: Fixed point theory plays a fundamental role in nonlinear analysis and has significant applications in differential equations, integral equations, optimization, and applied mathematics. Inspired by developments in generalized metric structures, we introduced and studied the Geraghty-type contractive conditions in the setting of cone θ-type multiplicative metric spaces defined on ordered Banach algebras. By extending the classical notions of θ-metric and multiplicative metric spaces, we constructed a framework based on solid multiplicative cones and established their essential topological and convergence properties. Within this generalized structure, we formulated new fixed point theorems for self-mappings satisfying cone-valued θ-type multiplicative Geraghty contractions, which weakened the traditional Lipschitz condition while preserving the existence and uniqueness of fixed points in complete spaces. The proposed results significantly extended and unified several well-known contraction principles, including those of Banach, Kannan, Chatterjea, and standard Geraghty contractions, under a broader cone-valued multiplicative setting. Furthermore, we developed extensions to higher-dimensional frameworks to enhance applicability in complex nonlinear systems. Illustrative examples are presented to substantiate the theoretical findings and to demonstrate the effectiveness of the introduced approach in generalized analytical environments.

Keywords: multiplicative metric; contraction; cone; partial order; integral equation
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1. Introduction

Fixed point theory is a central branch of modern mathematical analysis with deep and wide-ranging applications in differential equations, optimization, dynamic systems, game theory, computer science, and mathematical modeling in the natural sciences. Its core significance arises from the Banach Contraction Principle, formulated in 1922 by S. Banach [1], which guarantees the existence and uniqueness of fixed points for contraction mappings on complete metric spaces. This principle has since served as a foundation for countless generalizations.

The concept of θ-metric spaces, introduced by Khojasteh et al. [2], represents one such generalization, replacing the classical triangle inequality with a broader inequality involving a θ-function. This approach has proved fruitful in extending fixed point theorems beyond the scope of conventional metric spaces. In parallel, the development of multiplicative calculus, initiated by Grossman et al. [3] and formalized by Bashirov et al. [4], inspired the creation of multiplicative metric spaces where distances are defined multiplicatively rather than additively. This perspective aligns naturally with problems in fields such as economics, biology, and physics, where multiplicative structures emerge.

Further contributions by Özavsar et al. [5] and Abbas et al. [6] explored the topological properties and fixed point results within multiplicative metric spaces. These works established a foundation for generalizations of contraction principles in settings where multiplication replaces addition. Building on these insights, we introduce the notion of a cone θ-type multiplicative metrics and extend into a 2-multiplicative metric framework, thereby merging ideas from θ-metrics and multiplicative structures. Further studies on multiplicative metric spaces are available in [7–14].

The θ-type multiplicative metric spaces can be used to investigate equivalences between different classes of functional spaces that share similar metric topologies. While logarithmic transformations often relate multiplicative metrics to standard additive metrics, the θ-type and cone-valued structure preserves additional order and multiplicative information that is lost under such transformations.

This highlights the role of θ-type multiplicative metric spaces as a genuine extension rather than a mere reformulation of classical metric frameworks. The new generalizations introduced in this paper are important because they substantially enlarge the class of spaces and mappings for which fixed point results remain valid while preserving the existence and uniqueness of solutions. Classical metric fixed point theory is built on scalar-valued distances and strict linear contraction conditions. In contrast, this framework combines cone structures, multiplicative metrics, and θ-type control functions inside ordered Banach algebras. This creates a richer analytical setting capable of modeling nonlinear phenomena that cannot be adequately represented in ordinary metric spaces. The cone-valued structure is particularly significant because it enables the “distance” between points to carry algebraic and order information rather than being merely a nonnegative real number. In applications involving matrices, operators, vector-valued quantities, or coupled systems, scalar distances often lose important structural information. By taking values in a solid multiplicative cone of a Banach algebra, the metric preserves multiplicative behavior and partial ordering. This makes the theory suitable for multidimensional and operator-based nonlinear systems.

The proposed cone θ-type multiplicative framework provides a natural setting for the existence and uniqueness of solutions of rational power-based fractional differential equations, extending classical fixed point methods to nonlinear fractional control systems [15].

Inspired by recent developments, authors in [2] introduced the concept of a θ-metric.

Definition 1. *[2] Let X be a nonempty set, a mapping $d : X \times X \to [0, \infty)$ that satisfies the following conditions:*

1) $d(x, y) = 0 \Leftrightarrow x = y$.

2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$.
3) $d(x, z) \leq \theta(d(x, y), d(y, z))$,

where the function $\theta : [0, \infty) \times [0, \infty) \to [0, \infty)$ *satisfies the properties:*

1) θ *is continuous.*
2) $\theta(0, 0) = 0$.
3) $\theta(s, t) = \theta(t, s)$.
4) *For each* $m \in \Im(\theta) = \{m \in \mathbb{R}^+ : \theta(s, t) = m\}$ *and* $t \in [0, m]$ *there exists* $s \in [0, m]$ *such that* $\theta(s, t) = m$.
5) $\theta(s, t) \leq \theta(u, v)$ *for* $s < u$ *and* $t < v$.
6) $\theta(s, 0) \leq s$ *for all* $s > 0$.

Then, d is a θ*-metric on X with respect to the function* θ *and* (X, d) *is a* θ*-metric space.*

Remark 1. *If* (X, d) *is a* θ*-metric space, with* $\theta(s, t) = k(s + t)$, $k \in (0, 1]$ *then* (X, d) *is a metric space.*

2. Major results

We start this section by assuming that the θ-function factorizes into a product of functions, each depending on a single variable and we define the concept of a multiplicative cone which differs to the conventional definition found in [16].

Definition 2. *(Solid multiplicative Cone): Let* $(X, \|\cdot\|)$ *be a commutative unital Banach algebra over* $\mathbb{C}$ *or* $\mathbb{R}$ *with identity element* 1_X.

Let $P \subset Inv(X) = \{u \in X : \exists\, v \in X$ *such that* $uv = vu = 1_X\}$, *be a solid multiplicative cone, i.e.:*

1) *P is nonempty, closed.*
2) $\alpha P \subset P$ *for real* $\alpha \geq 1$;
3) $P \cdot P \subset P$;
4) *If* $u \in P$ *and* $u^{-1} \in P$ *then* $u = 1_X$.
5) $int(P) \neq \emptyset$.
6) *for all* $u \in P$, $\sigma(u) \subset \mathbb{C} \setminus (-\infty, 0]$,

where $\sigma(u) = \{\lambda \in \mathbb{C} : (u - \lambda 1_X)$ *is not invertible in* $X\}$ *denotes the spectrum of u. In the real case,* $\sigma(u) \subset (0, \infty)$.

The cone P induces a partial order on $\mathrm{Inv}(X)$ by

$$u \preceq v \iff vu^{-1} \in P.$$

$$u \succ v \iff uv^{-1} \in \mathrm{int}(P).$$

If $u \preceq v$, there exists a constant $k \geq 1$ such that $\|u\| \leq k\|v\|$ for $u, v \in X$.

Example 1. *(Cone:) Let*

$$X = D_n(\mathbb{R}) = \{\mathrm{diag}(z_1, z_2, \cdots, z_n) : z_i \in \mathbb{R} \textit{ for all } i = 1, 2, \cdots, n\}$$

be the commutative unital Banach algebra of $n \times n$ diagonal matrices with norm

$$\|A\| = \max_{1 \le i \le n} |z_i|$$

and

$$\mathrm{Inv}(X) = \{A \in D_n(\mathbb{R}) : \det(A) \neq 0\}.$$

If

$$P = \{\mathrm{diag}(z_1, z_2, \cdots, z_n) : z_i \geq 1 \text{ for all } i = 1, 2, \cdots, n\}$$

then it can be shown that P is a cone.

Definition 3. *(Cone θ-type multiplicative metric): Let X be a nonempty set and $(X, \|\cdot\|)$ be a commutative unital Banach algebra over $\mathbb{C}$ or $\mathbb{R}$ with identity element 1_X. A mapping*

$$d : X \times X \longrightarrow P \subset \mathrm{Inv}(X)$$

is called a Cone θ-type multiplicative metric if there exist continuous, order-preserving mappings

$$\theta_i : P \to P, \quad i = 1, 2,$$

satisfying the following conditions for all $x, y, z \in X$:

(M1) (Identity and Positivity)

$$d(x, y) = 1_X \Longleftrightarrow x = y,$$

and $d(x, y) \succ 1_X$ whenever $x \neq y$.

(M2) (Symmetry)

$$d(x, y) = d(y, x).$$

(M3) (Cone θ-multiplicative triangle inequality)

$$d(x, z) \preceq \begin{cases} \theta_1(d(x, y))\, \theta_2(d(y, z)), & \text{if } y \neq x, z, \\ d(x, y) d(y, z), & \text{if } y = x \text{ or } y = z. \end{cases}$$

where the product on the right-hand side is the algebra product in X, and $\preceq$ denotes the partial order induced by P.

Each function $\theta_i : P \to P$ satisfies:

1) *θ_i is continuous;*
2) *θ_i is monotone: if $u \preceq v$ then $\theta_i(u) \preceq \theta_i(v)$;*
3) *$\theta_i(1_X) = 1_X$;*
4) *$\theta_i(u) \preceq u$ for all $u \in P$.*

If the mapping d satisfies (M1)–(M3), the pair (X, d) is called a Cone θ-type multiplicative metric space over X.

Remark 2. *If the control function $\theta_i(u) = u$, then the θ-type multiplicative metric reduces to a multiplicative metric.*

The significance of this example is that it establishes that the existence of an element in P is sufficient to guarantee the existence of a cone-metric on X.

Example 2. *(Metric:) Let X be a nonempty set and $(X, \|\cdot\|)$ be a commutative unital Banach algebra over $\mathbb{C}$ or $\mathbb{R}$ with identity element 1_X and P a cone in X. Define a mapping*

$$d : X \times X \longrightarrow P$$

$$d(x,y) = \begin{cases} 1_X, & x = y, \\ c, & x \neq y \end{cases}$$

where $c \in P$. The mapping satisfies axioms (M1)–(M3). The triangle inequality follows, since

$$c = c^{1/3} \cdot c^{2/3} \preceq c^{1/2} \cdot c^{2/3}$$

by the partial order $c^{1/2+2/3} \cdot c^{-1} = c^{1/6} \in P$, it follows that

$$d(x,z) \preceq \theta_1(d(x,y))\theta_2(d(y,z))$$

where $\theta_1(t) = t^{1/2}$ and $\theta_2(t) = t^{2/3}$.

Example 3. *(Metric:) Let $(\mathbb{R}^n, \|\cdot\|)$ be a commutative unital Banach algebra over $\mathbb{R}$ with sup-norm $\|x\| = \max_i |x_i|$. Multiplication is defined coordinatewise by $x \cdot y = (x_1y_1, x_2y_2, \cdots, x_ny_n)$, identity element $1_X = (1, 1, \cdots, 1)$ with $\mathrm{Inv}(\mathbb{R}^n) = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_i \neq 0\}$ and $P = \{(x_1, x_2, \cdots, x_n) : x_i \geq 1\}$ a cone in $\mathbb{R}^n$. Define a mapping*

$$d : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow P$$

by

$$d(x,y) = (e^{|x_1-y_1|}, e^{|x_2-y_2|}, \cdots, e^{|x_n-y_n|})$$

where $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ and $\theta_i = t$. All properties (M1)– (M3) can be easily verified. Thus, $(\mathbb{R}^n, d)$ is a θ-type multiplicative metric space.

Definition 4. *(Convergence and Cauchy sequence:) A sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is said to converge to $x \in X$ if*

$$\|d(x_n, x) - 1_X\| \to 0 \quad \text{as } n \to \infty.$$

The sequence is Cauchy if

$$\|d(x_n, x_m) - 1_X\| \to 0 \quad \text{as } n, m \to \infty.$$

The space (X, d) is a complete cone θ-type multiplicative space if every Cauchy sequence converges in the norm.

Definition 5. *The fractional powers of $x \in P$ is*

$$x^\alpha := e^{\alpha \log x},$$

where $\alpha \in \mathbb{R}$ and $\log x$ is the logarithm defined by the Dunford-Taylor integral formula, where the contour is taken surrounding the spectrum of element $\sigma(x) \subset \mathbb{C} \setminus (-\infty, 0]$ [17, 18].

Lemma 1. *If $u, v \in P$ such that $u \preceq v$, implies that $vu^{-1} \in P$. Then for real $0 \leq c < 1$, it follows that $u^c \preceq v^c$.*

Proof. Since $u \preceq v$, we obtain that $vu^{-1} := w \in P$. It follows that $v = wu$. Because X is commutative, elements u and w are invertible and belong to P, with the spectra avoiding the branch cut $(-\infty, 0]$. Hence, functional Calculus gives

$$v^c = (wu)^c = e^{c\log(wu)} = e^{c(\log w + \log u)} = e^{c\log w}e^{c\log u} = w^c u^c.$$

Computing the product $v^c u^{-c} = (w^c u^c)u^{-c} = w^c$ and using the closure of P under fractional powers, for every $w \in P$ and $c \in [0, 1)$, the element

$$w^c = e^{c\log w}$$

is defined by the Dunford Taylor functional Calculus belonging to P. This computes the product $v^c u^{-c} = w^c \in P$. From the partial ordering relation, we get

$$u^c \preceq v^c.$$

□

Lemma 2. *(Continuity of partial map) Let (X, d) be a cone θ-type multiplicative metric space. If a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges to some $x \in X$ then for any $y \in X$ fixed,*

$$\lim_{n\to\infty} d(x_n, y) = d(x, y).$$

Proof. For the sequence $\{x_n\}_{n\in\mathbb{N}}$ and fixed $y \in X$, using the θ-multiplicative triangle inequality, we get

$$d(x_n, y) \preceq \theta_1(d(x_n, x))\theta_2(d(x, y)) \preceq \theta_1(d(x_n, x))d(x, y) \tag{2.1}$$

$$d(x, y) \preceq \theta_1(d(x_n, x))\theta_2(d(x_n, y)) \preceq \theta_1(d(x_n, x))d(x_n, y) \tag{2.2}$$

Combining inequalities (2.1) and (2.2), we get

$$\theta_1(d(x_n, x))^{-1}d(x, y) \preceq d(x_n, y) \preceq \theta_1(d(x_n, x))d(x, y),$$

$$\theta_1(d(x_n, x))^{-1} \preceq d(x_n, y)d(x, y)^{-1} \preceq \theta_1(d(x_n, x)). \tag{2.3}$$

Since θ_1 is continuous,

$$\theta_1(d(x_n, x)) \to \theta_1(1_X) = 1_X$$

and

$$\theta_1(d(x_n, x))^{-1} \to 1_X \quad \text{as } n \to \infty.$$

Taking the limit as $n \to \infty$ in (2.3), we obtain

$$d(x_n, y)d(x, y)^{-1} \to 1_X \quad \text{as } n \to \infty$$

i.e.,

$$\lim_{n\to\infty} d(x_n, y) = d(x, y).$$

□

Theorem 1. *Let (X, d) be a complete cone-valued θ-type multiplicative metric space and assume $T : X \to X$ satisfies the following contraction condition for all $x, y \in X$:*

$$d(Tx, Ty) \preceq (\theta_1(d(x, Tx)))^{\alpha}(\theta_2(d(y, Ty)))^{\beta}(\theta_3(d(x, y)))^{\gamma}, \tag{2.4}$$

where $\alpha, \beta, \gamma \geq 0$ are real numbers with

$$0 < \alpha + \beta + \gamma < 1.$$

where $P \subset X$ is a cone in an ordered Banach algebra X and $\theta_i : P \to P$ are continuous, monotone, and satisfy $\theta_i(u) \preceq u$, $\theta_i(1_X) = 1_X$. Then T has a unique fixed point $z \in X$, and for any $x_0 \in X$, the Picard iterates $x_{n+1} = Tx_n$ converge to z in norm.

Proof. Fix $x_0 \in X$ and define the Picard sequence $x_{n+1} = Tx_n$ for $n \geq 0$. Put

$$\delta_n := d(x_n, x_{n+1}) \in P, \quad n \geq 0.$$

Applying (2.4) with $x = x_{n-1}$ and $y = x_n$ gives

$$d(Tx_{n-1}, Tx_n) \preceq (\theta_1(d(x_{n-1}, Tx_{n-1})))^{\alpha}(\theta_2(d(x_n, Tx_n)))^{\beta}(\theta_3(d(x_{n-1}, x_n)))^{\gamma}$$
$$\preceq (d(x_{n-1}, Tx_{n-1}))^{\alpha}(d(x_n, Tx_n))^{\beta}(d(x_{n-1}, x_n))^{\gamma}.$$

In terms of δ, and using $\theta_i(u) \preceq u$, we get

$$\delta_n \preceq \delta_{n-1}^{\alpha}\delta_n^{\beta}\delta_{n-1}^{\gamma}.$$

Applying the partial order, we obtain

$$\delta_{n-1}^{\alpha}\delta_n^{\beta}\delta_{n-1}^{\gamma}\delta_n^{-1} = \delta_{n-1}^{\alpha+\gamma}\delta_n^{\beta}\delta_n^{-1} = \delta_{n-1}^{\alpha+\gamma}(\delta_n^{1-\beta})^{-1} \in P,$$

and

$$\delta_n^{1-\beta} \preceq \delta_{n-1}^{\alpha+\gamma},$$

and therefore

$$\delta_n \preceq \delta_{n-1}^{\xi}, \quad \text{where } \xi := \frac{\alpha + \gamma}{1 - \beta}.$$

Note that $\xi \in [0, 1)$ since $\alpha + \beta + \gamma < 1$.

Iterating, we obtain

$$\delta_n \preceq \delta_{n-1}^{\xi} \preceq \delta_{n-2}^{\xi^2} \preceq \cdots \preceq \delta_0^{\xi^n}.$$

Since $0 \leq \xi < 1$, we have $\xi^n \to 0$ as $n \to \infty$. For the multiplicative metric, raising $\delta_0 \in P$ to power $\xi^n \to 0$ yields the identity 1_X, i.e.,

$$\lim_{n\to\infty} \delta_0^{\xi^n} = 1_X.$$

Hence, by the normality property, we obtain

$$\lim_{n\to\infty} \delta_n = 1_X.$$

Next, we show that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy. Using the cone θ-triangle control inequality repeatedly and the monotonicity of the θ_i's, we obtain, for $m > n$,

$$d(x_n, x_m) \preceq \left(\prod_{k=n}^{m-2} \theta_1(d(x_k, x_{k+1}))\right)\theta_2(d(x_{m-1}, x_m))$$

$$\preceq \prod_{k=n}^{m-1} \delta_0^{\xi^k} \preceq \delta_0^{\frac{\xi^n}{1-\xi}}.$$

Hence, $d(x_n, x_m) \to 1_X$ as $n, m \to \infty$. Thus, $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy, and by completeness, it converges to some $z \in X$.

We now prove z is a fixed point. Apply (2.4) with $x = x_n$ and $y = z$:

$$d(x_{n+1}, Tz) \preceq (\theta_1(d(x_n, x_{n+1})))^{\alpha}(\theta_2(d(z, Tz)))^{\beta}(\theta_3(d(x_n, z)))^{\gamma}.$$

Letting $n \to \infty$ and using Lemma 2 yields

$$d(z, Tz) \preceq d(z, Tz)^{\beta}.$$

Using the partial ordering, it follows that $d(z, Tz)^{\beta} d(z, Tz)^{-1} \in P$ and

$$d(z, Tz)^{\beta-1} = d(z, Tz)^{-(1-\beta)}.$$

Since $0 \leq \beta < 1$, it follows that $d(z, Tz)^{1-\beta} \in P$. Thus, from the property of cone P, we get that $d(z, Tz) = 1_X$, i.e.

$$Tz = z.$$

For uniqueness, assume $y \in X$ is another fixed point. Then, applying (2.4) with $x = z$ and $y = y$, and using $d(z, Tz) = d(y, Ty) = 1_X$, we obtain

$$d(z, y) \preceq (\theta_3(d(z, y)))^{\gamma} \preceq (d(z, y))^{\gamma}.$$

Again, since $0 \leq \gamma < 1$, it follows that $d(z, y) = 1_X$ and so $z = y$. This completes the proof. □

Remark 3. *In the theorem, with $\gamma = 0$, we get $\alpha + \beta < 1$, resulting in a multiplicative Kannan-type contraction [19]*

$$d(Tx, Ty) \preceq (\theta_1(d(x, Tx)))^{\alpha}(\theta_2(d(y, Ty)))^{\beta}.$$

Example 4. *(Application:) Let*

$$X = D_n(\mathbb{R}) = \{\mathrm{diag}(x_1, \ldots, x_n) : x_i \in \mathbb{R}\}$$

be the set of all $n \times n$ diagonal matrices over $\mathbb{R}$, equipped with the norm

$$\|A\| = \max_{1\leq i\leq n} |x_i| \quad \textit{for } A = \mathrm{diag}(x_1, \ldots, x_n) \in X.$$

Define a cone

$$P = \{\mathrm{diag}(z_1, \ldots, z_n) : z_i \geq 1 \textit{ for all } i\} \subset X.$$

Define the algebra multiplication operation by

$$\operatorname{diag}(x_1,\ldots,x_n)\cdot\operatorname{diag}(y_1,\ldots,y_n)=\operatorname{diag}(x_1y_1,\ldots,x_ny_n),$$

and the norm is submultiplicative:

$$\|AB\|=\max_i|x_iy_i|\le\left(\max_i|x_i|\right)\left(\max_i|y_i|\right)=\|A\|\|B\|.$$

It can be shown that $(D_n(\mathbb{R}),\|\cdot\|)$ *is a unital commutative Banach algebra.*

Define a mapping $d: X\times X\to P$ *by*

$$d(A,B)=\operatorname{diag}\left(e^{|x_1-y_1|},\ldots,e^{|x_n-y_n|}\right),$$

for $A=\operatorname{diag}(x_1,\ldots,x_n)$ *and* $B=\operatorname{diag}(y_1,\ldots,y_n)$.

Then it can be shown that (X,d) *is a complete cone-type metric space.*

Let $g_i:\mathbb{R}\to\mathbb{R}$ *be Lipschitz continuous functions with constants* $L_i\in(0,1)$*, and let* $\lambda=\max_i L_i<1$. *Assume each* g_i *is bounded below by* c_i*, and choose* $a_i\ge 1-c_i$ *to ensure* $a_i+g_i(x)\ge 1$ *for all* x*. Define the operator*

$$T: X\to X$$

by

$$T(\operatorname{diag}(x_1,\ldots,x_n))=\operatorname{diag}(a_1+g_1(x_1),\ldots,a_n+g_n(x_n)).$$

Then for any $A=\operatorname{diag}(x_1,\ldots,x_n)$ *and* $B=\operatorname{diag}(y_1,\ldots,y_n)$*, we have*

$$|[T(A)]_{ii}-[T(B)]_{ii}|=|g_i(x_i)-g_i(y_i)|\le L_i|x_i-y_i|\le\lambda|x_i-y_i|.$$

so

$$d(T(A),T(B))=\operatorname{diag}\left(e^{|g_1(x_1)-g_1(y_1)|},\ldots,e^{|g_n(x_n)-g_n(y_n)|}\right)\preceq d(A,B)^{\lambda}.$$

Set $\gamma\in(\lambda,1)$ *and* $s=\lambda/\gamma\in(0,1)$*, and define*

$$\theta_3(\operatorname{diag}(z_1,\ldots,z_n))=\operatorname{diag}(z_1^s,\ldots,z_n^s),\quad z_i\ge 1.$$

Then θ_3 *is continuous, order-preserving, and satisfies* $\theta_3(1_X)=1_X$. *The contractive condition of Theorem 1 holds:*

$$d(T(A),T(B))\preceq(\theta_3(d(A,B)))^{\gamma}.$$

By Theorem 1, T *has a unique fixed point*

$$A^*=\operatorname{diag}(x_1^*,\ldots,x_n^*)\in X,$$

satisfying

$$x_i^*=a_i+g_i(x_i^*),\quad i=1,\ldots,n.$$

Remark 4. *For each* $i\in\{1,2,\cdots,n\}$, $S_i:\mathbb{R}\to\mathbb{R}$ *defined by* $S_i(x)=a_i+g(x)$ *is a Banach contraction since* g_i *is Lipschitz with* $L_i<1$. *Hence, by the Banach Contraction Principle,* S_i *admits a unique fixed point* $x_i^*\in\mathbb{R}$.

In the theorem that follows, the arguments of the control functions were changed to include cross terms, resulting in a different contraction.

Theorem 2. *Let (X, d) be a complete cone-valued θ-type multiplicative metric space, where $P \subset X$ is a solid multiplicative cone in a commutative unital Banach algebra X. Suppose $T : X \to X$ satisfies, for all $x, y \in X$:*

$$d(Tx, Ty) \preceq (\theta_1(d(x, Ty)))^{\alpha}(\theta_2(d(y, Tx)))^{\beta}(\theta_3(d(x, y)))^{\gamma},$$

where $\alpha, \beta, \gamma \geq 0$, with $\alpha + \beta + \gamma < 1$, and $\theta_i : P \to P$ are continuous, monotone, and satisfy $\theta_i(u) \preceq u$, $\theta_i(1_X) = 1_X$. Then T has a unique fixed point $z \in X$, and for any $x_0 \in X$, the Picard iterates $x_{n+1} = Tx_n$ converge to z in norm.

Proof. (Picard sequence:) Fix $x_0 \in X$ and define $x_{n+1} = Tx_n$, $n \geq 0$. Set

$$\delta_n := d(x_n, x_{n+1}) = d(x_n, Tx_n) \in P.$$

(Contractive condition:) For $x = x_{n-1}$, $y = x_n$, we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \preceq (\theta_1(d(x_{n-1}, Tx_n)))^{\alpha}(\theta_2(d(x_n, Tx_{n-1})))^{\beta}(\theta_3(d(x_{n-1}, x_n)))^{\gamma}.$$

Note that $d(x_{n-1}, Tx_n) = d(x_{n-1}, x_{n+1})$ and $d(x_n, Tx_{n-1}) = d(x_n, x_n) = 1_X$. Using $\theta_i(1_X) = 1_X$ and $\theta_i(u) \preceq u$, we obtain

$$\delta_n \preceq (d(x_{n-1}, x_{n+1}))^{\alpha}(\delta_{n-1})^{\gamma}.$$

(Estimate using triangle inequality:) By the cone θ-triangle inequality,

$$d(x_{n-1}, x_{n+1}) \preceq \theta_1(d(x_{n-1}, x_n))\theta_2(d(x_n, x_{n+1})) \preceq \delta_{n-1}\delta_n.$$

Thus,

$$\delta_n \preceq (\delta_{n-1}\delta_n)^{\alpha}(\delta_{n-1})^{\gamma} = \delta_{n-1}^{\alpha+\gamma}\delta_n^{\alpha} \Rightarrow \delta_n^{1-\alpha} \preceq \delta_{n-1}^{\alpha+\gamma} \Rightarrow \delta_n \preceq \delta_{n-1}^{\frac{\alpha+\gamma}{1-\alpha}}.$$

Set $\xi := \frac{\alpha+\gamma}{1-\alpha} \in [0, 1)$. Iterating,

$$\delta_n \preceq \delta_{n-1}^{\xi} \preceq \delta_{n-2}^{\xi^2} \preceq \cdots \preceq \delta_0^{\xi^n} \to 1_X.$$

(Cauchy property:) For $m > n$, repeated application of the cone θ- triangle inequality gives

$$d(x_n, x_m) \preceq \delta_n\delta_{n+1}\cdots\delta_{m-1} \to 1_X \text{ as } n, m \to \infty,$$

so $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy. By completeness, there exists $z \in X$ such that $x_n \to z$.
(Fixed point:) Applying the contractive condition with $x = x_n$, $y = z$:

$$d(x_{n+1}, Tz) = d(Tx_n, Tz) \preceq (\theta_1(d(x_n, Tz)))^{\alpha}(\theta_2(d(z, x_{n+1})))^{\beta}(\theta_3(d(x_n, z)))^{\gamma}.$$

Taking $n \to \infty$, $d(x_n, z) \to 1_X$, so

$$d(z, Tz) \preceq d(z, Tz)^{\alpha} \text{ if and only if } d(z, Tz) = 1_X.$$

Hence, $Tz = z$.

(Uniqueness:) If y is another fixed point, then

$$d(z,y) = d(Tz,Ty) \preceq (\theta_3(d(z,y)))^{\gamma} \preceq d(z,y)^{\gamma}$$

if and only if $d(z,y) = 1_X$ if and only if $z = y$.

Therefore, T has a unique fixed point $z \in X$, and the Picard iterates converge to z.

□

Remark 5. *In Theorem 2, taking $\gamma = 0$ and $\beta = \alpha$, from $0 \leq \alpha + \beta < 1$, we get $0 \leq \alpha < \frac{1}{2}$ with contraction*

$$d(Tx,Ty) \preceq (\theta_1(d(x,Ty)))^{\alpha}(\theta_2(d(y,Tx)))^{\alpha},$$

which is a generalized form of the Chatterjea contraction in the multiplicative form [20].

Example 5. *(θ-metric space:) Let X be a nonempty set and $(\mathbb{R}, |\cdot|)$ be the unital Banach algebra and the natural cone $P = [1,\infty)$. A mapping*

$$d : X \times X \to [1,\infty)$$

satisfies the following conditions:

(i) $d(x,y) > 1$ *for all* $x, y \in X$ *and* $x \neq y$.
(ii) $d(x,y) = 1 \Leftrightarrow x = y$.
(iii) $d(x,y) = d(y,x)$ *for all* $x, y \in X$.
(iv) $d(x,z) \leq \theta_1(d(x,y))\theta_2(d(y,z))$ *for* $y \neq x, z \in X$,

where the function $\theta_i : [1,\infty) \to [1,\infty)$, $i = 1,2$ satisfies the following properties:

(i) θ_i *is continuous.*
(ii) $\theta_i(1) = 1$.
(iii) $\theta_i(s) \leq \theta_i(t)$ *if* $s \leq t$.
(iv) $\theta_i(t) \leq t$ *for all* $t \geq 1$.

Then, d is a θ-type multiplicative metric on X and (X,d) is a θ-type multiplicative metric space. The θ-type multiplicative metric is a generalization of the multiplicative metric.

Remark 6. *If d is a θ-type multiplicative metric with $\theta_i = \theta$ for $i = 1,2$, then by using logarithms, metric d can be transformed to a θ-metric if we assume that θ satisfies an additional property:*

For $m = \theta(a)\theta(b)$ and $t \in [1,m]$, there exists $s \in \Im(\theta)$ with $s \in [1,m]$, such that $\theta(s)\theta(t) = m$. Let $\rho(x,y) = \log(d(x,y))$; then, from the multiplicative triangular inequality, we get

$$d(x,z) \leq \theta(d(x,y))\theta(d(y,z)).$$

Then, taking logarithms, we get

$$\begin{aligned}\rho(x,z) = \log(d(x,z)) &\leq \log(\theta(d(x,y))\theta(d(y,z))) \\ &= \log(\theta(e^{\rho(x,y)})) + \log(\theta(e^{\rho(y,z)})) \\ &= \tilde{\theta}(\rho(x,y),\rho(y,z)),\end{aligned}$$

where $\tilde{\theta}(s,t) = \log(\theta(e^s)) + \log(\theta(e^t))$.

It can be shown that $\tilde{\theta}$ satisfies properties (i)–(vi) in definition 1.

Example 6. *(θ-type metric space :) Let $\mathbb{R}_+^n$ be the set of all positive n-tuples of numbers and define $d : \mathbb{R}_+^n \times \mathbb{R}_+^n \to [1, \infty)$ by*

$$d(x, y) = \left|\frac{x_1}{y_1}\right|^* \cdot \left|\frac{x_2}{y_2}\right|^* \cdots \left|\frac{x_n}{y_n}\right|^*,$$

where $x = (x_1, x_2, \cdots x_n)$, $y = (y_1, y_2, \cdots y_n) \in \mathbb{R}_+^n$,

and

$$|x|^* = \begin{cases} x, & \text{if } x \geq 1, \\ \frac{1}{x}, & \text{if } 0 < x < 1. \end{cases}$$

Then, $(\mathbb{R}_+^n, d)$ is a θ- type multiplicative metric space with $\theta_i = \theta(t) = t$ for $i = 1, 2$.

Example 7. *(θ- type metric space:) Let (X, ρ) be a metric space. Fix the constant $a > 1$, then define*

$$d(x, y) = a^{\rho(x,y)}.$$

Define $\theta : [1, \infty) \to [1, \infty)$ by $\theta_i = \theta(t) = t$ for $i = 1, 2$.

It can be shown that properties $(i) - (iii)$ of example 5 are satisfied. We verify only property (iv) of example 5: For $y, x, z \in X$,

$$d(x, z) = a^{\rho(x,z)} \leq a^{\rho(x,y)+\rho(y,z)} = a^{\rho(x,y)} a^{\rho(y,z)} = d(x, y)d(y, z) = \theta(d(x, y))\theta(d(y, z)).$$

Thus, (X, d) is a θ-type metric space.

Example 8. *(θ- type metric space:) Let (X, ρ) be a metric space. Define*

$$d(x, y) = 1 + c\rho(x, y)$$

and choose constant $0 < c < 1$ and let $\theta_i : [1, \infty) \to [1, \infty)$ be given by

$$\theta_i(t) = t.$$

Properties $(i) - (iii)$ can be verified as in example 5. We verify only property (iv) of example 5. For $x, y, z \in X$

$$\begin{aligned} &\theta_1(d(x, y))\theta_2(d(y, z)) \\ &= (1 + c\rho(x, y))(1 + c\rho(y, z)) \\ &= 1 + c\rho(x, y) + c\rho(y, z) + c^2\rho(x, y)\rho(y, z), \end{aligned}$$

$$\rho(x, z) \leq \rho(x, y) + \rho(y, z), \text{ we obtain}$$

$$\theta_1(d(x, y))\theta_2(d(y, z)) \qquad \geq 1 + c(\rho(x, y) + \rho(y, z)) \geq 1 + c\rho(x, z) = d(x, z).$$

Thus,

$$d(x, z) \leq \theta_1(d(x, y))\theta_2(d(y, z)).$$

It follows that (X, d) is a θ-type metric space.

Let (X, d) be a θ-multiplicative metric space and define the multiplicative open balls

$$B_r(x) = \{y \in X : d(x, y) < r\},$$

for $r > 1$. Then for every $x \in X$, the countable family

$$\left\{B_{1+\frac{1}{n}}(x) : n \in \mathbb{N}\right\}$$

is a local base at x. Fix $x \in X$, let U be any open neighborhood of x as a multiplicative topology, by definition, there exists some radius $\epsilon > 1$ with $B_\epsilon(x) \subset U$. We find $n \in \mathbb{N}$ such that $B_{1+\frac{1}{n}}(x) \subset B_\epsilon(x)$. Since $\epsilon - 1 > 0$ by Archimedean property choose $n \in \mathbb{N}$ with $\frac{1}{n} < \epsilon - 1$. Then $1 + \frac{1}{n} < \epsilon$. Therefore, every point y with $d(x, y) < 1 + \frac{1}{n}$ also satisfies $d(x, y) < \epsilon$, so

$$B_{1+\frac{1}{n}}(x) \subset B_\epsilon(x) \subset U.$$

Since U is an arbitrary neighborhood of x, this shows that for every neighborhood U, there exists n with $B_{1+\frac{1}{n}}(x) \subset U$.

Hence,

$$\left\{B_{1+\frac{1}{n}}(x) : n \in \mathbb{N}\right\}$$

is a local base at x.

We now prove a fixed point theorem for a Geraghty type contraction.

Theorem 3. *Let (X, d) be a complete cone-valued θ-type multiplicative metric space and assume $T : X \to X$ satisfies the following Geraghty contraction condition if there exists a function $\mu \in \mathfrak{F}$ such that for all $x, y \in X$:*

$$\begin{aligned} &d(Tx, Ty) \\ &\preceq (\theta_1(d(x, Tx)))^{\alpha\mu(d(x,Tx))}(\theta_2(d(y, Ty)))^{\beta\mu(d(y,Ty))}(\theta_3(d(x, y)))^{\gamma\mu(d(x,y))}, \end{aligned} \tag{2.5}$$

where $\alpha, \beta, \gamma \geq 0$ are real numbers with

$$0 < \alpha + \beta + \gamma < 1,$$

where $P \subset X$ is a cone in an ordered Banach algebra X and $\theta_i : P \to P$ are continuous, monotone satisfying $\theta_i(u) \preceq u$, $\theta_i(1_X) = 1_X$, and $\mathfrak{F}$ is a set of all functions $\mu : P \to (0, 1]$ such that $\lim_{n\to\infty} \mu(t_n) = 1$ *implies that* $\lim_{n\to\infty} t_n = 1_X$.

Then, T has a unique fixed point $z \in X$, and for any $x_0 \in X$, the Picard iterates $x_{n+1} = Tx_n$ converge to z in norm.

Proof. Fix $x_0 \in X$ and define the Picard sequence $x_{n+1} = Tx_n$ for $n \geq 0$. Put

$$\delta_n := d(x_n, x_{n+1}) \in P, \quad n \geq 0.$$

Applying (2.5) with $x = x_{n-1}$ and $y = x_n$ gives

$$\begin{aligned} &d(Tx_{n-1}, Tx_n) \\ &\preceq (\theta_1(d(x_{n-1}, Tx_{n-1})))^{\alpha\mu(d(x_{n-1},Tx_{n-1}))}(\theta_2(d(x_n, Tx_n)))^{\beta\mu(d(x_n,Tx_n))} \end{aligned}$$

$$
\begin{aligned}
&\times (\theta_3(d(x_{n-1}, x_n)))^{\gamma\mu(d(x_{n-1},x_n))} \\
&\leq (d(x_{n-1}, Tx_{n-1}))^{\alpha\mu(d(x_{n-1},Tx_{n-1}))}(d(x_n, Tx_n))^{\beta\mu(d(x_n,Tx_n))} \\
&\times (d(x_{n-1}, x_n))^{\gamma\mu(d(x_{n-1},x_n))} \\
&\leq (d(x_{n-1}, Tx_{n-1}))^{\alpha}(d(x_n, Tx_n))^{\beta}(d(x_{n-1}, x_n))^{\gamma}.
\end{aligned}
$$

In terms of δ, and using $\theta_i(u) \leq u$, we get

$$\delta_n \leq \delta_{n-1}^{\alpha}\delta_n^{\beta}\delta_{n-1}^{\gamma}.$$

Applying the partial order, we obtain

$$\delta_{n-1}^{\alpha}\delta_n^{\beta}\delta_{n-1}^{\gamma}\delta_n^{-1} = \delta_{n-1}^{\alpha+\gamma}\delta_n^{\beta-1} = \delta_{n-1}^{\alpha+\gamma}(\delta_n^{1-\beta})^{-1} \in P,$$

and

$$\delta_n^{1-\beta} \leq \delta_{n-1}^{\alpha+\gamma},$$

and therefore

$$\delta_n \leq \delta_{n-1}^{\xi}, \quad \text{where } \xi := \frac{\alpha+\gamma}{1-\beta}.$$

Note that $\xi \in [0, 1)$ since $\alpha + \beta + \gamma < 1$. Iterating, we obtain

$$\delta_n \leq \delta_{n-1}^{\xi} \leq \delta_{n-2}^{\xi^2} \leq \cdots \leq \delta_0^{\xi^n}.$$

Since $0 \leq \xi < 1$, we have $\xi^n \to 0$ as $n \to \infty$. For the multiplicative metric, raising $\delta_0 \in P$ to power $\xi^n \to 0$ yields the identity 1_X, i.e., $\lim_{n\to\infty} \delta_0^{\xi^n} = 1_X$.

Since $1_X \leq \delta_n \leq \delta_0^{\xi^n}$, it follows that

$$0 \leq \|\delta_n - 1_X\| \leq \|\delta_0^{\xi^n} - 1_X\|.$$

It follows that $\lim_{n\to\infty} \delta_n = 1_X$.

Next, we show that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy. Using the cone θ-triangle control inequality repeatedly and the monotonicity of the θ_i's, we obtain, for $m > n$,

$$d(x_n, x_m) \leq \prod_{k=n}^{m-2} \theta_1(d(x_k, x_{k+1}))\, \theta_2(d(x_{m-1}, x_m)) \leq \prod_{k=n}^{m-1} \delta_0^{\xi^k} \leq \delta_0^{\xi^n/(1-\xi)}.$$

Hence, $d(x_n, x_m) \to 1_X$ as $n, m \to \infty$. Thus, $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy, and by completeness, it converges to some $z \in X$.

We now prove that z is a fixed point. Apply (2.5) with $x = x_n$ and $y = z$:

$$
\begin{aligned}
&d(x_{n+1}, Tz) \\
&\leq (\theta_1(d(x_n, x_{n+1})))^{\alpha\mu(d(x_n,x_{n+1}))}(\theta_2(d(z, Tz)))^{\beta\mu(d(z,Tz))}(\theta_3(d(x_n, z)))^{\gamma\mu(d(x_n,z))}.
\end{aligned}
$$

Letting $n \to \infty$, yields

$$d(z, Tz) \leq d(z, Tz)^{\beta\mu(d(z,Tz))}.$$

Using the partial ordering, it follows that

$$d(z, Tz)^{1-\beta\mu(d(z,Tz))} \in P.$$

Since $0 \le \beta < 1$, and $0 < \mu(t) \le 1$, it follows $\beta\mu(t) \le \beta < 1$, which implies that $d(z, Tz)^{1-\beta\mu(d(z,Tz))} \in P$. Thus, from the property of cone P, we get that

$$d(z, Tz)^{1-\beta\mu(d(z,Tz))} = 1_X.$$

This implies that $d(z, Tz) = 1_X$, i.e., $z = Tz$.

For uniqueness, assume $y \in X$ is another fixed point. Then, applying (2.5) with $x = z$ and $y = y$, and using $d(z, Tz) = d(y, Ty) = 1_X$, we obtain

$$d(z, y) \preceq (\theta_3(d(z, y)))^{\gamma\mu(d(z,y))} \preceq (d(z, y))^{\gamma}.$$

Again, since $0 \le \gamma < 1$, it follows that $d(z, y) = 1_X$ and so $z = y$. This completes the proof. □

We now extend the topological notions of convergence, Cauchy sequences, and continuity to the setting of the 3-variable cone θ-type multiplicative metric.

Definition 6. *Let $(X, \|\cdot\|)$ be a commutative unital Banach algebra over $\mathbb{C}$ or $\mathbb{R}$ with identity element 1_X. Let $P \subset \mathrm{Inv}(X) = \{u \in X : u$ is invertible$\}$ be a solid multiplicative cone, i.e.:*

1) *P is nonempty, closed.*
2) *$\alpha P \subset P$ for all real $\alpha \ge 1$;*
3) *$P \cdot P \subset P$;*
4) *If $u \in P$ and $u^{-1} \in P$ then $u = 1_X$.*
5) $\mathrm{int}(P) \neq \emptyset$.
6) *For all $u \in P$, $\sigma(u) \subset \mathbb{C} \setminus (-\infty, 0]$, where $\sigma(u) = \{\lambda \in \mathbb{C} : u - \lambda 1_X$ is not invertible in $X\}$ denotes the spectrum of u. In the real case, $\sigma(u) \subset (0, \infty)$.*

The cone P induces a partial order on $\mathrm{Inv}(X)$ *by*

$$u \preceq v \iff vu^{-1} \in P.$$

If $u \preceq v$, there exists a constant $k \ge 1$ such that $\|u\| \le k\|v\|$ for $u, v \in X$.

Definition 7. *Cone θ-type multiplicative metric: Let $\mathbb{X}$ be a nonempty set and $(X, \|\cdot\|)$ be a commutative unital Banach algebra over $\mathbb{C}$ or $\mathbb{R}$ with identity element 1_X. A mapping*

$$d : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \longrightarrow P \subset \mathrm{Inv}(X)$$

is called a Cone θ-type multiplicative metric *if there exist continuous, order-preserving mappings*

$$\theta_i : P \to P, \quad i = 1, 2, 3.$$

Cone θ-type multiplicative metric space

Let $d : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to P \subset X$ be a mapping satisfying the following conditions for all $x, y, z \in \mathbb{X}$:

(M1) **(Identity and Positivity)** $d(x, y, z) = 1_X$ if and only if at least two of the three points are the same, and $d(x, y, z) \succ 1_X$ for all $x, y, z \in \mathbb{X}$ with $x, y \neq z \in \mathbb{X}$.
(M2) **(Symmetry)** $d(x, y, z) = d(y, x, z) = \cdots$.
(M3) **(Cone θ-multiplicative triangle inequality)**

$$d(x,y,z) \preceq \begin{cases} \theta_1(d(x,y,t))\theta_2(d(y,z,t))\theta_3(d(z,x,t)), & t \neq x,y,z, \\ d(x,y,t)d(y,z,t)d(z,x,t), & t = x,y,z, \end{cases}$$

where the product on the right-hand side is the algebra product in X, and $\preceq$ denotes the partial order induced by P.

Each function $\theta_i : P \to P$ for $i = 1, 2, 3$ satisfies:

1) θ_i is continuous;
2) θ_i is monotone: if $u \preceq v$ then $\theta_i(u) \preceq \theta_i(v)$;
3) $\theta_i(1_X) = 1_X$;
4) $\theta_i(u) \preceq u$ for all $u \in P$.

If the mapping d satisfies (M1)–(M3), the pair $(\mathbb{X}, d)$ is called a *cone θ-type multiplicative metric space* over $\mathbb{X}$.

Definition 8. *Let $(\mathbb{X}, d)$ be a cone θ-type multiplicative metric space, where $P \subset X$ is a cone in a Banach algebra X. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{X}$ and $x \in \mathbb{X}$. The sequence is* convergent *to x if and only if*

$$\|d(x_n, x, z) - 1_X\| \to 0 \quad \text{as } n \to \infty$$

for all $z \in \mathbb{X}$, where $\|\cdot\|$ is the norm on the Banach algebra X.

Definition 9. *Let $(\mathbb{X}, d)$ be a cone θ-type multiplicative metric space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{X}$ is a* Cauchy sequence *if and only if*

$$\|d(x_n, x_m, z) - 1_X\| \to 0 \quad \text{as } n, m \to \infty$$

for all $z \in \mathbb{X}$.

Definition 10. *The cone θ-type multiplicative metric space $(\mathbb{X}, d)$ is* complete *if every Cauchy sequence in $\mathbb{X}$ converges to some $x \in \mathbb{X}$.*

Definition 11. *Let $(\mathbb{X}, d)$ be a cone θ-type multiplicative metric space. A mapping $T : \mathbb{X} \to \mathbb{X}$ is a* multiplicative contraction mapping *if there exists a real number $0 < \alpha < 1$ such that*

$$d(Tx, Ty, z) \preceq (\theta(d(x, y, z)))^{\alpha}$$

for all $x, y, z \in \mathbb{X}$, where $P \subset X$ is a cone in a Banach algebra X and $\theta : P \to P$.

Theorem 4. *Let $(\mathbb{X}, d)$ be a complete cone θ-multiplicative metric space. If a mapping $T : \mathbb{X} \to \mathbb{X}$ satisfies the condition*

$$d(Tx, Ty, z) \preceq (\theta_1(d(x, Tx, z)))^{\alpha}(\theta_2(d(y, Ty, z)))^{\beta}(\theta_3(d(x, y, z)))^{\gamma}, \tag{2.6}$$

with α, β, γ non-negative real numbers such that $0 < \alpha + \beta + \gamma < 1$, where $P \subset X$ is a cone in a Banach algebra $(X, \|\cdot\|)$ and $\theta_i : P \to P$ are continuous, monotone, and satisfy $\theta_i(u) \preceq u$, $\theta_i(1_X) = 1_X$, then T has a unique fixed point $z \in \mathbb{X}$, and for any $x_0 \in \mathbb{X}$, the Picard iterates $x_{n+1} = Tx_n$ converge to z in norm.

Proof. Let x_0 be an arbitrary element in $\mathbb{X}$. For each $n \in \mathbb{N}$, define the sequence $x_{n+1} = Tx_n$. For the sequence $\{x_n\}_{n\in\mathbb{N}}$, we get

$$\begin{aligned} d(x_{n+1}, x_n, z) &= d(Tx_n, Tx_{n-1}, z) \\ &\preceq (\theta_1(d(x_n, Tx_n, z)))^{\alpha}\, (\theta_2(d(x_{n-1}, Tx_{n-1}, z)))^{\beta} \\ &\quad (\theta_3(d(x_{n-1}, x_n, z)))^{\gamma} \\ &\preceq (d(x_n, x_{n+1}, z))^{\alpha}(d(x_{n-1}, x_n, z))^{\beta}(d(x_{n-1}, x_n, z))^{\gamma}. \end{aligned}$$

If $\delta_{n,z} = d(x_{n+1}, x_n, z)$ for all $z \in \mathbb{X}$, then

$$\delta_{n,z} \preceq \delta_{n,z}^{\alpha}\delta_{n-1,z}^{\beta+\gamma}.$$

Since

$$\delta_n^{\alpha}\delta_{n-1}^{\beta+\gamma}\delta_n^{-1} = \delta_{n-1}^{\beta+\gamma}(\delta_n^{1-\alpha})^{-1} \in P,$$

it follows that

$$\delta_{n,z}^{1-\alpha} \preceq \delta_{n-1,z}^{\beta+\gamma}.$$

Taking the $(\alpha - 1)^{-1}$-th power yields

$$\delta_{n,z} \preceq \delta_{n-1,z}^{\frac{\beta+\gamma}{1-\alpha}} = \delta_{n-1,z}^{\xi}, \tag{2.7}$$

where $\xi = \frac{\beta+\gamma}{1-\alpha}$. Since $\beta + \gamma < 1 - \alpha$, it follows that $\xi < 1$.

Repeated use of (2.7) yields

$$\delta_{n,z} \preceq \delta_{n-1,z}^{\xi} \preceq \delta_{n-2,z}^{\xi^2} \cdots \preceq \delta_{0,z}^{\xi^n}.$$

We claim that the sequence $\{x_n\}$ is a Cauchy sequence in $\mathbb{X}$. For $n, m \in \mathbb{N}$, and $t \in \mathbb{X}$, we get

$$\begin{aligned} d(x_n, x_{n+m}, z) &\preceq \theta_1(d(x_n, x_{n+m}, t))\theta_2(d(x_{n+m}, z, t))\theta_3(d(z, x_n, t)) \\ &\preceq d(x_n, x_{n+m}, t)d(x_{n+m}, z, t)d(z, x_n, t). \end{aligned} \tag{2.8}$$

Taking $t = x_{n+1}$ in (2.8), we obtain

$$\begin{aligned} d(x_n, x_{n+m}, z) &\preceq d(x_n, x_{n+m}, x_{n+1})d(x_{n+m}, z, x_{n+1})d(z, x_n, x_{n+1}) \\ &\preceq (d(x_1, x_0, x_{n+m}))^{\xi^n} d(x_{n+m}, z, x_{n+1})(d(x_1, x_0, z))^{\xi^n}. \end{aligned} \tag{2.9}$$

Similarly,

$$\begin{aligned}d(x_{n+m},z,x_{n+1}) &\preceq \theta_1(d(x_{n+m},z,x_{n+2}))\theta_2(d(z,x_{n+1},x_{n+2}))\theta_3(d(x_{n+1},x_{n+m},x_{n+2}))\\ &\preceq d(x_{n+m},z,x_{n+2})d(z,x_{n+1},x_{n+2})d(x_{n+1},x_{n+m},x_{n+2})\\ &\preceq (d(x_1,x_0,x_{n+m}))^{\xi^{n+1}}d(x_{n+m},z,x_{n+2})(d(x_1,x_0,z))^{\xi^{n+1}}.\end{aligned} \tag{2.10}$$

Substituting into (2.9), we get

$$d(x_n,x_{n+m},z) \preceq (d(x_1,x_0,x_{n+m}))^{\xi^n+\xi^{n+1}}d(x_{n+m},z,x_{n+2})(d(x_1,x_0,z))^{\xi^n+\xi^{n+1}}. \tag{2.11}$$

Continuing similarly, we get

$$\begin{aligned}d(x_n,x_{n+m},z) &\preceq (d(x_1,x_0,x_{n+m}))^{\xi^n(1+\xi+\cdots+\xi^m)}\,(d(x_1,x_0,z))^{\xi^n(1+\xi+\cdots+\xi^m)}\\ &= (d(x_1,x_0,x_{n+m}))^{\xi^n\frac{1-\xi^{m+1}}{1-\xi}}\,(d(x_1,x_0,z))^{\xi^n\frac{1-\xi^{m+1}}{1-\xi}}.\end{aligned} \tag{2.12}$$

Taking $n\to\infty$, we obtain

$$\|d(x_n,x_{n+m},z)-1_X\|\to 0,$$

so $\{x_n\}$ is Cauchy.

Since $\mathbb{X}$ is complete, there exists $x\in\mathbb{X}$ such that

$$\|d(x_n,x,z)-1_X\|\to 0.$$

We claim x is a fixed point:

$$d(x,Tx,z)\preceq d(x,Tx,t)d(Tx,z,t)d(z,x,t).$$

Taking $t=x_n$,

$$d(x,Tx,z)\preceq d(x,Tx,x_n)d(Tx,z,x_n)d(z,x,x_n).$$

Using previous estimate (2.12), we get

$$\begin{aligned}&d(x,Tx,z)\\ &\preceq d(x,Tx,x_n)d(Tx,z,Tx_{n-1})d(z,x,x_n)\\ &\preceq d(x,Tx,x_n)\Big(\theta_1(d(x_{n-1},Tx_{n-1},z))\Big)^{\alpha}\Big(\theta_2(d(x,Tx,z))\Big)^{\beta}\\ &\times\Big(\theta_3(d(x_{n-1},x,z))\Big)^{\gamma}d(z,x,x_n)\\ &= d(x,Tx,x_n)\Big(d(x_{n-1},x_n,z)\Big)^{\alpha}\Big(d(x,Tx,z)\Big)^{\beta}\Big(d(x_{n-1},x,z)\Big)^{\gamma}d(z,x,x_n)\\ &\preceq d(x,Tx,x_n)\Big((d(x_1,x_0,x_n))^{\xi^{n-1}}\Big)^{\alpha}\Big((d(x_1,x_0,z))^{\xi^{n-1}}\Big)^{\alpha}\\ &\Big(d(x,Tx,z)\Big)^{\beta}\Big(d(x_{n-1},x,z)\Big)^{\gamma}d(z,x,x_n)\end{aligned}$$

Thus,

$$1_X\preceq d(x,Tx,z)\preceq (d(x,Tx,x_n))^{\frac{1}{1-\beta}}(d(x_1,x_0,x_n))^{\frac{\alpha\xi^{n-1}}{1-\beta}}$$

$$(d(x_1, x_0, z))^{\frac{\alpha\xi^{n-1}}{1-\beta}}(d(x_{n-1}, x, z))^{\frac{\gamma}{1-\beta}}(d(z, x, x_n))^{\frac{1}{1-\beta}}.$$

Taking limits gives $d(x, Tx, z) = 1_X$.
For uniqueness, suppose $y^* \neq x$ with $Ty^* = y^*$. Then

$$\begin{aligned} d(x, y^*, z) &= d(Tx, Ty^*, z) \\ &\leq (d(x, Tx, z))^{\alpha}(d(y^*, Ty^*, z))^{\beta}(d(x, y^*, z))^{\gamma} \\ &\leq (d(x, y^*, z))^{\gamma}. \end{aligned}$$

This is a contradiction unless $d(x, y^*, z) = 1_X$, hence $x = y^*$. □

Example 9. *(θ-type metric space:) Let $\mathbb{X}$ be a nonempty set. Fix a constant $a > 1$ and define $\theta_i : [1, \infty) \to [1, \infty)$ for $i = 1, 2, 3$ by*

$$\theta_i(t) = t^{1/2}.$$

The function θ_i is continuous and non-decreasing on $[1, \infty)$ and $\theta_i(t) \leq t$ for $t \geq 1$.
Define $d : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [1, \infty)$ by

$$d(x, y, z) = \begin{cases} 1, & \text{if at least two of the points are the same,} \\ a^{\frac{\rho(x,y)+\rho(y,z)+\rho(z,x)}{3}}, & \text{if } x, y, z \text{ are distinct,} \end{cases}$$

where $(\mathbb{X}, \rho)$ is a usual metric. Properties $(i) - -(iii)$ can be verified.
The remaining property (iv) of Definition 7: Let

$$s = \rho(x, y) + \rho(y, z) + \rho(z, x).$$

Using the triangle inequality of ρ, for arbitrary t, we get

$$\rho(x, y) \leq \rho(x, t) + \rho(t, y), \quad \rho(y, z) \leq \rho(y, t) + \rho(t, z), \quad \rho(z, x) \leq \rho(z, t) + \rho(t, x).$$

Adding the inequalities, we get

$$s \leq 2T \quad \Longleftrightarrow \quad \frac{s}{3} \leq \frac{s + 2T}{6},$$

where

$$T = \rho(t, y) + \rho(t, x) + \rho(t, z).$$

Finally,

$$d(x, y, z) = a^{s/3} \leq a^{(s+2T)/6} = \theta_1(d(x, y, t))\theta_2(d(y, z, t))\theta_3(d(z, x, t)).$$

Thus, d is a θ-type metric on X.

Example 10. *(θ-type metric space:) Let $X = C([a, b], \mathbb{R}^+)$, then define $d : X \times X \times X \to [1, \infty)$ by*

$$d(x, y, z) = \begin{cases} \max_{t \in [a,b]} \left\{ \left|\frac{x(t)}{y(t)}\right|_*, \left|\frac{y(t)}{z(t)}\right|_*, \left|\frac{z(t)}{x(t)}\right|_* \right\}, & \text{if all functions are distinct,} \\ 1, & \text{if at least two functions are equal for all } t. \end{cases}$$

For $x \in C([a,b], \mathbb{R}^+)$*,*

$$|x(t)|_* = \begin{cases} x(t), & x(t) \geq 1, \\ \dfrac{1}{x(t)}, & 0 < x(t) < 1. \end{cases}$$

Properties $(i) - -(iii)$ *follow from Definition 7. We verify property (iv).*

Let $x, y, z, s \in C([a,b], \mathbb{R}^+)$*. For* $t \in [a,b]$*, we get*

$$\left|\frac{x(t)}{y(t)}\right|_* \leq \max_{t\in[a,b]} \left\{ \left|\frac{x(t)}{y(t)}\right|_*, \left|\frac{x(t)}{s(t)}\right|_*, \left|\frac{s(t)}{y(t)}\right|_* \right\},$$

$$\left|\frac{y(t)}{z(t)}\right|_* \leq \max_{t\in[a,b]} \left\{ \left|\frac{y(t)}{z(t)}\right|_*, \left|\frac{y(t)}{s(t)}\right|_*, \left|\frac{s(t)}{z(t)}\right|_* \right\},$$

$$\left|\frac{z(t)}{x(t)}\right|_* \leq \max_{t\in[a,b]} \left\{ \left|\frac{z(t)}{x(t)}\right|_*, \left|\frac{z(t)}{s(t)}\right|_*, \left|\frac{s(t)}{x(t)}\right|_* \right\}.$$

With $\theta(t) = t$*, taking maximum over* $t \in [a,b]$*, we get*

$$\max_{t\in[a,b]} \left\{ \left|\frac{x(t)}{y(t)}\right|_*, \left|\frac{y(t)}{z(t)}\right|_*, \left|\frac{z(t)}{x(t)}\right|_* \right\} \leq \theta(d(x,y,s))\theta(d(y,z,s))\theta(d(z,x,s)).$$

Thus, (X,d) *is a* θ*-type multiplicative metric space. It can easily be shown that* (X,d) *is complete.*

Definition 12. *Let* $(\mathbb{X}, d)$ *be a* θ*-type metric space. The open ball* $B_\varepsilon(x,y)$ *with centre* $x, y \in \mathbb{X}$ *and radius* $\varepsilon > 1$ *is defined by*

$$B_\varepsilon(x,y) = \{z \in \mathbb{X} : d(x,y,z) < \varepsilon\}.$$

Theorem 5. *Let* $(\mathbb{X}, d)$ *be a complete multiplicative metric space. If a mapping* $T : \mathbb{X} \to \mathbb{X}$ *satisfies*

$$d(Tx, Ty, z) \leq (\theta_1(d(x,Tx,z)))^\alpha (\theta_2(d(y,Ty,z)))^\beta (\theta_3(d(x,y,z)))^\gamma, \tag{2.13}$$

where $\alpha, \beta, \gamma \geq 0$ *and* $0 < \alpha+\beta+\gamma < 1$*, and* $\theta_i : [1,\infty) \to [1,\infty)$ *satisfies:*

(i) θ_i *is continuous,*
(ii) $\theta_i(1) = 1$*,*
(iii) $\theta_i(s) \leq \theta_i(t)$ *if* $s \leq t$*,*
(iv) $\theta_i(t) \leq t$ *for all* $t \geq 1$*,*

then T *has a unique fixed point.*

Proof. Follows in a similar manner to Theorem 4. □

3. Application

Functional equations can be effectively studied in the setting of θ-type multiplicative metric spaces by rewriting them as fixed point problems for suitably defined operators. By endowing the function space with a θ-type multiplicative metric, one can construct functionals that measure multiplicative deviations between functions. If the associated operator satisfies a θ-type multiplicative contractive condition, existence and uniqueness of solutions follow from the corresponding fixed point theorems. This approach is particularly useful for functional equations with exponential, scaling, or product-type structures. Hence, θ-type multiplicative metric spaces provide a natural and powerful framework for solving broad classes of functional equations.

Example 11. *Let $[a,b] \subset \mathbb{R}$ with $a < b$ and define*

$$X = C([a,b],\mathbb{R}^+), \quad \mathbb{R}^+ = (0,\infty).$$

For $x,y,z \in X$, define the multiplicative metric

$$d(x,y,z) = \max_{t\in[a,b]} \left\{ \left|\frac{x(t)}{y(t)}\right|_*, \left|\frac{y(t)}{z(t)}\right|_*, \left|\frac{z(t)}{x(t)}\right|_* \right\},$$

where the multiplicative modulus $|\cdot|_*$ is defined by

$$|x|_* = \begin{cases} x, & x \geq 1, \\ \dfrac{1}{x}, & 0 < x < 1. \end{cases}$$

It can be shown that (X,d) is a complete θ–multiplicative metric space.
Let

$$k : [a,b] \times [a,b] \times \mathbb{R}^+ \to \mathbb{R}^+$$

be a continuous function such that $k(t,s,x) > 0$ for all $t,s \in [a,b]$ and $x > 0$.
We consider the Fredholm multiplicative integral equation

$$x(t) = \prod_a^b k(t,s,x(s))^{ds}, \quad t \in [a,b],$$

where the product integral is defined by

$$\prod_a^b f(s)^{ds} = \exp\left(\int_a^b \ln f(s)\,ds\right), \quad f(s) > 0.$$

Define the operator $T : X \to X$ by

$$(Tx)(t) = \prod_a^b k(t,s,x(s))^{ds}, \quad t \in [a,b].$$

Remark 7. *Since k is continuous and positive, the function defined, $s \mapsto k(t,s,x(s))$ is continuous and positive for each $t \in [a,b]$. Hence, the product integral is well defined, and $Tx \in X$.*

For all $t,s \in [a,b]$ and all $x,y \in X$,

$$\left|\frac{k(t,s,x(s))}{k(t,s,y(s))}\right|_* \leq \left(\left|\frac{x(s)}{y(s)}\right|_*\right)^{\zeta},$$

where $\zeta > 0$ is a Lipschitz exponent.
Verification of the contractive condition. *For any $x,y,z \in X$ and $t \in [a,b]$, we have*

$$\left|\frac{(Tx)(t)}{(Ty)(t)}\right|_* = \left|\prod_a^b \frac{k(t,s,x(s))}{k(t,s,y(s))}^{ds}\right|_*$$

$$= \left|\exp\left(\int_a^b \ln\frac{k(t,s,x(s))}{k(t,s,y(s))}\right)ds\right|_*$$
$$\le \exp\left(\int_a^b \left|\ln\frac{k(t,s,x(s))}{k(t,s,y(s))}\right| ds\right)$$
$$= \exp\int_a^b \ln\left|\frac{k(t,s,x(s))}{k(t,s,y(s))}\right|_* ds.$$

Applying the assumption, it follows that

$$\left|\frac{(Tx)(t)}{(Ty)(t)}\right|_* \le \exp\left(\int_a^b \zeta \ln\left|\frac{x(s)}{y(s)}\right|_* ds\right).$$

Since $\left|\frac{x(s)}{y(s)}\right|_* \le d(x,y,z)$ *for all* $s \in [a,b]$, *we obtain*

$$\left|\frac{(Tx)(t)}{(Ty)(t)}\right|_* \le \exp\left(\int_a^b \zeta \ln(d(x,y,z))\, ds\right) = (d(x,y,z))^{\zeta(b-a)}.$$

We assume that $\zeta(b-a) > 1$. *Choose a constant* $\mu > 0$ *such that*

$$\zeta < \mu, \quad (b-a) < \gamma\mu, \quad 0 < \gamma < 1,$$

and take $\alpha = \beta = 0$.

Taking the maximum over $t \in [a,b]$, *we conclude that*

$$d(Tx,Ty,z) \le (d(x,y,z))^{\frac{\zeta}{\mu}(b-a)} \le (\theta(d(x,y,z)))^{\gamma},$$

where $\theta(t) = t^{\frac{\zeta}{\mu}}$ *and* $\zeta < \mu$.

Thus, T *is a* θ*–multiplicative contraction on* (X,d).

Since (X,d) *is complete, the Banach-type fixed point theorem in multiplicative metric spaces guarantees that* T *admits a unique fixed point* $x^* \in X$. *Consequently, the Fredholm multiplicative integral equation has a unique positive continuous solution on* $[a,b]$.

Remark 8. *If* $\zeta \ge \mu$, *the inequality does not force a contraction. Thus, the multiplicative metric with control function* $\theta(t) = t^{\frac{\zeta}{\mu}}$ *fails to produce a contraction. We choose* θ *that compresses values, for instance pick* μ, *so that* $\frac{\zeta}{\mu} < 1$.

Example 12. *Consider the multidimensional nonlinear Hammerstein-Fredholm integral system*

$$(Tu)_i(t) = a_i(t)\exp\left(\sum_{j=1}^{n}\int_a^b K_{ij}(t,s)\,F_j(s,u_j(s))\,ds\right), \quad i = 1,\dots,n,$$

where $u(t) = (u_1(t),\dots,u_n(t)) \in C([a,b],\mathbb{R}_+^n)$. *We assume that*

(i) $a_i(t) \ge 1$ *are continuous on* $[a,b]$,
(ii) $K_{ij}(t,s) \ge 0$ *are continuous kernels,*

(iii) $F_j : [a, b] \times \mathbb{R}_+ \to \mathbb{R}_+$ *are continuous and satisfy the Lipschitz condition*

$$|F_j(s, x) - F_j(s, y)| \le L_j|x - y|, \quad L_j > 0.$$

Let

$$X = C([a, b], \mathbb{R}^n_+), \quad \|u_i\|_\infty = \sup_{t\in[a,b]} |u_i(t)|.$$

Define the multiplicative metric $d : X \times X \to P \subset D_n(\mathbb{R})$ *by*

$$d(u, v) = \operatorname{diag}\left(e^{\|u_1-v_1\|_\infty}, \ldots, e^{\|u_n-v_n\|_\infty}\right),$$

where

$$P = \{\operatorname{diag}(z_1, \ldots, z_n) : z_i \ge 1\}.$$

The order is defined componentwise:

$$\operatorname{diag}(x_1, \ldots, x_n) \le \operatorname{diag}(y_1, \ldots, y_n) \iff x_i \le y_i \ \forall i.$$

Let $T : X \to X$ *be defined by*

$$(Tu)_i(t) = a_i(t) \exp\left(\sum_{j=1}^{n} \int_a^b K_{ij}(t, s)\, F_j(s, u_j(s))\, ds\right).$$

and define

$$M_{ij} = L_j \sup_{t\in[a,b]} \int_a^b K_{ij}(t, s) ds,$$

and $A_i(u) = \sum_{j=1}^{n} \int_a^b K_{ij}(t, s)\, F_j(s, u_j(s))\, ds$. *Thus, it follows that*

$$\|A_i(u) - A_i(v)\|_\infty \le \sum_{j=1}^{n} M_{ij}\|u_j - v_j\|_\infty.$$

Let

$$q = \max_i \sum_{j=1}^{n} M_{ij}.$$

Then

$$\|A_i(u) - A_i(v)\|_\infty \le q \max_j \|u_j - v_j\|_\infty.$$

Using exponential Lipschitz behavior on bounded sets, there exists $\xi > 0$*, such that*

$$\|(Tu)_i - (Tv)_i\|_\infty \le \xi\|A_i(u) - A_i(v)\|_\infty.$$

Hence,

$$\|(Tu)_i - (Tv)_i\|_\infty \le \xi q \max_j \|u_j - v_j\|_\infty.$$

$$d_i(Tu, Tv) = \exp(\|(Tu)_i - (Tv)_i\|_\infty).$$

Thus,

$$d_i(Tu, Tv) \leq \left(\max_j e^{\|u_j - v_j\|_\infty} \right)^{\xi q} = \left(\max_j d_j(u, v) \right)^{\xi q}.$$

If $\xi q < 1$, choose $\mu > 1$, such that $\xi q < 1/\mu$. Then

$$d(Tu, Tv) \leq d(u, v)^{1/\mu}.$$

Define

$$\theta_3(Z) = Z^{1/\mu}.$$

Thus,

$$d(Tu, Tv) \leq \theta_3(d(u, v)).$$

By the cone θ-type multiplicative contraction theorem, T admits a unique fixed point $u^* \in X$, which is the unique positive solution of the coupled nonlinear integral system.

4. Conclusions

In this work, we proposed and thoroughly analyzed a new class of metric type structures, namely cone θ-type multiplicative metric spaces, formulated in the setting of ordered commutative Banach algebras. This framework blends multiplicative distance notions with θ-control functions and cone induced partial orders, providing a unifying extension of several existing generalized metric spaces. By enabling the distance function to take values in a solid multiplicative cone, the theory captures richer algebraic, order, and topological features than those available in classical scalar-valued settings. We established the notion of solid multiplicative cones and examined their basic properties, highlighting their role in ensuring consistency between the multiplicative order, algebraic operations, and convergence concepts. Building on this foundation, we defined cone θ-type multiplicative metrics in two-variable and three-variable forms, thereby extending the multiplicative triangle inequality through flexible θ-type conditions. These generalized metrics enable the study of nonlinear contractive mappings beyond the reach of standard metric techniques. Within this setting, we proved several fixed point theorems for mappings satisfying θ-type multiplicative contractive conditions, generalizing well-known Banach, Kannan, and Chatterjea principles to cone-valued multiplicative contexts. The proofs employ iterative schemes adapted to the multiplicative order structure and convergence relative to the identity element of the Banach algebra. Illustrative examples and an application to a Fredholm-type integral equation demonstrate the effectiveness and nontrivial nature of the proposed framework. Overall, the results show that cone θ-type multiplicative metric spaces offer a genuinely new analytical approach with strong potential for further theoretical development and applications.

Author contributions

P. Singh: Conceptualization, methodology, writing-original draft ; S. Singh: Investigation, writing-review and editing; S. Salisu, V. Singh: Investigation, editing the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that no Artificial Intelligence (AI) tools were used in the conception, writing, analysis, editing, or preparation of this manuscript.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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