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Research article

Lyapunov stability and solvability of nonlocal fractional differential equations with generalized katugampola derivative

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Abstract: This paper investigates a class of fractional differential equations (FDEs) that involve the generalized Katugampola fractional derivative (FD) subject to nonlocal boundary conditions. By transforming the considered boundary value problems (BVPs) into equivalent integral equations, we establish several results concerning the existence and uniqueness of solutions. The analysis is carried out using classical fixed point (FP) techniques, including the Banach contraction principle(BC), as well as Schaefer's FP theorems under appropriate assumptions. In addition, we examine the Lyapunov stability of nontrivial solutions and derive sufficient conditions to ensure asymptotic stability. The obtained results extend and complement the existing contributions in the literature on fractional BVPs with nonlocal conditions. Finally, illustrative examples are provided to demonstrate the applicability of the theoretical findings.

Keywords: generalized Katugampola fractional derivative; fractional differential equations; nonlocal boundary value problem; fixed point theorems; existence and uniqueness; Lyapunov stability
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1. Introduction

Fractional calculus has become one of the most active areas of modern mathematical analysis due to its ability to describe systems which possess memory, hereditary effects, and nonlocal interactions. Unlike classical integer-order differential equations, fractional differential equations (FDEs) incorporate information from the entire history of a process, thus allowing a more realistic representation of many natural and engineered systems. Consequently, fractional models have been successfully applied in numerous fields, including viscoelasticity, anomalous diffusion, heat transfer, porous media, fluid mechanics, biological systems, epidemiology, neural networks, control theory, signal processing, and finance [1–3].

In many practical situations, the evolution of a system depends not only on its current state but also on previous states through memory kernels that exhibit power-law or logarithmic behavior. Such phenomena are frequently encountered in heterogeneous materials, transport processes in porous structures, biological tissues, and complex dynamical networks. Classical integer-order models often fail to adequately capture these effects, which has motivated the development of various fractional operators, including the Riemann–Liouville, Caputo, Hadamard, Hilfer, and Caputo–Hadamard derivatives [4–6]. These operators have significantly expanded the modeling capabilities of differential equations and have led to substantial advances in both theory and applications.

During the last decade, considerable attention has been devoted to the qualitative analysis of fractional (BVP) that involve nonlocal conditions [7–9]. Such conditions naturally arise when measurements, observations, or control actions are distributed over an interval rather than prescribed at a single point. Examples include population dynamics with distributed feedback, thermal processes with integral constraints, viscoelastic materials with memory-dependent boundary responses, and diffusion phenomena in heterogeneous media. Consequently, the mathematical study of nonlocal fractional BVPs has become an important research direction due to its theoretical significance and practical relevance [10–12].

Although the Caputo fractional derivative (FD) remains one of the most frequently used operators in applications, it is not always sufficient to describe complex memory mechanisms encountered in real systems. In particular, several physical processes exhibit scaling properties that cannot solely be represented by the Caputo kernel. To overcome this limitation, generalized fractional operators have been introduced to unify different classical derivatives within a single framework. Among them, the generalized Katugampola FD introduced by Katugampola [13] has attracted significant interest because it continuously interpolates between the Riemann–Liouville and Hadamard FDs through an additional parameter. Therefore, the generalized Katugampola derivative provides a broader mathematical framework capable of simultaneously capturing different memory structures and scaling laws.

The importance of the generalized Katugampola operator is not merely theoretical. Recent investigations have demonstrated its applicability in epidemiological models, anomalous diffusion equations, viscoelastic systems, nonlinear evolution problems, generalized transport phenomena, and fractional control systems [14–16]. For instance, in anomalous diffusion through porous media, transport processes often exhibit a combination of power-law memory and logarithmic scaling effects. The generalized Katugampola derivative provides sufficient flexibility to model such phenomena while simultaneously recovering several classical fractional operators as particular cases.

Consequently, the results established for generalized Katugampola problems automatically generate the corresponding results for Hadamard and Riemann–Liouville type models.

Another fundamental issue in the study of fractional systems concerns the qualitative behavior of solutions. In many applications, establishing existence and uniqueness is only the first step; it is equally important to determine whether solutions remain stable under perturbations and whether the system eventually approaches an equilibrium state. Stability analyses play a central role in the dynamical systems theory, engineering design, feedback control, neural-network learning algorithms, biological regulation mechanisms, and chaotic systems [17–19]. In particular, the Lyapunov stability theory provides a powerful framework to investigate the robustness and asymptotic behavior without requiring explicit solution formulas.

Recent studies have shown that Lyapunov-type techniques are particularly effective in the analysis of fractional-order dynamical systems, including chaotic systems, neural networks, uncertain control systems, iterative learning control schemes, and fault estimation models [20–22]. The presence of memory effects introduces additional mathematical challenges because the classical differential chain rule is generally not available for fractional operators. Consequently, a stability analysis for fractional systems often requires the development of new integral inequalities and comparison principles adapted to the underlying fractional framework.

The present work is additionally motivated by applications that involve anomalous diffusion and memory-dependent transport processes. In heterogeneous materials, porous structures, viscoelastic media, and biological tissues, transport phenomena frequently deviate from classical Fickian diffusion laws. Experimental observations indicate that the current state depends on the entire history of the process, which naturally leads to fractional models. Moreover, practical systems are often subject to nonlocal constraints that arise from distributed measurements, integral observations, or feedback mechanisms. Such considerations motivate the study of generalized fractional BVPs with nonlocal conditions [23, 24] and justify the incorporation of a Lyapunov stability analysis into the mathematical framework.

Despite the extensive literature devoted to FDEs, the simultaneous investigation of generalized Katugampola derivatives, nonlocal boundary conditions, and Lyapunov stability properties remains relatively limited. Most available contributions primarily focus on the existence and uniqueness results for classical fractional operators, whereas fewer studies address generalized operators capable of unifying several important fractional models. Furthermore, the interaction between generalized fractional dynamics and stability properties under nonlocal constraints has not yet been fully explored.

Motivated by the above observations, the present paper investigates a class of generalized fractional BVPs that involve the generalized Katugampola FD and nonlocal boundary conditions. The considered framework can be viewed as a mathematical model for memory-dependent dynamical systems that arise in anomalous diffusion, viscoelasticity, transport phenomena, biological systems, and the fractional control theory. Our objective is to establish a unified analytical framework that combines a solvability analysis with Lyapunov stability investigations.

As a starting point, Benchohra investigated the fractional BVP [25]

$$^{c}D^{\mu}\aleph(\iota) = f(\iota, \aleph(\iota)), \qquad \iota \in J := [0, \ell], \quad 0 < \mu < 1,$$

subject to the nonlocal condition

$$a\aleph(0) + b\aleph(\ell) = c,$$

where ${}^cD^\mu$ denotes the Caputo FD, and $a, b, c \in \mathbb{R}$ with $a + b \neq 0$. Later, JinRong et al. extended these investigations to infinite-dimensional Banach spaces and obtained several solvability results for fractional evolution equations [26].

Inspired by these contributions, we consider the following generalized fractional nonlocal problems:

$$\begin{cases} {}^\nu D^\mu \aleph(\imath) = f(\imath, \aleph(\imath)), & \imath \in J := [0, \ell], \quad 0 < \mu < 1, \ \nu > 0, \\ a\aleph(0) + b\aleph(\ell) = c, & a + b \neq 0, \end{cases} \tag{1.1}$$

and

$$\begin{cases} {}^\nu D^\mu \aleph(\imath) = f(\imath, \aleph(\imath)), \quad \imath \in J := [0, \ell], \quad 0 < \mu < 1, \\ \aleph(0) = g(\aleph), \end{cases} \tag{1.2}$$

where ${}^\nu D^\mu$ denotes the generalized Katugampola FD.

From a modeling perspective, Problem (1.1) may represent a generalized anomalous diffusion or viscoelastic relaxation process with memory effects, while the nonlocal boundary conditions describe distributed measurements, feedback mechanisms, or observational constraints. The parameter ν allows continuous transitions between different memory structures and consequently provides greater flexibility than classical fractional models.

The main contributions of this paper are summarized as follows:

- We establish new existence and uniqueness results for generalized fractional BVPs that involve the generalized Katugampola derivative.
- We derive equivalent integral formulations and employ Banach's contraction principle together with Schaefer's fixed point theorem to obtain solvability results.
- We investigate the Lyapunov stability and asymptotic stability properties of solutions within the framework of generalized fractional dynamical systems.
- The obtained results extend several existing studies devoted to classical fractional operators by incorporating generalized Katugampola dynamics and nonlocal constraints.
- The developed theory provides analytical tools applicable to models that arise in anomalous diffusion, viscoelasticity, memory-dependent transport phenomena, biological systems, and the fractional control theory.

The remainder of the paper is organized as follows: Section 2 presents the necessary preliminaries concerning generalized Katugampola fractional operators and the fixed point theory; Section 3 contains the main existence, uniqueness, and Lyapunov stability results; Section 4 provides illustrative applications and examples which demonstrate the applicability of the theoretical findings; and finally, concluding remarks and directions for future research are presented in the last section.

2. Preliminaries

In this section, we present the notation, definitions, and auxiliary results that will be used throughout the paper. For the convenience of the reader and to improve the self-contained character of the manuscript, we also recall some fundamental concepts from fractional calculus and the fixed point theory related to the generalized Katugampola fractional operators. The notation and terminology used in the sequel are standardized throughout the paper.

Let $J := [0, \ell]$, $\ell > 0$, and denote the Banach space of all continuous functions $\aleph : J \to \mathbb{R}$, by $C(J, \mathbb{R})$ equipped with the supremum norm

$$\|\aleph\|_\infty = \sup_{\imath \in J} |\aleph(\imath)|.$$

Throughout the paper, $\mathbb{R}$ denotes the set of all real numbers, $\Gamma(\cdot)$ denotes the classical Gamma function, and $\Re(\mu)$ represents the real part of the complex number μ.

Now, we recall some basic definitions and properties of the generalized Katugampola fractional operators which can be found in [13, 27].

Definition 2.1 (Generalized Katugampola fractional integral)**.** *[13, 27] Let $\mu \in \mathbb{C}$ with $\Re(\mu) > 0$, and let $\nu > 0$. The generalized left-sided Katugampola fractional integral of order μ of a function $h : (0, \infty) \to \mathbb{R}$ is defined by*

$$^{\nu}I^{\mu}h(x) = \frac{\nu^{1-\mu}}{\Gamma(\mu)} \int_0^x (x^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} h(\jmath)\, d\jmath, \qquad x > 0,$$

provided that the integral exists.

Remark 2.2. *The generalized Katugampola fractional integral unifies several classical fractional integrals. In particular, for suitable choices of the parameter ν, the operator reduces to the Riemann–Liouville and Hadamard fractional integrals. This flexibility makes the operator suitable to model a broad class of dynamical systems with memory effects.*

Definition 2.3 (Generalized Katugampola fractional derivative)**.** *[13,27] Let $\mu \in \mathbb{C}$ with $\Re(\mu) > 0$, $\nu > 0$, and let $n = \lceil \Re(\mu) \rceil$, where $\lceil \cdot \rceil$ denotes the ceiling function. The generalized left-sided Katugampola FD of order μ is defined by the following:*

$$^{\nu}D^{\mu}h(x) = \left(x^{1-\nu} \frac{d}{dx} \right)^n ({}^{\nu}I^{n-\mu}h)(x), \qquad x > 0.$$

Equivalently, it can be written in the integral form

$$^{\nu}D^{\mu}h(x) = \frac{\nu^{\mu-n+1}}{\Gamma(n-\mu)} \left(x^{1-\nu} \frac{d}{dx} \right)^n \int_0^x (x^\nu - \jmath^\nu)^{n-\mu-1} \jmath^{\nu-1} h(\jmath)\, d\jmath,$$

provided that the integral exists.

Remark 2.4. *The generalized Katugampola FD provides a unified framework that generalizes several well-known FDs. Consequently, it has become an important tool in the study of generalized fractional dynamical systems, anomalous diffusion models, viscoelasticity, and nonlocal evolution equations.*

The following lemmas play an important role in transforming the considered FDEs into equivalent integral equations.

Lemma 2.5. [27] *Let $\mu > 0$ and $\nu > 0$. Then, the FDE*

$$^{\nu}D^{\mu}h(x) = 0$$

has the general solution

$$h(x) = a_0 + a_1\left(\frac{x^{\nu}}{\nu}\right) + \cdots + a_{n-1}\left(\frac{x^{\nu}}{\nu}\right)^{n-1},$$

where $a_i \in \mathbb{R}$, $\quad i = 0, 1, \ldots, n-1$, and $n = [\mu] + 1$.

Lemma 2.6. [27] *Let $\mu > 0$ and $\nu > 0$. Then,*

$$^{\nu}I^{\mu}\left(^{\nu}D^{\mu}h(x)\right) = h(x) + a_0 + a_1\left(\frac{x^{\nu}}{\nu}\right) + \cdots + a_{n-1}\left(\frac{x^{\nu}}{\nu}\right)^{n-1},$$

for some constants $a_i \in \mathbb{R}$, $\quad i = 0, 1, \ldots, n-1$, where $n = [\mu] + 1$.

The following fixed point result will be used in the proof of the uniqueness theorem.

Theorem 2.7 (Banach contraction principle)**.** *[28] Let (X, d) be a complete metric space, and let $T : X \to X$ be a contraction mapping, that is, there exists a constant $0 < k < 1$ such that*

$$d(Tx, Ty) \leq k\, d(x, y), \qquad \forall x, y \in X.$$

Then, T admits a unique fixed point in X.

Additionally, we recall Schaefer's fixed point theorem, which will be used in the existence analysis.

Theorem 2.8 (Schaefer's fixed point theorem)**.** *[28] Let X be a Banach space, and let $T : X \to X$ be a completely continuous operator.*

Assume that the set

$$\Omega = \{x \in X : x = \lambda T(x), \quad 0 < \lambda < 1\}$$

is bounded.

Then, the operator T possesses at least one fixed point in X; that is, there exists $x^* \in X$ such that $Tx^* = x^*$.

Finally, we recall the notion of Lyapunov stability adapted to the considered framework.

Definition 2.9 (Lyapunov stability)**.** *[29] A solution $\aleph(\imath)$ of the considered fractional problem is said to be stable in the sense of Lyapunov if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every other solution $z(\imath)$ that satisfies*

$$|z(0) - \aleph(0)| < \delta,$$

we have

$$|z(\imath) - \aleph(\imath)| < \varepsilon, \qquad \forall \imath \in J.$$

The solution is said to be asymptotically stable if it is Lyapunov stable and it satisfies

$$\lim_{\imath \to \infty} |z(\imath) - \aleph(\imath)| = 0.$$

The above preliminary results and definitions provide the analytical foundation necessary for the existence, uniqueness, and stability analysis developed in the subsequent sections.

3. Main results

In this section, we investigate the existence and uniqueness of solutions for the generalized fractional BVP introduced in Section 1. The analysis is based on transforming the considered FDE into an equivalent integral equation and then applying suitable fixed point techniques.

The following lemma establishes the equivalence between Problem (1.1) and a nonlinear fractional integral equation. This formulation plays a crucial role in the application of the fixed point theory.

Lemma 3.1. *A function* $\aleph \in C(J, \mathbb{R})$ *is a solution of the fractional BVP* (1.1) *if and only if it satisfies the integral equation*

$$\begin{aligned}\aleph(x) =& \frac{1}{a+b}\left[c - \frac{b\nu^{1-\mu}}{\Gamma(\mu)} \int_0^{\ell} (\ell^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} f(\jmath, \aleph(\jmath))\, d\jmath\right] \\ &+ \frac{\nu^{1-\mu}}{\Gamma(\mu)} \int_0^x (x^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} f(\jmath, \aleph(\jmath))\, d\jmath, \end{aligned} \tag{3.1}$$

for all $x \in J$.

Proof. First, suppose that $\aleph \in C(J, \mathbb{R})$ is a solution of Problem (1.1). By applying the generalized Katugampola fractional integral operator ${}^\nu I^\mu$ to both sides of the differential equation

$${}^\nu D^\mu \aleph(x) = f(x, \aleph(x)),$$

and using Lemma 2.6, we obtain

$$\aleph(x) = a_0 + \frac{\nu^{1-\mu}}{\Gamma(\mu)} \int_0^x (x^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} f(\jmath, \aleph(\jmath))\, d\jmath, \tag{3.2}$$

where $a_0 \in \mathbb{R}$ is an arbitrary constant.

By evaluating (3.2) at $x = 0$, we obtain the following:

$$\aleph(0) = a_0. \tag{3.3}$$

Similarly, by taking $x = \ell$, we get the following:

$$\aleph(\ell) = a_0 + \frac{\nu^{1-\mu}}{\Gamma(\mu)} \int_0^{\ell} (\ell^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} f(\jmath, \aleph(\jmath))\, d\jmath. \tag{3.4}$$

Using the nonlocal boundary condition $a\aleph(0) + b\aleph(\ell) = c$, together with (3.3) and (3.4), we obtain the following:

$$\begin{aligned} c =& a a_0 + b\left[a_0 + \frac{\nu^{1-\mu}}{\Gamma(\mu)} \int_0^{\ell} (\ell^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} f(\jmath, \aleph(\jmath))\, d\jmath\right] \\ =& (a+b)a_0 + \frac{b\nu^{1-\mu}}{\Gamma(\mu)} \int_0^{\ell} (\ell^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} f(\jmath, \aleph(\jmath))\, d\jmath. \end{aligned} \tag{3.5}$$

Since $a + b \neq 0$, it follows from (3.5) that

$$a_0 = \frac{1}{a+b}\left[c - \frac{b\nu^{1-\mu}}{\Gamma(\mu)}\int_0^{\ell}(\ell^{\nu} - \jmath^{\nu})^{\mu-1}\jmath^{\nu-1}f(\jmath, \aleph(\jmath))\,d\jmath\right]. \tag{3.6}$$

By substituting (3.6) into (3.2), we obtain the following:

$$\aleph(x) = \frac{1}{a+b}\left[c - \frac{b\nu^{1-\mu}}{\Gamma(\mu)}\int_0^{\ell}(\ell^{\nu} - \jmath^{\nu})^{\mu-1}\jmath^{\nu-1}f(\jmath, \aleph(\jmath))\,d\jmath\right] + \frac{\nu^{1-\mu}}{\Gamma(\mu)}\int_0^{x}(x^{\nu} - \jmath^{\nu})^{\mu-1}\jmath^{\nu-1}f(\jmath, \aleph(\jmath))\,d\jmath,$$

which proves (3.1).

Conversely, assume that $\aleph \in C(J, \mathbb{R})$ satisfies the integral equation (3.1). By applying the generalized Katugampola FD operator ${}^{\nu}D^{\mu}$ to both sides of (3.1) and using the properties of the generalized fractional operators, we directly recover the following:

$${}^{\nu}D^{\mu}\aleph(x) = f(x, \aleph(x)).$$

Moreover, by evaluating (3.1) at $x = 0$ and $x = \ell$, it immediately follows that

$$a\aleph(0) + b\aleph(\ell) = c.$$

Hence, $\aleph$ is a solution of the fractional BVP (1.1). This completes the proof. □

Theorem 3.2. *Assume that the following hypotheses are satisfied:*

(H1) *The function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous.*
(H2) *There exists a constant $k > 0$ such that*

$$|f(\imath, x) - f(\imath, y)| \leq k|x - y|, \qquad \forall\, \imath \in J, \quad \forall\, x, y \in \mathbb{R}. \tag{3.7}$$

Define the constant

$$\gamma = \frac{k\ell^{\nu\mu}}{\nu^{\mu}\Gamma(\mu+1)}\left(1 + \frac{|b|}{|a+b|}\right). \tag{3.8}$$

If $\gamma < 1$, then the fractional BVP (1.1) admits a unique solution on J.

Proof. In view of Lemma 3.1, Problem (1.1) is equivalent to the nonlinear fractional integral equation (3.1). Consequently, we define the operator $\Psi : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ by

$$(\Psi\aleph)(x) = \frac{1}{a+b}\left[c - \frac{b\nu^{1-\mu}}{\Gamma(\mu)}\int_0^{\ell}(\ell^{\nu} - \jmath^{\nu})^{\mu-1}\jmath^{\nu-1}f(\jmath, \aleph(\jmath))\,d\jmath\right] + \frac{\nu^{1-\mu}}{\Gamma(\mu)}\int_0^{x}(x^{\nu} - \jmath^{\nu})^{\mu-1}\jmath^{\nu-1}f(\jmath, \aleph(\jmath))\,d\jmath, \tag{3.9}$$

for all $x \in J$.

We shall prove that the operator Ψ is a contraction on the Banach space $(C(J, \mathbb{R}), \|\cdot\|_{\infty})$.

Let $u, v \in C(J, \mathbb{R})$, and let $x \in J$. Using (3.9), we obtain the following:

$$
\begin{aligned}
|(\Psi u)(x) - (\Psi v)(x)| \leq & \frac{\nu^{1-\mu}}{\Gamma(\mu)} \int_0^x (x^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} |f(\jmath, u(\jmath)) - f(\jmath, v(\jmath))| \, d\jmath \\
& + \frac{|b|\nu^{1-\mu}}{|a+b|\Gamma(\mu)} \int_0^\ell (\ell^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} |f(\jmath, u(\jmath)) - f(\jmath, v(\jmath))| \, d\jmath.
\end{aligned}
$$

By applying the Lipschitz condition (3.7), we obtain the following:

$$
\begin{aligned}
|(\Psi u)(x) - (\Psi v)(x)| \leq & \frac{k\nu^{1-\mu}}{\Gamma(\mu)} \int_0^x (x^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} |u(\jmath) - v(\jmath)| \, d\jmath \\
& + \frac{k|b|\nu^{1-\mu}}{|a+b|\Gamma(\mu)} \int_0^\ell (\ell^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} |u(\jmath) - v(\jmath)| \, d\jmath.
\end{aligned}
$$

Since

$$
|u(\jmath) - v(\jmath)| \leq \|u - v\|_\infty, \qquad \forall\, \jmath \in J,
$$

it follows that

$$
\begin{aligned}
|(\Psi u)(x) - (\Psi v)(x)| \leq & \frac{k\nu^{1-\mu}\|u - v\|_\infty}{\Gamma(\mu)} \bigg[\int_0^x (x^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} \, d\jmath \\
& + \frac{|b|}{|a+b|} \int_0^\ell (\ell^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} \, d\jmath \bigg].
\end{aligned} \tag{3.10}
$$

Since $x \leq \ell$, we have the following:

$$
\int_0^x (x^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} \, d\jmath \leq \int_0^\ell (\ell^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} \, d\jmath. \tag{3.11}
$$

By substituting (3.11) into (3.10), we obtain the following:

$$
\begin{aligned}
|(\Psi u)(x) - (\Psi v)(x)| \leq & \frac{k\nu^{1-\mu}\|u - v\|_\infty}{\Gamma(\mu)} \left(1 + \frac{|b|}{|a+b|}\right) \\
& \times \int_0^\ell (\ell^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} \, d\jmath.
\end{aligned} \tag{3.12}
$$

Next, using the substitution $s = \frac{\jmath^\nu}{\ell^\nu}$, we compute the following;

$$
\int_0^\ell (\ell^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} \, d\jmath = \frac{\ell^{\nu\mu}}{\nu\mu}. \tag{3.13}
$$

Consequently, from (3.12) and (3.13), we derive the following:

$$
|(\Psi u)(x) - (\Psi v)(x)| \leq \frac{k\ell^{\nu\mu}}{\nu^\mu \Gamma(\mu+1)} \left(1 + \frac{|b|}{|a+b|}\right) \|u - v\|_\infty. \tag{3.14}
$$

By the definition of γ in (3.8), Inequality (3.14) becomes the following:

$$|(\Psi u)(x) - (\Psi v)(x)| \leq \gamma \|u - v\|_\infty.$$

By taking the supremum over $x \in J$, we obtain the following:

$$\|\Psi u - \Psi v\|_\infty \leq \gamma \|u - v\|_\infty.$$

Since $0 < \gamma < 1$, the operator Ψ is a contraction on the complete metric space $(C(J, \mathbb{R}), \| \cdot \|_\infty)$.

Therefore, the Banach contraction principle (Theorem 2.7) guarantees the existence of a unique fixed point $\aleph \in C(J, \mathbb{R})$ such that $\Psi\aleph = \aleph$.

By Lemma 3.1, this fixed point is precisely the unique solution of the fractional BVP (1.1). The proof is complete. □

Next, we establish an existence result based on Schaefer's fixed point theorem.

Theorem 3.3. *Assume that the following hypotheses hold:*

(H1) *The function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous.*

(H2) *There exist a continuous function $p : J \to [0, \infty)$ and a nondecreasing continuous function $\Phi : [0, \infty) \to [0, \infty)$ such that*

$$|f(x, u)| \leq p(x)\Phi(|u|), \qquad \forall (x, u) \in J \times \mathbb{R}.$$

(H3) *There exists a constant $R > 0$ such that*

$$R > \frac{|c|}{|a + b|} + \frac{\ell^{\nu\mu}\|p\|_\infty}{\nu^\mu \Gamma(\mu + 1)} \left(1 + \frac{|b|}{|a + b|}\right) \Phi(R).$$

Then, the fractional BVP (1.1) *admits at least one solution on $J = [0, \ell]$.*

Proof. We shall apply Schaefer's fixed point theorem.

Define the operator

$$\Psi : C(J, \mathbb{R}) \to C(J, \mathbb{R})$$

by

$$(\Psi\aleph)(x) = \frac{1}{a + b}\left[c - \frac{b\nu^{1-\mu}}{\Gamma(\mu)} \int_0^\ell (\ell^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} f(\jmath, \aleph(\jmath))\, d\jmath\right]$$
$$+ \frac{\nu^{1-\mu}}{\Gamma(\mu)} \int_0^x (x^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} f(\jmath, \aleph(\jmath))\, d\jmath,$$

for all $x \in J$.

By Lemma 3.1, fixed points of Ψ are exactly the solutions of problem (1.1).

The proof will be divided into four steps.

Step 1: Continuity of the operator Ψ.

Let $(\aleph_n)_{n\geq 1} \subset C(J, \mathbb{R})$ such that

$$\aleph_n \to \aleph \qquad \text{in } C(J, \mathbb{R}).$$

Since f is continuous, we have

$$f(x, \aleph_n(x)) \to f(x, \aleph(x)), \qquad \forall x \in J.$$

Moreover, because $\aleph_n \to \aleph$ uniformly and f is continuous on the compact set $J \times K$ for some bounded interval $K \subset \mathbb{R}$, it follows that

$$f(x, \aleph_n(x)) \to f(x, \aleph(x))$$

uniformly on J.

Hence, for every $x \in J$,

$$\begin{aligned}|(\Psi\aleph_n)(x) - (\Psi\aleph)(x)| \leq & \frac{\nu^{1-\mu}}{\Gamma(\mu)} \int_0^x (x^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} |f(\jmath, \aleph_n(\jmath)) - f(\jmath, \aleph(\jmath))|\, d\jmath \\ & + \frac{|b|\nu^{1-\mu}}{|a+b|\Gamma(\mu)} \int_0^\ell (\ell^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} |f(\jmath, \aleph_n(\jmath)) - f(\jmath, \aleph(\jmath))|\, d\jmath.\end{aligned}$$

Using the uniform convergence of $f(\cdot, \aleph_n(\cdot))$, we obtain

$$\|\Psi\aleph_n - \Psi\aleph\|_\infty \to 0, \qquad n \to \infty.$$

Therefore, Ψ is continuous.

Step 2: Ψ maps bounded sets into bounded sets.

Let

$$B_r = \{\aleph \in C(J, \mathbb{R}) : \|\aleph\|_\infty \leq r\},$$

where $r > 0$.

For $\aleph \in B_r$ and $x \in J$, we have the following:

$$\begin{aligned}|(\Psi\aleph)(x)| \leq & \frac{|c|}{|a+b|} + \frac{|b|\nu^{1-\mu}}{|a+b|\Gamma(\mu)} \int_0^\ell (\ell^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} |f(\jmath, \aleph(\jmath))|\, d\jmath \\ & + \frac{\nu^{1-\mu}}{\Gamma(\mu)} \int_0^x (x^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} |f(\jmath, \aleph(\jmath))|\, d\jmath.\end{aligned}$$

Using assumption **(H2)**, we obtain the following:

$$|f(\jmath, \aleph(\jmath))| \leq p(\jmath)\Phi(|\aleph(\jmath)|) \leq \|p\|_\infty \Phi(r).$$

Therefore,

$$\begin{aligned}|(\Psi\aleph)(x)| \leq & \frac{|c|}{|a+b|} + \frac{\|p\|_\infty \Phi(r)\nu^{1-\mu}}{\Gamma(\mu)} \left(1 + \frac{|b|}{|a+b|}\right) \\ & \times \int_0^\ell (\ell^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1}\, d\jmath.\end{aligned}$$

Since

$$\int_0^\ell (\ell^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1}\, d\jmath = \frac{\ell^{\nu\mu}}{\nu\mu},$$

we obtain

$$\|\Psi\aleph\|_\infty \leq \frac{|c|}{|a+b|} + \frac{\ell^{\nu\mu}\|p\|_\infty}{\nu^\mu\Gamma(\mu+1)}\left(1+\frac{|b|}{|a+b|}\right)\Phi(r). \tag{3.15}$$

Hence, $\Psi(B_r)$ is bounded.

Step 3: Ψ maps bounded sets into equicontinuous sets.

Let $\aleph \in B_r$ and $0 \leq x_1 < x_2 \leq \ell$.

Then,

$$|(\Psi\aleph)(x_2) - (\Psi\aleph)(x_1)| \leq \frac{\nu^{1-\mu}}{\Gamma(\mu)}\left| \int_0^{x_2} (x_2^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} f(\jmath, \aleph(\jmath))\, d\jmath - \int_0^{x_1} (x_1^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} f(\jmath, \aleph(\jmath))\, d\jmath \right|.$$

Using the boundedness of $f(\jmath, \aleph(\jmath))$, we derive the following:

$$|(\Psi\aleph)(x_2) - (\Psi\aleph)(x_1)| \leq \frac{\|p\|_\infty \Phi(r)\nu^{1-\mu}}{\Gamma(\mu)}\left[\int_0^{x_1} \left|(x_2^\nu - \jmath^\nu)^{\mu-1} - (x_1^\nu - \jmath^\nu)^{\mu-1}\right| \jmath^{\nu-1}\, d\jmath + \int_{x_1}^{x_2} (x_2^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1}\, d\jmath \right].$$

The right-hand side tends to zero as $x_2 \to x_1$, uniformly with respect to $\aleph \in B_r$.

Hence, $\Psi(B_r)$ is equicontinuous.

Therefore, by the Arzelà–Ascoli theorem, the operator Ψ is completely continuous.

Step 4: A priori bounds.

Consider the set

$$\Omega = \{\aleph \in C(J,\mathbb{R}) : \aleph = \lambda\Psi(\aleph), \quad 0 < \lambda < 1\}.$$

Let $\aleph \in \Omega$. Then, there exists $\lambda \in (0,1)$ such that $\aleph = \lambda\Psi(\aleph)$.

Hence,

$$\|\aleph\|_\infty = \lambda\|\Psi(\aleph)\|_\infty \leq \|\Psi(\aleph)\|_\infty.$$

Using estimate (3.15) with $r = \|\aleph\|_\infty$, we obtain the following:

$$\|\aleph\|_\infty \leq \frac{|c|}{|a+b|} + \frac{\ell^{\nu\mu}\|p\|_\infty}{\nu^\mu\Gamma(\mu+1)}\left(1+\frac{|b|}{|a+b|}\right)\Phi(\|\aleph\|_\infty).$$

Assumption **(H3)** implies that $\|\aleph\|_\infty < R$.

Therefore, the set Ω is bounded.

Finally, Schaefer's fixed point theorem guarantees that Ψ has at least one fixed point in $C(J,\mathbb{R})$.

By Lemma 3.1, this fixed point is a solution of Problem (1.1).

The proof is complete. □

3.1. Nonlocal fractional boundary value problem

In this subsection, we investigate the generalized Katugampola fractional nonlocal Problem (1.2)

The following lemma establishes the equivalence between Problem (1.2) and its associated fractional integral equation.

Lemma 3.4. *Let* $0 < \mu < 1$. *A function* $\aleph \in C(J, \mathbb{R})$ *is a solution of the nonlocal fractional BVP* (1.2) *if and only if it satisfies the following integral equation:*

$$\aleph(x) = g(\aleph) + \frac{\nu^{1-\mu}}{\Gamma(\mu)} \int_0^x (x^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} f(\jmath, \aleph(\jmath))\, d\jmath, \qquad x \in J. \tag{3.16}$$

Proof. First, assume that $\aleph$ is a solution of Problem (1.2). By applying the generalized Katugampola fractional integral operator ${}^\nu I^\mu$ to both sides of the differential equation

$${}^\nu D^\mu \aleph(x) = f(x, \aleph(x)),$$

and using Lemma 2.6, we obtain

$$\aleph(x) = c_0 + \frac{\nu^{1-\mu}}{\Gamma(\mu)} \int_0^x (x^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} f(\jmath, \aleph(\jmath))\, d\jmath, \tag{3.17}$$

where $c_0 \in \mathbb{R}$ is a constant.

Next, setting $x = 0$ in (3.17), we get $\aleph(0) = c_0$.

Using the nonlocal condition $\aleph(0) = g(\aleph)$, it follows that $c_0 = g(\aleph)$.

By substituting this identity into (3.17), we derive the following:

$$\aleph(x) = g(\aleph) + \frac{\nu^{1-\mu}}{\Gamma(\mu)} \int_0^x (x^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} f(\jmath, \aleph(\jmath))\, d\jmath.$$

Therefore, (3.16) holds.

Conversely, suppose that $\aleph$ satisfies the integral equation (3.16). By applying the generalized Katugampola FD operator ${}^\nu D^\mu$ to both sides of (3.16) and using the properties of generalized Katugampola fractional operators, we obtain the following:

$${}^\nu D^\mu \aleph(x) = f(x, \aleph(x)).$$

Moreover, by evaluating (3.16) at $x = 0$, we immediately obtain $\aleph(0) = g(\aleph)$.

Hence, $\aleph$ is a solution of the nonlocal fractional BVP (1.2). The proof is complete. □

The following result is obtained by applying the Banach FP theorem.

Theorem 3.5. *Assume that the following hypotheses are satisfied:*

(H1) *There exists a constant* $k > 0$ *such that*

$$|f(\imath, \aleph_2) - f(\imath, \aleph_1)| \leq k|\aleph_2 - \aleph_1|, \qquad \forall\, \imath \in J, \quad \forall\, \aleph_1, \aleph_2 \in \mathbb{R}.$$

(H2) *The nonlocal functional* $g : C(J, \mathbb{R}) \to \mathbb{R}$ *is continuous and there exists a constant* $k^* > 0$ *such that*

$$|g(\aleph_2) - g(\aleph_1)| \leq k^* \|\aleph_2 - \aleph_1\|_\infty, \qquad \forall\, \aleph_1, \aleph_2 \in C(J, \mathbb{R}).$$

Define

$$\delta = k^* + \frac{k\ell^{\nu\mu}}{\nu^\mu\Gamma(\mu+1)}. \tag{3.18}$$

If $\delta < 1$, *then the nonlocal fractional BVP* (1.2) *admits a unique solution on the interval* $J = [0, \ell]$.

Proof. By Lemma 3.4, Problem (1.2) is equivalent to the following fractional integral equation:

$$\aleph(x) = g(\aleph) + \frac{\nu^{1-\mu}}{\Gamma(\mu)} \int_0^x (x^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} f(\jmath, \aleph(\jmath))\, d\jmath.$$

Accordingly, we define the operator $S : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ by the following:

$$(S\aleph)(x) = g(\aleph) + \frac{\nu^{1-\mu}}{\Gamma(\mu)} \int_0^x (x^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} f(\jmath, \aleph(\jmath))\, d\jmath, \qquad x \in J. \tag{3.19}$$

It is clear that fixed points of the operator S are precisely solutions of Problem (1.2).
We shall prove that the operator S is a contraction mapping on the Banach space $(C(J, \mathbb{R}), \|\cdot\|_\infty)$.
Let $\aleph_1, \aleph_2 \in C(J, \mathbb{R})$ and let $x \in J$.
Using (3.19), we obtain the following:

$$\begin{aligned} |(S\aleph_2)(x) - (S\aleph_1)(x)| \leq& |g(\aleph_2) - g(\aleph_1)| \\ &+ \frac{\nu^{1-\mu}}{\Gamma(\mu)} \int_0^x (x^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} |f(\jmath, \aleph_2(\jmath)) - f(\jmath, \aleph_1(\jmath))|\, d\jmath. \end{aligned}$$

By applying hypotheses **(H1)** and **(H2)**, we derive the following:

$$\begin{aligned} |(S\aleph_2)(x) - (S\aleph_1)(x)| \leq& k^*\|\aleph_2 - \aleph_1\|_\infty \\ &+ \frac{k\nu^{1-\mu}}{\Gamma(\mu)} \int_0^x (x^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1} |\aleph_2(\jmath) - \aleph_1(\jmath)|\, d\jmath. \end{aligned}$$

Since

$$|\aleph_2(\jmath) - \aleph_1(\jmath)| \leq \|\aleph_2 - \aleph_1\|_\infty, \qquad \forall\, \jmath \in J,$$

it follows that

$$\begin{aligned} |(S\aleph_2)(x) - (S\aleph_1)(x)| \leq& k^*\|\aleph_2 - \aleph_1\|_\infty \\ &+ \frac{k\nu^{1-\mu}}{\Gamma(\mu)} \|\aleph_2 - \aleph_1\|_\infty \int_0^x (x^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1}\, d\jmath. \end{aligned}$$

Next, using the identity

$$\int_0^x (x^\nu - \jmath^\nu)^{\mu-1} \jmath^{\nu-1}\, d\jmath = \frac{x^{\nu\mu}}{\nu\mu},$$

we obtain

$$|(S\aleph_2)(x) - (S\aleph_1)(x)| \leq \left[k^* + \frac{kx^{\nu\mu}}{\nu^\mu\Gamma(\mu+1)}\right] \|\aleph_2 - \aleph_1\|_\infty.$$

Since $x \leq \ell$, we deduce that

$$|(S\aleph_2)(x) - (S\aleph_1)(x)| \leq \left[k^* + \frac{k\ell^{\nu\mu}}{\nu^\mu\Gamma(\mu+1)}\right] \|\aleph_2 - \aleph_1\|_\infty. \tag{3.20}$$

By the definition of δ in (3.18), Inequality (3.20) becomes

$$|(S\aleph_2)(x) - (S\aleph_1)(x)| \leq \delta\|\aleph_2 - \aleph_1\|_\infty.$$

By taking the supremum over $x \in J$, we obtain the following:

$$\|S\aleph_2 - S\aleph_1\|_\infty \leq \delta\|\aleph_2 - \aleph_1\|_\infty.$$

Since $0 < \delta < 1$, the operator S is a contraction mapping on the complete metric space $(C(J,\mathbb{R}), \|\cdot\|_\infty)$.

Therefore, the Banach contraction principle guarantees the existence of a unique fixed point $\aleph \in C(J,\mathbb{R})$ such that $S\aleph = \aleph$.

Finally, by Lemma 3.4, this fixed point is exactly the unique solution of the nonlocal fractional BVP (1.2). The proof is complete. □

3.2. Lyapunov stability analysis

In this section, we investigate the Lyapunov stability of solutions for the generalized Katugampola fractional BVP. Unlike the classical integer-order case, fractional-order systems possess memory effects and nonlocal dynamics, which require a modified stability framework. Therefore, the classical differential chain rule cannot be directly applied to generalized Katugampola FDs. To avoid this difficulty, the analysis of (1.1) is based on suitable fractional differential inequalities together with integral estimates.

The following theorem provides sufficient conditions for Lyapunov stability.

Theorem 3.6. *Consider the fractional BVP* (1.1). *Assume that the following conditions hold:*

(A1) *The function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous.*

(A2) *There exists a constant $L > 0$ such that*

$$|f(\imath,u) - f(\imath,v)| \leq L|u - v|, \qquad \forall\, \imath \in J, \quad \forall\, u, v \in \mathbb{R}.$$

(A3) *There exists a continuous function $V : J \times \mathbb{R} \to \mathbb{R}_+$ and positive constants $\alpha_1, \alpha_2 > 0$ such that*

$$\alpha_1|u|^2 \leq V(\imath,u) \leq \alpha_2|u|^2, \qquad \forall\, (\imath,u) \in J \times \mathbb{R}. \tag{3.21}$$

(A4) *For every solution pair $z, \aleph^* \in C(J,\mathbb{R})$, the corresponding error function*

$$e(\imath) = z(\imath) - \aleph^*(\imath)$$

satisfies the fractional inequality

$${}^{\nu}D^{\mu}V(\imath, e(\imath)) \leq -\eta\, V(\imath, e(\imath)), \qquad \eta > 0.$$

Then, the solution $\aleph^*$ is Lyapunov stable on J.

Furthermore, if $J = [0, \infty)$, then $\aleph^*$ is asymptotically stable.

Proof. Let $\aleph^* \in C(J, \mathbb{R})$ be a solution of Problem (1.1), and let $z \in C(J, \mathbb{R})$ be another solution that satisfies

$$|z(0) - \aleph^*(0)| < \delta.$$

Define the error function

$$e(\imath) = z(\imath) - \aleph^*(\imath).$$

Using the FDE satisfied by both solutions, we obtain the following:

$$^{\nu}D^{\mu} e(\imath) = f(\imath, z(\imath)) - f(\imath, \aleph^*(\imath)). \tag{3.22}$$

By applying the generalized Katugampola fractional integral operator to (3.22), we derive the following equivalent integral representation:

$$e(\imath) = e(0) + \frac{\nu^{1-\mu}}{\Gamma(\mu)} \int_0^{\imath} (\imath^{\nu} - \jmath^{\nu})^{\mu-1} \jmath^{\nu-1} \Big[f(\jmath, z(\jmath)) - f(\jmath, \aleph^*(\jmath)) \Big] \, d\jmath.$$

By taking absolute values and using hypothesis **(A2)**, we obtain the following:

$$|e(\imath)| \leq |e(0)| + \frac{L\nu^{1-\mu}}{\Gamma(\mu)} \int_0^{\imath} (\imath^{\nu} - \jmath^{\nu})^{\mu-1} \jmath^{\nu-1} |e(\jmath)| \, d\jmath.$$

By applying the fractional Grönwall inequality associated with the generalized Katugampola operator, we deduce that

$$|e(\imath)| \leq |e(0)| E_{\mu} \left(\frac{L}{\nu^{\mu}} \imath^{\nu\mu} \right), \qquad \forall \, \imath \in J, \tag{3.23}$$

where $E_{\mu}(\cdot)$ denotes the Mittag–Leffler function.

Since the Mittag–Leffler function is continuous and positive on bounded intervals, there exists a constant $M > 0$ such that

$$E_{\mu} \left(\frac{L}{\nu^{\mu}} \imath^{\nu\mu} \right) \leq M, \qquad \forall \, \imath \in J. \tag{3.24}$$

By combining (3.23) and (3.24), we obtain the following:

$$|e(\imath)| \leq M |e(0)|, \qquad \forall \, \imath \in J. \tag{3.25}$$

Let $\varepsilon > 0$ be arbitrary and choose $\delta = \frac{\varepsilon}{M}$.

Then, whenever

$$|e(0)| < \delta,$$

from (3.25), it follows that

$$|e(\imath)| < \varepsilon, \qquad \forall \, \imath \in J.$$

Therefore,

$$|z(\imath) - \aleph^*(\imath)| < \varepsilon, \qquad \forall \, \imath \in J,$$

which proves that $\aleph^*$ is Lyapunov stable.

Next, assume that $J = [0, \infty)$.

From hypothesis **(A4)**, we have the following:

$$^{\nu}D^{\mu}V(\iota, e(\iota)) \leq -\eta V(\iota, e(\iota)).$$

Applying the fractional comparison principle yields the following:

$$V(\iota, e(\iota)) \leq V(0, e(0))E_{\mu}\left(-\eta\frac{\iota^{\nu\mu}}{\nu^{\mu}}\right).$$

Since $E_{\mu}(-\theta) \to 0 \quad$ as $\theta \to \infty$, we obtain the following:

$$\lim_{\iota\to\infty} V(\iota, e(\iota)) = 0.$$

Finally, using the lower bound in (3.21), we derive the following:

$$\alpha_1|e(\iota)|^2 \leq V(\iota, e(\iota)).$$

Consequently,

$$\lim_{\iota\to\infty} |e(\iota)| = 0,$$

that is,

$$\lim_{\iota\to\infty} |z(\iota) - \aleph^*(\iota)| = 0.$$

Hence, $\aleph^*$ is asymptotically stable. The proof is complete. □

4. Illustrative examples

Example 4.1. ***Physical motivation and model derivation.***

Consider the spread of an infectious disease (e.g., measles or influenza) in a closed population of size N. In a classical SIR compartment model, after standard rescaling, the infected fraction $\aleph(\iota) = I(\iota)/N$ *satisfies an equation of the following form:*

$$\aleph'(\iota) = \beta S(\iota)\aleph(\iota) - \gamma\aleph(\iota),$$

where $\beta > 0$ *is the transmission rate, and* $\gamma > 0$ *is the recovery rate. Two well-documented empirical facts motivate the use of the* generalized Katugampola FD $^{\nu}D^{\mu}$ *in place of the classical integer-order derivative* $\frac{d}{d\iota}$ *(or even the Caputo derivative* $^{c}D^{\mu}$*):*

i ***Power-law and logarithmic memory.*** *Epidemiological data for diseases with long incubation periods or super-spreading events exhibit memory kernels that combine* power-law *and* logarithmic *scaling [30]. The Caputo derivative only uses a power-law kernel* $(\imath - \jmath)^{\mu-1}$*, and the Hadamard derivative only uses* $\ln(\imath/\jmath)^{\mu-1}$*. The generalized Katugampola kernel* $(\imath^{\nu} - \jmath^{\nu})^{\mu-1}$ *continuously interpolates between both by tuning the parameter* $\nu > 0$*, thus providing strictly greater flexibility. Choosing* $\nu = 1$ *recovers the Riemann–Liouville/Caputo setting, while* $\nu \to 0^+$ *approaches the Hadamard setting. For diseases such as fascioliasis, optimal values of* $\nu \in (0, 1)$ *have been identified by fitting to clinical incidence data [31], a fitting that* cannot *be achieved with the Caputo derivative alone.*

ii ***Nonlinear saturation and nonlocal initial state.*** *Field observations indicate that (a) the effective contact rate decreases as the infected fraction grows (saturation / Michaelis–Menten nonlinearity), and (b) the initial infected fraction at time $\iota = 0$ depends on past observations rather than a single point measurement (nonlocal boundary condition). Together, these lead to the BVP* (4.1) *below.*

The fractional epidemic BVP.

We consider the following generalized Katugampola fractional BVP:

$$\begin{cases} {}^{\frac{1}{3}}D^{\frac{1}{2}}\aleph(\iota) = \dfrac{e^{-2\iota}\,\aleph(\iota)}{(1+e^{\iota})(1+\aleph(\iota))}, & \iota \in J := [0,1], \\ \aleph(0) + \aleph(1) = 0. \end{cases} \tag{4.1}$$

Here, $\aleph(\iota)$ represents the normalized infected fraction *at epidemiological time $\iota \in [0,1]$ (scaled so that $\iota = 1$ corresponds to the end of the observation window, e.g., one epidemic season). The nonlocal two-point condition $\aleph(0) + \aleph(1) = 0$ encodes a* balance constraint*: The net change in the infected population over the observation window is prescribed, as occurs when the total incidence data are available from disease-surveillance systems rather than instantaneous point measurements.*

The fractional order $\mu = \frac{1}{2}$ and parameter $\nu = \frac{1}{3}$ are selected so that the Katugampola kernel captures the observed sub-diffusive spread of the disease front in a heterogeneous population [30]. The Caputo derivative ($\nu = 1$) would require $\mu < \frac{1}{2}$ to match the same memory profile, but fitting to incidence data shows that $(\mu, \nu) = (\frac{1}{2}, \frac{1}{3})$ provides a significantly better empirical fit (see [30], Table 2, and [32] for analogous parameter identification in the Hilfer–Katugampola setting).

Define the following nonlinear incidence function:

$$f(\iota,\aleph) = \frac{e^{-2\iota}\,\aleph}{(1+e^{\iota})(1+\aleph)}, \qquad (\iota,\aleph) \in J \times [0,\infty). \tag{4.2}$$

The three factors carry precise epidemiological meaning:

- $e^{-2\iota}$*: exponentially decaying* transmission rate*, which models the effect of awareness campaigns, seasonal forcing, or waning immunity that reduce the effective contact rate as the epidemic progresses [33].*
- $(1+e^{\iota})^{-1}$*: a time-varying* susceptible depletion *factor; as more individuals are infected, fewer remain susceptible, thus reducing a further spread.*
- $(1+\aleph)^{-1}$*:* Michaelis–Menten (Holling type II) saturation *of the incidence, which prevents unbounded growth at high infection levels and is widely used in epidemic modeling [34, 35].*

We verify that all assumptions of Theorem 3.2 are satisfied.

Step 1: Continuity of the nonlinear term.

The function $f : J \times [0,\infty) \to \mathbb{R}$ defined in (4.2) *is continuous on $J \times [0,\infty)$. Hence, assumption **(H1)** is fulfilled.*

Step 2: Verification of the Lipschitz condition.

Let $\aleph_1, \aleph_2 \in [0,\infty)$, $\iota \in J$. Then,

$$|f(\iota,\aleph_1) - f(\iota,\aleph_2)| = \frac{e^{-2\iota}}{1+e^{\iota}}\left|\frac{\aleph_1}{1+\aleph_1} - \frac{\aleph_2}{1+\aleph_2}\right|$$

$$= \frac{e^{-2\iota}}{(1+e^{\iota})(1+\aleph_1)(1+\aleph_2)}|\aleph_1 - \aleph_2|. \quad (4.3)$$

Since $\aleph_1, \aleph_2 \geq 0$, *we have* $(1+\aleph_1)(1+\aleph_2) \geq 1$. *Moreover,*

$$e^{-2\iota} \leq 1, \qquad 1+e^{\iota} \geq 2, \qquad \forall\, \iota \in [0,1].$$

Therefore, from (4.3),

$$|f(\iota,\aleph_1) - f(\iota,\aleph_2)| \leq \frac{1}{2}|\aleph_1 - \aleph_2|.$$

Hence, the Lipschitz condition holds with $k = \frac{1}{2}$.

Step 3: Verification of the contraction constant.

For Problem (4.1), *we have* $a = b = 1$, $\ell = 1$, $\mu = \frac{1}{2}$, *and* $\nu = \frac{1}{3}$. *The contraction constant introduced in Theorem 3.2 as follows:*

$$\gamma = \frac{k\ell^{\nu\mu}}{\nu^{\mu}\Gamma(\mu+1)}\left(1 + \frac{|b|}{|a+b|}\right). \quad (4.4)$$

By substituting the parameters into (4.4),

$$\gamma = \frac{\frac{1}{2}}{\left(\frac{1}{3}\right)^{1/2}\Gamma\left(\frac{3}{2}\right)}\left(1+\tfrac{1}{2}\right) = \frac{3}{4\sqrt{3}\,\Gamma\left(\frac{3}{2}\right)}.$$

Since $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2} \approx 0.8862$, *it follows that* $\gamma \approx 0.489 < 1$.

Therefore, all assumptions of Theorem 3.2 are satisfied, and Problem (4.1) *admits a* unique solution *on* $J = [0,1]$.

Step 4: Lyapunov stability of the epidemic equilibrium.

The unique solution $\aleph^*$ *of* (4.1) *represents the* endemic equilibrium trajectory *of the fractional epidemic model. The stability of this trajectory is not merely a mathematical nicety: it determines whether the disease prevalence is* robust *to measurement errors in the initial infected fraction, a question of central importance in public-health decision-making.*

From a dynamical-systems perspective, the lack of an integer-order chain rule for the Katugampola operator means that classical Lyapunov arguments must be adapted, exactly as required for fractional chaotic systems such as the fractional Lorenz and Chen attractors [36]. In those systems, Lyapunov functions which satisfy the fractional decay condition ${}^{\nu}D^{\mu}V(\iota, e(\iota)) \leq -\eta V(\iota, e(\iota))$ *(Assumption (A4) of Theorem 3.6) have been employed to certify stability without requiring explicit solution formulas. Our fractional epidemic BVP falls into the same class.*

Let $z(\iota)$ *be any other continuous solution of* (4.1) *and define the error* $e(\iota) = z(\iota) - \aleph^*(\iota)$. *Applying the generalized Katugampola fractional integral and the Lipschitz estimate with* $L = k = \frac{1}{2}$, *together with the fractional Grönwall inequality, yields the following:*

$$|e(\iota)| \;\leq\; |e(0)|\, E_{1/2}\!\left(\frac{1}{2\,3^{1/2}}\,\iota^{1/6}\right), \qquad \iota \in [0,1], \quad (4.5)$$

where $E_{1/2}(\cdot)$ is the Mittag–Leffler function. Since $E_{1/2}$ is continuous and positive on the compact interval $[0, \imath^{1/6}]$*, there exists a constant* $M > 0$ *such that* $E_{1/2}(\frac{1}{2\sqrt{3}}\imath^{1/6}) \leq M$ *for all* $\imath \in [0, 1]$*. Concretely, using the known bound* $E_{1/2}(s) \leq e^{s^2}$*, one obtains the following:*

$$M = \exp\left(\frac{1}{12}\right) \approx 1.0869.$$

Hence,

$$|z(\imath) - \aleph^*(\imath)| \leq M\,|z(0) - \aleph^*(0)|, \qquad \forall\, \imath \in [0, 1]. \tag{4.6}$$

Given any epidemiological tolerance $\varepsilon > 0$ *(e.g, 1% of population), choose* $\delta = \varepsilon/M \approx 0.920\,\varepsilon$*. Then, whenever the initial infected fractions of two epidemic trajectories differ by less than* δ*, their entire evolutions remain within* ε *of each other. This is precisely* Lyapunov stability *in the sense of Definition 2.9, and it guarantees that the prevalence predictions derived from* $\aleph^*$ *are robust to surveillance noise in the initial count.*

Remark on the advantage of Katugampola over Caputo.

Had the Caputo derivative ($\nu = 1$) *been used in place of the Katugampola derivative* ($\nu = \frac{1}{3}$)*, the contraction constant would become*

$$\gamma_{\text{Caputo}} = \frac{k\ell^\mu}{\Gamma(\mu+1)}\left(1 + \frac{|b|}{|a+b|}\right) = \frac{\frac{1}{2}}{\Gamma(\frac{3}{2})} \cdot \frac{3}{2} \approx 0.846,$$

which is still less than 1 but significantly larger*, thus indicating a weaker contraction and slower convergence of fixed-point iterations. More importantly, the parameter* $\nu = \frac{1}{3}$ *is not arbitrarily chosen: it is the value that best fits the sub-diffusive spread exponent* $\alpha \approx 0.33$ *identified in the fascioliasis incidence data of [30]. With* $\nu = 1$ *(Caputo), reproducing the same memory exponent would require lowering* μ *to roughly* 0.17*, which loses the connection to the half-order derivative and complicates the functional-analytic framework. Therefore, the Katugampola operator provides both a* better physical fit *and a* more tractable *mathematical structure.*

Example 4.2. *Consider the following generalized Katugampola fractional nonlocal problem:*

$$\begin{cases} {}^{\frac{1}{3}}D^{\frac{1}{2}}\aleph(\imath) = \dfrac{5e^{-3\imath}\,\aleph(\imath)}{(1+e^{\imath})(3+\aleph(\imath))}, & \imath \in J := [0,1], \\ \aleph(0) = \displaystyle\sum_{k=1}^{m} a_k \aleph(x_k), & 0 < x_1 < x_2 < \cdots < x_m < 1, \end{cases} \tag{4.7}$$

where $a_k \geq 0, \qquad k = 1, 2, \ldots, m.$

Define

$$f(\imath, \aleph) = \frac{5e^{-3\imath}\,\aleph}{(1+e^{\imath})(3+\aleph)}, \qquad (\imath, \aleph) \in J \times [0, \infty), \tag{4.8}$$

and

$$g(\aleph) = \sum_{k=1}^{m} a_k \aleph(x_k). \tag{4.9}$$

We verify the assumptions of Theorem 3.5.

Step 1: Continuity of the nonlinear terms.

The functions $f : J \times [0,\infty) \to \mathbb{R}$ *and* $g : C(J,\mathbb{R}) \to \mathbb{R}$ *defined by* (4.8) *and* (4.9) *are continuous. Hence, the continuity assumptions of the theorem are satisfied.*

Step 2: Verification of the Lipschitz condition for f**.**

Let $\aleph_1, \aleph_2 \in [0,\infty),\ \imath \in J$*. Then,*

$$\begin{aligned} |f(\imath,\aleph_1) - f(\imath,\aleph_2)| &= \frac{5e^{-3\imath}}{1+e^{\imath}}\left|\frac{\aleph_1}{3+\aleph_1} - \frac{\aleph_2}{3+\aleph_2}\right| \\ &= \frac{5e^{-3\imath}}{1+e^{\imath}}\left|\frac{3(\aleph_1-\aleph_2)}{(3+\aleph_1)(3+\aleph_2)}\right| \\ &= \frac{15e^{-3\imath}}{(1+e^{\imath})(3+\aleph_1)(3+\aleph_2)}|\aleph_1-\aleph_2|. \end{aligned}$$

Since $\aleph_1, \aleph_2 \geq 0$, *we have* $(3+\aleph_1)(3+\aleph_2) \geq 9$, *and*

$$1 + e^{\imath} \geq 2, \qquad e^{-3\imath} \leq 1, \qquad \forall\, \imath \in [0,1].$$

Therefore,

$$\begin{aligned} |f(\imath,\aleph_1) - f(\imath,\aleph_2)| &\leq \frac{15}{18}|\aleph_1-\aleph_2| \\ &= \frac{5}{6}|\aleph_1-\aleph_2|. \end{aligned} \tag{4.10}$$

Hence, f *is globally Lipschitz continuous with Lipschitz constant* $k = \frac{5}{6}$.

Step 3: Verification of the Lipschitz condition for g**.**

For any $\aleph_1, \aleph_2 \in C(J,\mathbb{R})$, *we obtain the following:*

$$\begin{aligned} |g(\aleph_2) - g(\aleph_1)| &= \left|\sum_{k=1}^{m} a_k(\aleph_2(x_k) - \aleph_1(x_k))\right| \\ &\leq \sum_{k=1}^{m} a_k\,|\aleph_2(x_k) - \aleph_1(x_k)| \\ &\leq \left(\sum_{k=1}^{m} a_k\right)\|\aleph_2 - \aleph_1\|_{\infty}. \end{aligned}$$

Hence, g *is Lipschitz continuous with constant* $k^* = \sum_{k=1}^{m} a_k$.

Step 4: Verification of the contraction condition.

The contraction constant introduced in Theorem 3.5 is as follows:

$$\delta = \frac{k^*}{2} + \frac{k\ell^{\nu\mu}\nu^{-\mu}}{\Gamma(\mu+1)}. \tag{4.11}$$

For the present example, $\ell = 1, \qquad \mu = \frac{1}{2}, \qquad \nu = \frac{1}{3},$ *and* $k = \frac{5}{6}$.

By substituting these values into (4.11)*, we obtain the following:*

$$\delta = \frac{k^*}{2} + \frac{\frac{5}{6}}{\left(\frac{1}{3}\right)^{1/2}\Gamma\left(\frac{3}{2}\right)}$$

$$= \frac{k^*}{2} + \frac{5\sqrt{3}}{3\sqrt{\pi}}.$$

Therefore, if

$$\frac{k^*}{2} + \frac{5\sqrt{3}}{3\sqrt{\pi}} < 1,$$

then all assumptions of Theorem 3.5 are satisfied.

Consequently, Problem (4.7) *admits a unique solution on* $J = [0, 1]$.

Lyapunov stability analysis.

Next, we verify the stability properties of the obtained solution.

Let $\aleph(\imath)$ *and* $z(\imath)$ *be two solutions of Problem* (4.7)*, and define the error function*

$$e(\imath) = z(\imath) - \aleph(\imath).$$

Using (4.7)*, we derive the following:*

$${}^{\frac{1}{3}}D^{\frac{1}{2}}e(\imath) = f(\imath, z(\imath)) - f(\imath, \aleph(\imath)).$$

By applying the Lipschitz estimate (4.10)*, we obtain the following:*

$$\left|{}^{\frac{1}{3}}D^{\frac{1}{2}}e(\imath)\right| \le \frac{5}{6}|e(\imath)|.$$

Integrating both sides by means of the generalized Katugampola fractional integral operator yields the following:

$$|e(\imath)| \le |e(0)| + \frac{5\nu^{1-\mu}}{6\Gamma(\mu)} \int_0^{\imath} (\imath^{\nu} - \jmath^{\nu})^{\mu-1} \jmath^{\nu-1} |e(\jmath)|\, d\jmath.$$

By the generalized fractional Grönwall inequality, it follows that

$$|e(\imath)| \le |e(0)| E_{\frac{1}{2}}\left(\frac{5}{6\sqrt{3}}\imath^{1/6}\right), \qquad \forall\, \imath \in [0, 1].$$

Since the Mittag–Leffler function is bounded on compact intervals, there exists a constant $M > 0$ *such that*

$$E_{\frac{1}{2}}\left(\frac{5}{6\sqrt{3}}\imath^{1/6}\right) \le M, \qquad \forall\, \imath \in [0, 1].$$

Hence,

$$|e(\imath)| \le M|e(0)|, \qquad \forall\, \imath \in [0, 1]. \tag{4.12}$$

Let $\varepsilon > 0$*. Choose* $\delta = \frac{\varepsilon}{M}$.

Then, whenever $|e(0)| < \delta$, *Relation* (4.12) *implies that*

$$|e(\iota)| < \varepsilon, \qquad \forall\, \iota \in [0, 1].$$

Therefore,

$$|z(\iota) - \aleph(\iota)| < \varepsilon, \qquad \forall\, \iota \in [0, 1],$$

which proves that the solution is stable in the sense of Lyapunov.

Physical interpretation and application.

The fractional nonlocal model (4.7) *describes a dynamical process with hereditary memory and distributed initial information. The generalized Katugampola derivative captures nonlocal temporal effects, which naturally arise in anomalous diffusion, viscoelasticity, biological systems, and the fractional control theory.*

The nonlinear term

$$\frac{5e^{-3\iota}\,\aleph(\iota)}{(1 + e^{\iota})(3 + \aleph(\iota))}$$

models a dissipative growth mechanism with saturation effects. In particular:

- *the exponential factor* $e^{-3\iota}$ *represents a decaying external influence;*
- *the denominator* $3 + \aleph(\iota)$ *prevents unbounded growth and introduces nonlinear stabilization and*
- *the fractional operator reflects the dependence of the present state on the previous history of the system.*

The nonlocal condition

$$\aleph(0) = \sum_{k=1}^{m} a_k \aleph(x_k)$$

indicates that the initial state depends on observations collected at several intermediate times. Such conditions appear in population dynamics, epidemiological models, heat transfer with memory, and viscoelastic systems.

Moreover, the Lyapunov stability result guarantees robustness of the model with respect to perturbations in the initial data, which is important in practical applications that involve measurement errors and external disturbances.

Therefore, this example demonstrates the applicability of the theoretical results established in this paper to realistic fractional systems governed by memory-dependent and nonlocal dynamics.

5. Conclusions

In this work, we studied a class of nonlocal fractional BVPs that involve the generalized Katugampola FD. The considered framework combines generalized fractional operators with nonlocal boundary conditions, which allows the modeling of dynamical systems that possess hereditary memory effects and nonlocal interactions. Such models naturally arise in several areas of applied sciences, including viscoelasticity, anomalous diffusion, the control theory, population dynamics, and systems with distributed memory.

The main objective of this paper was to establish a rigorous analytical framework for the solvability and stability analysis of the proposed fractional problems. By transforming the considered boundary value problems into equivalent fractional integral equations, we derived new existence and uniqueness results under suitable assumptions on the nonlinear terms. The analysis was carried out using classical fixed point techniques, including the Banach contraction principle and Schaefer's fixed point theorem. The obtained conditions guarantee the well-posedness of the considered problems in the Banach space $C(J, \mathbb{R})$ endowed with the supremum norm.

A further contribution of this work concerns the qualitative analysis of solutions. In particular, we investigated Lyapunov-type stability for the considered fractional models. Instead of relying on the classical integer-order chain rule, the stability analysis was developed through suitable fractional integral inequalities and comparison-type arguments adapted to the generalized Katugampola framework. Sufficient conditions that ensured Lyapunov stability and the asymptotic stability of solutions were established. These results provide important information regarding the robustness and long-term behavior of the considered systems under perturbations.

Compared with several existing studies in the literature, the present paper offers the following improvements:

- the use of the generalized Katugampola FD, which unifies different classical fractional operators within a single framework;
- the treatment of nonlocal boundary conditions together with generalized fractional dynamics;
- the combination of a solvability analysis and a Lyapunov stability investigation in the same setting and
- the derivation of explicit estimates and sufficient conditions guaranteeing the existence, uniqueness, boundedness, and stability of solutions.

In addition, illustrative examples were presented to verify the applicability of the theoretical results. The examples demonstrate that the obtained assumptions can be effectively checked and that the developed theory can be applied to fractional models which arise in real applications with memory-dependent dynamics and nonlocal effects.

Although the present work is mainly theoretical, the obtained results may provide a mathematical basis for future numerical simulations and parameter analysis of generalized fractional systems. The developed framework can also be useful in the analysis of fractional control systems, neural-network-based dynamical models, and iterative learning processes that involve memory effects.

Several interesting research directions remain open for future investigation, including the following:

- numerical approximation schemes for generalized Katugampola fractional problems;
- systems of coupled FDEs;
- impulsive and stochastic fractional models;
- optimal control problems that involve generalized fractional operators;
- fractional models with delays, distributed nonlocal conditions, or state-dependent parameters and
- applications to neural network dynamics, iterative learning control, and fault estimation models.

We believe that the results obtained in this paper contribute to the growing theory of generalized FDEs and provide a useful foundation for further analytical and applied investigations in this active research area.

Author contributions

(M.S.S.): Writing-original draft, Project administration, Supervision; **(Z.B.)**: Data curation, Methodology; **(H.G.)**: Investigation, Visualization, Validation; **(K.M.)**: Software, Formal analysis; **(M.B.)**: Conceptualization, Writing-review & editing; **(M.A.)**: Resources, Funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declares they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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