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Research article

A multidimensional first-encounter time model for random walk search based on a two-stage stochastic transport process

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Abstract: This paper develops a two-stage stochastic transport model for the detection and tracking of a moving target governed by a multidimensional random walk. In the first stage, the target's initial location is estimated through an optimal search strategy over independent regions, where a truncated bivariate distribution is used to describe prior uncertainty. Explicit analytical expressions for the optimal search distances minimizing the expected detection time are derived. In the second stage, the tracking problem is formulated along intersecting trajectories, where coordinated searchers aim to intercept a stochastically moving target. Closed-form expressions for the first-encounter (first-passage) time and the corresponding tracking distances are obtained. Moreover, sufficient conditions ensuring the finiteness of the expected first-encounter time are established for a bounded controlled tracking framework, where unbiased random walk motion is coupled with coordinated recurrent coverage of the admissible tracking trajectories and a uniform positive encounter probability. The proposed model provides a unified analytical framework for first-passage phenomena in multidimensional stochastic transport systems.

Keywords: stochastic transport; multidimensional random walks; first-passage time; target search; analytical modeling
Mathematics Subject Classification: 60G50, 60J10, 60J45, 60K35, 90B40, 49J20

1. Introduction

In recent years, increasing attention in statistical physics has been directed toward understanding first-passage and encounter processes in systems governed by randomness. When both the searchers and targets move under uncertainty, detection is no longer a purely geometric task is but rather a stochastic event shaped by the underlying motion dynamics. Random walk models, in particular, have provided a natural and widely used framework for describing such behavior across different spatial scales. Within this perspective, the time required for a searcher to encounter a moving target becomes a fundamental quantity, closely linked to first-passage phenomena. These processes capture how motion, probability, and spatial structure interact, and they play a central role in determining search efficiency in uncertain environments.

Motivated by this viewpoint, recent research has increasingly modeled search and detection problems as two coupled transport phases: An initial localization phase that identifies the most probable starting position of a target, followed by a tracking phase governed by the stochastic transport of the target itself. Such a decomposition has been shown to yield systematic performance improvements over single-stage formulations, both in terms of the expected detection time and stability of the pursuit process, as reported in Liu et al. [1] and Wang et al. [2]. From a computational perspective, this separation enables the design of distinct decision mechanisms for spatial allocation and subsequent transport-driven pursuit, allowing each phase to be optimized under its own probabilistic structure.

Within the first phase, the problem naturally reduces to the probabilistic allocation of transport effort across independent regions, where the sensing or search resources are distributed according to prior uncertainty. Dynamic probability maps concentrate effort in high-probability regions, reducing the expected localization time; see Zhang and Duan [3] and Moshagen et al. [4]. Variational and adaptive discretizations of these probability maps further refine the spatial resolution in proportion to belief concentration, while Monte Carlo and Markov-chain based constructions offer tractable approximations of the underlying transport uncertainty. Comparative studies consistently show that probability-driven allocation policies outperform uniform or purely geometric coverage strategies, particularly in large or heterogeneous domains [3,4].

These developments motivate a formal optimization framework for the first stage, in which region-wise effort (e.g., transport distances or path lengths) is selected to minimize the expected localization cost under a prescribed prior distribution. This abstraction removes dependence on any specific platform and allows the search phase to be interpreted as a stochastic transport optimization problem, providing a natural theoretical foundation for the two-stage model developed in this work.

The second (tracking) stage introduces a fundamentally different challenge, as the target's motion is inherently uncertain and is naturally modeled as a multidimensional independent random walk. From a transport theoretic perspective, this setting corresponds to a stochastic propagation process along intersecting trajectories, where the primary objective is to characterize and minimize the expected first-encounter (or first-passage) time between moving agents and the randomly evolving target. Robust performance under such stochastic dynamics requires coordinated tracking strategies capable of adapting to uncertainty in both motion and observation. Recent developments emphasize the role of cooperative agents equipped with online information fusion and adaptive control mechanisms, including learning-based predictors for target relocalization after loss, which demonstrate sustained tracking performance under occlusions and complex geometries. Related studies further introduced stochastic multi-target transport models and coordinated pursuit strategies, highlighting the necessity of control policies that explicitly account for motion uncertainty and constrained sensing, as discussed in Feng et al. [5] and Wu et al. [6]. These results motivate the formulation of a dedicated second stage in which tracking distances and switching rules are optimized with respect to the expected first-encounter time, conditioned on an estimated initial position and a stochastic motion model. Within this two-stage perspective, linking the localization and tracking processes requires adaptive allocation of the search and tracking effort as probabilistic beliefs evolve, a topic that has been widely studied in multi-agent systems [7–9].

From a broader statistical physics viewpoint, a substantial body of literature has established rigorous analytical foundations for first-passage and encounter processes in stochastic transport systems. Intermittent search strategies and their optimization properties were systematically analyzed in the classical review of Bénichou et al. [10], while diffusion with stochastic resetting was shown to generate finite mean first-passage times and non-equilibrium stationary states by Evans and Majumdar [11]. The universal aspects and optimization principles of stochastic restarts were further clarified in Reuveni [12] and Pal and Reuveni [13]. Complementary studies addressed universality classes of first-passage distributions in confined geometries [14], scaling behavior in complex media [15], and anomalous or fractional transport effects on networks [16,17]. Transport properties under stochastic resetting were also examined in Masó-Puigdellosas et al. [18], highlighting the interplay between resetting rates and search efficiency. More recently, Rubio-Gómez et al. [19] derived analytical expressions for mean first-encounter times under resetting mechanisms, providing explicit optimality conditions. In a related direction, Flaquer-Galmés et al. [20] developed a fully analytical treatment of intermittent random walks under stochastic resetting, establishing the existence of a non-equilibrium steady state and proving the finiteness of the mean first arrival time in unbounded domains together with the emergence of optimal reset rates. To place the present work more clearly within the standard first-passage and stochastic search literature, we emphasize that the proposed model is related to several classical lines of research, including intermittent searches, stochastic resetting, recurrence and transience of random walks, and first-passage processes in confined or complex media. These established frameworks provide the theoretical background for understanding how dimensionality, confinement, search-path geometry, and the randomness of motion affect encounter times. The present contribution builds on this broader context by considering a controlled two-stage transport setting in which probabilistic localization is coupled with coordinated tracking along intersecting Euclidean trajectories.

These results collectively demonstrate that finiteness and optimization of encounter times depend precisely on the dimensionality, recurrence properties, and restart mechanisms. The present work complements this line of research by extending first-encounter analysis to a two-stage stochastic transport framework in multidimensional settings $(d \geq 2)$, where region-level probabilistic localization is explicitly coupled with encounter-time analysis for independent random walk motions. In doing so, the model situates its existence results within the broader theory of first-passage and resetting processes, while addressing a structurally different problem involving coordinated transport and multidimensional encounter geometry. Unlike resetting-based models, intermittent strategies, or network random walks, the present framework does not rely on restart mechanisms, switching dynamics, or graph topology to ensure the finiteness of first-passage times. The finiteness result is derived from the bounded controlled tracking structure, the recurrent coverage of the admissible trajectories, finite tracking distance functions, and a uniform positive encounter probability, with unbiased motion serving only as a balance condition rather than a standalone guarantee of recurrence. The analytical existence result therefore follows from the intrinsic structure of the two-stage transport formulation itself, rather than from externally imposed control mechanisms. This structural coupling between probabilistic localization and coordinated stochastic pursuit is not addressed in the cited statistical physics literature and defines the specific contribution of the present study.

Within this line of research, El-Hadidy et al. [21–23] have developed a sequence of theoretical results addressing stochastic search and transport problems that combine probabilistic allocation of effort with random target motion. Related theoretical developments were further reported in [24–26] and [27,28]. In the context of the search phase, a multi zone framework with k searchers was introduced, where the allocation of effort across independent regions is optimized through a utility function based on the expected detection time; see El-Hadidy and Fakharany [29]. This framework was subsequently generalized to N heterogeneous regions with cooperative searchers, leading to an explicit optimization problem that minimizes the expected detection time under a prescribed time budget, as presented in El-Hadidy et al. [30]. Related contributions have examined detection over a finite number of intervals under truncated probability distributions, including numerical optimization of the expected detection time; see Fakharany et al. [31]. Collectively, these results provide a rigorous foundation for modeling the first stage as a probabilistic transport problem over independent regions with prior weighted effort allocation.

For the tracking phase, subsequent studies have focused on stochastic transport induced by multidimensional independent random walk motions in complex environments, deriving existence conditions and reductions of first-encounter times; see El-Hadidy [32]. Closely related analyses address random searches for a randomly moving particle and investigate first-passage and encounter phenomena under stochastic dynamics; see El-Hadidy [33]. These developments build upon an earlier foundational result on coordinated search for a random walk target on the real line with two symmetric searchers starting from a common origin, as established in El-Hadidy [34], which forms a central theoretical component of the present tracking formulation.

Unlike existing formulations, the proposed framework explicitly couples probabilistic region-level transport with multidimensional random walk encounter analysis within a unified two-stage model. Motivated by this body of work, we formulate a two-stage stochastic transport model for the search and first-encounter analysis of a multidimensional random walk target. In Stage 1, optimal search distances are derived by minimizing the expected detection time of the target's initial position through the probabilistic allocation of effort across independent regions. Conditioned on this initial localization, Stage 2 analyzes the transport of the target along intersecting trajectories governed by a multidimensional random walk, and establishes tracking distances that give the expected first-encounter time. By coupling region-level probabilistic transport with stochastic motion-driven encounter analysis, the proposed framework yields analytically tractable existence results, while remaining compatible with the cooperative multi agent tracking architectures reported in related studies [1,3−5,7−9].

The mathematical novelty of this work lies in the way the two stages are coupled within a single stochastic transport framework. While two-stage search procedures and random walk first-passage models have been studied separately, the present model connects them by using the probabilistic localization outcome of Stage 1 as the geometric reference for the Stage 2 encounter process. Thus, the optimal allocation of search distances over independent regions is directly linked to the subsequent multidimensional first-encounter analysis along intersecting Euclidean trajectories. In addition, the finiteness of the expected first-encounter time is obtained under the bounded controlled tracking conditions, including coordinated recurrent coverage, finite tracking distances, and a uniform positive encounter probability, rather than from the unbiased random walk structure alone. This distinction clarifies how the proposed formulation differs from previous stochastic search models that treat localization and random walk encounter analysis as separate or independently optimized problems.

The main contributions of this paper are as follows.

(1) A two-stage stochastic transport model is developed by coupling probabilistic localization with random walk tracking.

(2) Optimal search distances are derived for the initial localization stage under a truncated bivariate distribution.

(3) A multidimensional first-encounter framework is formulated along intersecting trajectories.

(4) Sufficient conditions are established for the finiteness of the expected first-encounter time under a bounded controlled tracking framework with recurrent coverage and a uniform positive encounter probability.

The remainder of this paper is organized as follows. Section 2 formulates the stochastic transport problem, introduces the notation and modeling assumptions, and presents the two-stage analytical framework. In this section, a first-stage model is developed for the probabilistic allocation of search effort across independent regions, optimal search distances are derived, and the conditions that minimize the expected detection time are established. Section 3 investigates the second-stage transport and pursuit problem, where the target follows a multidimensional random walk along a set of intersecting trajectories. For this stage, finite tracking distances and a recursive characterization of the expected first-encounter time are obtained within a coordinated multiagent search framework. Section 4 establishes the existence of a finite first-encounter time in the proposed model. Section 5 demonstrates the effectiveness and practical applicability of the proposed model. Section 6 concludes the paper.

2. Problem description

We consider a stochastic transport and search problem in which a moving target evolves according to a multidimensional independent random walk and is pursued by a coordinated set of mobile search agents. The detection process is structured as a two-stage strategy. In the first stage, search effort is probabilistically allocated over a collection of independent and non-overlapping regions, each associated with a known prior probability for the target's initial location. The objective of this stage is to minimize the expected time required to identify the region containing the target's starting position through optimal allocation of the search distances.

Conditioned on the outcome of the first stage, the second stage addresses the transport and pursuit of the target after localization. During this stage, the target performs a multidimensional independent random walk constrained to a set of intersecting trajectories passing through the estimated initial position. The search agents move symmetrically along these trajectories according to a coordinated tracking policy. The primary objective of the second stage is to establish and minimize the expected first-encounter time between any search agent and the randomly moving target. Collectively, the two stages are designed to ensure the existence of a finite search transport strategy that balances probabilistic allocation of effort with stochastic transport dynamics.

The proposed two-stage strategy consists of two sequential but connected phases. In Stage 1, the search domain is divided into independent non-overlapping regions, and search effort is allocated probabilistically to estimate the initial location of the target. In Stage 2, conditioned on the estimated initial location obtained from Stage 1, coordinated search agents track the target as it moves according to a multidimensional random walk along intersecting trajectories. Thus, the first stage addresses the initial localization, while the second stage addresses stochastic tracking and first-encounter time analysis.

For clarity, the variables used throughout the paper are listed in Table 1.

Table 1. Description of the main variables used in the proposed model.

Variable	Description
N	Number of independent non-overlapping regions
i	Index of a search region
A_i	Area of region i
O_i	Center point of region i
X	Random variable representing the target position in the x-direction
Y	Random variable representing the target position in the y-direction
$f(x,y)$	Joint probability density function
$F(x,y)$	Joint cumulative distribution function
$\aleph$	Number of deleted areas
$D(\Delta_i,F)$	Expected detection time in region i
$D(\Delta,F)$	Total expected detection time
d	Dimension of the random walk
$X(t)$	The vector of the target position at time t

2.1. Stage-1: Probabilistic allocation of search effort over independent transport regions

In search problems arising from large-scale uncertain environments, such as those encountered in aerial disappearance scenarios over extended domains, several stochastic and optimization based frameworks have been proposed to minimize detection time through coordinated search effort. Representative models include the collaborative optimization approach of Xie et al. [35,36], which is based on the artificial physical optimization (APO) paradigm and formulates the search task as a distributed exploration process over a spatial domain. Motivated by this line of work, we consider a generalized transport-based search model in which the search space is partitioned into N independent, non-overlapping square regions. Each region is explored through a structured family of expanding search trajectories originating from its center point $(h_i,k_i), i=1,2,\ldots,N$, and progressing in successive stages until detection occurs.

Region i is modeled as a square with an area of $A_i=4b_i^2$, with a side length of $2b_i, i=1,2...,N$, allowing a consistent geometric representation across all sub-domains. To improve efficiency and reduce the expected detection time, the search process is executed in parallel across regions by a collection of coordinated mobile searchers, each assigned to a single region and following the same deterministic expansion rule. This parallel and region-wise independent deployment significantly reduces the global search time and enhances spatial coverage, in line with the coordinated multi-agent search frameworks studied in Yang et al. [37] and Arshid et al. [38].

Effective coordination among searchers is achieved through information exchange, task allocation, and overlap avoidance mechanisms, which ensure non-redundant coverage and efficient utilization of available resources, as discussed in Liu et al. [39]. Furthermore, optimization-driven motion planning techniques, including particle swarm optimization and motion-encoded strategies, have been shown to enhance adaptability and robustness in stochastic search environments, particularly when the target dynamics are uncertain or time-varying; see Ghassemi and Chowdhury [40] and Phung and Ha [41]. From a broader perspective, the proposed formulation aligns with contemporary studies on coordinated mobile agent systems and swarm-based search dynamics, as surveyed in Alqudsi and Makaraci [42].

Figure 1 illustrates the local search mechanism within a representative region. Starting from the region center $(h_i, k_i), i = 1, 2, \ldots, N$, the searcher follows an expanding square trajectory along the positive x-direction. Each square $c_{ij}, i = 1, 2 \ldots, N, j = 1, 2, \ldots$, corresponds to a successive search loop with a step-length $\frac{a_{ij} - a_{i(j-1)}}{2}$, ensuring systematic and complete coverage of the region before transitioning to the subsequent stage of the overall two-stage framework. Each region $i, i = 1, 2, \ldots N$, is partitioned into an ordered collection of structured search paths, as schematically illustrated in Figure 1. The geometric reference of region i is defined by its center point (h_i, k_i), which serves as the origin of the local search process. From this reference point, a search agent initiates a continuous exploration along a prescribed expanding trajectory.

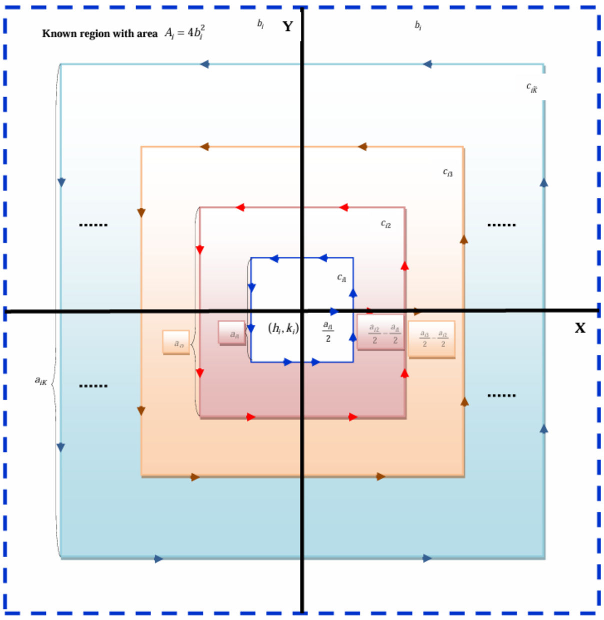

Figure 1. The expanding square search trajectory within a representative search region $i, i = 1, 2, \ldots N$, during Stage 1 localization.

The initial position of the target within region i is assumed to be uncertain and is modeled by a bivariate probability distribution with two independent truncated random variables X and Y, incorporating $\aleph$ deleted areas. This probabilistic formulation reflects incomplete spatial information and induces a transport-driven search problem within each region. Along the local *x*-direction, the region is further decomposed into a sequence of nested square sub-regions to ensure systematic and exhaustive coverage.

The search strategy within region i proceeds as follows. The search starts at the reference point (h_i, k_i) and advances along the positive local *x*-axis to the point $\left(h_i + \frac{a_{i1}}{2}, k_i\right)$. The first square subregion c_{i1}, together with its associated search track, is completely explored during this phase. If the target is not detected, the search continues outward along the same axis by an incremental distance $\frac{a_{i2} - a_{i1}}{2}$, leading to the exploration of the next square c_{i2}. This procedure is repeated iteratively with successive step-lengths $\frac{a_{i3} - a_{i2}}{2}, \frac{a_{i4} - a_{i3}}{2}, \ldots$, thereby generating an expanding sequence of square subregions $c_{i3}, c_{i4}, \ldots$. The process continues until the target is detected within region i, ensuring full spatial coverage consistent with the underlying probabilistic transport model. Since the N regions are independent and non-overlapping, each search agent operates within its assigned region according to an identical motion policy initiated from the reference point (h_i, k_i). The search processes are executed concurrently across all regions, which guarantees full spatial coverage while significantly reducing the overall expected detection time through parallel exploration. Without loss of generality, the motion of each search agent is modeled along the positive *x*-direction, following the predefined sequence of expanding steps. At a constant unit speed, the agent sequentially scans the corresponding family of search squares $c_{ij}, j = 1, 2, \ldots$, together with their associated coverage paths, thereby inducing a structured transport process over the region.

Let (X, Y) denote two independent random variables with $\aleph$ deleted areas, a regular symmetric bivariate distribution, and with a joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} \Omega^{-1} g_{X,Y}(x,y), & \begin{array}{l} \text{if } x \in (-\infty, a_1) \cup (\beta_1, a_2) \cup (\beta_2, a_3) \cup \ldots \cup (\beta_{\aleph-1}, a_{\aleph}) \cup (\beta_{\aleph}, \infty) \\ \text{and } y \in (-\infty, \vartheta_1) \cup (\zeta_1, \vartheta_2) \cup (\zeta_2, \vartheta_3) \cup \ldots \cup (\zeta_{\aleph-1}, \vartheta_{\aleph}) \cup (\zeta_{\aleph}, \infty), \end{array} \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

where $g(.)$, $G(.)$ refers to the original probability density and cumulative distribution functions of the target's position.

$$\Omega = 1 - \sum_{\bar{j}=1}^{\aleph+1} \left[G_{X,Y}(\alpha_{\bar{j}}, \vartheta_{\bar{j}}) - G_{X,Y}(\beta_{\bar{j}-1}, \vartheta_{\bar{j}}) - G_{X,Y}(\alpha_{\bar{j}}, \zeta_{\bar{j}-1}) + G_{X,Y}(\beta_{\bar{j}-1}, \zeta_{\bar{j}-1}) \right],$$
$$\alpha_{\aleph+1} = \vartheta_{\aleph+1} = \infty, \ \beta_0 = \zeta_0 = -\infty.$$

Moreover,

$$\ell = \{-\infty < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_{\aleph-1} < \beta_{\aleph-1} < \alpha_{\aleph} < \beta_{\aleph} < +\infty\},$$
$$\wp = \{-\infty < \vartheta_1 < \zeta_1 < \vartheta_2 < \zeta_2 < \ldots < \vartheta_{\aleph-1} < \zeta_{\aleph-1} < \vartheta_{\aleph} < \zeta_{\aleph} < +\infty\},$$

are finite sequences of the mesh points on the original coordinates respectively; see El-Hadidy and Alraddadi [43]. These variables represent the target's position within these search regions and have a symmetric distribution with the cumulative distribution function

$$F_{X,Y}(x,y)=\begin{cases}\Omega^{-1}\left(\Gamma_0+G_{X,Y}(x,y)\right), & if\ x\in(-\infty,\alpha_1),\ y\in(-\infty,\vartheta_1)\\ \Omega^{-1}\left(\Gamma_1+G_{X,Y}(x,y)-G_{X,Y}(\beta_1,y)-G_{X,Y}(x,\zeta_1)+G_{X,Y}(\beta_1,\zeta_1)\right), & if\ x\in(\beta_1,\alpha_2),\ y\in(\zeta_1,\vartheta_2)\\ \Omega^{-1}\left(\Gamma_2+G_{X,Y}(x,y)-G_{X,Y}(\beta_2,y)-G_{X,Y}(x,\zeta_2)+G_{X,Y}(\beta_2,\zeta_2)\right), & if\ x\in(\beta_2,\alpha_3),\ y\in(\zeta_2,\vartheta_3)\\ \Omega^{-1}\left(\Gamma_3+G_{X,Y}(x,y)-G_{X,Y}(\beta_3,y)-G_{X,Y}(x,\zeta_3)+G_{X,Y}(\beta_3,\zeta_3)\right), & if\ x\in(\beta_3,\alpha_4),\ y\in(\zeta_3,\vartheta_4)\\ \quad\cdots & \quad\cdots\\ \Omega^{-1}\left(\Gamma_{\aleph-1}+G_{X,Y}(x,y)-G_{X,Y}(\beta_{\aleph-1},y)-G_{X,Y}(x,\zeta_{\aleph-1})+G_{X,Y}(\beta_{\aleph-1},\zeta_{\aleph-1})\right), & if\ x\in(\beta_{\aleph-1},\alpha_{\aleph}),\ y\in(\zeta_{\aleph-1},\vartheta_{\aleph})\\ \Omega^{-1}\left(\Gamma_{\aleph}+G_{X,Y}(x,y)-G_{X,Y}(\beta_{\aleph},y)-G_{X,Y}(x,\zeta_{\aleph})+G_{X,Y}(\beta_{\aleph},\zeta_{\aleph})\right), & if\ x\in(\beta_{\aleph},\infty),\ y\in(\zeta_{\aleph},\infty)\end{cases}, \tag{2.2}$$

where

$$\Gamma_{\aleph}=\sum_{\tilde{j}=1}^{\aleph}\left[G_{X,Y}(\alpha_{\tilde{j}},\vartheta_{\tilde{j}})-G_{X,Y}(\beta_{\tilde{j}-1},\vartheta_{\tilde{j}})-G_{X,Y}(\alpha_{\tilde{j}},\zeta_{\tilde{j}-1})+G_{X,Y}(\beta_{\tilde{j}-1},\zeta_{\tilde{j}-1})\right],\ \Gamma_0=0,$$

and

$$G_{X,Y}(-\infty,y)=G_{X,Y}(x,-\infty)=G_{X,Y}(-\infty,-\infty)=0.$$

Each search region is modeled as a bounded square domain to ensure a tractable geometric representation of the transport process. For region i, the half side length b_i is defined by its spatial boundaries $b_i=\beta_i-\alpha_{i-1}=\vartheta_i-\zeta_{i-1}$, where α_{i-1} and β_i denote the horizontal limits of the region, and ζ_{i-1} and ϑ_i represent the corresponding vertical limits. The center of the region i is therefore located at $(h_i,k_i)=\left(\dfrac{\beta_i+\alpha_{i-1}}{2},\dfrac{\vartheta_i+\zeta_{i-1}}{2}\right)$, and the total area of this region is given by $A_i=4b_i^2$. This formulation provides a consistent geometric framework for a collection of independent, non-overlapping transport domains within the search space. Within each region $i, i=1,2,...N$, the search process is represented by a sequence of expanding square contours $c_{ij}, i=1,2..., N, j=1,2,...$ each associated with a transport path of effective width $\dfrac{a_{ij}-a_{i(j-1)}}{2}$. As the search agent traverses the contour C_{ij}, it covers a strip of this width, thereby inducing a progressive spatial exploration of the domain. Owing to the stochastic nature of the target's initial position, the total search cost in this stage is treated as a random variable.

It is assumed that detection occurs during the *K-th* expansion cycle. Starting from the regional center (h_i,k_i), the search agent follows a predefined sequence of transport paths $\Delta_{ij}, i=1,2,...,N, j=1,2,...,$ where the initial path Δ_{i1} corresponds to the first expansion loop, and each subsequent path $\Delta_{i(j+1)}$ preserves the same geometric structure while being initiated after an outward displacement of length $\dfrac{a_{ij}-a_{i(j-1)}}{2}, i=1,2,...,N, j=1,2,...,$ along the positive transport direction. Consequently, the complete search strategy $\Delta_{ij}, i=1,2,...,N, j=1,2,...,$ in region *i* is fully characterized by the ordered sequence $\left\{\dfrac{a_{ij}-a_{i(j-1)}}{2}\right\}_{i=1,2,...,N,\, j=1,2,...,}$, with $a_{i0}=0\le\dfrac{a_{i1}}{2}\le\dfrac{a_{i2}}{2}\le\dfrac{a_{i3}}{2}\le...,$ with $\dfrac{a_{ij}}{2}\to b_i$ This transport-based search construction is applied independently over all regions. A single search agent is assigned to each region, and all agents initiate their trajectories simultaneously from their respective centers. The resulting parallel deployment ensures complete spatial coverage of the entire domain and yields a substantial reduction in the expected detection time within the stochastic transport framework.

For a collection of independent and non-overlapping search regions, the total search duration is defined as the cumulative time expended along the corresponding coordinate directions within each region. In region $i = 1, 2, ..., N$, the time required to fully scan any expanding square $c_{ij}, i = 1, 2..., N, j = 1, 2, ...$, together with its associated trajectory, is given by $4a_i$.

On the basis of the probability distribution of the target's initial position specified in (2.2), let Q_i denote the class of admissible transport paths within region *i*, and let $D(\Delta_i, F)$ represent the total transport time needed to reach a point (X, Y) from the regional reference point (h_i, k_i) along a given path $\Delta_{ij}, i = 1, 2, ..., N, j = 1, 2, ...$ As the target location is random, $D(\Delta_i, F)$ is naturally modeled as a random variable for each region.

Since the regions are independent, the global search duration is obtained by aggregating these independent transport times. Accordingly, the objective of this stage is to evaluate the expected total transport cost required to reach the target when its initial position follows a symmetric truncated bivariate probability distribution with $\aleph$ deleted areas.

We now derive the expected value of the total search time (or cost) required to detect the target when the search process is performed simultaneously over *N* independent and non-overlapping regions. Within this stochastic transport framework, each search agent operates locally within its assigned region according to the same expanding square search policy, and the detection time in each region is modeled as an independent random variable $D(\Delta_i, F),\ i = 1, 2, ..., N$. The overall expected detection time is then obtained by aggregating these independent regional contributions. The resulting expression represents the global expected cost of first detection for a target whose initial position follows a symmetric truncated bivariate probability distribution with $\aleph$ deleted areas.

Theorem 1. *Let* $\Delta_i = \{\Delta_{ij}, j = 1, 2, ...\} \in Q_i$ *denote the admissible expanding-square transport paths in region i. The expected total cost of first detection over N independent regions is given by*

$$D(\mathbf{\Delta}, F) = \sum_{i=1}^{N} D(\Delta_i, F) \le 2\sum_{i=1}^{N}\sum_{j=1}^{\infty}\left(9a_{ij} - a_{i(j-1)}\right)\left(\int_0^{k_i+\frac{a_{ij}}{2}}\int_0^{h_i+\frac{a_{ij}}{2}} f(x,y)dxdy - \int_0^{k_i+\frac{a_{i(j-1)}}{2}}\int_0^{h_i+\frac{a_{i(j-1)}}{2}} f(x,y)dxdy\right), \tag{2.3}$$

where $f(x, y)$ *is defined in (2.1) and* a_{ij} *represents the associated expansion lengths.*

Proof. Consider *N* independent and non-overlapping regions, and let the initial position of the target be distributed according to the joint probability density function $f(x, y)$ defined in (2.1). For each region *i*, we introduce an increasing sequence of non-negative parameters $\{a_{ij}\}_{i=1,2,...,N, j=1,2,...}$, such that $a_{i0} = 0 \le \frac{a_{i1}}{2} \le \frac{a_{i2}}{2} \le \frac{a_{i3}}{2} \le ...$, with $\frac{a_{ij}}{2} \to b_i$, which characterizes the successive expansion of the local search domain within region *i*. Let $\Delta_i = \{\Delta_{ij}, j = 1, 2, ...\} \in Q_i$ denote the admissible sequence of expanding search paths executed in region *i*, where Q_i is the class of admissible search trajectories. The corresponding detection time $D(\Delta_i)$ is modeled as a random variable that depends on the subregion in which the target is first encountered. Specifically, if the target lies within the first square c_{i1}, the detection time is given by $D(\Delta_i)=\frac{1}{2}a_{i1}+4a_{i1}$. If the target is located within the second square c_{i2}, then $D(\Delta_i)=\frac{1}{2}(a_{i2}-a_{i1})+4a_{i2}$, and, more generally, if the target lies in the *j-th* square $c_{ij}, i=1,2...,N, j=1,2,...$, we obtain $D(\Delta_i)=\frac{1}{2}\left(a_{ij}-a_{i(j-1)}\right)+4a_{ij}$. Accordingly, the expected detection time within region *i* depends on the cumulative transport distances associated with these expanding search paths. Consequently, we have

$$
\begin{aligned}
D(\Delta_i,F)\le \Bigg[&\left(\frac{1}{2}a_{i1}+4a_{i1}\right)\left(\int_{k_i-\frac{a_{i1}}{2}}^{k_i+\frac{a_{i1}}{2}}\int_{h_i-\frac{a_{i1}}{2}}^{h_i+\frac{a_{i1}}{2}} f(x,y)dxdy\right)\\
&+\left(\frac{1}{2}(a_{i2}-a_{i1})+4a_{i2}\right)\left(\int_{k_i-\frac{a_{i2}}{2}}^{k_i+\frac{a_{i2}}{2}}\int_{h_i-\frac{a_{i2}}{2}}^{h_i+\frac{a_{i2}}{2}} f(x,y)dxdy-\int_{k_i-\frac{a_{i1}}{2}}^{k_i+\frac{a_{i1}}{2}}\int_{h_i-\frac{a_{i1}}{2}}^{h_i+\frac{a_{i1}}{2}} f(x,y)dxdy\right)\\
&+\left(\frac{1}{2}(a_{i3}-a_{i2})+4a_{i3}\right)\left(\int_{k_i-\frac{a_{i3}}{2}}^{k_i+\frac{a_{i3}}{2}}\int_{h_i-\frac{a_{i3}}{2}}^{h_i+\frac{a_{i3}}{2}} f(x,y)dxdy-\int_{k_i-\frac{a_{i2}}{2}}^{k_i+\frac{a_{i2}}{2}}\int_{h_i-\frac{a_{i2}}{2}}^{h_i+\frac{a_{i2}}{2}} f(x,y)dxdy\right)+\ldots\Bigg].
\end{aligned}
$$

Integrating with respect to the symmetric truncated bivariate distribution of the target's initial position yields the expected regional detection cost. Owing to the symmetry of $f(x,y)$ about the regional center, this expectation admits a simplified closed form representation as follows:

$$
D(\Delta_i,F)\le 2\sum_{j=1}^{\infty}\left(9a_{ij}-a_{i(j-1)}\right)\left(\int_0^{k_i+\frac{a_{ij}}{2}}\int_0^{h_i+\frac{a_{ij}}{2}} f(x,y)dxdy-\int_0^{k_i+\frac{a_{i(j-1)}}{2}}\int_0^{h_i+\frac{a_{i(j-1)}}{2}} f(x,y)dxdy\right).
$$

Finally, since the *N* regions are probabilistically independent and spatially non-overlapping, the global expected cost of first detection is obtained by summing the corresponding expectations over all regions $i=1,2,...,N$. This aggregation yields the total expected detection cost stated as in (2.3).□

After the optimal allocation of search effort has been determined and the target's initial position has been localized through the Stage 1 model, the problem naturally transitions to the analysis of the target's subsequent motion. In this second stage, the search process is reformulated as a stochastic transport problem, where the target evolves according to a multidimensional independent random walk originating from the estimated initial location. The coordinated searchers then switch from region-based exploration to a transpor-driven tracking regime along prescribed trajectories. The main objective of this stage is to characterize and optimize the tracking policy that minimizes the expected first-encounter time between the moving target and the searchers, conditioned on the information provided by Stage 1.

2.2. Optimal allocation of search effort for initial target localization

The selection of the step-lengths a_{ij} within each independent region is formulated as an optimization problem whose objective is to minimize the expected detection time $D(\Delta,F)$ of the target's initial location. This formulation treats the search process as a stochastic transport mechanism, where the expanding search fronts govern the rate at which the probability mass is intercepted. It is well established that the overall efficiency of such transport-driven search processes is highly sensitive to the choice of the inter expansion distances, as these distances directly control the trade-off between coverage speed and redundant traversal.

Previous studies have shown that optimal expansion policies emerge from balancing probabilistic gain against geometric cost. In particular, Xie et al. [35,36] demonstrated that, within artificial physics-based search models, the optimal spacing between successive expansion loops results from an equilibrium between attraction toward regions of high probability density and repulsive effects that prevent excessive overlap. From a transport perspective, this balance ensures efficient propagation of the search front through the domain. Similarly, Ghassemi and Chowdhury [40] proposed an informative path planning framework with local penalization, where the step-lengths adapt to spatial gradients of the posterior probability map: Shorter expansions are used in regions with a concentrated probability mass, while longer steps are used in low-density regions, leading to accelerated convergence toward the target.

More recent cooperative search frameworks extend this principle by embedding step-length optimization within dynamic allocation and control schemes. Arshid et al. [38] and Liu et al. [39] showed that casting the distance selection problem as a constrained optimization either through one-dimensional convex minimization in each region or via dynamic programming over successive expansions yields solutions that are both analytically tractable and computationally efficient. In addition, Yang et al. [37] established that the influence of environmental uncertainty on the optimal distance control policy is relatively limited with respect to the overall expected detection time, reinforcing the robustness of transport-based optimization strategies in multi-agent search settings.

From a probabilistic transport perspective, the optimal step-length a_{ij} is defined as the value that minimizes an upper bound on the expected detection time established in Theorem 1. This criterion is naturally linked to the distribution of probability mass over the search domain and can be implemented by evaluating the marginal probability contained within each expanding square c_{ij}. Accordingly, the optimal choice of a_{ij} corresponds to a balance between the incremental gain in detection probability and the additional transport cost induced by spatial expansion such that

$$\frac{\partial D(\Delta, F)}{\partial a_{ij}} = \frac{\sum_{i=1}^{N} \partial D(\Delta_i, F)}{\partial a_{ij}} = 0.$$

When a closed-form characterization of the optimal step-length is not available, the problem reduces to a one-dimensional numerical optimization over admissible distances. In this case, standard computational techniques, such as golden section searching or gradient-based methods, can be used to determine the step-length that minimizes the expected detection time under full regional coverage. For environments exhibiting temporal variability or structural complexity, adaptive distance selection mechanisms may be introduced, allowing the step-length a_{ij} to be updated dynamically in response to observational feedback and system-related constraints. Such adaptive schemes, including particle swarm optimization and learning-based controllers, have been shown to achieve an effective compromise between detection efficiency, transport effort, and computational cost (Phung and Ha [41]; Alqudsi and Makaraci [42]).

Within this framework, let $\Delta_i^* = \left\{\Delta_{ij}^*, j = 1,2,...\right\}$ denote the optimal transport search trajectory associated with region $i = 1,2,...,N$. The path Δ_i^* is said to be optimal if the induced sequence of transport step-lengths $\frac{a_{ij} - a_{i(j-1)}}{2}$ achieves the minimum expected detection time among all admissible paths. Equivalently, optimality holds when the sequence $\frac{a_{ij}^* - a_{i(j-1)}^*}{2}, i = 1,2,...,N, j = 1,2,...,$ satisfies the probabilistic optimality condition derived from Theorem 1.

The problem therefore reduces to determining the optimal sequence of turning points $\frac{a_{ij}}{2}$ that minimizes the expected total detection time associated with a given joint probability density function $f(x, y)$ in (2.1) over each independent region. From a theoretical perspective, this formulation can be viewed as a stochastic transport problem in which a searching agent traverses a spatial domain according to a prescribed expansion rule, and detection corresponds to a first-hitting event. The resulting optimization framework naturally extends the classical linear search model studied in Mohamed et al. [44], where a single searcher moves along a one-dimensional line at a unit of speed and seeks to minimize the expected time to encounter a randomly located target governed by a specific probability distribution. In contrast to the one-dimensional setting, the present model generalizes this transport-based search paradigm to bounded two-dimensional regions, where coordinated searchers operate over independent square domains following expanding trajectories. The objective is to determine an optimal sequence of transport distances that balances geometric expansion with probabilistic gain, thereby minimizing the expected search cost associated with the initial localization of the target.

Hence, if $\Delta_i^* \in Q_i$ denotes the optimal search path associated with region *i*, the globally optimal search configuration is given by $\Delta^* = \left\{\Delta_1^*, \Delta_2^*, ..., \Delta_N^*\right\} \in Q$, where $Q = Q_1 \times Q_2 \times ... \times Q_N$. In this formulation, each regional search path is governed by two distinct sources of uncertainty: The probabilistic description of the target's initial location, represented by the distribution $F(x, y)$ in (2.2), and the sequence of decision variables $\left\{a_{ij}\right\}_{i=1,2,...,N, j=1,2,...}$ that determine the spatial allocation of search effort. The interaction between these stochastic and decision components renders the optimization problem inherently nonlinear and multidimensional. Accordingly, determining the globally optimal allocation of search effort across independent regions can be cast as the following nonlinear optimization problem (NLOP1), whose solution characterizes the optimal transport of search resources that minimizes the expected time to initial target localization.

NLOP1:

$$\min_{a_{ij}} \ \mathrm{D}(\Delta, F)$$

$$\text{subject to } 4\sum_{i=1}^{N} \int_{k_i}^{k_i+b_i} \int_{h_i}^{h_i+b_i} f(x, y)dxdy = 1,$$

$$\text{subject to } 4\sum_{i=1}^{N} \int_{k_i}^{k_i+b_i} \int_{h_i}^{h_i+b_i} f(x, y)dxdy = 1, \frac{a_{ij}}{2} - \frac{a_{i(j-1)}}{2} > 0,\ \frac{a_{ij}}{2} \le b_i, i = 1,2,...,N, j = 1,2,...,$$

where $a_{i0} = 0 \le \frac{a_{i1}}{2} \le \frac{a_{i2}}{2} \le \frac{a_{i2}}{2} \le .....$

These constraints jointly characterize the admissible geometric and probabilistic structure of each region $i=1,2,...,N$, guaranteeing full spatial coverage, the mutual independence of the search trajectories, and minimization of the overall expected detection time. The resulting optimization problem (NLOP1) therefore represents a constrained stochastic transport problem, where the decision variables a_{ij} govern the propagation of search effort through the spatial domain.

To determine the optimal values of a_{ij} that minimize the expected detection cost $D(\Delta, F)$, the Kuhn-Tucker optimality conditions were used. Accordingly, the Lagrangian function was formulated by combining the objective functional with the associated geometric and probabilistic constraints as follows:

$$L(a_{ij}, \lambda_i, \mu_{ij}) = D(\mathbf{\Delta}, F) + \sum_{i=1}^{N} \lambda_i \tilde{g}_i(a_{ij}) - \sum_{i=1}^{N} \sum_{j=1}^{\infty} \left[\mu_{ij}^{(1)} \left(a_{i(j-1)} - a_{ij} \right) + \mu_{ij}^{(2)} \left(a_{ij} - 2b_i \right) \right],$$

where λ_i, $\mu_{ij}^{(1)} \geq 0$, and $\mu_{ij}^{(2)} \geq 0$ denote the Lagrange multipliers associated with the equality and inequality constraints, respectively.

The necessary conditions for optimality are obtained by differentiating the Lagrangian with respect to each a_{ij}

$$\frac{\partial L(a_{ij}, \lambda_i, \mu_{ij})}{\partial a_{ij}} = \frac{\partial D(\mathbf{\Delta}, F)}{\partial a_{ij}} + \frac{\partial \lambda_i \tilde{g}_i(a_{ij})}{\partial a_{ij}} + \mu_{ij}^{(2)} - \mu_{ij}^{(1)} = 0,$$

where $\tilde{g}_i(a_{ij}) = 1 - \int_{k_i}^{k_i + a_{ij}} \int_{h_i}^{h_i + a_{ij}} f(x, y) dx dy,$ together with the feasibility and complementary slackness relations for all i, j

$$a_{ij} - a_{i(j-1)} \geq 0,\ 2b_i - a_{ij} \geq 0,\ \mu_{ij}^{(1)} \geq 0,\ \mu_{ij}^{(2)} \geq 0,\ \mu_{ij}^{(1)} \left(a_{ij} - a_{i(j-1)} \right) = 0 \text{ and } \mu_{ij}^{(2)} \left(2b_i - a_{ij} \right) = 0.$$

By substituting the analytical form of $D(\Delta, F)$ given by Eq (2.3) and calculating its derivative, one gets,

$$\frac{\partial D(\mathbf{\Delta}, F)}{\partial a_{ij}} = 18 a_{ij} P_i(a_{ij}),$$

where $P_i(a_{ij}) = \int_{k_i}^{k_i + a_{ij}} \int_{h_i}^{h_i + a_{ij}} f(x, y) dx dy,$ is the cumulative probability mass of the target initial position within the expanding square c_{ij} of side a_{ij}, $i = 1, 2, ..., N$, $j = 1, 2, ...$ Hence, the stationary condition becomes

$$18 a_{ij} P_i(a_{ij}) + \mu_{ij}^{(2)} - \mu_{ij}^{(1)} = 0.$$

Solving this system under the Karush-Kuhn-Tucker (KKT) framework [45,46] characterizes the equilibrium governing the stochastic transport of search effort across independent regions. The resulting conditions formalize the trade-off between the marginal gain in detection probability and the associated cost induced by spatial expansion of the search domain. Consequently, the KKT structure establishes a rigorous analytical criterion for global optimality, ensuring that the allocation of search distances minimizes the expected transport time to initial target localization.

3. Stage 2: Stochastic transport and first-encounter analysis for a multidimensional random walk target

Conditioned on the localization outcome obtained in Stage 1, the second stage formulates a stochastic transport and tracking problem in which coordinated searchers pursue a target whose motion is governed by a multidimensional independent random walk. The search space consists of a finite collection of straight trajectories intersecting at the estimated initial position of the target, which serves as the reference point of the tracking framework. The target evolves along one of these trajectories according to an independent d-dimensional random walk, and the probability of its presence exactly at the intersection point is assumed to be zero.

Within this setting, each searcher patrols a bounded segment of its assigned trajectory by executing symmetric outbound and return motions. The collective objective of this stage is to minimize the expected value of the first-encounter time between any searcher and the randomly moving target. The theoretical basis of this formulation builds upon classical linear search models for random walk targets, as developed in El-Hadidy [33] and El-Hadidy and Abou-Gabal [34], where conditions ensuring the existence of finite optimal encounter times were established in one-dimensional settings. The present framework generalizes these results to intersecting trajectories embedded in higher-dimensional Euclidean space, yielding a multidimensional random walk transport model for pursuit and detection.

The primary objective of this stage is to establish the existence of a feasible and finite tracking strategy, namely, to verify that the expected first-encounter time remains finite under appropriate geometric and probabilistic conditions. In particular, attention is given to the role of balance in the target's random walk dynamics, where the absence of directional drift contributes to reachability, but finiteness is guaranteed only when it is combined with bounded tracking geometry, finite tracking distances, recurrent coverage, and a uniform positive encounter probability. This existence result provides the theoretical foundation for the tracking distances derived in the subsequent analysis and confirms the internal consistency of the two-stage stochastic transport framework.

From an applied standpoint, the proposed formulation captures a broad class of search-and-pursuit scenarios arising in uncertain environments. For instance, in search-and-rescue operations following aviation or maritime incidents, coordinated sensing agents may be deployed along intersecting paths centered around an estimated location, while the target's motion is influenced by stochastic environmental effects such as wind or currents. In such contexts, the probability of immediate localization is negligible, and performance is determined by the expected time to first encounter rather than instantaneous detection.

Recent studies strongly support the effectiveness of multi-stage and multidirectional tracking strategies under stochastic motion uncertainty. Liu et al. [1] and Wang et al. [2] demonstrated that adaptive second-stage pursuit policies significantly reduce expected detection times, while Feng et al. [5] and Wu et al. [6] showed that cooperative tracking with probabilistic prediction yields robust performance under limited sensing and motion uncertainty. Experimental validation of probability-driven search strategies was also reported in Dumenčić et al. [9], confirming their practical relevance in complex environments. Accordingly, the proposed Stage 2 model constitutes both a mathematical extension of linear random walk search theory and a unified stochastic transport framework for first-encounter analysis on intersecting trajectories. The existence results established in this section guarantee the finiteness of the expected first-encounter time under the bounded controlled tracking conditions stated above, rather than under unbiased random walk dynamics alone.

Remark 1. In Stage 1, encounter events are excluded so that all first-encounter analysis is confined to the Stage 2 stochastic transport model.

In the second stage, the problem is formulated as a stochastic transport and tracking framework in a d-dimensional Euclidean space composed of n mutually intersecting straight trajectories, which represent the admissible transport paths of the moving target. All trajectories intersect at a single reference point $\Theta_{\tilde{i}0}^{(u)}$, $\tilde{i}=1,2,\dots,n$, corresponding to the estimated initial location obtained from Stage 1. Conditioned on this estimate, the target evolves along one of these trajectories according to an independent d-dimensional random walk $\{S(t),\ t\in I^{+}\}$, (I^{+} is the set of +*ve* integer numbers), indexed by discrete time t. The target position at time t is described by a random vector $\boldsymbol{X(t)}=(X_1(t),X_2(t),\dots,X_d(t))$ whose increments $\tilde{\Delta}X_u(t)=X_{u+1}(t)-X_u(t)$ are independent and identically distributed random variables, governed by a probability vector $\boldsymbol{P}=(p^{(1)},p^{(2)},\dots,p^{(d)})$ that specifies the transition probabilities along each admissible direction, where $p^{(u)}$ denotes the probability of successfully detecting the target in the direction u, $u=1,2,\dots,d$.

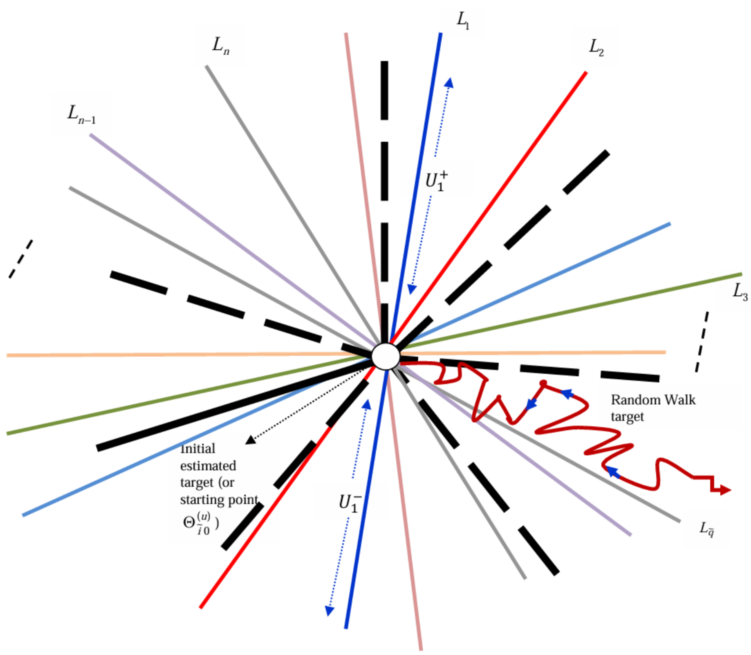

Figure 2. Tracking scheme of the multidimensional random walk target along intersecting trajectories in Stage 2.

The initial position of the target $\boldsymbol{X(0)}=(X_1(0),X_2(0),\dots,X_d(0))$ is modeled as a random vector whose distribution is inherited from the localization outcome of Stage-1. A coordinated set of $2n$ unit-speed search agents (see, Figure 2) is deployed along the intersecting trajectories, such that for each trajectory (or line) $L_{\tilde{i}},\tilde{i}=1,2,\dots,n$, two agents denoted by $U_{\tilde{i}}^{+}$ and $U_{\tilde{i}}^{-}$ start from a common point on $L_{\tilde{i}}$ and move symmetrically in opposite directions with respect to the intersection point. Each agent executes a sequence of alternating outbound and return motions, characterized by prescribed transport distances $\{H_{\tilde{i}1},H_{\tilde{i}2},\dots\}$, such that $H_{\tilde{i}0}=0$ and $H_{\tilde{i}\tilde{k}}>0,\tilde{i}=1,2,\dots n;\tilde{k}=1,2,\dots$, that define successive tracking segments along the assigned trajectory $L_{\tilde{i}}$. A first-encounter event is declared when at least one search agent comes within a fixed detection radius $\tilde{\rho} > 0$ of the moving target. The following assumptions define the stochastic transport and tracking framework of Stage 2. All search agents move with a unit of speed and follow symmetric transport trajectories about the intersection point. The target evolves independently according to a multidimensional random walk, whose dynamics are statistically decoupled from the motion of the search agents. The probability that the target is located exactly at the intersection point is assumed to be zero. The transport domain is bounded, so that each search agent patrols a finite segment of its assigned trajectory, and detection occurs instantaneously when at least one search agent enters the fixed detection radius $\tilde{\rho} > 0$ of the target.

Let $\tau_{\tilde{i},\tilde{\rho}}$ denote the first-encounter time between the target and the search agent $U_{\tilde{i}}$ within the detection radius $\tilde{\rho}$, and let $\tau_{\tilde{\rho}} = \min_{\tilde{i}} \tau_{\tilde{i},\tilde{\rho}}$ be the earliest encounter time. The central objective of this stage is to establish the existence of a finite tracking strategy by proving that the expected first-encounter time $E(\tau_{\tilde{\rho}})$ remains finite under the imposed geometric and probabilistic transport conditions. Formally, the complete search configuration is $\boldsymbol{\Theta} = \{\Theta_1, \Theta_2, \ldots, \Theta_n\} \in \tilde{\boldsymbol{Q}}$ ($\tilde{\boldsymbol{Q}} = \tilde{Q}_1 \times \tilde{Q}_2 \times \ldots \times \tilde{Q}_n$ where $\tilde{Q}_{\tilde{i}}$ is the class of admissible search paths within region $\tilde{i}, \tilde{i} = 1,2,\ldots n$), where each sequence $\Theta_{\tilde{i}} = \left\{H_{\tilde{i}1}^{(u)}, H_{\tilde{i}2}^{(u)}, \ldots\right\}$ depends on the random variables $\xi_{\tilde{i}\tilde{k}}^{(u)}$ and a set of unknown parameters $\boldsymbol{\theta} = \{\theta_1, \theta_2, \ldots, \theta_n\}$ (e.g., the diffusion rate or expected target velocity) such that $H_{\tilde{i}\tilde{k}}^{(u)} = \tilde{f}(\theta_{\tilde{i}}, \xi_{\tilde{i}\tilde{k}}^{(u)})$. Therefore, we have a probability vector

$$\left[\sum_{u=1}^{d} P(\xi_{I1}^{(u)} = -1) = p_n^{(1)}, \ldots, \sum_{u=1}^{d} P(\xi_{In}^{(u)} = -1) = p_n^{(u)}\right] + \left[\sum_{u=1}^{d} P(\xi_{I1}^{(u)} = +1) = q_1^{(u)}, \ldots, \sum_{u=1}^{d} P(\xi_{In}^{(u)} = +1) = q_n^{(u)}\right]$$
$$= \left[\sum_{u=1}^{d} p_1^{(u)} + q_1^{(u)}, \ldots, \sum_{u=1}^{d} p_n^{(u)} + q_n^{(u)}\right],$$

which holds where

$$[p_1, p_2, \ldots, p_d] > [0, 0, \ldots, 0].$$

Moreover, $S(t) = (S_1(t), S_2(t), \ldots, S_d(t)) = \left(\sum_{i=1}^{t} \xi_{\tilde{i}1}^{(1)}, \sum_{i=1}^{t} \xi_{\tilde{i}1}^{(2)}, \ldots, \sum_{i=1}^{t} \xi_{\tilde{i}n}^{(d)}\right)$ is an independent d-dimensional random walk where $t > 0,\ S_u(0) = 0$ and $[\xi_0^{(1)}, \xi_0^{(2)}, \ldots, \xi_0^{(d)}]$ is the initial random vector of the initial target position.

Assuming that $\Theta_0^{(u)}$ represents the distance measured from the intersection point (the origin of the tracking framework) to the starting position of $U_{\tilde{i}}^{+}$ and $U_{\tilde{i}}^{-}$ along the line $L_{\tilde{i}}, \tilde{i} = 1,2,\ldots,n$ in the direction of motion. Also, the tracking plans of these search agents are $\Theta_{\tilde{i}}^{+} : \Re^{+} \to \Re$ and $\Theta_{\tilde{i}}^{-} : \Re^{+} \to \Re,\ \tilde{i} = 1,2,\ldots,n,$ respectively. The corresponding tracking plans can be expressed as follows

$$\left|\Theta_{\tilde{i}}^{+}(t_1) - \Theta_{\tilde{i}}^{+}(t_2)\right| = \left|\Theta_{\tilde{i}}^{-}(t_1) - \Theta_{\tilde{i}}^{-}(t_2)\right| \le \left|t_1 - t_2\right| \forall t_1, t_2 \in I^{+}, \tag{3.1}$$

the expected first-encounter time is then minimized as $E(\tau(\Theta)) = \min_{\boldsymbol{\Theta} \in \tilde{\boldsymbol{Q}}} E(\tau)$ (i.e., finding the random vector $\boldsymbol{\Theta}^{*} \in \tilde{\boldsymbol{Q}}$ that minimizes the expected detection time). We let the probability space $(\tilde{\Omega}, \Lambda, \tilde{P})$ represent all possible realizations of the target's and the search agents' trajectories, and each event (belonging to Λ) corresponds to a potential first detection configuration with probability measure $\tilde{P}$, where the first-encounter time and the target's location on the random path are described in $\tilde{\Omega}$. Under this stochastic framework, the existence of a finite expected encounter time ensures the theoretical validity and practical feasibility of the proposed multidimensional search agents tracking model, noting that in Stage 1 the target was assumed not to have encountered any search agent.

Following the approach introduced by El-Hadidy and Abou-Gabal [34], we define the total tracking distance at time t as the cumulative sum of these alternating displacements. To do this, we consider two alternating sequences, $\{W_{\tilde{i}\tilde{k}}^{(u)}\}_{\tilde{k}\geq 1,\tilde{i}=1,2,\dots,n,u=1,2,\dots,d}$ and $\{H_{\tilde{i}\tilde{k}}^{(u)}\}_{\tilde{k}\geq 1,\tilde{i}=1,2,\dots,n,u=1,2,\dots,d}$, where each element $H_{\tilde{i}\tilde{k}}^{(u)}$ represents the distance covered by the search agents along the line $L_{\tilde{i}}, \tilde{i}=1,2,\dots,n$ during the $\tilde{k}^{th}$ forward or return tracking cycle around the intersection point. As described above, these distances are expressed as functions of a set of independent random variables $\xi_{\tilde{i}\tilde{k}}^{(u)}$ and a group of unknown parameters $\boldsymbol{\theta}=\{\theta_1,\theta_2,\dots,\theta_n\}$, such that $H_{\tilde{i}\tilde{k}}^{(u)}=\tilde{f}(\theta_{\tilde{i}},\lambda_{\tilde{i}\tilde{k}}^{(u)})$, where $\left|\lambda_{\tilde{i}\tilde{k}}^{(u)}\right|>1, \tilde{i}=1,2,\dots,n; \tilde{k}=1,2,\dots$ reflects the stochastic variation caused by the random walk motion of the target, and $\theta_{\tilde{i}}>0$ denotes the physical and probabilistic parameters governing the search agents' tracking dynamics. Through these sequences, each search agent alternately expands its search range symmetrically about the intersection point, ensuring continuous coverage of the target's possible random walk trajectory.

Here, we consider

$$W_{\tilde{i}\tilde{k}}^{(u)}=2^{\frac{1}{2}[1-(-1)^{\tilde{k}+1}]}\lambda_{\tilde{i}\tilde{k}}^{(u)}\left(\left|\theta_{\tilde{i}}^{(u)}\right|^{\frac{\tilde{k}}{2}+\frac{1}{4}-(-1)^{\tilde{k}}\frac{1}{4}}-1\right). \tag{3.2}$$

Since the starting point $\Theta_{\tilde{i}0}^{(u)}$ of the search may lie on either side of the intersection point, two symmetric cases must be considered.

Case a: When the starting point lies on the positive side of the intersection point, the search agent scans the right-hand segment of the line $L_{\tilde{i}}, \tilde{i}=1,2,\dots,n$, such that

$$\dots<-H_{\tilde{i}(\tilde{k}+2)}^{(u)}<-\Theta_{\tilde{i}0}^{(u)}<-H_{\tilde{i}\tilde{k}}^{(u)}<-H_{\tilde{i}(\tilde{k}-1)}^{(u)}<\dots<-H_{\tilde{i}2}^{(u)}<-H_{\tilde{i}1}^{(u)}<0<H_{\tilde{i}1}^{(u)}<H_{\tilde{i}2}^{(u)}<H_{\tilde{i}3}^{(u)}<\dots,$$

and $W_{\tilde{i}\tilde{k}}^{(u)}\leq t\leq W_{\tilde{i}(\tilde{k}+1)}^{(u)}$ accordingly, the tracking trajectory on the right side of $L_{\tilde{i}}, \tilde{i}=1,2,\dots,n$ from the intersection point is

$$\Theta_{\tilde{i}}^{(u)+}(t)=\left\{\Theta_{\tilde{i}0}^{(u)+}+\frac{1}{2}(1+(-1)^{\tilde{k}+1})H_{\frac{\tilde{i}(\tilde{k}+1)}{2}}^{(u)}+(-1)^{\tilde{k}}(t-W_{\tilde{i}\tilde{k}}^{(u)})\right\}, \tag{3.3}$$

and the tracking path along the left-hand side of $L_{\tilde{i}}, \tilde{i}=1,2,\dots,n$ is

$$\Theta_{\tilde{i}}^{(u)-}(t)=\left\{\Theta_{\tilde{i}0}^{(u)-}-\frac{1}{2}(1+(-1)^{\tilde{k}+1})H_{\frac{\tilde{i}(\tilde{k}+1)}{2}}^{(u)}+(-1)^{\tilde{k}}(t-W_{\tilde{i}\tilde{k}}^{(u)})\right\}. \tag{3.4}$$

Case b: When the starting point lies on the negative side of the intersection point, the search agent scans the left-hand segment of $L_{\tilde{i}}, \tilde{i}=1,2,\dots,n$, such that

$$\ldots < -H_{\tilde{i}2}^{(u)} < -H_{\tilde{i}1}^{(u)} < 0 < H_{\tilde{i}1}^{(u)} - 2\Theta_{\tilde{i}0}^{(u)} < H_{\tilde{i}2}^{(u)} - 2\Theta_{\tilde{i}0}^{(u)} < \ldots < H_{\tilde{i}\tilde{k}}^{(u)} - 2\Theta_{\tilde{i}0}^{(u)} < -\Theta_{\tilde{i}0}^{(u)} < H_{\tilde{i}(\tilde{k}+2)}^{(u)} - 2\Theta_{\tilde{i}0}^{(u)} < \ldots,$$

and $W_{\tilde{i}\tilde{k}}^{(u)} \le t \le W_{\tilde{i}(\tilde{k}+1)}^{(u)}$, then the tracking trajectory in the right part of $L_{\tilde{i}}, \tilde{i} = 1,2,\ldots,n$ from the intersection point is:

$$\Theta_{\tilde{i}}^{(u)+}(t) = \left\{ -\Theta_{\tilde{i}0}^{(u)+} + \frac{1}{2}(1+(-1)^{\tilde{k}+1}) H_{\tilde{i}\frac{(\tilde{k}+1)}{2}}^{(u)} + (-1)^{\tilde{k}}(t - W_{\tilde{i}\tilde{k}}^{(u)}) \right\}, \tag{3.5}$$

and in the left part is:

$$\Theta_{\tilde{i}}^{(u)-}(t) = \left\{ -\Theta_{\tilde{i}0}^{(u)-} - \frac{1}{2}(1+(-1)^{\tilde{k}+1}) H_{\tilde{i}\frac{(\tilde{k}+1)}{2}}^{(u)} + (-1)^{\tilde{k}}(t - W_{\tilde{i}\tilde{k}}^{(u)}) \right\}. \tag{3.6}$$

The total search time in both cases equals the sum of the forward and return times, since all search agents move with a unit of speed. Then, the total time of searching and returning to the intersection point $\Theta_{\tilde{i}0}^{(u)}$, $\tilde{i} = 1,2,\ldots,n$, $u = 1,2,\ldots,d$ is $t = H_{\tilde{i}\tilde{k}}^{(u)} = W_{\tilde{i}(2\tilde{k}-1)}^{(u)} = \frac{1}{2} W_{\tilde{i}(2\tilde{k})}^{(u)}$. This structure provides a model for repeated bidirectional tracking of multiple search agents around the intersection point, ensuring complete coverage of the target's potential random walk trajectory in the Stage 2 framework.

Remark 2. Let $v_{\tilde{i}u}$, $\tilde{i} = 1,2,\ldots,n, u = 1,2,\ldots,d$ be defined on the d-dimensional Euclidean space $\boldsymbol{R}^d$ as a random vector $V_{\tilde{i}u} = [\xi_0^{(1)}, \xi_0^{(2)}, \ldots, \xi_0^{(d)}]$, where $\sum_{u=1}^{d} [v_{1u}, v_{2u}, \ldots, v_{nu}] = \left[\frac{\wp_1}{n}, \frac{\wp_2}{n}, \ldots, \frac{\wp_d}{n} \right]$, represents the normalized tracking velocity components of the search agents along one of the intersecting lines $L_{\tilde{i}}, \tilde{i} = 1,2,\ldots,n$ such that $\sum_{u=1}^{d} \wp_u = 1$.

Here, each $\wp_u$ denotes the instantaneous displacement of either the search agent or the randomly moving target in the u^{th} spatial direction. This vector formulation allows the search agents' tracking motion and the random walk dynamics of the target to be expressed within the same stochastic d-dimensional tracking framework, ensuring consistent analysis of their interaction during the second stage of the search process. The random vector $V_{\tilde{i}u}$ defined above can be directly used to evaluate the instantaneous Euclidean distance between the moving target and the search agents during the Stage 2 tracking process.

Let $\boldsymbol{r}_{U_{\tilde{i}}}(t)$ and $\boldsymbol{X}(\mathrm{t})$ denote the position vectors of the $\tilde{i}^{th}$ search agent and the random walk of the target's position, respectively, at time t. The instantaneous distance between them is then expressed as

$$D_{\tilde{i}}(t) = \left\| \boldsymbol{X}(\mathrm{t}) - \boldsymbol{r}_{U_{\tilde{i}}}(t) \right\|^2 = \sqrt{\sum_{u=1}^{d} \left(tV_{\tilde{i}u}(t) - X_u(t) \right)^2}.$$

Since each search agent moves with the normalized velocity components $[V_{1u}, V_{2u}, \ldots, V_{du}]$ as given in Remark 2, the random behavior of the target encoded in $V_{\tilde{i}u}$ governs the stochastic evolution of $D_{\tilde{i}}(t)$. The first-encounter time $\tau_{\tilde{i},\tilde{\rho}}$ for the $\tilde{i}^{th}$ search agent is therefore defined as the first instance when the search agent enters the detection radius of the target, i.e., $\tau_{\tilde{i},\tilde{\rho}} = \inf\{t > 0 : D_{\tilde{i}}(t) \le \tilde{\rho}\}$, $\tilde{\rho} > 0$. The expected value $E(\tau_{\tilde{i},\tilde{\rho}})$ over all realizations of $V_{\tilde{i}u}$ and the target's random walk paths provides a quantitative measure of the tracking efficiency. Minimizing $E(\tau_{\tilde{i},\tilde{\rho}})$ for all $\tilde{i}=1,2,...,n$ yields the finite tracking technique that ensures the minimum expected detection time within the multidimensional search agent framework.

This formulation places the tracking problem within a stochastic transport framework that rigorously captures the probabilistic interaction between moving searchers and a randomly evolving target. This formulation naturally models the multidimensional random walk dynamics of the target together with the corresponding transport driven responses of the searchers under uncertainty. Moreover, it provides the mathematical basis for establishing the existence and finiteness of the expected first-passage (first-encounter) time, thereby confirming the theoretical validity of the proposed tracking technique within the multidimensional random walk transport setting.

3.1. Preliminary theorems for proving the existence of the finite tracking technique

In this section, we establish a collection of analytical results that form the theoretical basis for the existence of a finite and optimal tracking strategy within the proposed Stage 2 stochastic transport framework. Building on the stochastic structure introduced previously, which specifies the random vectors, motion dynamics, and distance functions governing the interaction between the search agents and the target, we present a sequence of theorems that characterize the probabilistic behavior of the system. These results provide a rigorous foundation for demonstrating that the expected first-encounter (first-passage) time between any search agent and a target performing a multidimensional independent random walk is finite. Consequently, the analysis confirms both the theoretical validity of the proposed transport driven-tracking technique.

To improve the readability of the analytical derivation, Figure 3 presents a schematic summary of the main steps leading to the expected first-encounter time. The flowchart shows how the Stage 1 localization output is used as the reference point for the Stage 2 tracking model, and how the main assumptions, encounter quantities, and preliminary analytical results are combined to establish the finiteness of the expected first-encounter time.

For clarity, we briefly summarize the probabilistic notation used in the Stage 2 analysis. The target motion is considered on a probability space $(\tilde{\Omega},\Lambda,\tilde{P})$, where the possible realizations describe the random trajectories of the target and the corresponding tracking paths of the search agents. The target's position and its successive displacements are represented by a sequence of independent and identically distributed random variables. The probability vector specifies the probabilities of the admissible random walk directions, while the tracking distances describe the finite outward and return motions performed by the search agents along the intersecting trajectories. The first-encounter time is interpreted as the first time at which the distance between the target and at least one search agent becomes less than or equal to the detection radiusThis notation is used throughout the $\tilde{\rho}$ following assumptions and theorems.

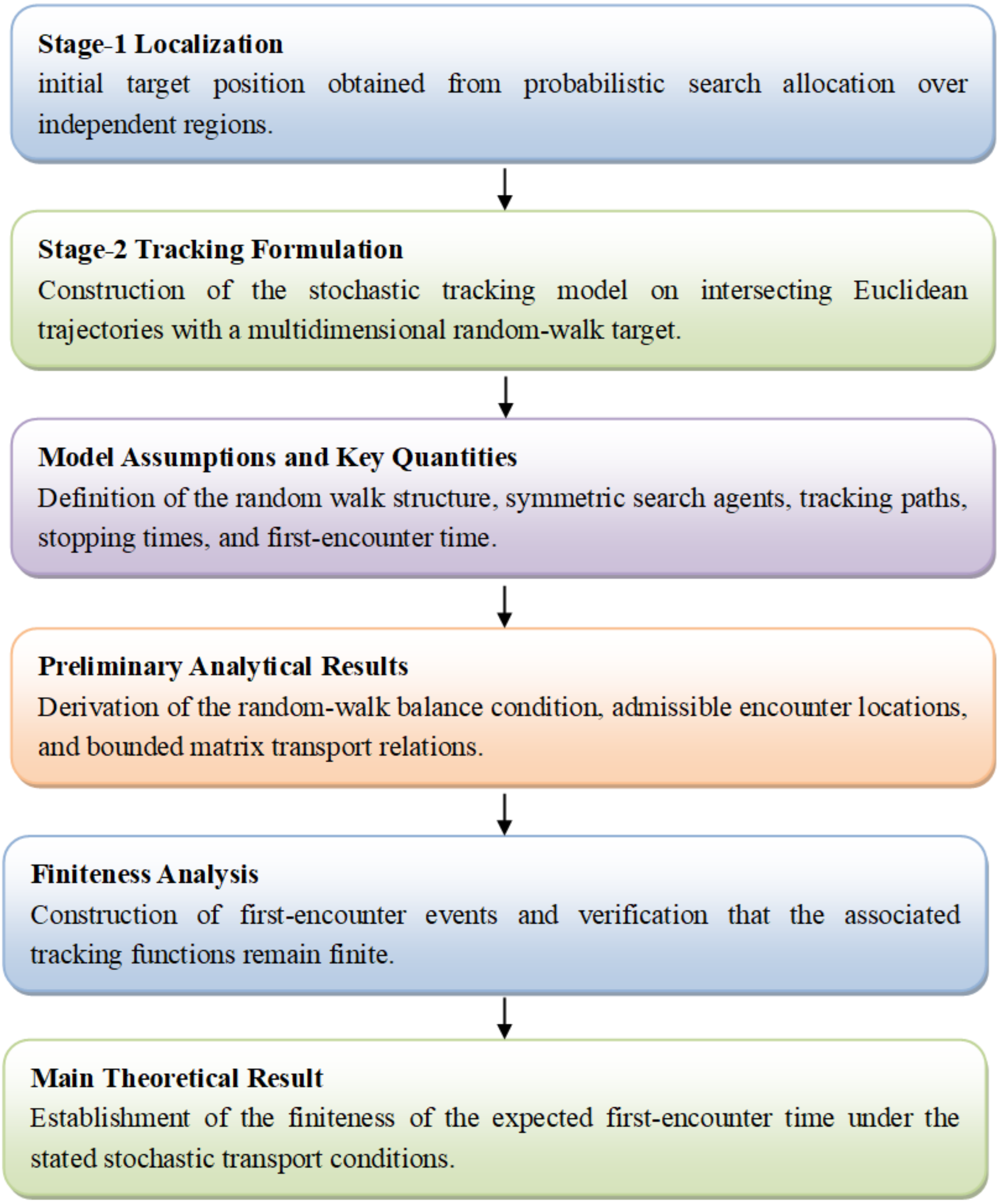

Figure 3. Schematic flowchart summarizing the main derivation path leading to the finiteness of the expected first-encounter time.

In the following assumption, the balance condition corresponds to an unbiased random walk, meaning that the target's motion has no preferred drift direction along the admissible tracking trajectories.

Assumption 1. Let the vector $[m_1, m_2, \ldots, m_d]$ be strictly greater than the zero vector $[0,0,\ldots,0]$ such that $0 < K_1^{(u)} = \dfrac{m_u + y^{(u)}}{2} \le m_u$, where $K_1^{(u)} \in I, u = 1,2,\ldots,d$, I is a set of $+ve$ integer number and m_u, $u = 1,2,\ldots,d$ denote the total number of random steps taken by the target up to known time (these steps are governed by a sequence of independent and identically distributed random variables). Under these conditions, the probability of success for the target's movement in the u^{th} direction is defined by

$$P(S_u(m_u) = K_1^{(u)}) = \begin{cases} \begin{pmatrix} m_u \\ K_1^{(u)} \end{pmatrix} p_u^{K_1^{(u)}} (1-p_u)^{K_1^{(u)}} \\ 0, \qquad otherwise \end{cases},$$

where $K_1^{(u)}$ represents a positive scaling coefficient associated with the success rate of the random step in that spatial dimension.

In the context of the Stage 2 tracking framework, the target motion is governed by a random walk process defined in the preceding assumptions. Each search agent alternates its tracking path symmetrically about the intersection point, while the target performs an independent d-dimensional random walk with a probabilistic step distribution characterized by the vector $[p_1, p_2, ..., p_d]$. Under this stochastic setting, it becomes essential to establish theoretical guarantees for the existence of a finite expected first meeting time between the search agents and the moving target. To achieve this, the following theorem formulates the probabilistic relationship between the random variables that represent the search agents' tracking distances, the random walk dynamics of the target, and the success probabilities associated with motion in each spatial direction. The derived result provides a mathematical foundation for demonstrating that the multidimensional search agent-target interaction leads to a bounded and realizable tracking process, thereby confirming that a finite tracking technique exists within the Stage 2 model.

Under Assumption 1, the following result characterizes the balance condition required for the random walk to remain statistically reachable by the symmetric search agents.

Theorem 2. *If* $\tilde{\xi}_{\tilde{i}\tilde{k}}^{(u)}$ *is the values of the random variables* $\xi_{\tilde{i}\tilde{k}}^{(u)}$, $\tilde{i} = 1,2,...,n;\ u = 1,2,...d;\ \tilde{k} \geq 1$ *and* $\{\tilde{X}_{\tilde{i}\tilde{k}}^{(u)}\}_{\tilde{i}=1,2,...,n,\tilde{k}\geq 1,u=1,2,...,d}$ *be a sequence of independently and identically distributed random variables, whose probability distribution is concentrated on the set of integers* $E(\tilde{X}_{\tilde{i}\tilde{k}}^{(u)}) = (\theta_{\tilde{k}}^{(u)}/2)(E(\xi_{\tilde{i}\tilde{k}}^{(u)}) - \tilde{c}_{\tilde{k}}^{(u)})$ *and* $P(X_{\tilde{i}\tilde{k}}^{(u)} = \gamma_{\tilde{k}} > 0)$, *where*

$$\left|\tilde{c}_{\tilde{k}}^{(u)}\right| \leq 1 \quad and \quad \varsigma_{\tilde{k}}^{(u)} \geq 1$$

are constants, then $V_u(\varsigma_{\tilde{k}}^{(u)}) = \dfrac{S(\varsigma_{\tilde{k}}^{(u)}\theta_{\tilde{k}}^{(u)}) - \tilde{c}_{\tilde{k}}^{(u)}\varsigma_{\tilde{k}}^{(u)}\theta_{\tilde{k}}^{(u)}}{2} = \sum_{\tilde{i}=1}^{\varsigma_{\tilde{k}}^{(u)}} \tilde{X}_{\tilde{i}\tilde{k}}^{(u)}$ *if and only if* $\left|\gamma_{\tilde{k}}\right| \leq \dfrac{\theta_{\tilde{k}}^{(u)}(1-\tilde{c}_{\tilde{k}}^{(u)})}{2}$.

Proof. Assuming that,

$$\begin{aligned} &\left[\sum_{u=1}^{d} P(\tilde{X}_{\tilde{i}1}^{(u)} = \tilde{\xi}_1^{(u)}), ..., \sum_{u=1}^{d} P(\tilde{X}_{\tilde{i}n}^{(u)} = \tilde{\xi}_n^{(u)})\right] \\ &= \left[\sum_{u=1}^{d} P(S_u(\theta_1^{(u)}) = 2\tilde{\xi}_1^{(u)} + \theta_1^{(u)}\tilde{c}_1^{(u)}), ..., \sum_{u=1}^{d} P(S_u(\theta_n^{(u)}) = 2\tilde{\xi}_n^{(u)} + \theta_n^{(u)}\tilde{c}_n^{(u)})\right] > [0,0,...,0], \end{aligned}$$

and $\tilde{X}_{\tilde{i}\tilde{k}}^{(u)} = \sum_{\tilde{i}=1}^{\theta_{\tilde{k}}^{(u)}} \dfrac{\xi_{\tilde{i}+(\tilde{k}-1)\theta_{\tilde{k}}^{(u)}}^{(u)} - \tilde{c}_{\tilde{k}}^{(u)}}{2}$ $\forall\ \tilde{i} = 1,2,...,n; u = 1,2,...d;\ \tilde{k} \geq 1$, then under Assumption 1, we get $\tilde{\xi}_{\tilde{k}}^{(u)} + \theta_{\tilde{k}}^{(u)}(1+\tilde{c}_{\tilde{k}}^{(u)})/2$ which becomes an integer. Hence the target's position remains integer-valued at each step $\tilde{k}$. At this stage, the target performs discrete jumps along one of the directions $u = 1,2,...,d$, moving from one integer valued position to another on the real line $L_{\tilde{i}}, \tilde{i} = 1,2,...,n$ embedded in the d-dimensional space. One can then get $0 \le \tilde{\xi}_{\tilde{k}}^{(u)} + \theta_{\tilde{k}}^{(u)}(1+\tilde{c}_{\tilde{k}}^{(u)})/2 \le \theta_{\tilde{k}}^{(u)}$ where $\left|\theta_{\tilde{k}}^{(u)}\right| > 1$ and $\tilde{\xi}_{\tilde{k}}^{(u)}$ are integers which lead to $\left|\tilde{\xi}_{\tilde{k}}^{(u)}\right| \le \theta_{\tilde{k}}^{(u)}(1-\tilde{c}_{\tilde{k}}^{(u)})/2$. Moreover, under Assumption 1 and for $\tilde{X}_{\tilde{i}\tilde{k}}^{(u)} = \sum_{\tilde{i}=1}^{\theta_{\tilde{k}}^{(u)}} \frac{\xi_{\tilde{i}+(\tilde{k}-1)\theta_{\tilde{k}}^{(u)}}^{(u)} - \tilde{c}_{\tilde{k}}^{(u)}}{2}$, the result $E\left[\tilde{X}_{\tilde{i}\tilde{k}}^{(u)}\right] = E\left[\sum_{\tilde{i}=1}^{\theta_{\tilde{k}}^{(u)}} \frac{\xi_{\tilde{i}+(\tilde{k}-1)\theta_{\tilde{k}}^{(u)}}^{(u)} - \tilde{c}_{\tilde{k}}^{(u)}}{2}\right] = (\theta_{\tilde{k}}^{(u)}/2)(E(\xi_{\tilde{i}}^{(u)}) - \tilde{c}_{\tilde{k}}^{(u)})$ can be obtained. Finally, if $\xi_{\tilde{k}}^{(u)} = \gamma_{\tilde{k}}$, then the vector $\left[\sum_{u=1}^{d} P(\tilde{X}_{\tilde{i}1}^{(u)} = \gamma_1), \sum_{u=1}^{d} P(\tilde{X}_{\tilde{i}2}^{(u)} = \gamma_2), \ldots, \sum_{u=1}^{d} P(\tilde{X}_{\tilde{i}n}^{(u)} = \gamma_n)\right]$ will be obtained, where $\left|\gamma_{\tilde{k}}\right| \le \frac{\theta_{\tilde{k}}^{(u)}(1-\tilde{c}_{\tilde{k}}^{(u)})}{2}$. □

This condition characterizes the balanced random walk structure and contributes to the statistical reachability of the target by the symmetric search agents. However, the finiteness of the expected encounter time is guaranteed only when this balance condition is combined with a bounded tracking geometry, finite tracking distances, recurrent coverage, and a uniform positive encounter probability. Without it, a directional bias arises, leading to diverging paths and the failure of the tracking model in the second stage. For all $\tilde{i} = 1,2,\ldots,n,\ u = 1,2,\ldots,d,$ the probability of the first meeting point $K_1^{(u)} = \frac{m_u + \tilde{\xi}^{(u)}}{2}$, where $K_1^{(u)} \in I, u = 1,2,\ldots,d$ between any search agent and the target depends on the net number of successful random movements performed by the target within the search environment. The resulting displacement of the target along the real search line $L_{\tilde{i}}, \tilde{i} = 1,2,\ldots,n$ therefore depends explicitly on the statistical balance of these movements. In the context of searching for a lost target under uncertainty, Theorem 2 establishes that the probability of the first meeting is maximized when the underlying random walk of the target is unbiased, that is, when the expected value of each step is zero. This condition eliminates directional drift and helps keep the target statistically reachable by the search agents operating symmetrically in opposite directions. Accordingly, the following corollary derives the precise condition under which the probability of first encounter between a search agent and the lost target is maximized.

Corollary 1. For all integers $\tilde{\xi}_{\tilde{i}}^{(u)},\ \tilde{i} = 1,2,\ldots,n,\ u = 1,2,\ldots,d$, we have $\left|\tilde{\xi}_{\tilde{i}}^{(u)}\right| \le \frac{-\varsigma_{\tilde{k}}^{(u)}\theta_{\tilde{i}}^{(u)}(1-\tilde{c}_{\tilde{i}}^{(u)})}{2}$ if $\left[P(V_u(\varsigma_1^{(u)}) = \tilde{\xi}_1^{(u)}), P(V_u(\varsigma_2^{(u)}) = \tilde{\xi}_2^{(u)}), \ldots, P(V_u(\varsigma_n^{(u)}) = \tilde{\xi}_n^{(u)})\right] > [0,0,\ldots,0]$.

Proof. If we consider $V_u(\varsigma_{\tilde{i}}^{(u)}) = \frac{S(\varsigma_{\tilde{i}}^{(u)}\theta_{\tilde{i}}^{(u)}) - \tilde{c}_{\tilde{i}}^{(u)}\varsigma_{\tilde{i}}^{(u)}\theta_{\tilde{i}}^{(u)}}{2} = \sum_{i=1}^{\varsigma_{\tilde{i}}^{(u)}} X_{\tilde{i}\tilde{k}}^{(u)}$, then from Theorem 2, we have

$$\left[\sum_{u=1}^{d} P\left(\frac{S(\varsigma_1^{(u)}\theta_1^{(u)}) - \tilde{c}_1^{(u)}\varsigma_1^{(u)}\theta_1^{(u)}}{2} = \tilde{\xi}_1^{(u)}\right), \ldots, \sum_{u=1}^{d} P\left(\frac{S(\varsigma_n^{(u)}\theta_n^{(u)}) - \tilde{c}_n^{(u)}\varsigma_n^{(u)}\theta_n^{(u)}}{2} = \tilde{\xi}_n^{(u)}\right)\right]$$
$$= \left[\sum_{u=1}^{d} P\left(S_u(\varsigma_1^{(u)}\theta_1^{(u)}) = 2\tilde{\xi}_1^{(u)} + \tilde{c}_1^{(u)}\varsigma_1^{(u)}\theta_1^{(u)}\right), \ldots, \sum_{u=1}^{d} P\left(S_u(\varsigma_n^{(u)}\theta_n^{(u)}) = 2\tilde{\xi}_n^{(u)} + \tilde{c}_n^{(u)}\varsigma_n^{(u)}\theta_n^{(u)}\right)\right].$$

Then, from Assumption 1, we have $\left|\tilde{\xi}_{\tilde{i}}^{(u)}\right| \le \frac{-\varsigma_{\tilde{k}}^{(u)}\theta_{\tilde{i}}^{(u)}(1-\tilde{c}_{\tilde{i}}^{(u)})}{2}$. □

That is a direct implication of Theorem 2 and corresponds to the formulation of the unbiased random walk condition as a concrete rule that controls the probability of the first encounter. Although

Theorem 2 gives the overall theoretical condition for balance, Corollary 1 confirms this balance condition within the stochastic tracking framework and supports, together with the bounded controlled tracking assumptions, the proof of the existence of a finite tracking strategy.

Theorem 3. *If* $c_{\tilde{i}}^{(u)} \neq p_{\tilde{i}}^{(u)} - q_{\tilde{i}}^{(u)}$, $\tilde{\xi}_{\tilde{i}}^{(u)} \geq 0$, *and* $\varsigma_{\tilde{i}}^{(u)} > \tilde{r}_{\tilde{i}1}^{(u)} \tilde{\xi}_{\tilde{i}}^{(u)} + \tilde{r}_{\tilde{i}2}^{(u)}$, *where* $\tilde{\xi}_{\tilde{i}}^{(u)}$ *is a real number and* $\tilde{r}_{\tilde{i}1}^{(u)} > 0$ *and* $\tilde{r}_{\tilde{i}2}^{(u)} > 0$ *depend on* $\tilde{c}_{\tilde{i}}^{(u)}$, $p_{\tilde{i}}^{(u)}$, *then the probability vector*

$$\left[\sum_{u=1}^{d} P(0 \leq V_u(\varsigma_1^{(u)}+1) \leq \tilde{\xi}_1^{(u)}), \dots, \sum_{u=1}^{d} P(0 \leq V_u(\varsigma_n^{(u)}+1) \leq \tilde{\xi}_n^{(u)})\right] \leq \left[\sum_{u=1}^{d} P(0 \leq V_u(\varsigma_1^{(u)}) \leq \tilde{\xi}_1^{(u)}, \dots, P(0 \leq V_u(\varsigma_n^{(u)}) \leq \tilde{\xi}_n^{(u)})\right],$$

will be obtained and if $\tilde{\xi}_{\tilde{i}}^{(u)} \leq 0$ *then we have*

$$\begin{aligned}&\left[\sum_{u=1}^{d} P(\tilde{\xi}_1^{(u)} \leq V_u(\varsigma_1^{(u)}+1) \leq 0), \dots, \sum_{u=1}^{d} P(\tilde{\xi}_n^{(u)} \leq V_u(\varsigma_n^{(u)}+1) \leq 0)\right] \\ &\leq \left[\sum_{u=1}^{d} P(\tilde{\xi}_1^{(u)} \leq V_u(\varsigma_1^{(u)}) \leq 0), \dots, \sum_{u=1}^{d} P(\tilde{\xi}_n^{(u)} \leq V_u(\varsigma_n^{(u)}) \leq 0)\right].\end{aligned}$$

Proof. There are infinitely many integer-valued positions $\tilde{\xi}_{\tilde{i}}^{(u)}$ at which the target may be encountered by one of the search agents along one of the real search lines. Consequently, a collection of mutually exclusive events can be constructed, each corresponding to a possible encounter location, with exactly one event representing the first meeting point between one of search agents and the target. Two distinct cases arise. In the first case, when $\tilde{\xi}_{\tilde{i}}^{(u)} \geq 0$, it follows directly from Corollary 1 that the probability of the first meeting event is strictly positive and the expected first meeting time remains finite, where

$$\begin{aligned}&\left[\bigcup_{u=1}^{d} P(0 \leq V_u(\varsigma_1^{(u)}) \leq \tilde{\xi}_1^{(u)}), \bigcup_{u=1}^{d} P(0 \leq V_u(\varsigma_2^{(u)}) \leq \tilde{\xi}_2^{(u)}), \dots, \bigcup_{u=1}^{d} P(0 \leq V_u(\varsigma_n^{(u)}) \leq \tilde{\xi}_n^{(u)})\right] \\ &= \left[\sum_{u=1}^{d} \sum_{\tilde{i}=0}^{[\tilde{\xi}_1^{(u)}]} P(V_u(\varsigma_1^{(u)}) = \tilde{i}), \sum_{u=1}^{d} \sum_{\tilde{i}=0}^{[\tilde{\xi}_2^{(u)}]} P(V_u(\varsigma_2^{(u)}) = \tilde{i}), \dots, \sum_{u=1}^{d} \sum_{\tilde{i}=0}^{[\tilde{\xi}_n^{(u)}]} P(V_u(\varsigma_n^{(u)}) = \tilde{i})\right].\end{aligned}$$

In the second case, the first meeting may occur when $\tilde{\xi}_{\tilde{i}}^{(u)} \leq 0$, and then we have,

$$\left[\bigcup_{u=1}^{d} P(\tilde{\xi}_1^{(u)} \leq V_u(\varsigma_1^{(u)}) < 0), \dots, \bigcup_{u=1}^{d} P(\tilde{\xi}_n^{(u)} \leq V_u(\varsigma_n^{(u)}) < 0)\right] = \left[\sum_{u=1}^{d} \sum_{i=[\tilde{\xi}_1^{(u)}]}^{0} P(V_u(\varsigma_1^{(u)}) = \tilde{i}), \dots, \sum_{u=1}^{d} \sum_{i=[\tilde{\xi}_n^{(u)}]}^{0} P(V_u(\varsigma_n^{(u)}) = \tilde{i})\right],$$

where $[.]$ denotes the greatest integer (floor) function, representing the maximum admissible integer value. Since, $\tilde{r}_{\tilde{i}1}^{(u)}$ and $\tilde{r}_{\tilde{i}2}^{(u)}$ are positive real numbers , we aim to prove that

$$\begin{aligned}&\left[\sum_{u=1}^{d} P(V_u(\varsigma_1^{(u)}+1) = 1), \sum_{u=1}^{d} P(V_u(\varsigma_2^{(u)}+1) = 2), \dots, \sum_{u=1}^{d} P(V_u(\varsigma_n^{(u)}+1) = n\right] \\ &\leq \left[\sum_{u=1}^{d} P(V_u(\varsigma_1^{(u)}) = 1), \sum_{u=1}^{d} P(V_u(\varsigma_2^{(u)}) = 2), \dots, \sum_{u=1}^{d} P(V_u(\varsigma_n^{(u)}) = n)\right],\end{aligned}$$

if $\varsigma_{\tilde{i}}^{(u)} > \tilde{r}_{\tilde{i}1}^{(u)} |\tilde{i}| + \tilde{r}_{\tilde{i}2}^{(u)}$, $\tilde{i} = 1, 2, \dots, n$, $u = 1, 2, \dots, d$. For these reasons, the following results are obtained.

Case I. This case corresponds to the symmetric reference configuration in which the target's initial position is located on the positive side of the intersection point, and the search agents initiate their tracking motion accordingly. It constitutes the baseline setting of the Stage 2 model, where the search geometry is fully aligned with the direction of motion. Within this configuration, the influence of an unbiased random walk on the finiteness of the expected first meeting time can be analyzed in its most transparent form. Consequently, invoking Theorem 2 directly establishes the condition under which the expected first meeting time remains finite. Now, if $c_{\tilde{i}}^{(u)} < -1$, then Corollary 1 implies that

$$\left[\sum_{u=1}^{d} P(V_u(\varsigma_1^{(u)}) = 1), \sum_{u=1}^{d} P(V_u(\varsigma_2^{(u)}) = 2), ..., \sum_{u=1}^{d} P(V_u(\varsigma_n^{(u)}) = n)\right] > [0,0,...,0],$$

and only if $\dfrac{2\tilde{i}}{\theta_{\tilde{i}}^{(u)}(1-\tilde{c}_{\tilde{i}}^{(u)})} \le \varsigma_{\tilde{i}}^{(u)} \le \dfrac{-2\tilde{i}}{\theta_{\tilde{i}}^{(u)}(1+\tilde{c}_{\tilde{i}}^{(u)})}$, $\tilde{i} = 1,2,...,n, u = 1,2,...,d$. If $\tilde{r}_{\tilde{i}1}^{(u)} = \dfrac{-2\tilde{i}}{\theta_{\tilde{i}}^{(u)}(1+\tilde{c}_{\tilde{i}}^{(u)})}$ and $\tilde{r}_{\tilde{i}2}^{(u)} = 0$ then $\varsigma_{\tilde{i}}^{(u)} > \tilde{r}_{\tilde{i}1}^{(u)} \left|\tilde{i}\right|$. Consequently, $\varsigma_{\tilde{i}}^{(u)} > \dfrac{-2\tilde{i}}{\theta_{\tilde{i}}^{(u)}(1+\tilde{c}_{\tilde{i}}^{(u)})}$. It follows immediately that

$$\left[\sum_{u=1}^{d} P(V_u(\varsigma_1^{(u)}) = 1), \sum_{u=1}^{d} P(V_u(\varsigma_2^{(u)}) = 2), ..., \sum_{u=1}^{d} P(V_u(\varsigma_n^{(u)}) = n)\right] = [0,0,...,0].$$

This yields

$$\left[\sum_{u=1}^{d} P(V_u(\varsigma_1^{(u)}+1) = 1), ..., \sum_{u=1}^{d} P(V_u(\varsigma_n^{(u)}+1) = n)\right] \le \left[\sum_{u=1}^{d} P(V_u(\varsigma_1^{(u)}) = 1, ..., \sum_{u=1}^{d} P(V_u(\varsigma_n^{(u)}) = n)\right].$$

Case II. In this case, we consider the symmetric counterpart of Case I, where the target's initial position is located on the negative side of the intersection point and the search agents initiate their tracking motion in the opposite direction. This configuration is introduced to complete the analysis and to confirm that the finiteness of the expected first meeting time does not depend on the sign of the initial displacement along any search line. Owing to the symmetry of the tracking geometry and the unbiased nature of the random walk, the same finiteness condition derived in Case I remains valid under this setting. Thus, if $c_{\tilde{i}}^{(u)} > 1$, then

$$\left[\sum_{u=1}^{d} P(V_u(\varsigma_1^{(u)}) = 1), \sum_{u=1}^{d} P(V_u(\varsigma_2^{(u)}) = 2), ..., \sum_{u=1}^{d} P(V_u(\varsigma_n^{(u)}) = n)\right] > [0,0,...,0],$$

and only if $\dfrac{2\tilde{i}}{\theta_{\tilde{i}}^{(u)}(1-\tilde{c}_{\tilde{i}}^{(u)})} \le \varsigma_{\tilde{i}}^{(u)} \le \dfrac{-2\tilde{i}}{\theta_{\tilde{i}}^{(u)}(1+\tilde{c}_{\tilde{i}}^{(u)})} \forall\ \tilde{i} = 1,2,...,n,\ u = 1,2,...,d$. At $\tilde{r}_{\tilde{i}1}^{(u)} = \dfrac{-2\tilde{i}}{\theta_{\tilde{i}}^{(u)}(1+\tilde{c}_{\tilde{i}}^{(u)})}$ and $\tilde{r}_{\tilde{i}2}^{(u)} = 0$, we have $\varsigma_{\tilde{i}}^{(u)} > \tilde{r}_{\tilde{i}1}^{(u)} \left|\tilde{i}\right|$. This implies to $\varsigma_{\tilde{i}}^{(u)} > \dfrac{2j}{\theta_{\tilde{i}}^{(u)}(1-c_{\tilde{i}}^{(u)})}$. Moreover

$$\left[\sum_{u=1}^{d} P(V_u(\varsigma_1^{(u)}) = 1), \sum_{u=1}^{d} P(V_u(\varsigma_2^{(u)}) = 2), ..., \sum_{u=1}^{d} P(V_u(\varsigma_n^{(u)}) = n)\right] = [0,0,...,0].$$

Consequently, the same result in Case I will hold.

Case III. This case represents the general geometric setting of the Stage 2 framework, where the initial positions and tracking distances are arbitrary positive values resulting from the outcome from Stage 1, without assuming symmetry or integer constraints. This case verifies that the balance condition stated in Theorem 2 and Corollary 1 contributes to statistical reachability under this general configuration, while finiteness follows only when it is combined with the fully bounded controlled tracking conditions. Accordingly, if $\left|\tilde{c}_{\tilde{i}}^{(u)}\right| \leq 1$ and $E(\xi_{\tilde{i}\tilde{k}}^{(u)}) \neq \tilde{c}_{\tilde{i}}^{(u)}\ \forall\ \tilde{i} = 1,2,...,n,\ u = 1,2,...,d,$ then to verify that the finiteness condition remains valid under this general geometric configuration, one can assume that $\tilde{r}_{\tilde{i}2}^{(u)} = \dfrac{1}{\tilde{h}_{\tilde{i}}^{(u)} - 1}$, $\tilde{h}_{\tilde{i}}^{(u)} > 1$ and $\tilde{r}_{\tilde{i}1}^{(u)} = \theta_{\tilde{i}}^{(u)} \tilde{h}_{\tilde{i}}^{(u)} (\tilde{h}_{\tilde{i}}^{(u)} - 1) \max\left[\dfrac{1}{\tilde{\alpha}_{\tilde{i}}^{(u)}}, \dfrac{1}{\tilde{\beta}_{\tilde{i}}^{(u)}}\right]$ where

$$\tilde{h}_{\tilde{i}}^{(u)} = \left(\frac{\tilde{\alpha}_{\tilde{i}}^{(u)}}{q_{\tilde{i}}^{(u)}}\right)^{\tilde{\alpha}_{\tilde{i}}^{(u)}} \left(\frac{\tilde{\beta}_{\tilde{i}}^{(u)}}{1 - \sum_{\tilde{i}=1}^{n} \sum_{\upsilon=1,\upsilon\neq u}^{d} q_{\tilde{i}}^{(\upsilon)}}\right)^{\tilde{\beta}_{\tilde{i}}^{(u)}},\ 0 < \tilde{\alpha}_{\tilde{i}}^{(u)} = \frac{1 - \tilde{c}_{\tilde{i}}^{(u)}}{2} < 1,\ \tilde{\beta}_{\tilde{i}}^{(u)} = 1 - \sum_{\tilde{i}=1}^{n} \sum_{\upsilon=1,\upsilon\neq u}^{d} \tilde{\alpha}_{\tilde{i}}^{(\upsilon)},\ {}^{u}\tilde{\alpha}_{\tilde{i}} \neq {}^{u}q_{\tilde{i}},$$

and $\tilde{r}_{\tilde{i}1}^{(u)},\ \tilde{r}_{\tilde{i}2}^{(u)} > 0$. Moreover, if $\varsigma_{\tilde{i}}^{(u)} > r_{\tilde{i}1}^{(u)} \left|\tilde{i}\right|$ then $-\varsigma_{\tilde{i}}^{(u)} \alpha_{\tilde{i}}^{(u)} \theta_{\tilde{i}}^{(u)} \leq \tilde{i} \leq \varsigma_{j}^{(u)} \tilde{\beta}_{\tilde{i}}^{(u)} \theta_{\tilde{i}}^{(u)}$. This implies that $\left[\sum_{u=1}^{d} P(V_u(\varsigma_1^{(u)}) = 1), \sum_{u=1}^{d} P(V_u(\varsigma_2^{(u)}) = 2), ..., \sum_{u=1}^{d} P(V_u(\varsigma_n^{(u)}) = n)\right] > [0,0,...,0]$ as given in Corollary 1. Moreover, if

$$\left[\sum_{u=1}^{d} P(V_u(\varsigma_1^{(u)}) = 1), \sum_{u=1}^{d} P(V_u(\varsigma_2^{(u)}) = 2), ..., \sum_{u=1}^{d} P(V_u(\varsigma_n^{(u)}) = n)\right] > [0,0,...,0]$$

and

$$\varsigma_{\tilde{i}}^{(u)} > \left|\tilde{i}\right| \theta_{\tilde{i}}^{(u)} \tilde{h}_{\tilde{i}}^{(u)} (\tilde{h}_{\tilde{i}}^{(u)} - 1) \max\left[\frac{1}{\tilde{\alpha}_{\tilde{i}}^{(u)}}, \frac{1}{\tilde{\beta}_{\tilde{i}}^{(u)}}\right] + \frac{1}{\tilde{h}_{\tilde{i}}^{(u)} - 1},$$

then it can be shown that

$$\begin{aligned}&\left[\sum_{u=1}^{d} P(V_u(\varsigma_1^{(u)} + 1) = 1), \sum_{u=1}^{d} P(V_u(\varsigma_2^{(u)} + 1) = 2), ..., \sum_{u=1}^{d} P(V_u(\varsigma_n^{(u)} + 1) = n)\right] \\ &\leq \left[\sum_{u=1}^{d} P(V_u(\varsigma_1^{(u)}) = 1), \sum_{u=1}^{d} P(V_u(\varsigma_2^{(u)}) = 2), ..., \sum_{u=1}^{d} P(V_u(\varsigma_n^{(u)}) = n)\right].\end{aligned}$$

If we let $Z_{\tilde{k}\tilde{i}}^{(u)} = (\xi_{\tilde{k}\tilde{i}}^{(u)} - \tilde{c}_{\tilde{i}}^{(u)})/2$, then

$$\begin{aligned}&\left[\sum_{u=1}^{d} P(Z_{\tilde{k}1}^{(u)} = \tilde{\alpha}_1^{(u)}), \sum_{u=1}^{d} P(Z_{\tilde{k}2}^{(u)} = \tilde{\alpha}_2^{(u)}), ..., \sum_{u=1}^{d} P(Z_{\tilde{k}n}^{(u)} = \tilde{\alpha}_n^{(u)})\right] = \left[\sum_{u=1}^{d} p_1^{(u)}, \sum_{u=1}^{d} p_2^{(u)}, ..., \sum_{u=1}^{d} p_n^{(u)}\right], \\ &\left[\sum_{u=1}^{d} P(Z_{\tilde{k}1}^{(u)} = -\tilde{\beta}_1^{(u)}), \sum_{u=1}^{d} P(Z_{\tilde{k}2}^{(u)} = -\tilde{\beta}_2^{(u)}), ..., \sum_{u=1}^{d} P(Z_{\tilde{k}n}^{(u)} = -\tilde{\beta}_n^{(u)})\right] = \left[\sum_{u=1}^{d} q_1^{(u)}, \sum_{u=1}^{d} q_2^{(u)}, ..., \sum_{u=1}^{d} q_n^{(u)}\right],\end{aligned}$$

and

$$\left[\sum_{u=1}^{d} P(V_u(\varsigma_1^{(u)})=1), \sum_{u=1}^{d} P(V_u(\varsigma_2^{(u)})=2), \ldots, \sum_{u=1}^{d} P(V_u(\varsigma_n^{(u)})=n)\right]$$
$$=\left[\sum_{u=1}^{d} P\left(\sum_{\tilde{k}=1}^{\varsigma_1^{(u)}\theta_1^{(u)}} Z_{\tilde{k}1}^{(u)}\right), \sum_{u=1}^{d} P\left(\sum_{\tilde{k}=1}^{\varsigma_2^{(u)}\theta_2^{(u)}} Z_{\tilde{k}2}^{(u)}\right), \ldots, \sum_{u=1}^{d} P\left(\sum_{\tilde{k}=1}^{\varsigma_n^{(u)}\theta_n^{(u)}} Z_{\tilde{k}n}^{(u)}\right)\right].$$

From Assumption 1, we obtain,

$$\left[\frac{\sum_{u=1}^{d} P(V_u(\varsigma_1^{(u)}+1)=1)}{\sum_{u=1}^{d} P(V_u(\varsigma_1^{(u)})=1)}, \frac{\sum_{u=1}^{d} P(V_u(\varsigma_2^{(u)}+1)=2)}{\sum_{u=1}^{d} P(V_u(\varsigma_2^{(u)})=2)}, \ldots, \frac{\sum_{u=1}^{d} P(V_u(\varsigma_n^{(u)}+1)=n)}{\sum_{u=1}^{d} P(V_u(\varsigma_n^{(u)})=n)}\right]$$
$$=\left[\prod_{\tilde{k}=1}^{\theta_1^{(u)}\tilde{\beta}_1^{(u)}} \frac{\theta_1^{(u)}\varsigma_1^{(u)}+\tilde{k}}{\tilde{h}_1^{(u)}\left(\theta_1^{(u)}\varsigma_1^{(u)}+\tilde{k}+\left(\frac{\tilde{k}\tilde{\alpha}_1^{(u)}+1}{\tilde{\beta}_1^{(u)}}\right)\right)} \prod_{\tilde{k}=1}^{\theta_1^{(u)}\tilde{\alpha}_1^{(u)}} \frac{\theta_1^{(u)}(\varsigma_1^{(u)}+\tilde{\beta}_1^{(u)})+\tilde{k}}{\tilde{h}_1^{(u)}\left(\theta_1^{(u)}\varsigma_1^{(u)}+\tilde{k}+\left(\frac{\tilde{k}\tilde{\beta}_1^{(u)}-1}{\tilde{\alpha}_1^{(u)}}\right)\right)}, \ldots,\right.$$
$$\left.\prod_{\tilde{k}=1}^{\theta_n^{(u)}\tilde{\beta}_n^{(u)}} \frac{\theta_n^{(u)}\varsigma_n^{(u)}+\tilde{k}}{\tilde{h}_n^{(u)}\left(\theta_n^{(u)}\varsigma_n^{(u)}+\tilde{k}+\left(\frac{\tilde{k}\tilde{\alpha}_n^{(u)}+1}{\tilde{\beta}_n^{(u)}}\right)\right)} \prod_{\tilde{k}=1}^{\theta_n^{(u)}\tilde{\alpha}_n^{(u)}} \frac{\theta_n^{(u)}(\varsigma_n^{(u)}+\tilde{\beta}_n^{(u)})+\tilde{k}}{\tilde{h}_n^{(u)}\left(\theta_n^{(u)}\varsigma_n^{(u)}+\tilde{k}+\left(\frac{\tilde{k}\tilde{\beta}_n^{(u)}-n}{\tilde{\alpha}_n^{(u)}}\right)\right)}\right].$$

Moreover, if $\varsigma_{\tilde{i}}^{(u)} < |\tilde{i}|\theta_{\tilde{i}}^{(u)}\tilde{h}_{\tilde{i}}^{(u)}(\tilde{h}_{\tilde{i}}^{(u)}-1)\max\left[\frac{1}{\tilde{\alpha}_{\tilde{i}}^{(u)}}, \frac{1}{\tilde{\beta}_{\tilde{i}}^{(u)}}\right]+\frac{1}{\tilde{h}_{\tilde{i}}^{(u)}-1}$, then for all $\tilde{i}=1,2,\ldots,n,\ u=1,2,\ldots,d$, we get

$$\prod_{\tilde{k}=1}^{\theta_{\tilde{i}}^{(u)}\tilde{\beta}_{\tilde{i}}^{(u)}} \frac{\theta_{\tilde{i}}^{(u)}\varsigma_{\tilde{i}}^{(u)}+\tilde{k}}{h_{\tilde{i}}^{(u)}\left(\theta_{\tilde{i}}^{(u)}\varsigma_{\tilde{i}}^{(u)}+\tilde{k}+\left(\frac{\tilde{k}\tilde{\alpha}_{\tilde{i}}^{(u)}+1}{\tilde{\beta}_{\tilde{i}}^{(u)}}\right)\right)}<1,$$

and

$$\prod_{\tilde{k}=1}^{\tilde{\theta}_{\tilde{i}}^{(u)}\tilde{\alpha}_{\tilde{i}}^{(u)}} \frac{\theta_{\tilde{i}}^{(u)}(\varsigma_{\tilde{i}}^{(u)}+\tilde{\beta}_{\tilde{i}}^{(u)})+\tilde{k}}{\tilde{h}_{\tilde{i}}^{(u)}\left(\theta_{\tilde{i}}^{(u)}\varsigma_{\tilde{i}}^{(u)}+\tilde{k}+\left(\frac{\tilde{k}\tilde{\beta}_{\tilde{i}}^{(u)}-1}{\tilde{\alpha}_{\tilde{i}}^{(u)}}\right)\right)}<1.$$

It follows that

$$\frac{\sum_{u=1}^{d} P(V_u(\varsigma_j^{(u)}+1)=\tilde{i})}{\sum_{u=1}^{d} P(V_u(\varsigma_j^{(u)})=\tilde{i})} << 1,$$

for all $\tilde{i}=1,2,\ldots,n,\ u=1,2,\ldots,d$. Thus, we have

$$\left[\sum_{u=1}^{d} P(V_u(\varsigma_1^{(u)}+1)=1), \ldots, \sum_{u=1}^{d} P(V_u(\varsigma_n^{(u)}+1)=n)\right] \le \left[\sum_{u=1}^{d} P(V_u\varsigma_1^{(u)})=1), \ldots, \sum_{u=1}^{d} P(V_u(\varsigma_n^{(u)})=n)\right]. \qquad \square$$

In the analysis of stochastic transport processes, matrix representations naturally arise in the description of transition mechanisms, propagation operators, and iterative transport dynamics. In particular, nonnegative matrices are commonly used to model transport kernels, transition probabilities, and cumulative displacement effects over successive stages of motion. When comparing alternative transport configurations or bounding the evolution of stochastic systems, it becomes essential to establish ordering properties that are preserved under repeated matrix multiplication. The following result formalizes a fundamental monotonicity property for the powers of non-negative matrices, which plays a key role in deriving the bounds and convergence properties of the transport dynamics considered in this study.

Theorem 4. *If* $E(\xi_{\tilde{i}}^{(u)}) < \tilde{c}_{\tilde{i}}^{(u)}$ *where* $\tilde{c}_{\tilde{i}}^{(u)} \in \mathbf{R}$ *for all* $\tilde{i} = 1,2,...,n,\ u = 1,2,...,d$*, then for* $0 < \varepsilon_{\tilde{i}u} < 1$*, we have*

$$\left[\sum_{u=1}^{d} P(S_1(\varsigma_1^{(u)}), \sum_{u=1}^{d} P(S_2(\varsigma_1^{(u)}),..., \sum_{u=1}^{d} P(S_n(\varsigma_n^{(u)})\right] \geq \left[\varepsilon_{1u}^{\varsigma_1^{(u)}}, \varepsilon_{2u}^{\varsigma_2^{(u)}},..., \varepsilon_{nu}^{\varsigma_n^{(u)}}\right].$$

Proof. Let, $\mathbf{A} = [a_{\tilde{i}u}]$ and $\mathbf{B} = [b_{\tilde{i}u}]$ are two non-negative matrices of the same order $n \times d$, where $a_{ju} = E(\xi_{\tilde{i}}^{(u)})$ and $b_{\tilde{i}u} = \tilde{c}_{\tilde{i}}^{(u)}$. If we suppose that $\mathbf{A} > \mathbf{B}$, then for any $k_{\tilde{i}}^{(u)} > 0$, we obtain

$$f_{\tilde{i}}(k_{\tilde{i}}^{(u)}) = E(\exp[k_{\tilde{i}}^{(u)}\{Y_{\tilde{i}}^{(u)} - \tilde{c}_{\tilde{i}}^{(u)}\}]) = p_{\tilde{i}}^{(u)}.\ \exp\{(k_{\tilde{i}}^{(u)}(1 - c_{\tilde{i}}^{(u)})) + q_{\tilde{i}}^{(u)}.\exp\{(k_{\tilde{i}}^{(u)}(-1 - \tilde{c}_{\tilde{i}}^{(u)}))\},$$

for any, $u = 1,2,...,d,\ \tilde{i} = 1,2,...,n$. It follows directly that

$$\begin{aligned}
&\left[\sum_{u=1}^{d} P(S_1(\varsigma_1^{(u)}) \geq \tilde{c}_1^{(u)}\varsigma_1^{(u)}),..., \sum_{u=1}^{d} P(S_n(\varsigma_n^{(u)}) \geq \tilde{c}_n^{(u)}\varsigma_n^{(u)})\right] \\
&= \left[\sum_{u=1}^{d} P(\exp\{k_1^{(u)}(S_1(\varsigma_1^{(u)}) - \tilde{c}_1^{(u)}\varsigma_1^{(u)})\} \geq 1),..., \sum_{u=1}^{d} P(\exp\{k_n^{(u)}(S_n(\varsigma_n^{(u)}) - \tilde{c}_2^{(u)}\varsigma_n^{(u)})\} \geq 1)\right] \\
&\leq \left[\sum_{u=1}^{d}\sum_{\tilde{i}=1}^{n} E(\exp\{{}^{u}k_{\tilde{i}}(S_{\tilde{i}}({}^{u}\varsigma_{\tilde{i}}) - {}^{u}\tilde{c}_{\tilde{i}}\,{}^{u}\varsigma_{\tilde{i}})),..., \sum_{u=1}^{d}\sum_{\tilde{i}=1}^{n} E(\exp\{{}^{u}k_{\tilde{i}}(S_{\tilde{i}}({}^{u}\varsigma_{\tilde{i}}) - {}^{u}\tilde{c}_{\tilde{i}}\,{}^{u}\varsigma_{\tilde{i}}))\right] = \left[f_1({}^{u}k_1),..., f_n({}^{u}k_n)\right].
\end{aligned}$$

Then, $\left[f_1'(0),..., f_n'(0)\right] < [0,...,0]$, where $f_{\tilde{i}}(0) = 1$ when ${}^{u}\varsigma_{\tilde{i}} = 0$. We then have

$$\min_{\substack{k_{\tilde{i}}^{(u)} > 0 \\ \forall \tilde{i} = 1,2,...,n, u = 1,2,...,d}} f_{\tilde{i}}(k_{\tilde{i}}^{(u)}) = \varepsilon_{\tilde{i}u} < 1.$$

Moreover, if $\mathbf{A} < \mathbf{B}$, then we get $\min\limits_{\substack{k_{\tilde{i}}^{(u)} > 0 \\ \forall \tilde{i} = 1,2,...,n, u = 1,2,...,d}} f_{\tilde{i}}(k_{\tilde{i}}^{(u)}) = \varepsilon_{\tilde{i}u} < 1$ for all $\varepsilon_{\tilde{i}u} > 0$ such that

$$\left[\sum_{u=1}^{d} P(S_1(\varsigma_1^{(u)}),..., \sum_{u=1}^{d} P(S_n(\varsigma_n^{(u)})\right] \geq \left[\varepsilon_{1u}^{\varsigma_1^{(u)}},..., \varepsilon_{nu}^{\varsigma_n^{(u)}}\right]\ \forall\ {}^{u}\varsigma_{\tilde{i}}, u = 1,2,...,d,\ \tilde{i} = 1,2,...,n. \qquad \square$$

4. Existence of the finite first-encounter time in the stochastic tracking model

Within the proposed stochastic transport framework, the first detection time is represented by a countable family of mutually exclusive events associated with admissible encounter locations along the transport domain. These events are rigorously defined on the probability space $(\tilde{\Omega}, \Lambda, \tilde{P})$

introduced in Section 3, which encodes the underlying transport dynamics and the random motion of the target. By invoking the analytical results established in Subsection 3.1, sufficient conditions for the finiteness of the expected tracking (first-encounter) time are derived. This construction places the tracking problem within a well-posed probabilistic transport setting and ensures the mathematical consistency of the model.

It is important to distinguish the present finiteness result from the classical recurrence theory of unrestricted multidimensional random walks. In the classical setting, recurrence and transience depend strongly on the spatial dimension, and an unbiased random walk in an unbounded domain does not by itself guarantee a finite mean hitting time. The present model is different: The target's motion is analyzed within a constrained tracking framework generated by the Stage 1 localization output, while the search agents move along prescribed intersecting trajectories in a bounded transport domain. Hence, the finiteness of the expected first-encounter time follows from the combined effect of the bounded tracking geometry, the recurrent coverage of the admissible trajectories, finite tracking-distance functions, and a uniform positive encounter probability, with the unbiased motion assumption serving only as a balance condition rather than as a standalone recurrence guarantee. The result should therefore be interpreted as a sufficient condition for the proposed controlled stochastic transport model, not as a general recurrence statement for arbitrary multidimensional random walks.

Assumption 2. We consider that $\Gamma_{\tilde{i}}^{(u)}(t)=S_u(t)-t$ and $\tilde{\Gamma}_{\tilde{i}}^{(u)}(t)=S_u(t)+t,\ u=1,2,\dots,d,\ \tilde{i}=1,2,\dots,n$, such that

$$\tilde{\Delta}_1^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})=\sum_{\tilde{k}=1}^{\infty}((\theta_{\tilde{i}}^{(u)})^{\tilde{k}}-1)P[\tilde{\Gamma}_{\tilde{i}}^{(u)}(W_{\tilde{i}(2\tilde{k}-1)}^{(u)})<-\tilde{\xi}_{\tilde{i}}^{(u)}],\ \tilde{\Delta}_2^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})=\sum_{\tilde{k}=1}^{\tilde{i}}((\theta_{\tilde{i}}^{(u)})^{\tilde{k}}-1)P[\Gamma_{\tilde{i}}^{(u)}(W_{\tilde{i}(2\tilde{k}-1)}^{(u)})>-\tilde{\xi}_{\tilde{i}}^{(u)}],$$

$$\tilde{\Delta}_3^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})=\sum_{\tilde{k}=1}^{\infty}((\theta_{\tilde{i}}^{(u)})^{\tilde{k}}(\theta_{\tilde{i}}^{(u)}-2)+1)P[\tilde{\Gamma}_{\tilde{i}}^{(u)}(W_{\tilde{i}(2\tilde{k})}^{(u)})<-\tilde{\xi}_{\tilde{i}}^{(u)}],\ \tilde{\Delta}_4^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})=\sum_{\tilde{k}=1}^{\tilde{i}}((\theta_{\tilde{i}}^{(u)})^{\tilde{k}}(\theta_{\tilde{i}}^{(u)}-2)+1)P[\Gamma_{\tilde{i}}^{(u)}(W_{\tilde{i}(2\tilde{k})}^{(u)})>-\tilde{\xi}_{\tilde{i}}^{(u)}],$$

$$\tilde{\Delta}_5^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})=\sum_{\tilde{k}=\tilde{i}+1}^{\infty}((\theta_{\tilde{i}}^{(u)})^{\tilde{k}}-1)P[\tilde{\Gamma}_{\tilde{i}}^{(u)}(W_{\tilde{i}(2\tilde{k}-1)}^{(u)})>-\tilde{\xi}_{\tilde{i}}^{(u)}],\ \tilde{\Delta}_6^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})=\sum_{\tilde{k}=\tilde{i}+1}^{\infty}((\theta_{\tilde{i}}^{(u)})^{\tilde{k}}(\theta_{\tilde{i}}^{(u)}-2)+1)P[\Gamma_{\tilde{i}}^{(u)}(W_{\tilde{i}(2\tilde{k}-1)}^{(u)})>-\tilde{\xi}_{\tilde{i}}^{(u)}],$$

$$\Xi_1^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})=\sum_{\tilde{k}=1}^{\tilde{i}}((\theta_{\tilde{i}}^{(u)})^{\tilde{k}}-1)P[\tilde{\Gamma}_{\tilde{i}}^{(u)}(W_{\tilde{i}(2\tilde{k}-1)}^{(u)})<-\tilde{\xi}_{\tilde{i}}^{(u)}],\ \Xi_2^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})=\sum_{\tilde{k}=1}^{\infty}((\theta_{\tilde{i}}^{(u)})^{\tilde{k}}-1)P[\Gamma_{\tilde{i}}^{(u)}(W_{\tilde{i}(2\tilde{k}-1)}^{(u)})>-\tilde{\xi}_{\tilde{i}}^{(u)}],$$

$$\Xi_3^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})=\sum_{\tilde{k}=1}^{\tilde{i}}((\theta_{\tilde{i}}^{(u)})^{\tilde{k}}(\theta_{\tilde{i}}^{(u)}-2)+1)P[\tilde{\Gamma}_{\tilde{i}}^{(u)}(W_{\tilde{i}(2\tilde{k})}^{(u)})<-\tilde{\xi}_{\tilde{i}}^{(u)}],\ \Xi_4^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})=\sum_{\tilde{k}=1}^{\infty}((\theta_{\tilde{i}}^{(u)})^{\tilde{k}}(\theta_{\tilde{i}}^{(u)}-2)+1)P[\Gamma_{\tilde{i}}^{(u)}(W_{\tilde{i}(2\tilde{k})}^{(u)})>-\tilde{\xi}_{\tilde{i}}^{(u)}],$$

$$\Xi_5^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})=\sum_{\tilde{k}=\tilde{i}+1}^{\infty}((\theta_{\tilde{i}}^{(u)})^{\tilde{k}}-1)P[\tilde{\Gamma}_{\tilde{i}}^{(u)}(W_{\tilde{i}(2\tilde{k}-1)}^{(u)})>-\tilde{\xi}_{\tilde{i}}^{(u)}],\ \Xi_6^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})=\sum_{\tilde{k}=\tilde{i}+1}^{\infty}((\theta_{\tilde{i}}^{(u)})^{\tilde{k}}(\theta_{\tilde{i}}^{(u)}-2)+1)P[\Gamma_{\tilde{i}}^{(u)}(W_{\tilde{i}(2\tilde{k})}^{(u)})>-\tilde{\xi}_{\tilde{i}}^{(u)}].$$

To establish the existence of a finite first-encounter time in the proposed stochastic tracking model, we examine the probabilistic structure of the initial detection event through a precise characterization of all admissible encounter locations between the search agents and the moving target. These locations are modeled as a collection of mutually exclusive and independent events defined on the underlying probability space governing the stochastic transport dynamics of the target. The behavior of the first-encounter time depends critically on the relative position of the starting point with respect to the reference point $(0,0,\dots,0)$, as the tracking process admits distinct formulations depending on whether the initial position lies on the negative or positive side of this reference. Under the conditions imposed by Assumption 2, sufficient criteria can be derived to guarantee the finiteness of the expected first detection time. The following theorems formalize this analysis by systematically treating both geometric configurations and establishing the corresponding finiteness conditions.

Remark 3. It is important to emphasize that the unbiasedness assumption is not used here as a standalone recurrence condition for an unrestricted multidimensional random walk. In classical random walk theory, recurrence depends strongly on the spatial dimension, and in more than two dimensions, an unbiased random walk may be transient. Therefore, the present result should not be interpreted as a claim that unbiasedness alone implies a finite mean first-passage time. Instead, finiteness is obtained only within the controlled bounded tracking framework considered in this paper, where the admissible target positions are restricted by the Stage 1 localization output, the search agents move along prescribed intersecting trajectories, and the tracking distances remain finite.

To make the finiteness argument explicit, the following proposition formulates the first-encounter problem as a hitting-time problem for the joint target-searcher process. The key point is that the bounded controlled tracking assumptions imply a uniform positive probability of reaching the detection set within a finite tracking block. This yields a geometric upper bound for the probability of non-detection and, consequently, a finite expected first-encounter time.

Proposition 1. (Finiteness of the expected first-encounter time under bounded controlled tracking) Let $\hat{\boldsymbol{Q}}_{\tilde{i},t} \in \mathbf{R}^d, \tilde{i} = 1,2,...,n$, denote the position vector of the $\tilde{i}$ *-th* search agent at time *t*, and since $\boldsymbol{X}(\mathrm{t}) \equiv \boldsymbol{X}_t \in \mathbf{R}^d$, we define the joint state vector of the target-searcher system by, $\tilde{\boldsymbol{Z}}_t = \left(\boldsymbol{X}_t, \hat{Q}_{1,t}, \hat{Q}_{2,t}, ..., \hat{Q}_{2,t}\right) \in \mathbf{R}^{d(n+1)}$. Moreover, let $\tilde{A}_{\tilde{\rho}}$ denote the detection set, defined by $\tilde{A}_{\tilde{\rho}} = \left\{\tilde{z} = (x, \hat{q}_1, \hat{q}_2, ..., \hat{q}_n) \in \mathbf{R}^{d(n+1)} : \min_{1 \le \tilde{i} \le n} \left\| x - \hat{q}_{\tilde{i}} \right\| \le \tilde{\rho}\right\}$, where $\tilde{\rho} > 0$ is the detection radius of the search agents. The first-encounter time is then defined as: $\tau_{\tilde{\rho}} = \inf\left\{t \ge 0 : \tilde{\boldsymbol{Z}}_t \in \tilde{A}_{\tilde{\rho}}\right\}$. Assume that the joint process $\left\{\tilde{\boldsymbol{Z}}_t : t \ge 0\right\}$ evolves in a bounded admissible tracking domain and that the tracking paths of the search agents provide recurrent coverage of this domain. Also assume that the tracking distances are finite, so that every tracking cycle is completed in finite time. Suppose further that the constants $\tilde{L} \in \boldsymbol{N}$ exists, where $\boldsymbol{N}$ denotes the set of natural numbers and $0 < \hat{\varepsilon} \le 1$ such that, for every admissible initial state, $\tilde{z} \notin \tilde{A}_{\tilde{\rho}}$, $P_{\tilde{z}}(\tau_{\tilde{\rho}} \le \tilde{L}) \ge \hat{\varepsilon}$. Equivalently, from any admissible non-detected state, the controlled tracking process has a uniformly positive probability of entering the detection set within a finite tracking block. Then the expected first-encounter time is finite. Moreover, $E_{\tilde{z}}[\tau_{\tilde{\rho}}] \le \dfrac{\tilde{L}}{\hat{\varepsilon}} < \infty$.

Proof. Since the process $\tilde{\boldsymbol{Z}}_t$ contains both the target position and all search agent positions, the event $\tilde{\boldsymbol{Z}}_t \in \tilde{A}_{\tilde{\rho}}$ means that at least one search agent is within detection distance $\tilde{\rho}$ from the target at time t. Therefore, $\tau_{\tilde{\rho}}$ is the first time at which the target is detected by one of the search agents. The following bounded controlled tracking assumptions are used. The bounded tracking domain prevents the joint target-searcher process from escaping the admissible region. The finiteness of the tracking distances ensures that each tracking block has a finite duration. The recurrent coverage condition ensures that the admissible tracking trajectories are repeatedly revisited by the search agents. Finally, the uniform positive encounter probability gives a state-independent lower bound for reaching the detection set within a finite number of tracking steps. By assumption, from every admissible state $\tilde{z} \notin \tilde{A}_{\tilde{\rho}}$, there is a uniformly positive probability $\hat{\varepsilon}$ that the joint process reaches the detection set $\tilde{A}_{\tilde{\rho}}$ within the next L time steps. Hence, $P_{\tilde{z}}(\tau_{\tilde{\rho}} > \tilde{L}) \le 1 - \hat{\varepsilon}$. The same estimate applies after each tracking block, because the lower bound $\hat{\varepsilon}$ is uniform over all admissible non-detected states. Thus, using the same argument recursively over consecutive time blocks of length $\tilde{L}$, and using the uniform conditional lower bound on the encounter probability over successive tracking blocks, we obtain $P_{\tilde{z}}(\tau_{\tilde{\rho}} > \hat{m}\tilde{L}) \le (1-\hat{\varepsilon})^{\hat{m}}$, $\hat{m} = 0,1,2,....$ This is the key geometric tail estimate. It shows that the probability of remaining undetected after successive finite tracking blocks decreases at least geometrically. Since $\tau_{\tilde{\rho}}$ is a integer-valued first-hitting time, its expectation can be written in terms of its tail probabilities as $E_{\tilde{z}}[\tau_{\tilde{\rho}}] = \sum_{t=0}^{\infty} P_{\tilde{z}}(\tau_{\tilde{\rho}} > t)$.

Since the interval $[\hat{m}\tilde{L}, (\hat{m}+1)\tilde{L}]$ contains at most $\tilde{L}$ integer time points, $E_{\tilde{z}}[\tau_{\tilde{\rho}}] = \tilde{L}\sum_{\hat{m}=0}^{\infty} P_{\tilde{z}}(\tau_{\tilde{\rho}} > \hat{m}\tilde{L}).$ Using the geometric tail bound, $E_{\tilde{z}}[\tau_{\tilde{\rho}}] \le \tilde{L}\sum_{\hat{m}=0}^{\infty}(1-\hat{\varepsilon})^{\hat{m}} = \dfrac{\tilde{L}}{\hat{\varepsilon}}$, where $0 < \hat{\varepsilon} \le 1$ i.e., $\sum_{\hat{m}=0}^{\infty}(1-\hat{\varepsilon})^{\hat{m}}$ is finite. Consequently, $E_{\tilde{z}}[\tau_{\tilde{\rho}}] \le \dfrac{\tilde{L}}{\hat{\varepsilon}} < \infty.$

This shows that the bounded tracking geometry, finite tracking distances, recurrent coverage, and a uniform positive encounter probability are sufficient to guarantee the finiteness of the expected first-encounter time under the proposed bounded controlled tracking framework. This proves the proposition. □

This proposition clarifies the mathematical mechanism behind the finiteness result. The bounded tracking domain prevents escape from the admissible region, the finite tracking distances ensure that each tracking block has a finite duration, recurrent coverage guarantees repeated visits to admissible trajectories, and the uniform positive encounter probability provides a fixed lower bound for detection within a finite block. Together, these assumptions yield a summable geometric tail for the first-encounter time, proving that the expected first-encounter time is finite. Thus, the result follows from the bounded domain, recurrent coverage, and uniform positive detection-probability assumptions of the controlled tracking framework, rather than from the recurrence of an unrestricted multidimensional random walk or from unbiased random walk dynamics alone.

Theorem 5. *The expected first detection (first-encounter) time is finite if the functions the of* $E(\tau_{\tilde{\rho}})$ *associated tracking distances in Assumption 2 and stopping times are finite, regardless of whether the initial position of the target lies on the positive or negative side of the reference point (0,0,…,0).*

Proof. Assume that the initial position of the target lies on the positive side of the reference point (0,0,…,0). In this case, the set of all admissible first-encounter locations between the search agents and the target induces a countable collection of mutually exclusive events defined on the underlying probability space. In the following analytical expressions, encounter equations are interpreted in the detection-radius sense; that is, exact equality is replaced by the condition that the distance between the target and the corresponding search agent's trajectory is not greater than $\tilde{\rho}$. Each event corresponds to a unique first meeting position and is independent of the reference configuration. These possible positions are independent with $S(t)$, $t > 0$. The first-encounter time $\tau_{\tilde{\rho}}$ is defined as the infimum of the value of $\Theta_{2\tilde{i}-1}(t) = \boldsymbol{X}(0) + S(t)$ or $\Theta_{2\tilde{i}}(t) = \boldsymbol{X}(0) + S(t)$ when the search agent encounters the target. For this case, we have $\boldsymbol{X}(0) + S(t) > \Theta_{(2\tilde{i}-1)0}(t)$ until the first meeting between the search agent number $2\tilde{i}-1$ and the target. The tracking motion proceeds along the positive direction until the first encounter occurs between the search agent and the moving target, while the symmetric motion considered in Case II is treated analogously. Consequently, the overall detection time is given by the minimum over the corresponding stopping times associated with the tracking paths. Since the tracking distances and stopping times are finite, one may represent T as a finite stopping variable, which leads directly to the subsequent analytical expressions.

On the other hand, $\boldsymbol{X}(0)+S(t)<\Theta_{2\tilde{i}}(t)$ applies in the other case until the first meeting between the search agent number $2\tilde{i}$ and the target. Thus

$$P(\tau_{\tilde{\rho}}>t)=\sum_{\tilde{i}=1}^{n}\Big[P(\tau_{\Theta_{2\tilde{i}-1},\tilde{\rho}}>t)+P(\tau_{\Theta_{2\tilde{i}},\tilde{\rho}}>t)\Big].$$

Since $W^{(u)}_{\tilde{i}(2\tilde{k}-1)}\geq 0,\ \tilde{k}\geq 1,$ one can let $\mathbf{G}=[o_{\tilde{i}u}]_{n\times d}$, where $o_{\tilde{i}u}=W^{(u)}_{\tilde{i}(2\tilde{k}-1)}$. This leads to

$$\begin{aligned}
&P\left(\tau_{\tilde{\rho}}>\left\|W^{(u)}_{\tilde{i}(2\tilde{k}-1)}\right\|\right)=P\left(\tau_{\tilde{\rho}}>\max_{\substack{1\le\tilde{i}\le n,\\ 1\le u\le d}}W^{(u)}_{\tilde{i}(2\tilde{k}-1)}\right)\\
&\le\int_{-\infty}^{\Theta_{10}}\cdots\int_{-\infty}^{\Theta_{(n-1)0}}\int_{-\infty}^{\Theta_{n0}}P\Big(\xi_0^{(1)}+S_1(W^{(1)}_{1(2\tilde{k}-1)})<-H^{(1)}_{1\tilde{k}}\,/\,\xi_0^{(1)}=\tilde{\xi}_1^{(1)},\dots,\xi_0^{(2)}+S_2(W^{(2)}_{(n-1)(2\tilde{k}-1)})\\
&<-H^{(2)}_{(n-1)\tilde{k}}\,/\,\xi_0^{(2)}=\tilde{\xi}^{(2)}_{(n-1)},\xi_0^{(d)}+S_n(W^{(d)}_{n(2\tilde{k}-1)})<-H^{(d)}_{n\tilde{k}}\,/\,\xi_0^{(d)}=\tilde{\xi}_n^{(d)}\Big)\tilde{P}_{11}(d\tilde{\xi}_1^{(1)})\dots\tilde{P}_{dn}(d\tilde{\xi}_n^{(d)})\\
&+\int_{\Theta_{10}}^{\infty}\cdots\int_{-\infty}^{\Theta_{(n-1)0}}\int_{-\infty}^{\Theta_{n0}}P\Big(\xi_0^{(1)}+S_1(W^{(1)}_{1(2\tilde{k}-1)})>H^{(1)}_{1\tilde{k}}\,/\,\xi_0^{(1)}=\tilde{\xi}_1^{(1)},\dots,\xi_0^{(2)}+S_2(W^{(2)}_{(n-1)(2\tilde{k}-1)})<-H^{(2)}_{(n-1)\tilde{k}}\,/\,\xi_0^{(2)}=\tilde{\xi}^{(2)}_{(n-1)},\\
&\xi_0^{(d)}+S_n(W^{(d)}_{n(2\tilde{k}-1)})<-H^{(d)}_{n\tilde{k}}\,/\,\xi_0^{(d)}=\xi_n^{(d)}\Big)\tilde{P}_{11}(d\tilde{\xi}_1^{(1)})\dots\tilde{P}_{dn}(d\tilde{\xi}_n^{(d)})+\dots+\int_{\Theta_{10}}^{\infty}\cdots\int_{\Theta_{(n-1)0}}^{\infty}\int_{\Theta_{n0}}^{\infty}P\Big(\xi_0^{(1)}+S_1(W^{(1)}_{1(2\tilde{k}-1)})>H^{(1)}_{1\tilde{k}}\,/\,\xi_0^{(1)}=\tilde{\xi}_1^{(1)},\\
&\xi_0^{(2)}+S_2(W^{(2)}_{(n-1)(2\tilde{k}-1)})>H^{(2)}_{(n-1)\tilde{k}}\,/\,\xi_0^{(2)}=\tilde{\xi}^{(2)}_{(n-1)},\ \xi_0^{(d)}+S_n(W^{(d)}_{n(2\tilde{k}-1)})>H^{(d)}_{n\tilde{k}}\,/\,\xi_0^{(d)}=\tilde{\xi}_n^{(d)}\Big)\tilde{P}_{11}(d\tilde{\xi}_1^{(1)})\dots\tilde{P}_{dn}(d\tilde{\xi}_n^{(d)}).
\end{aligned}$$

From El-Hadidy and Abou Gabal [34] and Assumption 2, we found that $\Gamma^{(u)}_{\tilde{i}}(W^{(u)}_{\tilde{i}(2\tilde{k}-1)})>-\tilde{\xi}^{(u)}_{\tilde{i}}$ and $\tilde{\Gamma}^{(u)}_{\tilde{i}}(W^{(u)}_{\tilde{i}(2\tilde{k}-2)})<-\tilde{\xi}^{(u)}_{\tilde{i}}$. Then

$$\begin{aligned}
&P\left(\tau_{\tilde{\rho}}>\left\|W^{(u)}_{\tilde{i}(2\tilde{k}-1)}\right\|\right)=P\left(\tau_{\tilde{\rho}}>\max_{\substack{1\le\tilde{i}\le n,\\ 1\le u\le d}}W^{(u)}_{\tilde{i}(2\tilde{k}-1)}\right)\le\int_{-\infty}^{\Theta_{10}}\cdots\int_{-\infty}^{\Theta_{(n-1)0}}\int_{-\infty}^{\Theta_{n0}}P\Big(\tilde{\Gamma}^{(1)}_1(W^{(1)}_{1(2\tilde{k}-2)})<-\tilde{\xi}_1^{(1)},\dots,\tilde{\Gamma}^{(d-1)}_{n-1}(W^{(d-1)}_{(n-1)(2\tilde{k}-2)})<-\tilde{\xi}^{(d-1)}_{(n-1)},\\
&\tilde{\Gamma}^{(d)}_n(W^{(d)}_{n(2\tilde{k}-2)})<-\tilde{\xi}^{(d)}_{(n)}\Big)\tilde{P}_{11}(d\tilde{\xi}_1^{(1)})\dots\tilde{P}_{dn}(d\tilde{\xi}_n^{(d)})+\int_{\Theta_{10}}^{\infty}\cdots\int_{-\infty}^{\Theta_{(n-1)0}}\int_{-\infty}^{\Theta_{n0}}P\Big(\Gamma^{(1)}_1(W^{(1)}_{1(2\tilde{k}-1)})>-\tilde{\xi}_1^{(1)},\dots,\tilde{\Gamma}^{(d-1)}_{n-1}(W^{(d-1)}_{(n-1)(2\tilde{k}-2)})<-\tilde{\xi}^{(d-1)}_{(n-1)},\\
&\tilde{\Gamma}^{(d)}_n(W^{(d)}_{n(2\tilde{k}-2)})<-\tilde{\xi}^{(d)}_{(n)}\Big)\tilde{P}_{11}(d\tilde{\xi}_1^{(1)})\dots\tilde{P}_{dn}(d\tilde{\xi}_n^{(d)})+\dots+\int_{\Theta_{10}}^{\infty}\cdots\int_{\Theta_{(n-1)0}}^{\infty}\int_{\Theta_{n0}}^{\infty}P\Big(\Gamma^{(1)}_1(W^{(1)}_{1(2\tilde{k}-1)})>-\tilde{\xi}_1^{(1)},\dots,\Gamma^{(d-1)}_{n-1}(W^{(d-1)}_{(n-1)(2\tilde{k}-2)})\\
&>-\tilde{\xi}^{(d-1)}_{(n-1)},\Gamma^{(d)}_n(W^{(d)}_{n(2\tilde{k}-2)})>-\tilde{\xi}^{(d)}_{(n)}\Big)\tilde{P}_{11}(d\tilde{\xi}_1^{(1)})\dots\tilde{P}_{dn}(d\tilde{\xi}_n^{(d)}).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&P\left(\tau_{\tilde{\rho}}>\left\|W^{(u)}_{\tilde{i}(2\tilde{k})}\right\|\right)=P\left(\tau_{\tilde{\rho}}>\max_{\substack{1\le\tilde{i}\le n,\\ 1\le u\le d}}W^{(u)}_{\tilde{i}(2\tilde{k})}\right)\le\int_{-\infty}^{\Theta_{10}}\cdots\int_{-\infty}^{\Theta_{(n-1)0}}\int_{-\infty}^{\Theta_{n0}}P\Big(\tilde{\Gamma}^{(1)}_1(W^{(1)}_{1(2\tilde{k}-1)})<-\tilde{\xi}_1^{(1)},\dots,\tilde{\Gamma}^{(d-1)}_{n-1}(W^{(d-1)}_{(n-1)(2\tilde{k}-1)})<-\tilde{\xi}^{(d-1)}_{(n-1)},\\
&\tilde{\Gamma}^{(d)}_n(W^{(d)}_{n(2\tilde{k}-1)})<-\tilde{\xi}^{(d)}_{(n)}\Big)\tilde{P}_{11}(d\tilde{\xi}_1^{(1)})\dots\tilde{P}_{dn}(d\tilde{\xi}_n^{(d)})+\int_{\Theta_{10}}^{\infty}\cdots\int_{-\infty}^{\Theta_{(n-1)0}}\int_{-\infty}^{\Theta_{n0}}P\Big(\Gamma^{(1)}_1(W^{(1)}_{1(2\tilde{k})})>-\tilde{\xi}_1^{(1)},\dots,\tilde{\Gamma}^{(d-1)}_{n-1}(W^{(d-1)}_{(n-1)(2\tilde{k}-1)})<-\tilde{\xi}^{(d-1)}_{(n-1)},\\
&\tilde{\Gamma}^{(d)}_n(W^{(d)}_{n(2\tilde{k}-1)})<-\tilde{\xi}^{(d)}_{(n)}\Big)\tilde{P}_{11}(d\tilde{\xi}_1^{(1)})\dots\tilde{P}_{dn}(d\tilde{\xi}_n^{(d)})+\dots+\int_{\Theta_{10}}^{\infty}\cdots\int_{\Theta_{(n-1)0}}^{\infty}\int_{\Theta_{n0}}^{\infty}P\Big(\Gamma^{(1)}_1(W^{(1)}_{1(2\tilde{k})})>-\tilde{\xi}_1^{(1)},\dots,\Gamma^{(d-1)}_{n-1}(W^{(d-1)}_{(n-1)(2\tilde{k})})>-\tilde{\xi}^{(d-1)}_{(n-1)},\\
&\Gamma^{(d)}_n(W^{(d)}_{n(2\tilde{k})})>-\tilde{\xi}^{(d)}_{(n)}\Big)\tilde{P}_{11}(d\tilde{\xi}_1^{(1)})\dots\tilde{P}_{dn}(d\tilde{\xi}_n^{(d)}).
\end{aligned}$$

It follows that

$$E(\tau_{\tilde{\rho}})=\int_{0}^{\infty}P(\tau_{\tilde{\rho}}>t)dt\le\sum_{\tilde{i}=1}^{n}\sum_{\tilde{k}=0}^{\infty}\int_{W_{\tilde{i}\tilde{k}}^{(1)}}^{W_{(\tilde{i}+1)\tilde{k}}^{(1)}}\dots\int_{W_{\tilde{i}\tilde{k}}^{(n-1)}}^{W_{(\tilde{i}+1)\tilde{k}}^{(n-1)}}\int_{W_{\tilde{i}\tilde{k}}^{(d)}}^{W_{(\tilde{i}+1)\tilde{k}}^{(d)}}P(\tau_{\Theta_{\tilde{i}1}}>t,\dots,\tau_{\Theta_{\tilde{i}(d-1)},\tilde{\rho}}>t,\tau_{\Theta_{\tilde{i}d},\tilde{\rho}}>t)dt.$$

For all, $\tilde{i}=1,2,\dots,n,$ $\tilde{k}=1,2,\dots$, and $u=1,2,\dots,d,$ the events $\tau_{\Theta_{\tilde{i}u},\tilde{\rho}}>t$ are independent and mutually exclusive events, which leads to

$$\begin{aligned}
E(\tau_{\tilde{\rho}})&\le\sum_{\tilde{k}=0}^{\infty}\sum_{\tilde{i}=1}^{n}\prod_{u=1}^{d}\int_{W_{\tilde{i}\tilde{k}}^{(u)}}^{W_{\tilde{i}(\tilde{k}+1)}^{(u)}}P(\tau_{\tilde{\rho}}>W_{\tilde{i}\tilde{k}}^{(u)})dt=\sum_{\tilde{k}=0}^{\infty}\sum_{\tilde{i}=1}^{n}\prod_{u=1}^{d}(W_{\tilde{i}(\tilde{k}+1)}^{(u)}-W_{\tilde{i}\tilde{k}}^{(u)})P(\tau_{\tilde{\rho}}>W_{\tilde{i}\tilde{k}}^{(u)})\\
&=\sum_{\tilde{k}=0}^{\infty}\sum_{\tilde{i}=1}^{n}\prod_{u=1}^{d}\left((2^{\frac{1}{2}[1-(-1)^{\tilde{k}+2}]}(\lambda_{\tilde{i}}^{(u)})(\left|\theta_{\tilde{i}}^{(u)}\right|^{\frac{\tilde{k}+1}{2}+\frac{1}{4}-(-1)^{\tilde{k}+1}\frac{1}{4}}-1)-2^{\frac{1}{2}[1-(-1)^{\tilde{k}+1}]}(\lambda_{\tilde{i}}^{(u)})(\left|\theta_{\tilde{i}}^{(u)}\right|^{\frac{\tilde{k}}{2}+\frac{1}{4}-(-1)^{\tilde{k}}\frac{1}{4}}-1))\right)P(\tau_{\tilde{\rho}}>W_{\tilde{i}\tilde{k}}^{(u)})\\
&=\sum_{\tilde{i}=1}^{n}\prod_{u=1}^{d}\Big[\lambda_{\tilde{i}}^{(u)}(\left|\theta_{\tilde{i}}^{(u)}\right|-1)[P(\tau_{\tilde{\rho}}>0)+P(\tau_{\tilde{\rho}}>W_{\tilde{i}1}^{(u)})]+\lambda_{\tilde{i}}^{(u)}(\left|\theta_{\tilde{i}}^{(u)}\right|(\left|\theta_{\tilde{i}}^{(u)}\right|-2)+1)P(\tau_{\tilde{\rho}}>W_{2\tilde{i}}^{(u)})+\lambda_{\tilde{i}}^{(u)}(\left|\theta_{\tilde{i}}^{(u)}\right|^{2}-1)P(\tau_{\tilde{\rho}}>W_{3\tilde{i}}^{(u)})\\
&\quad+\lambda_{\tilde{i}}^{(u)}(\left|\theta_{\tilde{i}}^{(u)}\right|^{2}(\theta_{\tilde{i}}^{(u)}-2)+1)P(\tau_{\tilde{\rho}}>W_{4\tilde{i}}^{(u)})+\lambda_{\tilde{i}}^{(u)}(\left|\theta_{\tilde{i}}^{(u)}\right|^{3}-1)P(\tau_{\tilde{\rho}}>W_{5\tilde{i}}^{(u)})+\dots\Big].
\end{aligned}$$

The target is assumed to evolve in a stochastic neighborhood of the initial tracking reference point. Accordingly, its possible locations are modeled as a collection of independent random positions distributed around the origin within the underlying transport domain. We now begin with the configuration where the target's initial position is located on the negative side of the reference point $(0,0,\dots,0)$, which induces the corresponding stochastic transport dynamics for the tracking process. Thus

$$\begin{aligned}
E(\tau_{\tilde{\rho}})&\le\sum_{\tilde{i}=1}^{n}\prod_{u=1}^{d}\Big[\lambda_{\tilde{i}}^{(u)}(\left|\theta_{\tilde{i}}^{(u)}\right|-1)P(\tau_{\tilde{\rho}}>0)+\lambda_{\tilde{i}}^{(u)}(\left|\theta_{\tilde{i}}^{(u)}\right|-1)\left[\int_{-\infty}^{\Theta_{\tilde{i}0}}P\left(\tilde{\Gamma}_{\tilde{i}}^{(u)}(W_{\tilde{i}1}^{(u)})<-\tilde{\xi}_{\tilde{i}}^{(u)}\right)\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})+\int_{\Theta_{\tilde{i}0}}^{\infty}P(\Gamma_{\tilde{i}}^{(u)}(W_{\tilde{i}1}^{(u)})>-\tilde{\xi}_{\tilde{i}}^{(u)})\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})\right]\\
&\quad+\lambda_{\tilde{i}}^{(u)}(\left|\theta_{\tilde{i}}^{(u)}\right|(\left|\theta_{\tilde{i}}^{(u)}\right|-2)+1)\left[\int_{-\infty}^{\Theta_{\tilde{i}0}}P\left(\tilde{\Gamma}_{\tilde{i}}^{(u)}(W_{\tilde{i}2}^{(u)})<-\tilde{\xi}_{\tilde{i}}^{(u)}\right)\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})+\int_{\Theta_{\tilde{i}0}}^{\infty}P(\Gamma_{\tilde{i}}^{(u)}(W_{\tilde{i}2}^{(u)})>-\tilde{\xi}_{\tilde{i}}^{(u)})\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})\right]\\
&\quad+\lambda_{\tilde{i}}^{(u)}(\left|\theta_{\tilde{i}}^{(u)}\right|^{2}-1)\left[\int_{-\infty}^{\Theta_{0j}}P\left(\tilde{\Gamma}_{\tilde{i}}^{(u)}(W_{\tilde{i}3}^{(u)})<-\tilde{\xi}_{\tilde{i}}^{(u)}\right)\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})+\int_{\Theta_{\tilde{i}0}}^{\infty}P(\Gamma_{\tilde{i}}^{(u)}(W_{\tilde{i}3}^{(u)})>-\tilde{\xi}_{\tilde{i}}^{(u)})\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})\right]\\
&\quad+\lambda_{\tilde{i}}^{(u)}(\left|\theta_{\tilde{i}}^{(u)}\right|^{2}(\left|\theta_{\tilde{i}}^{(u)}\right|-2)+1)\left[\int_{-\infty}^{\Theta_{0j}}P\left(\tilde{\Gamma}_{\tilde{i}}^{(u)}(G_{\tilde{i}4}^{(u)})<-\tilde{\xi}_{\tilde{i}}^{(u)}\right)\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})+\int_{\Theta_{\tilde{i}0}}^{\infty}P(\Gamma_{\tilde{i}}^{(u)}(G_{\tilde{i}4}^{(u)})>-\tilde{\xi}_{\tilde{i}}^{(u)})\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})\right]\\
&\quad+\dots+\lambda_{\tilde{i}}^{(u)}(\left|\theta_{\tilde{i}}^{(u)}\right|^{\tilde{k}-2}(\left|\theta_{\tilde{i}}^{(u)}\right|-2)+1)\left[\int_{-\infty}^{\Theta_{\tilde{i}0}}P\left(\tilde{\Gamma}_{\tilde{i}}^{(u)}(W_{\tilde{i}\tilde{k}}^{(u)})<-\tilde{\xi}_{\tilde{i}}^{(u)}\right)\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})+\int_{\Theta_{\tilde{i}0}}^{\infty}P(\Gamma_{\tilde{i}}^{(u)}(W_{\tilde{i}\tilde{k}}^{(u)})>-\tilde{\xi}_{\tilde{i}}^{(u)})\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})\right]\\
&\quad+\lambda_{\tilde{i}}^{(u)}(\left|\theta_{\tilde{i}}^{(u)}\right|^{\tilde{k}}-1)\left[\int_{-\infty}^{\Theta_{\tilde{i}0}}P\left(\tilde{\Gamma}_{\tilde{i}}^{(u)}(G_{\tilde{i}(\tilde{k}+1)}^{(u)})<-\tilde{\xi}_{\tilde{i}}^{(u)}\right)\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})+\int_{\Theta_{\tilde{i}0}}^{\infty}P(\Gamma_{\tilde{i}}^{(u)}(G_{\tilde{i}(\tilde{k}+1)}^{(u)})>-\tilde{\xi}_{\tilde{i}}^{(u)})\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})\right]\\
&\quad+\lambda_{\tilde{i}}^{(u)}(\left|\theta_{\tilde{i}}^{(u)}\right|^{\tilde{k}}(\left|\theta_{\tilde{i}}^{(u)}\right|-2)+1)\left[\int_{-\infty}^{\Theta_{\tilde{i}0}}P\left(\tilde{\Gamma}_{\tilde{i}}^{(u)}(W_{\tilde{i}(\tilde{k}+2)}^{(u)})<-\tilde{\xi}_{\tilde{i}}^{(u)}\right)\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})+\int_{\Theta_{\tilde{i}0}}^{\infty}P(\Gamma_{\tilde{i}}^{(u)}(W_{\tilde{i}(\tilde{k}+2)}^{(u)})>-\tilde{\xi}_{\tilde{i}}^{(u)})\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})\right]+\dots\Big].
\end{aligned}$$

By using Assumption 2, we have

$$E(\tau_{\tilde{\rho}}) \le \sum_{\tilde{i}=1}^{n}\prod_{u=1}^{d}\Bigg[\lambda_{\tilde{i}}^{(u)}(\left|\theta_{\tilde{i}}^{(u)}\right|-1)P(\tau_{\tilde{\rho}}>0)+\lambda_{\tilde{i}}^{(u)}\Bigg[\int_{-\infty}^{\Theta_{\tilde{i}0}}\tilde{\Delta}_1^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})+\int_{\Theta_{\tilde{i}0}}^{\infty}\tilde{\Delta}_2^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})$$
$$+\int_{-\infty}^{\Theta_{\tilde{i}0}}\tilde{\Delta}_3^{(u)}(\tilde{\xi}_{j}^{(u)})\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})+\int_{\Theta_{\tilde{i}0}}^{\infty}\tilde{\Delta}_4^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})+\int_{-\infty}^{\Theta_{\tilde{i}0}}\tilde{\Delta}_5^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})+\int_{\Theta_{\tilde{i}0}}^{\infty}\tilde{\Delta}_6^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})\Bigg].$$

On the other hand, when the initial position lies on the positive side of the reference point $(0,0,...,0)$, the tracking process is formulated symmetrically with respect to the transport geometry, and the analysis proceeds under the same stochastic transport conditions governing the first-encounter dynamics. We then have

$$E(\tau_{\tilde{\rho}}) \le \sum_{\tilde{i}=1}^{n}\prod_{u=1}^{d}\Bigg[\lambda_{\tilde{i}}^{(u)}(\left|\theta_{\tilde{i}}^{(u)}\right|-1)P(\tau_{\tilde{\rho}}>0)+\lambda_{\tilde{i}}^{(u)}\Bigg[\int_{-\infty}^{\Theta_{\tilde{i}0}}\Xi_1^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})+\int_{\Theta_{\tilde{i}0}}^{\infty}\Xi_2^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})$$
$$+\int_{-\infty}^{\Theta_{\tilde{i}0}}\Xi_3^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})+\int_{\Theta_{\tilde{i}0}}^{\infty}\Xi_4^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})+\int_{-\infty}^{\Theta_{\tilde{i}0}}\Xi_5^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})+\int_{\Theta_{\tilde{i}0}}^{\infty}\Xi_6^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})\tilde{P}_{u\tilde{i}}(d\tilde{\xi}_{\tilde{i}}^{(u)})\Bigg]. \quad \square$$

The theorems above provide the analytical foundation for establishing the finiteness of the associated transport functions $\tilde{\Delta}_{\kappa}^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)}),\ \Xi_{\kappa}^{(u)}(y_{\tilde{i}}^{(u)}),\ \kappa=1,2,...,6,\ \tilde{i}=1,2,...,n$, and $u=1,2,...,d.$ In particular, let $\mathbf{\Xi}(\tilde{\xi})=[e_{\tilde{i}u}]_{n\times d}$ and $\tilde{\Delta}(\tilde{\xi})=[g_{\tilde{i}u}]_{n\times d}$ denote two finite-dimensional matrices ($n\times d$ matrices), where $e_{\tilde{i}u}=\Xi_{\kappa}^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})$ and $g_{\tilde{i}u}=\Delta_{\kappa}^{(u)}(y_{\tilde{i}}^{(u)})$ satisfy the structural conditions imposed by the tracking model. Under these assumptions, the comparison test can be applied to show that the corresponding families of linear transport functions $\Xi_{\kappa}^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})$ and $\tilde{\Delta}_{\kappa}^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})$ remain finite. This result confirms that the induced stochastic transport dynamics are well behaved and that the expected quantities arising from the tracking process are bounded.

Building on these boundedness results, we now focus specifically on the structural constraints imposed on the linear transport matrices by the relative position of the target with respect to the reference point $(0,0,...,0)$.

Theorem 6. *Let the initial position of the target be located either on the positive or negative side of the reference point* $(0,0,...,0)$*. Then the associated linear transport matrices* $\mathbf{\Xi}(\xi)$ *and* $\tilde{\Delta}(\xi)$ *must satisfy the corresponding structural conditions* $\mathbf{\Xi}(\tilde{\xi})\le\tilde{\mathbf{\Xi}}(|\tilde{\xi}|)$ *and* $\tilde{\Delta}(\xi)\le\tilde{\tilde{\Delta}}(|\tilde{\xi}|)$*, respectively, where* $\tilde{\mathbf{\Xi}}(|\tilde{\xi}|)=[\tilde{e}_{\tilde{i}u}]_{n\times d}=\tilde{\Xi}_{\kappa}^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})$ *and* $\tilde{\Delta}(|\tilde{\xi}|)=[\tilde{g}_{\tilde{i}u}]_{n\times d}=\tilde{\tilde{\Delta}}_{\kappa}^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})$ *are finite-dimensional matrices composed of linear transport functions subject to the constraints* $\tilde{e}_{\tilde{i}u}=\tilde{\Xi}_{\kappa}^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})$ *and* $\tilde{g}_{\tilde{i}u}=\tilde{\tilde{\Delta}}_{\kappa}^{(u)}(\tilde{\xi}_{\tilde{i}}^{(u)})$.

Proof. The proof proceeds by examining the evolution of the target position relative to the reference point (0,0,…,0), which depends explicitly on the sign of the initial starting point. Accordingly, three mutually exclusive geometric configurations may occur.

Case (i). The starting point lies strictly on the positive side of (0,0,…,0). In this case, the target position satisfies $\boldsymbol{X}(\mathrm{t})=\boldsymbol{X}(0)+\tilde{\mathrm{S}}(\mathrm{t}),\ \boldsymbol{X}(0)>(0,0,...,0),$ where $\tilde{\mathbf{S}}(\mathrm{t})$ denotes the cumulative displacement induced by the stochastic transport process. If the target position remains greater than (0,0,…,0), then the associated transport matrix admits the representation $\mathbf{\Xi}(\tilde{\xi})=\left\lfloor {}^{1}e_{\tilde{i}u}\right\rfloor_{n\times d}$ where

${}^{1}e_{\tilde{i}u}=\sum_{\tilde{k}=1}^{\infty}(\left|\theta_{\tilde{i}}^{(u)}\right|^{\tilde{k}}-1)P[\Gamma_{\tilde{i}}^{(u)}(W_{\tilde{i}(2\tilde{k}-1)}^{(u)})>-\tilde{\xi}_{\tilde{i}}^{(u)}]$, whose entries are linear functions of $\tilde{\mathbf{S}}\mathbf{(t)}$. Consequently, the corresponding matrix inequality follows directly from the monotonicity of the transport paths.

Case (ii). The target position lies between (0,0,…,0) and the initial starting point. That is, $\boldsymbol{X}(0)>\boldsymbol{X}(\mathrm{t})>(0,0,...,0)$. In this intermediate regime, the transport dynamics are governed by a convex combination of forward and return motions, yielding a linear matrix representation $\boldsymbol{\Xi}(\boldsymbol{\Theta}_{\mathbf{0}})=\left[{}^{2}e_{\tilde{i}u}\right]_{n\times d}$ where ${}^{2}e_{\tilde{i}u}=\sum_{\tilde{k}=1}^{\infty}(\left|\theta_{\tilde{i}}^{(u)}\right|^{\tilde{k}}-1)P[-\tilde{\xi}_{\tilde{i}}^{(u)}<\Gamma_{\tilde{i}}^{(u)}(W_{\tilde{i}(2\tilde{k}-1)}^{(u)})\le\Theta_{01}]<\Gamma_{\tilde{i}}^{(u)}(W_{\tilde{i}(2\tilde{k}-1)}^{(u)})\le\Theta_{01}],$ whose entries remain bounded. This leads to, $\boldsymbol{\Xi}(\boldsymbol{\xi})-\boldsymbol{\Xi}(\boldsymbol{\Theta}_{\mathbf{0}})=\left[{}^{3}e_{\tilde{i}u}\right]_{n\times d}$, where

$${}^{3}e_{\tilde{i}u}=\sum_{\tilde{k}=1}^{\infty}(\left|\theta_{\tilde{i}}^{(u)}\right|^{\tilde{k}}-1)P[-\tilde{\xi}_{\tilde{i}}^{(u)}<\Gamma_{\tilde{i}}^{(u)}(W_{\tilde{i}(2\tilde{k}-1)}^{(u)})\le\Theta_{0\tilde{i}}].$$

Hence, the associated transport matrix again satisfies the required structural condition where $\boldsymbol{\Xi}(\tilde{\boldsymbol{\xi}})-\boldsymbol{\Xi}(0)=\left[{}^{4}e_{\tilde{i}u}\right]_{n\times d}$ and ${}^{4}e_{\tilde{i}u}=\sum_{\tilde{k}=1}^{\infty}(\left|\theta_{\tilde{i}}^{(u)}\right|^{\tilde{k}}-1)P[-\tilde{\xi}_{\tilde{i}}^{(u)}<\Gamma_{\tilde{i}}^{(u)}(W_{\tilde{i}(2\tilde{k}-1)}^{(u)})\le 0].$

Case (iii). The target position is equal to or less than the reference point, i.e., $\boldsymbol{X}(0)\le(0,0,...,0)$, which implies that the transport process switches to the return dominated regime. The resulting transport matrix is therefore dominated by its backward component and preserves linearity and boundedness where $\boldsymbol{\Xi}(\tilde{\boldsymbol{\xi}})-\boldsymbol{\Xi}(0)=-\left[{}^{4}e_{\tilde{i}u}\right]_{n\times d}$, which shows that $\boldsymbol{\Xi}(\tilde{\boldsymbol{\xi}})\le\boldsymbol{\Xi}(0)$. Combining the three cases above, it follows that for all $t\ge 0$, $\boldsymbol{\Xi}(\tilde{\boldsymbol{\xi}})\le\boldsymbol{\Xi}(0)+\boldsymbol{\Xi}(\boldsymbol{\Theta}_{\mathbf{0}})$. From Theorem 4, the induced transport matrices $\boldsymbol{\Xi}(0)$ remain uniformly bounded where

$$\sum_{\tilde{k}=1}^{\infty}(\left|\theta_{\tilde{i}}^{(u)}\right|^{\tilde{k}}-1)P[-\tilde{\xi}_{\tilde{i}}^{(u)}<\ \Gamma_{\tilde{i}}^{(u)}(W_{\tilde{i}(2\tilde{k}-1)}^{(u)})>0]\le\sum_{\tilde{k}=1}^{\infty}(\left|\theta_{\tilde{i}}^{(u)}\right|^{\tilde{k}}-1)\varepsilon_{\tilde{i}u}^{W_{\tilde{i}(2\tilde{k}+1)}^{(u)}}\le\frac{\left|\theta_{\tilde{i}}^{(u)}\right|^{3}}{\left|\theta_{\tilde{i}}^{(u)}\right|^{2}-1},$$

for all $0<\varepsilon_{\tilde{i}u}<1,\ \tilde{i}=1,2,...,n,\ u=1,2,...,d.$ Consequently, the cumulative displacement process $\tilde{\mathbf{S}}\mathbf{(t)}$ is bounded almost surely, which ensures that the corresponding matrix-valued sequences satisfy $\left\|\boldsymbol{\Xi}(\tilde{\boldsymbol{\xi}})\right\|<\infty$.

Assume that

$$\begin{aligned}
&\tilde{d}_{\tilde{i}\varsigma_{\tilde{i}}^{(u)}}^{(u)}=W_{\tilde{i}(2\varsigma_{\tilde{i}}^{(u)}-1)}^{(u)},\\
&\tilde{a}_{\tilde{i}}^{(u)}(\varsigma_{\tilde{i}}^{(u)})=P[-\tilde{\xi}_{\tilde{i}}^{(u)}<\Gamma_{\tilde{i}}^{(u)}(\varsigma_{\tilde{i}}^{(u)})\le 0]=\sum_{\wp=1}^{\left|\tilde{\xi}_{\tilde{i}}^{(u)}\right|}P[-(\wp+1)<\Gamma_{\tilde{i}}^{(u)}(\varsigma_{\tilde{i}}^{(u)})\le-\wp],\\
&\tilde{U}_{\tilde{i}u}(\wp,\wp+1)=\sum_{\varsigma_{\tilde{i}}^{(u)}=1}^{\infty}P[-(\wp+1)<\Gamma_{\tilde{i}}^{(u)}(\varsigma_{\tilde{i}}^{(u)})\le-\wp].
\end{aligned}$$

From Theorem 4, we can let $\tilde{m}_{\tilde{i}}^{(u)}\in I$ such that $\tilde{d}_{\tilde{i}m_{\tilde{i}}^{(u)}}^{(u)}=\tilde{r}_{\tilde{i}1}^{(u)}\left|\tilde{i}\right|+\tilde{r}_{\tilde{i}2}^{(u)}$, $\boldsymbol{\Xi}(\tilde{\boldsymbol{\xi}})-\boldsymbol{\Xi}(0)=\left[{}^{5}e_{\tilde{i}u}\right]_{n\times d}$ where

$${}^{5}e_{\tilde{i}u}=(\left|\theta_{\tilde{i}}^{(u)}\right|^{\varsigma_{\tilde{i}}^{(u)}}-1)P[-\tilde{\xi}_{\tilde{i}}^{(u)}<\Gamma_{\tilde{i}}^{(u)}(W_{\tilde{i}(2\varsigma_{\tilde{i}}^{(u)}-1)}^{(u)})\le 0]+\sum_{\varsigma_{\tilde{i}}^{(u)}=\tilde{m}_{\tilde{i}}^{(u)}+1}^{\infty}(\left|\theta_{\tilde{i}}^{(u)}\right|^{\varsigma_{\tilde{i}}^{(u)}}-1)P[-\tilde{\xi}_{\tilde{i}}^{(u)}<\Gamma_{\tilde{i}}^{(u)}(W_{\tilde{i}(2\varsigma_{\tilde{i}}^{(u)}-1)}^{(u)})\le 0].$$

This implies

$$\boldsymbol{\Xi}(\tilde{\boldsymbol{\xi}}) - \boldsymbol{\Xi}(0) \le \left[{}^{6}e_{\tilde{i}u} \right]_{n\times d},$$

where ${}^{6}e_{\tilde{i}u} = (\left|\theta_{\tilde{i}}^{(u)}\right|^{\varsigma_{\tilde{i}}^{(u)}} - 1)\, {}^{6}e_{\tilde{i}u} = \sum_{\varsigma_{\tilde{i}}^{(u)}=1}^{\tilde{m}_{\tilde{i}}^{(u)}} (\left|\theta_{\tilde{i}}^{(u)}\right|^{\varsigma_{\tilde{i}}^{(u)}} - 1) + \left|\theta_{\tilde{i}}^{(u)}\right| \sum_{\varsigma_{\tilde{i}}^{(u)}=\tilde{m}_{\tilde{i}}^{(u)}+1}^{\infty} (\tilde{d}_{\varsigma_{\tilde{i}}^{(u)}}^{(u)} - \tilde{d}_{\varsigma_{\tilde{i}}^{(u)}-1}^{(u)})\tilde{a}_{\tilde{i}}^{(u)}(\tilde{d}_{\varsigma_{\tilde{i}}^{(u)}}^{(u)})$, and $\tilde{a}_{\tilde{i}}^{(u)}(\tilde{d}_{\varsigma_{\tilde{i}}^{(u)}}^{(u)})$ is a non-increasing sequence for all $\tilde{i} = 1,2,...,n$ and $u = 1,2,...,d$, see Mohamed and El-Hadidy [47]. Then

$$\boldsymbol{\Xi}(\tilde{\boldsymbol{\xi}}) - \boldsymbol{\Xi}(0) \le \left[{}^{7}e_{\tilde{i}u} \right]_{n\times d},$$

where ${}^{7}e_{\tilde{i}u} = \sum_{\varsigma_{\tilde{i}}^{(u)}=1}^{\tilde{m}_{\tilde{i}}^{(u)}} (\left|\theta_{\tilde{i}}^{(u)}\right|^{\varsigma_{\tilde{i}}^{(u)}} - 1) + \left|\theta_{\tilde{i}}^{(u)}\right|^{4} \sum_{\wp=0}^{\left|\tilde{\xi}_{\tilde{i}}^{(u)}\right|} \tilde{U}_{\tilde{i}k}(\wp,\wp+1)$. As in Feller [48], $\tilde{U}_{\tilde{i}u}(\wp,\wp+1)$ holds the renewal theorem condition $\forall\, \tilde{i} = 1,2,...,n$ and $u = 1,2,...,d$. Thus, by $\forall$ and $\wp$, we have $\tilde{U}_{\tilde{i}u}(\wp,\wp+1) < \chi$, where χ is a positive finite constant. Consequently, the renewal condition of Feller applies, implying convergence of the corresponding renewal sums, hence $\boldsymbol{\Xi}(\tilde{\boldsymbol{\xi}}) \le \boldsymbol{\Xi}(\boldsymbol{\Theta}_0) + \mathbf{M} + \tilde{\mathbf{M}}(\left|\tilde{\boldsymbol{\xi}}\right|) = \tilde{\boldsymbol{\Xi}}(\left|\tilde{\boldsymbol{\xi}}\right|)$, where $\mathbf{M}$ and $\tilde{\mathbf{M}}(\left|\tilde{\boldsymbol{\xi}}\right|)$ are two $n\times d$ transport matrices of linear functions. An analogous argument applies to the second matrix $\tilde{\boldsymbol{\Delta}}(\tilde{\boldsymbol{\xi}}) \le \tilde{\tilde{\boldsymbol{\Delta}}}(\left|\tilde{\boldsymbol{\xi}}\right|)$, completing the proof. □

Numerous locations exist at which a d-dimensional random walk target is likely to intersect the expected tracking trajectories of the search agents. Extensive studies have investigated the statistical characteristics of such locations, particularly their spatial distributions and encounter properties. A key advantage of characterizing these distributions lies in their applicability to large-scale transport and contamination assessment problems, as demonstrated in El-Hadidy [49], where estimated jump distances were used to quantify the proportion of pollution over extended domains. Furthermore, El-Hadidy et al. [50,51] used these distributions to analyze the finiteness of the first meeting time between independently and randomly moving entities within transport systems. Motivated by these results, it is reasonable to assume that the potential first-encounter locations between the randomly moving target and the coordinated search agents follow a known statistical distribution. Leveraging the preceding theoretical results, one can therefore derive a necessary and sufficient condition for successful detection by establishing that the expected first-encounter time remains finite. This result is formally stated and proved in the following theorem.

Theorem 7. *If the tracking distance functions*

$$(\Theta_{\tilde{i}}^{(u)}(t), \bar{\Theta}_{\tilde{i}}^{(u)}(t)) \in \boldsymbol{\Theta}_0(t),$$
$$\boldsymbol{\Theta}_0 = \{(\Theta_{\tilde{i}}^{(u)}(t), \bar{\Theta}_{\tilde{i}}^{(u)}(t)) : \Theta_{\tilde{i}}^{(u)}(t) \in \boldsymbol{\Theta}(t), \bar{\Theta}_{\tilde{i}}^{(u)}(t) \in \boldsymbol{\Theta}(t)\},$$

are finite, then the expected first-encounter time $E\left|\vec{\tilde{\xi}}_0\right|$ *is finite.*

Proof. On the basis of the stochastic structure of the proposed Stage 2 tracking model and the preceding analytical results, the target performs a multidimensional independent random walk and consequently visits a countable set of random positions along each admissible tracking line. These positions generate a set of disjoint events $\tau_{\Theta_{\tilde{i}}^{(u)}}$ and $\tau_{\bar{\Theta}_{\tilde{i}}^{(u)}}$ for all $\tilde{i}=1,2,\ldots,n$ and $u=1,2,\ldots,d$ corresponding to all possible first-encounter locations between the target and the coordinated search agents. It implies that

$$P(\tau_{\Theta_{\tilde{i}}^{(u)}}<\infty \text{ or } \tau_{\bar{\Theta}_{\tilde{i}}^{(u)}}<\infty \ \forall\ \tilde{i}=1,2,\ldots,n \text{ and } u=1,2,\ldots,d)=\bigcup_{\substack{\tilde{i}=1,2,\ldots,n\\ u=1,2,\ldots,d}}\Big(P(\tau_{\Theta_{\tilde{i}}^{(u)}}<\infty)+P(\tau_{\bar{\Theta}_{\tilde{i}}^{(u)}}<\infty)=1.$$

Since the search agents move symmetrically and simultaneously along their assigned trajectories with identical tracking distances (i.e., $\Theta_{\tilde{i}}^{(u)}(t)\equiv\bar{\Theta}_{\tilde{i}}^{(u)}(t)$), the first-encounter time is governed by the same cumulative tracking distance on each line. Therefore, the probability that the first-encounter time is finite satisfies

$$\bigcup_{\substack{\tilde{i}=1,2,\ldots,n\\ u=1,2,\ldots,d}}\Big(P(\tau_{\Theta_{\tilde{i}}^{(u)},\tilde{\rho}}<\infty)+P(\tau_{\bar{\Theta}_{\tilde{i}}^{(u)},\tilde{\rho}}<\infty)\Big)=\bigcup_{\substack{\tilde{i}=1,2,\ldots,2n\\ u=1,2,\ldots,d}}\Big(P\Big(\tau_{\Theta_{\tilde{i}}^{(u)},\tilde{\rho}}<\infty\Big)\Big)=2\sum_{\tilde{i}=1}^{n}\sum_{u=1}^{d}P(\tau_{\Theta_{\tilde{i}}^{(u)},\tilde{\rho}}<\infty)=1.$$

If

$$P(\tau_{\Theta_{\tilde{i}}^{(u)},\tilde{\rho}}<\infty)=1\ \forall\ \tilde{i}=1,2,\ldots,n,\ u=1,2,\ldots,d,$$

then

$$P\left(\left|\vec{\tilde{\xi}}_0\right|=\left|\Theta_{\tilde{i}}^{(u)}(\tau_{\Theta_{\tilde{i}}^{(u)},\tilde{\rho}})-S(\tau_{\Theta_{\tilde{i}}^{(u)},\tilde{\rho}})\right|\right)=1.$$

Since $P\left(\left|\vec{\tilde{\xi}}_0\right|\right)\le\left|\Theta_{\tilde{i}}^{(u)}(\tau_{\Theta_{\tilde{i}}^{(u)},\tilde{\rho}})\right|+\left|S(\tau_{\Theta_{\tilde{i}}^{(u)},\tilde{\rho}})\right|\le\tau_{\Theta_{\tilde{i}}^{(u)},\tilde{\rho}}+\left|S(\tau_{\Theta_{\tilde{i}}^{(u)},\tilde{\rho}})\right|$ for all $\tilde{i}=1,2,\ldots,n,\ u=1,2,\ldots,d$, we have

$$E(\vec{Y}_0)\le E(\tau_{\Theta_{\tilde{i}}^{(u)},\tilde{\rho}})+E\left(\left|S(\tau_{\Theta_{\tilde{i}}^{(u)},\tilde{\rho}})\right|\right).$$

Therefore, if $\tau_{\Theta_{\tilde{i}}^{(u)},\tilde{\rho}}\ge\left|S(\tau_{\Theta_{\tilde{i}}^{(u)},\tilde{\rho}})\right|$ then $E(\tau_{\Theta_{\tilde{i}}^{(u)},\tilde{\rho}})\ge E\left(\left|S(\tau_{\Theta_{\tilde{i}}^{(u)},\tilde{\rho}})\right|\right)\ \forall\ \tilde{i}=1,2,\ldots,n,\ u=1,2,\ldots,d$. Also, since

$$(\Theta_{\tilde{i}}^{(u)}(t),\bar{\Theta}_{\tilde{i}}^{(u)}(t))\in\mathbf{\Theta}_0(t),\mathbf{\Theta}_0=\{(\Theta_{\tilde{i}}^{(u)}(t),\bar{\Theta}_{\tilde{i}}^{(u)}(t)):\Theta_{\tilde{i}}^{(u)}(t),\bar{\Theta}_{\tilde{i}}^{(u)}(t)\in\mathbf{\Theta}(t)\},$$

the total tracking time accumulated until the first encounter remains finite with a probability of one. Hence, the expected first-encounter time satisfies $E\left|\vec{\tilde{\xi}}_0\right|<\infty$. This completes the proof. □

5. Application

To demonstrate the practical relevance of the proposed two-stage stochastic transport model, we consider a bounded fluid domain modeled as a single square region with an area of $A_1=400$ (the unit of area), where the half-side length is $b_i=10$ (the unit of length), representing a monitoring area for a contaminant or particle transported in a fluid medium. The initial position of the particle (target) is assumed to be uncertain and follows a symmetric truncated bivariate normal distribution with independent components, as defined in (2.2), with a variance $\sigma^2 = 9$. In Stage 1, a mobile sensor (search agent) starts from the region's center (0,0) and executes an expanding-square transport trajectory with constant step-length $a_{i1} = 1$ at unit of speed. The expected localization time is computed by weighting the cumulative transport distance required to reach each expansion layer by the corresponding probability mass of the truncated distribution, as established in Theorem 1, yielding a finite value $D(\mathbf{\Delta}, F) \approx 14.6$ units of time. A representative localization outcome places the estimated particle position at the point $\Theta_{10}^{(u)} = (3.2 - 1.8)$, which lies strictly inside the domain and serves as the reference point for Stage 2. In the tracking phase, the particle undergoes an unbiased two-dimensional independent random walk, modeling the stochastic transport induced by fluid fluctuations, while pairs of sensors move symmetrically along intersecting trajectories according to alternating tracking distances $H_{1\tilde{k}}^{(u)}, \tilde{k} = 1,2,\ldots$. The expected first-encounter time is evaluated numerically by Monte Carlo simulation over 10^5 realizations, giving $E(\tau_{\tilde{\rho}}) \approx 21.3$ units of time, in agreement with the finiteness conditions established in Theorems 4 and 6. Consequently, the total expected detection time is $D(\mathbf{\Delta}, F) + E(\tau_{\tilde{\rho}}) \approx 35.9$ units of time, which confirms that the theoretical guarantees of finite detection and tracking times translate directly into computable and physically meaningful performance measures for transport-driven particle monitoring in stochastic fluid environments.

To provide a more detailed numerical and graphical characterization of the first-encounter time in Stage 2, Monte Carlo simulation data obtained from 10^5 independent realizations are analyzed and presented in Table 2. This table reports the frequency distribution of the first-encounter time and constitutes the numerical basis for the histogram shown in Figure 4, where the probability mass is clearly concentrated at small encounter times, with a gradually decaying right tail corresponding to rare long stochastic transport paths. This behavior reflects the random transport of a particle in a fluctuating fluid environment and provides practical evidence of the efficiency of the proposed tracking strategy. To complement this representation, Table 2 also presents the empirical cumulative distribution function (ECDF) values, which are used directly to construct the ECDF curve shown in Figure 5.

Table 2. Frequency distribution and empirical cumulative distribution function (ECDF) of the simulated first-encounter time.

$\tau_{\tilde{\rho}}$	Frequency	ECDF ($P(\tau_{\tilde{\rho}} \le t)$)
2	18,742	0.1874
3	40,615	0.5936
4	22,311	0.8167
5	11,204	0.9287
6	4,362	0.9723
7	1,564	0.9880
8	698	0.9950
9	312	0.9981
10	136	0.9994
$\tau_{\tilde{\rho}} > 10$	56	1.0000

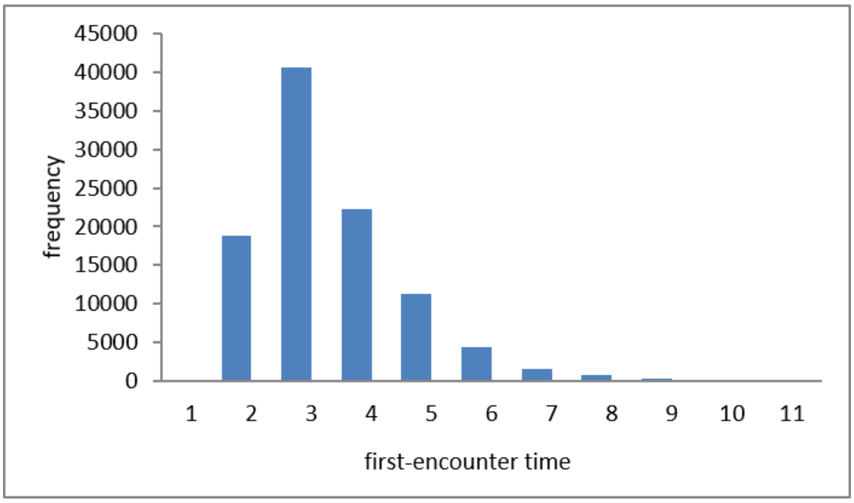

Figure 4. Histogram of the simulated first-encounter time.

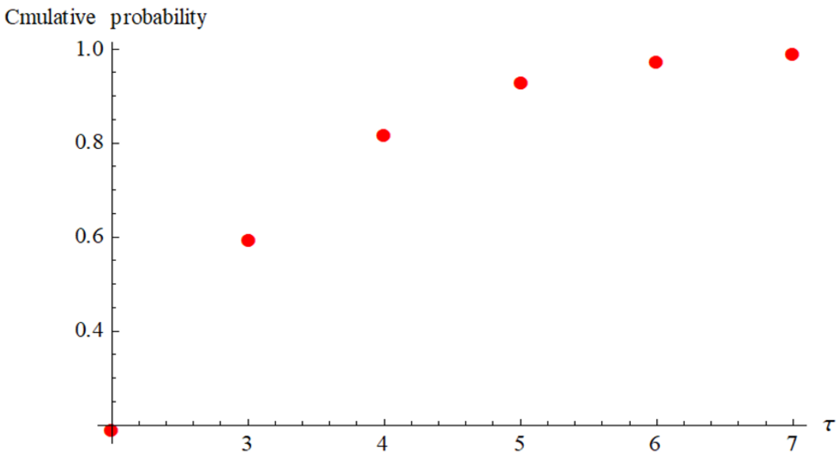

Figure 5. The ECDF of the simulated first-encounter time.

This figure indicates that more than 80% of encounter events occur within a relatively short time, and that the cumulative probability rapidly approaches unity, implying a distribution with a limited tail. These numerical and graphical observations are in direct agreement with the theoretical results established in Section 4, which guarantee the finiteness of the expected first-encounter time under the bounded controlled tracking conditions stated in Section 4. Consequently, the reported tables and figures not only illustrate the numerical outcomes of the model, but also confirm that its theoretical guarantees are realized in a practical stochastic transport setting involving particle motion and tracking in fluid media.

The numerical example presented in this section is intended as an illustrative validation of the theoretical finiteness result rather than as a complete computational benchmark. A broader numerical assessment, including comparisons with baseline search policies and a sensitivity analysis with respect to the tracking distances, domain size, and random walk parameters, would provide a more detailed evaluation of the practical performance of the proposed framework. Such extensions are therefore identified as natural directions for future numerical investigation.

6. Conclusions and future work

This work presented a two-stage analytical framework based on stochastic transport to address the problems of the search and first-encounter time for a target moving according to a multidimensional independent random walk. In the first stage, the search problem was formulated as a probabilistic transport problem over independent and non overlapping regions, where the optimal search distances were derived by minimizing the expected detection time under a truncated bivariate probability distribution describing the target's initial position. Conditioned on the localization outcome, the second stage was formulated as a stochastic tracking model in which the target's motion along intersecting trajectories is governed by an independent random walk, and coordinated tracking strategies for the search agents were analyzed.

The main theoretical result of this study is the establishment of sufficient conditions ensuring the finiteness of the expected first-encounter time. In particular, it was shown that the absence of directional drift is not sufficient by itself to guarantee a finite expected encounter time in the classical unrestricted random walk sense. In the proposed model, finiteness follows from the combined effect of the unbiased target motion, the bounded tracking geometry, finite tracking-distance functions, recurrent coverage of the admissible trajectories, and a uniform positive probability of an encounter within a finite number of tracking cycles. This result extends classical linear search and transport models to a multidimensional setting with intersecting trajectories, while preserving analytical rigor and tractability. Numerical results based on Monte Carlo simulations further confirmed that the distribution of the first-encounter time is concentrated at small values and exhibits a rapidly decaying tail, in full agreement with the derived theoretical conditions.

The proposed framework provides a coherent mathematical link between probabilistic search effort allocation in the first stage and stochastic transport-driven tracking analysis in the second stage. In this sense, the model offers a general analytical foundation for a broad class of stochastic transport and first-passage problems, particularly those related to the detection and tracking of randomly moving particles in complex environments.

Several promising directions may be pursued to further develop the present framework. One possible extension is to consider biased or drift-driven random walks, which would allow the model to describe transport processes influenced by external fields, environmental flows, or directional preferences. Another direction is to incorporate uncertainty propagation between the localization and tracking stages, leading to a more comprehensive description of the transition from initial detection to stochastic pursuit. The framework may also be generalized to heterogeneous search agents with different speeds, sensing capabilities, or operational constraints. In addition, further numerical studies in more complex geometries and realistic transport environments would provide deeper insight into the robustness and applicability of the proposed two-stage strategy.

Author contributions

All authors jointly worked on the results, and they read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflicts of interest.

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