Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(6): 17635–17652.
DOI: 10.3934/math.2026720
Received: 23 January 2026
Revised: 14 May 2026
Accepted: 28 May 2026
Published: 17 June 2026

*Research article*

# Observer design for practical exponential stabilization of polynomial fuzzy time-delay models

**Slim Dhahri[1], Essia Ben Alaia[1], Afrah Alanazi[2,*], Hamdi Gassara[3] and Sahar Almenwer[4]**

[1] Department of Computer Engineering and Networks, College of Computer and Information Sciences, Jouf University, Saudi Arabia

[2] Department of Information System, College of Computer and Information Sciences, Jouf University, Saudi Arabia

[3] Laboratory of Sciences and Technology of Automatic Control and Computer Engineering, National School of Engineering of Sfax, University of Sfax, Post Box 1173, 3038 Sfax, Tunisia

[4] Department of Computer Science, College of Computer Science and Information, Jouf University, Saudi Arabia

* **Correspondence:** Email: aoalenzy@ju.edu.sa.

**Abstract:** In this paper, we study the practical exponential stability (PES) of nonlinear time-delay (TD) systems with bounded exogenous input (BEI). Unlike existing results, this work is the first to provide a sum-of-squares (SOS)-based design to ensure the PES of the nonlinear TD system. First, polynomial fuzzy (PF) modeling is used to characterize the nonlinear behavior of the system. Specifically, we consider a class of models in which the premise variables are measurable, and the polynomial matrices depend on any measurable variable. Second, we propose a PF observer-based (PFOB) control strategy that guarantees the system's PES under the condition that the system is observable, even when some state variables are not directly measurable. Next, the PFOB controller is designed via the SOS approach to ensure the PES of the augmented system formed by the state and the state estimation error. This study is the first to incorporate recently proposed relaxed conditions for parameterized linear matrix inequalities formulated in a double-sum representation into the SOS approach. Finally, a numerical case study validates the PFOB control strategy.

**Keywords:** global uniform practical exponential stability; sum-of-squares approach; polynomial fuzzy models; polynomial fuzzy observer; time delay systems
**Mathematics Subject Classification:** 93C10, 93C43, 93D23

## 1. Introduction

Exogenous inputs are unavoidable in any dynamic system or control framework, as no system is entirely isolated from external factors. In particular, the presence of bounded exogenous input (BEI) prevents the state trajectories from converging to zero. Instead, the states tend to remain within a bounded region after a sufficiently long time. In control terminology, this stability is known as practical stability (PS), which stands in contrast to classical notions such as asymptotic or exponential stability, where convergence to the equilibrium point is required. As a result, PS has attracted considerable research interest, leading to thorough studies across diverse types of delay-free dynamic systems. For instance, the PS issue of hybrid and switched systems is addressed in [1]. The researchers in [2] examined the global uniform practical exponential stability (GUPES) of nonlinear time-varying cascaded systems. In [3], the authors studied the practical exponential stability (PES) of switched positive systems with nonlinear homogeneous dynamics, including cases with partially unstable modes and exogenous input. An improved max-separable Lyapunov function was proposed in [4], leading to enhanced results compared with those reported in [3].

The stability of dynamic systems is often affected not only by exogenous inputs but also by the presence of time delays (TDs). Therefore, investigating PS in TD systems becomes critically important. The GUPES problem for general nonlinear non autonomous delay differential equations was studied in [5]. The authors of [6] investigated $\mu$-PS for a category of nonlinear dynamic systems subject to multiple delays and BEI. The authors in [7] examined PES analysis and controller design for conformable fractional-order linear systems with TD. The issue of PS analysis for stochastic differential delay equations is treated in [8]. On the other hand, the ability of the Takagi-Sugeno fuzzy (TA-SUF) model to approximate complex nonlinear dynamics via piecewise linear representations makes it widely applicable to TD control systems. By employing an Nth-order affine integral inequality combined with a newly proposed Lyapunov-Krasovskii functional, the authors in [9] investigated the asymptotic stability and stabilization of TA-SUF model with time-varying delay. Zhang et al. [10] explored PS for a class of nonlinear TD systems described by TA-SUF models. The authors used an improved Jensen's inequality to derive less conservative stability conditions.

Notably, the polynomial fuzzy (PF) modeling and control approach extends the traditional TA-SUF model and offers improved capabilities for representing nonlinear control systems. Indeed, incorporating polynomial nonlinear terms allows us to streamline the model's structure and significantly reduce the number of fuzzy rules, which in turn eases the complexity of the analysis [11,12]. The key distinction between PF models and TA-SUF models lies in the nature of their local submodels. While TA-SUF models use linear submodels, PF models use polynomial expressions to capture the local dynamics. Given this aforementioned propriety, the linear matrix inequality (LMI) approach is unsuitable. As an alternative, the sum-of-squares (SOS) approach is taken, which provides a computationally effective framework for analyzing and controlling such models. Asymptotic stability has been extensively studied in the literature for delay-free and delayed nonlinear systems represented by PF models. The SOS method serves as a fundamental tool for deriving stability conditions and designing various types of stabilizing controllers, including memory control [13], tracking control using static output feedback [14] and event-triggered control [15, 16]. In our recent work [17], we examined the stabilization under finite-time boundedness of PF models with TD. Significantly, our review indicates that the PES of delayed PF models represents an unexplored research area.

Guided by the preceding discussion, in this work, we investigate the PES of a class of polynomial fuzzy time-delay (PFTD) models with BEI. The key contributions of this study are as follows.

- Utilizing the Lyapunov method and SOS approach, this study marks the first effort to investigate the PES of a class of PFTD models. Not only has this class of PFTD models not been investigated in this context before, but, to the best of our knowledge, even the corresponding particular case of TA-SUF models with TD systems under an LMI framework has not been addressed in this setting. This work therefore goes beyond classical TA-SUF formulations and considers the more general PF framework, which leads to a more challenging problem due to its higher nonlinear representation capability. This choice enables us to leverage the advantages of PF modeling, particularly for reducing the number of fuzzy rules required to represent complex nonlinear systems while improving modeling flexibility.
- An observer is used to ensure the PES of the augmented system, both when all states are measurable and when only a subset is available.
- Recent advances that introduce relaxed conditions for parameterized LMI expressed through a double-sum structure are applied here to the SOS framework for the first time.
- In the absence of BEI, novel SOS conditions are developed to secure the exponential stability (ES) for the considered class of PFTD models using observer-based (OB) control.

Section 2 presents the necessary preliminaries related to the SOS approach, and the system model, which includes a class of PFTD models together with an OB control. This section also states the main objective of the paper, which consists of ensuring the PES of the augmented system formed by the estimated state and the state estimation error. Section 3 presents the design conditions based on the SOS approach. Initially, sufficient bilinear polynomial matrix inequalities (PMI) conditions are obtained via the Lyapunov method. These conditions are then reformulated into linear PMI using a singular value decomposition (SVD) decoupling technique. Ultimately, an algorithmic description of the OB controller design process is given. In Section 4, our theoretical results are validated through a numerical example. In Section 5, the main conclusions drawn from the study are presented. Several avenues for future work are also suggested.

**Notations.** *The following notations are adopted for* $i, j, v_1, v_2 \in \mathbb{N}$, *and* $\varphi \in \mathbb{R}_{>0}$.

- $\mathbb{R}^i$ ($\mathbb{R}^{i\times j}$) *denotes the set of vectors of dimension* $i$ *(matrices of size* $i \times j$*).*
- *For* $\vartheta \in \mathbb{R}^{v_1}$, *we define* $\|\vartheta\| := \sqrt{\vartheta^T\vartheta}$, $\mathbb{P}(\vartheta)$ *and* $\mathbb{P}_{\geq 0}(\vartheta)$ *as the sets of polynomials and non-negative polynomials that are strictly positive for* $\vartheta \neq 0$, *respectively;* $\mathbb{P}^{i\times j}(\vartheta)$ *as the set of* $i \times j$ *matrices with polynomial entries in* $\vartheta$.
- $\wp_{i,\varphi}$ *refers to the Banach space consisting of continuous functions mapping* $[-\varphi, 0]$ *to* $\mathbb{R}^i$. *For* $d(t) \in \wp_{i,\varphi}$, $\|d\|_s = sup_{-\varphi\leq t\leq 0}\|d(t)\|$.
- $I_{i\times j}$ *and* $0_{i\times j}$ *denote the* $i \times j$ *identity matrix and the* $i \times j$ *zero matrix, respectively.*
- *For* $M \in \mathbb{R}^{i\times i}$, *we use* $M \prec 0$ ($M \succ 0$) *to indicate that* $M$ *is negative (positive) definite. The symbols* $\lambda_{\max}(M)$ *and* $\lambda_{\min}(M)$ *denote the maximum and minimum eigenvalues of* $M$, *respectively. Finally,* $\mathbf{Sys}(M)$ *is defined as* $M + M^T$.

- $\mathbb{I}^i$ *and* $\mathbb{S}^{i\times i}$ *denote the integer set* $\{1,\ldots,\ i\}$ *and the field of* $i\times i$ *symmetrical matrices, respectively.*
- *For* $\vartheta_1 \in \mathbb{R}^{v_1}$ *and* $\vartheta_2 \in \mathbb{R}^{v_1}$, $\Sigma(\vartheta_1)$ *denotes the set of SOS in* $\vartheta_1$*;* $\vartheta_1 \neq \vartheta_2$ *means that* $\vartheta_1$ *and* $\vartheta_2$ *are independent.*
- *The symbol* † *denotes, when applicable, a symmetric block element.*

## 2. Foundational concepts and problem statement

Consider the nonlinear TD system defined by the following state-space representation:

$$\begin{cases} \dot{\vartheta}(t) = \tilde{q}_1(\vartheta(t))\vartheta(t) + \tilde{q}_2(\vartheta(t))\vartheta(t-\varphi) + \tilde{q}_3(\vartheta(t))r(t) + \Delta(t,\vartheta(t)), \\ \eta(t) = \Theta\vartheta(t), \\ \vartheta(t) = d(t), \quad t \in [-\varphi, 0], \end{cases} \tag{2.1}$$

where $\vartheta(t)$, $\vartheta(t-\varphi)$, and $d(t) \in \mathbb{R}^{v_1}$ are the state, the state delay, and the initial condition vectors, respectively; $r(t) \in \mathbb{R}^{v_2}$ is the control input vector; $\eta(t) \in \mathbb{R}^{v_3}$ is the measured output vector; $\varphi \in \mathbb{R}_{>0}$ is the TD; $\tilde{q}_1(\vartheta(t))$, $\tilde{q}_2(\vartheta(t))$, and $\tilde{q}_3(\vartheta(t))$ are matrices whose entries are nonlinear functions of $\vartheta(t)$, with an appropriate size; $\Delta(t,\vartheta(t)) \in \mathbb{R}^{v_1}$ is an unknown BEI, satisfying

$$||\Delta(t,\vartheta(t))|| \leq \boldsymbol{\nu}, \tag{2.2}$$

and $\Theta \in \mathbb{R}^{v_3\times v_1}$ such that

$$rank(\Theta) = v_3 \leq v_1. \tag{2.3}$$

Using the SVD technique, it follows from (2.3) that $\Theta_1 \in \mathbb{R}^{v_3\times v_3}$ and $\Theta_2 \in \mathbb{R}^{v_1\times v_1}$ exist such that

$$\Theta_1^T\Theta_1 = I_{v_3\times v_3},\ \Theta_2^T\Theta_2 = I_{v_1\times v_1},\ \Theta = \Theta_1 \begin{bmatrix} \mathcal{M} & 0_{v_3\times(v_1-v_3)} \end{bmatrix} \Theta_2, \tag{2.4}$$

where $\mathcal{M} \in \mathbb{R}^{v_3\times v_3}$ is diagonal.

Depending on modeling requirements, the system in (2.1) can be expressed through different classes of PFTD models with BE. In our work, we assume that system in (2.1) can be modeled by the following class of PFTD model with BE:

Rule $\iota$: If $\varepsilon_1(t)$ is $\mathcal{E}_{\iota 1}$, ..., and $\varepsilon_f(t)$ is $\mathcal{E}_{\iota f}$, then

$$\begin{cases} \dot{\vartheta}(t) = q_{1\iota}(\zeta(t))\vartheta(t) + q_{2\iota}(\zeta(t))\vartheta(t-\varphi) + q_{3\iota}(\zeta(t))r(t) + \Delta(t,\vartheta(t)), \\ \eta(t) = \Theta\vartheta(t), \\ \vartheta(t) = d(t), \quad t \in [-\varphi, 0], \end{cases} \tag{2.5}$$

where $\{\varepsilon_j(t),\ \mathcal{E}_{\iota j}\}$ is defined for $j \in \mathbb{I}^f$ and $\iota \in \mathbb{I}^n$ following standard fuzzy system formulation in which $\varepsilon_j(t)$ is measurable; $q_{1\iota}(\zeta(t)), q_{2\iota}(\zeta(t)) \in \mathbb{P}^{v_1\times v_1}(\zeta(t))$, and $q_{3\iota}(\zeta(t)) \in \mathbb{P}^{v_1\times v_2}(\zeta(t))$, in which $\zeta(t) \in \mathbb{R}^w$ is a measurable vector that may comprise $\eta(t)$, $t$, and any other measurable external variables.

Let $\varepsilon(t) = \begin{bmatrix} \varepsilon_1(t)\ldots\varepsilon_f(t) \end{bmatrix}$. The global representation of the model is

$$\begin{cases} \dot{\vartheta}(t) = \sum_{\iota=1}^{n} \rho_\iota(\varepsilon(t))\{q_{1\iota}(\zeta(t))\vartheta(t) + q_{2\iota}(\zeta(t))\vartheta(t-\varphi) + q_{3\iota}(\zeta(t))r(t) + \Delta(t,\vartheta(t))\}, \\ \eta(t) = \Theta\vartheta(t), \\ \vartheta(t) = d(t), \quad t \in [-\varphi, 0], \end{cases} \tag{2.6}$$

where $\rho_\iota(\varepsilon(t))$ is the membership function normalized such that

$$0 \leq \rho_\iota(\varepsilon(t)) \leq 1, \ \sum_{\iota=1}^{n} \rho_\iota(\varepsilon(t)) = 1. \tag{2.7}$$

Consider an observer that has the same rules as the PFTD model (2.5). Its overall output is defined as follows:

$$\begin{cases} \dot{\hat{\vartheta}}(t) = \sum_{\iota=1}^{n} \rho_\iota(\varepsilon(t))\{q_{1\iota}(\zeta(t))\hat{\vartheta}(t) + q_{3\iota}(\zeta(t))r(t) + q_{2\iota}(\zeta(t))\hat{\vartheta}(t-\varphi) + M_\iota(\zeta(t), \hat{\vartheta}(t))\tilde{\eta}(t)\}, \\ \tilde{\eta}(t) = \eta(t) - \hat{\eta}(t), \\ \hat{\eta}(t) = \Theta\hat{\vartheta}(t), \\ \hat{\vartheta}(t) = \hat{d}(t), \quad t \in [-\varphi, 0], \end{cases} \tag{2.8}$$

where $\hat{\vartheta}(t)$, $\hat{\vartheta}(t-\varphi)$, and $\hat{\eta}(t)$ represent the estimates of $\vartheta(t)$, $\vartheta(t-\varphi)$, and $\eta(t)$, respectively; $\hat{d}(t)$ and $M_\iota(\zeta(t), \hat{\vartheta}(t))$ are the initial condition and the polynomial gains of the observer, respectively.

On the basis of the observer's output $\hat{\vartheta}(t)$, we consider the overall output of a PF controller that employs the same rules as the the model (2.5), formulated as follows:

$$r(t) = \sum_{\iota=1}^{n} \rho_\iota(\varepsilon(t))N_\iota(\zeta(t), \hat{\vartheta}(t))\hat{\vartheta}(t), \tag{2.9}$$

where $N_\iota(\zeta(t)$ and $\hat{\vartheta}(t))$ are its gains.

Hereafter, for simplicity, we omit the time argument $t$ in the variables $\zeta(t)$, $\vartheta(t)$, $\hat{\vartheta}(t)$, $r(t)$, $d(t)$, $\hat{d}(t)$, $\eta(t)$, $\hat{\eta}(t)$, and $\tilde{\eta}(t)$; $\vartheta(t-\varphi)$ is denoted by $\vartheta_\varphi$, $\hat{\vartheta}(t-\varphi)$ by $\hat{\vartheta}_\varphi$, and $\rho_\iota(\varepsilon(t))$ by $\rho_\iota$.

Let $\tilde{\vartheta} = \vartheta - \hat{\vartheta}$, $\tilde{\vartheta}_\varphi = \vartheta_\varphi - \hat{\vartheta}_\varphi$, and $\tilde{d} = d - \hat{d}$.

By substituting the controller in (2.9) into the system in (2.8), we obtain

$$\dot{\hat{\vartheta}} = \sum_{\iota=1}^{n} \rho_\iota\rho_\jmath\{(q_{1\iota}(\zeta) + q_{3\iota}(\zeta)N_\jmath(\zeta, \hat{\vartheta}))\hat{\vartheta} + q_{2\iota}(\zeta)\hat{\vartheta}_\varphi + M_\iota(\zeta, \hat{\vartheta})\Theta\tilde{\vartheta}\}. \tag{2.10}$$

Furthermore, it follows from (2.6) and (2.8) that the dynamics of the estimation error can be expressed as follows:

$$\dot{\tilde{\vartheta}} = \sum_{\iota=1}^{n} \rho_\iota\{(q_{1\iota}(\zeta) - M_\iota(\zeta, \hat{\vartheta})\Theta)\tilde{\vartheta} + q_{2\iota}(\zeta)\tilde{\vartheta}_\varphi + \Delta(t, \vartheta(t))\}. \tag{2.11}$$

On the basis of (2.10) and (2.11), we then obtain

$$\begin{cases} \dot{\chi} = \sum_{\iota=1}^{n}\sum_{\jmath=1}^{n} \rho_\iota\rho_\jmath\{\mathcal{A}_{\iota\jmath}(\zeta, \hat{\vartheta})\chi + \mathcal{B}_\iota(\zeta)\chi_\varphi\} + \tilde{\Delta}(t, \vartheta(t)), \\ \chi = D, \quad t \in [-\varphi, 0], \end{cases} \tag{2.12}$$

where

$$\chi = \begin{bmatrix} \hat{\vartheta} \\ \tilde{\vartheta} \end{bmatrix}, \ \chi_\varphi = \begin{bmatrix} \hat{\vartheta}_\varphi \\ \tilde{\vartheta}_\varphi \end{bmatrix}, \ \mathcal{A}_{\iota\jmath}(\zeta, \hat{\vartheta}) = \begin{bmatrix} q_{1\iota}(\zeta) + q_{3\iota}(\zeta)N_\jmath(\zeta, \hat{\vartheta}) & M_\iota(\zeta, \hat{\vartheta})\Theta \\ 0 & q_{1\iota}(\zeta) - M_\iota(\zeta, \hat{\vartheta})\Theta \end{bmatrix},$$

$$\mathcal{B}_\iota(\zeta) = \begin{bmatrix} q_{2\iota}(\zeta) & 0 \\ 0 & q_{2\iota}(\zeta) \end{bmatrix}, \ \tilde{\Delta}(t, \vartheta(t)) = \begin{bmatrix} 0 \\ \Delta(t, \vartheta(t)) \end{bmatrix}, \ D = \begin{bmatrix} \hat{d} \\ \tilde{d} \end{bmatrix}.$$

**Definition 2.1.** *[3] The PES of the system in (2.12) is ensured with a rate $\sigma$ and a radius $\delta$ if $a$, $\sigma$, and $\delta \in \mathbb{R}_{>0}$ exist such that all solutions $\chi$ meet the following requirement:*

$$\|\chi\| \leq a\|D\|_s e^{-\sigma t} + \delta, \ \forall t \in \mathbb{R}_{>0}. \tag{2.13}$$

**Remark 2.1.** *Definition 2.1 reduces to the classical exponential stability definition [18] when $\delta = 0$ and to the practical stability definition [6] when $a = 0$.*

The following provides the essential preliminaries on SOS. SOSTOOLS [19], in conjunction with semidefinite programming solvers like SeDuMi [20], is used to formulate and solve SOS problems symbolically.

**Definition 2.2.** *[21] Let $\vartheta \in \mathbb{R}^v$. $p(\vartheta) \in \Sigma(\vartheta)$, if $p_1(\vartheta), \ldots, p_n(\vartheta) \in \mathbb{P}(\vartheta)$ exist such that*

$$p(\vartheta) = \sum_{\imath=1}^{n} p_\imath^2(\vartheta). \tag{2.14}$$

*It is clear that $p(\vartheta) \in \Sigma(\vartheta)$ implies $p(\vartheta) \geq 0$.*

**Lemma 2.1.** *[21] Consider $\vartheta_1 \in \mathbb{R}^{v_1}$ and $\vartheta_2 \in \mathbb{R}^{v_2}$ such that $\vartheta_1 \neq \vartheta_2$. For a given $p(\vartheta_1) \in \mathbb{P}_{\geq 0}(\vartheta_1)$, if $P(\vartheta_1) \in \mathbb{P}^{v_2 \times v_2}(\vartheta_1)$ satisfies*

$$\vartheta_2^T (P(\vartheta_1) - p(\vartheta_1) I_{v_2 \times v_2}) \vartheta_2 \in \Sigma(\vartheta_1, \vartheta_2), \tag{2.15}$$

*then $P(\vartheta_1) \succ 0$.*

**Lemma 2.2.** *Let $\vartheta \in \mathbb{R}^v$, $\Psi_{\imath\jmath}(\vartheta) \in \mathbb{P}^{w \times w}(\vartheta)$ such that $\Psi_{\imath\jmath}(\vartheta) = \Psi_{\imath\jmath}(\vartheta)^T$ for all $(\imath, \jmath) \in \mathbb{I}^n \times \mathbb{I}^n$. If*

$$\Psi_{\imath\imath}(\vartheta) + \frac{1}{2} \sum_{\jmath=1, \jmath<\imath}^{n} \xi_\jmath (\Psi_{\imath\jmath}(\vartheta) + \Psi_{\jmath\imath}(\vartheta)) + \frac{1}{2} \sum_{\jmath=1, \jmath>\imath}^{n} \xi_{\jmath-1} (\Psi_{\imath\jmath}(\vartheta) + \Psi_{\jmath\imath}(\vartheta)) \prec 0, \tag{2.16}$$

*$\forall \imath \in \mathbb{I}^n$ and $\forall (\xi_1, \ldots, \xi_{n-1}) \in \underbrace{\{0, 1\} \times \cdots \times \{0, 1\}}_{(n-1)\text{-times}}$. One then has*

$$\sum_{\imath=1}^{n} \sum_{\jmath=1}^{n} \rho_\imath \rho_\jmath \Psi_{\imath\jmath}(\vartheta) \prec 0, \tag{2.17}$$

*for $\rho_\imath$ satisfying (2.7).*

*Proof.* The proof proceeds in the same manner as that given in [22], based on the convexity property of $\rho_\imath$. Here, $\Psi_{\imath\jmath}(\vartheta)$ is a polynomial matrix instead of the constant matrix considered in [22], which does not influence the proof.

The central purpose of the next section is to establish sufficient SOS constraints for the design of both the PF observer and controller gains. These conditions aim to ensure the PES of the system in (2.12).

## 3. Design via SOS conditions

**Theorem 3.1.** *For given $\sigma \in \mathbb{R}_{>0}$, suppose that $\beta \in \mathbb{R}$, $\mathcal{K}, \mathcal{L} \in \mathbb{S}^{2v_1 \times 2v_1}$, $M_\imath(\zeta, \hat{\vartheta}) \in \mathbb{P}^{v_1 \times v_3}(\zeta, \hat{\vartheta})$ and $N_\jmath(\zeta, \hat{\vartheta}) \in \mathbb{P}^{v_2 \times v_1}(\zeta, \hat{\vartheta})$ exist such that*

$$\beta > 0,\ \mathcal{K} \succ 0,\ \mathcal{L} \succ 0, \tag{3.1}$$

$$\begin{aligned}&\Psi_{\imath\imath}(\zeta, \hat{\vartheta}) + \frac{1}{2} \sum_{\jmath=1, \jmath<\imath}^{n} \xi_\jmath (\Psi_{\imath\jmath}(\zeta, \hat{\vartheta}) + \Psi_{\jmath\imath}(\zeta, \hat{\vartheta})) + \frac{1}{2} \sum_{\jmath=1, \jmath>\imath}^{n} \xi_{\jmath-1} (\Psi_{\imath\jmath}(\zeta, \hat{\vartheta}) + \Psi_{\jmath\imath}(\zeta, \hat{\vartheta})) \prec 0,\\ &\forall \imath \in \mathbb{I}^n \text{ and } \forall (\xi_1, \ldots, \xi_{n-1}) \in \underbrace{\{0, 1\} \times \cdots \times \{0, 1\}}_{(n-1)\text{-times}},\end{aligned} \tag{3.2}$$

*where*

$$\Psi_{\imath\jmath}(\zeta, \hat{\vartheta}) = \begin{bmatrix} \mathbf{Sys}(\mathcal{K}\mathcal{A}_{\imath\jmath}(\zeta, \hat{\vartheta}) + \sigma\mathcal{K}) + \mathcal{L} + \beta\mathcal{K}\mathcal{K} & \mathcal{K}\mathcal{B}_\imath(\zeta) \\ \dagger & -e^{-2\sigma\varphi}\mathcal{L} \end{bmatrix}.$$

*The PES of the system in (2.12) is the ensured with rate $\sigma$ and radius $\delta = \frac{\nu}{\sqrt{2\sigma\beta\lambda_m}}$, where*

$$\lambda_m = \lambda_{min}(\mathcal{K}).$$

*Proof.* Let us consider the Lyapunov-Krasovskii functional defined as follows:

$$\mathcal{F}(t) = \mathcal{F}_1(t) + \mathcal{F}_2(t), \tag{3.3}$$

where

$$\mathcal{F}_1(t) = \chi^T \mathcal{K} \chi,\ \mathcal{F}_2(t) = \int_t^{t+\varphi} e^{2\sigma(s-t-\varphi)} \chi^T(s-\varphi)\mathcal{L}\chi(s-\varphi)ds. \tag{3.4}$$

We have

$$\dot{\mathcal{F}}_1(t) = 2\chi^T\mathcal{K}\dot{\chi} = 2\chi^T\mathcal{K}\Big(\sum_{\imath=1}^{n}\sum_{\jmath=1}^{n} \rho_\imath\rho_\jmath \{\mathcal{A}_{\imath\jmath}(\zeta, \hat{\vartheta})\chi + \mathcal{B}_\imath(\zeta)\chi_\varphi\} + \tilde{\Delta}(t, \vartheta(t))\Big), \tag{3.5}$$

$$\begin{aligned}\dot{\mathcal{F}}_2(t) &= \Big(\chi^T\mathcal{L}\chi - e^{-2\sigma\varphi}\chi_\varphi^T\mathcal{L}\chi_\varphi - 2\sigma\int_t^{t+\varphi} e^{2\sigma(s-t-\varphi)}\chi^T(s-\varphi)\mathcal{L}\chi(s-\varphi)ds\Big)\\ &\leq \chi^T\mathcal{L}\chi - e^{-2\sigma\varphi}\chi_\varphi^T\mathcal{L}\chi_\varphi - 2\sigma\mathcal{F}_2.\end{aligned} \tag{3.6}$$

For any positive scalar $\beta$, and in light of (2.2), it follows that

$$2\chi^T\mathcal{K}\tilde{\Delta}(t, \vartheta(t)) \leq \beta\chi^T\mathcal{K}\mathcal{K}\chi + \frac{\nu^2}{\beta}. \tag{3.7}$$

From the combination of (3.5)–(3.7), we derive

$$\dot{\mathcal{F}}(t) + 2\sigma\mathcal{F}(t) \leq \sum_{\imath=1}^{n}\sum_{\jmath=1}^{n} \rho_\imath\rho_\jmath \tilde{\chi}^T \Psi_{\imath\jmath}(\zeta, \hat{\vartheta})\tilde{\chi} + \frac{\nu^2}{\beta},$$

where $\tilde{\chi} = \begin{bmatrix} \chi \\ \chi_\varphi \end{bmatrix}$.

In light of Lemma 2.2, the conditions in (3.2) lead to

$$\dot{\mathcal{F}}(t) \quad \leq \quad -2\sigma\mathcal{F}(t) + \frac{\boldsymbol{\nu}^2}{\beta}.$$

Let

$$\mathcal{G}(t) = \mathcal{F}(t) - \frac{\boldsymbol{\nu}^2}{2\sigma\beta}. \tag{3.8}$$

We then get

$$\dot{\mathcal{G}}(t) \leq -2\sigma\mathcal{G}(t), \tag{3.9}$$

which implies that

$$\dot{\mathcal{H}}(t) \leq 0, \tag{3.10}$$

where $\mathcal{H}(t) = e^{2\sigma t}\mathcal{G}(t)$.

By integrating (3.10) over $\begin{bmatrix} 0 & t \end{bmatrix}$, we get

$$\mathcal{G}(t) \leq \mathcal{G}(0)e^{-2\sigma t}. \tag{3.11}$$

Therefore, we obtain

$$\mathcal{F}(t) \leq \mathcal{F}(0)e^{-2\sigma t} + (1 - e^{-2\sigma t})\frac{\boldsymbol{\nu}^2}{2\sigma\beta} \leq \mathcal{F}(0)e^{-2\sigma t} + \frac{\boldsymbol{\nu}^2}{2\sigma\beta}. \tag{3.12}$$

In view of (3.3), we get

$$\mathcal{F}(t) \geq \lambda_m\|\chi\|, \tag{3.13}$$

and

$$\mathcal{F}(t) \quad = \quad \chi(0)^T\mathcal{K}\chi(0) + \int_0^\varphi e^{2\sigma(s-\varphi)}\chi^T(s-\varphi)\mathcal{L}\chi(s-\varphi)ds \leq \lambda_M\|D\|_s^2, \tag{3.14}$$

where $\lambda_M = \lambda_{max}(\mathcal{K}) + \frac{\lambda_{max}(\mathcal{L})}{2\sigma}(1 - e^{-2\sigma\varphi})$.

The combination of (3.12)–(3.14) yields

$$\|\chi\|^2 \leq \frac{\lambda_M}{\lambda_m}e^{-2\sigma t}\|D\|_s^2 + \delta^2, \tag{3.15}$$

which implies that

$$\|\chi\| \leq \sqrt{\frac{\lambda_M}{\lambda_m}}e^{-\sigma t}\|D\|_s + \delta. \tag{3.16}$$

Therefore, the system in (2.12) is PES with a rate $\sigma$ and a radius $\delta$.

It can be observed that the conditions in (3.2) consist of polynomial matrix inequalities where the polynomial gains associated with the OB controller, $M_\iota(\zeta, \hat{\vartheta})$ and $N_\iota(\zeta, \hat{\vartheta})$, are coupled with the Lyapunov-Krasovskii matrices $\mathcal{K}$ and $\mathcal{L}$. In the next theorem, we propose a reformulation that decouples these variables by transforming the bilinear PMI into a linear PMI, which are more suitable for numerical handling using SOSTOOLS.

**Theorem 3.2.** *For given $\sigma \in \mathbb{R}_{>0}$, suppose that $\beta \in \mathbb{R}$, $P_1$, $X_1$, $X_2 \in \mathbb{S}^{\nu_1\times\nu_1}$, $P_{211} \in \mathbb{S}^{\nu_3\times\nu_3}$, $P_{222} \in \mathbb{S}^{(\nu_1-\nu_3)\times(\nu_1-\nu_3)}$, $\tilde{M}_\iota(\zeta,\hat{\vartheta}) \in \mathbb{P}^{\nu_1\times\nu_3}(\zeta,\hat{\vartheta})$, and $\tilde{N}_\jmath(\zeta,\hat{\vartheta}) \in \mathbb{P}^{\nu_2\times\nu_1}(\zeta,\hat{\vartheta})$ exist such that*

$$\beta > 0,\ P_1 \succ 0,\ P_{211} \succ 0,\ P_{222} \succ 0,\ X_1 \succ 0,\ X_2 \succ 0, \tag{3.17}$$

$$F_{\iota\iota}(\zeta,\hat{\vartheta}) \prec 0,\ \ \forall\iota \in \mathbb{I}^n \text{ and } \forall(\xi_1,\ldots,\xi_{n-1}) \in \underbrace{\{0,1\}\times\cdots\times\{0,1\}}_{(n-1)\text{-times}}. \tag{3.18}$$

*Within this theorem, we have*

$$F_{\iota\iota}(\zeta,\hat{\vartheta}) = \Lambda_{\iota\iota}(\zeta,\hat{\vartheta}) + \frac{1}{2}\sum_{\jmath=1,\jmath<\iota}^{n}\xi_\jmath(\Lambda_{\iota\jmath}(\zeta,\hat{\vartheta}) + \Lambda_{\jmath\iota}(\zeta,\hat{\vartheta})) + \frac{1}{2}\sum_{\jmath=1,\jmath>\iota}^{n}\xi_{\jmath-1}(\Lambda_{\iota\jmath}(\zeta,\hat{\vartheta}) + \Lambda_{\jmath\iota}(\zeta,\hat{\vartheta})), \tag{3.19}$$

*where*

$$\Lambda_{\iota\jmath}(\zeta,\hat{\vartheta}) = \begin{bmatrix} \Lambda_{1\iota\jmath}(\zeta,\hat{\vartheta}) & \tilde{M}_\iota(\zeta,\hat{\vartheta})\Theta & q_{2\iota}(\zeta)P_1 & 0 \\ \dagger & \Lambda_{2\iota}(\zeta,\hat{\vartheta}) & 0 & q_{2\iota}(\zeta)P_2 \\ \dagger & \dagger & -e^{-2\sigma\varphi}X_1 & 0 \\ \dagger & \dagger & \dagger & -e^{-2\sigma\varphi}X_2 \end{bmatrix},$$

*in which*

$$\begin{aligned} \Lambda_{1\iota\jmath}(\zeta,\hat{\vartheta}) &= \mathbf{Sys}(q_{1\iota}(\zeta)P_1 + q_{3\iota}(\zeta)\tilde{N}_\jmath(\zeta,\hat{\vartheta}) + \sigma P_1) + X_1 + \beta I_{\nu_1\times\nu_1}, \\ \Lambda_{2\iota}(\zeta,\hat{\vartheta}) &= \mathbf{Sys}\big(q_{1\iota}(\zeta)P_2 - \tilde{M}_\iota(\zeta,\hat{\vartheta})\Theta + \sigma P_2\big) + X_2 + \beta I_{\nu_1\times\nu_1}, \end{aligned}$$

*with*

$$P_2 = \Theta_2^T \begin{bmatrix} P_{211} & 0_{\nu_1\times(\nu_1-\nu_3)} \\ \dagger & P_{222} \end{bmatrix} \Theta_2. \tag{3.20}$$

*Thus, conditions (3.1) and (3.2) in Theorem 3.1 are satisfied for*

$$N_\iota(\zeta,\hat{\vartheta}) = \tilde{N}_\iota(\zeta,\hat{\vartheta})P_1^{-1},\ \ M_\iota(\zeta,\hat{\vartheta}) = \tilde{M}_\iota(\zeta,\hat{\vartheta})\Theta_1\mathcal{M}P_{211}^{-1}\mathcal{M}^{-1}\Theta_1^T, \tag{3.21}$$

$$\mathcal{K} = P^{-1},\ \mathcal{L} = P^{-1}XP^{-1}, \tag{3.22}$$

*with*

$$P = \begin{bmatrix} P_1 & 0_{\nu_1\times\nu_1} \\ \dagger & P_2 \end{bmatrix},\ X = \begin{bmatrix} X_1 & 0_{\nu_1\times\nu_1} \\ \dagger & X_2 \end{bmatrix}. \tag{3.23}$$

*Proof.* On the one hand, condition (3.17) evidently leads to condition (3.1).

On the other side, from (2.4), (3.20), and (3.21), we obtain

$$\begin{aligned} \tilde{M}_\iota(\zeta,\hat{\vartheta})\Theta &= M_\iota(\zeta,\hat{\vartheta})\Theta_1\mathcal{M}P_{211}\mathcal{M}^{-1}\Theta_1^T\Theta_1\begin{bmatrix} \mathcal{M} & 0_{\nu_3\times(\nu_1-\nu_3)} \end{bmatrix}\Theta_2 \\ &= M_\iota(\zeta,\hat{\vartheta})\Theta_1\begin{bmatrix} \mathcal{M}P_{211} & 0_{\nu_3\times(\nu_1-\nu_3)} \end{bmatrix}\Theta_2 \\ &= M_\iota(\zeta,\hat{\vartheta})\Theta_1\begin{bmatrix} \mathcal{M} & 0_{\nu_3\times(\nu_1-\nu_3)} \end{bmatrix}\Theta_2\Theta_2^T\begin{bmatrix} P_{211} & 0_{\nu_1\times(\nu_1-\nu_3)} \\ \dagger & P_{222} \end{bmatrix}\Theta_2 = M_\iota(\zeta,\hat{\vartheta})\Theta P_2, \end{aligned}$$

$$\tilde{N}_\iota(\zeta,\hat{\vartheta}) \;=\; N_\iota(\zeta,\hat{\vartheta})P_1. \tag{3.24}$$

According to the form of $P$ and $X$ in (3.23), $\Lambda_{\iota\jmath}(\zeta,\hat{\vartheta})$ can be reformulated as

$$\Lambda_{\iota\jmath}(\zeta,\hat{\vartheta}) \;=\; \begin{bmatrix} \mathbf{Sys}(\mathcal{A}_{\iota\jmath}(\zeta,\hat{\vartheta})P+\sigma P)+X+\beta I_{2\nu_1\times 2\nu_1} & \mathcal{B}_\iota(\zeta)P \\ \dagger & -e^{-2\sigma\varphi}X \end{bmatrix}. \tag{3.25}$$

Thus

$$\Psi_{\iota\jmath}(\zeta,\hat{\vartheta}) \;=\; diag(P^{-1},P^{-1})\Lambda_{\iota\jmath}(\zeta,\hat{\vartheta})diag(P^{-1},P^{-1}). \tag{3.26}$$

It follows from (3.26) that condition (3.18) implies condition (3.2).

**Algorithm 3.1.** **Step 1:** Using the sector-nonlinearity method [23] and its Taylor series-based extension developed for polynomial case in [24], the nonlinear TD system (2.1) can be exactly transformed into the PFTD model (2.6).

**Step 2:** Let $\sigma\in\mathbb{R}_{>0}$, $\{\varpi_k\in\mathbb{R}_{>0}$ for $k\in\mathbb{I}^6$, $\varpi_{7\iota\iota}(\zeta,\hat{\vartheta})\in\mathbb{P}_{\geq 0}(\zeta,\hat{\vartheta})$ for $\iota\in\mathbb{I}^n\}$ and the independent vectors $\{\zeta\in\mathbb{R}^w,\ \hat{\vartheta}\in\mathbb{R}^{\nu_1},\ \varsigma_1\in\mathbb{R}^1,\ \varsigma_2\in\mathbb{R}^{\nu_1},\ \varsigma_3\in\mathbb{R}^{\nu_3},\ \varsigma_4\in\mathbb{R}^{\nu_1-\nu_3}$, and $\varsigma_5\in\mathbb{R}^{4\nu_1}\}$ be given.

Find $\beta\in\mathbb{R}$, $P_1,\ X_1,\ X_2\in\mathbb{S}^{\nu_1\times\nu_1}$, $P_{211}\in\mathbb{S}^{\nu_3\times\nu_3}$, $P_{222}\in\mathbb{S}^{(\nu_1-\nu_3)\times(\nu_1-\nu_3)}$, $M_\iota(\zeta,\hat{\vartheta})\in\mathbb{P}^{\nu_1\times\nu_3}(\zeta,\hat{\vartheta})$, and $N_\jmath(\zeta,\hat{\vartheta})\in\mathbb{P}^{\nu_2\times\nu_1}(\zeta,\hat{\vartheta})$ for $(\iota,\jmath)\in\mathbb{I}^n\times\mathbb{I}^n$ such that

$$\varsigma_1(\beta-\varpi_1)\varsigma_1\in\Sigma(\varsigma_1),\ \varsigma_2^T(P_1-\varpi_2 I_{\nu_1\times\nu_1})\varsigma_2\in\Sigma(\varsigma_2),\ \varsigma_2^T(X_1-\varpi_3 I_{\nu_1\times\nu_1})\varsigma_2\in\Sigma(\varsigma_2),$$
$$\varsigma_2^T(X_2-\varpi_4 I_{\nu_1\times\nu_1})\varsigma_2\in\Sigma(\varsigma_2),\ \varsigma_3^T(P_{211}-\varpi_5 I_{\nu_3\times\nu_3})\varsigma_3\in\Sigma(\varsigma_3),$$
$$\varsigma_4^T(P_{222}-\varpi_6 I_{(\nu_1-\nu_3)\times(\nu_1-\nu_3)})\varsigma_4\in\Sigma(\varsigma_4),\ \varsigma_5^T\Big(\mathsf{F}_{\iota\iota}(\zeta,\hat{\vartheta})-\varpi_{7\iota\iota}(\zeta,\hat{\vartheta})\Big)\varsigma_5\in\Sigma(\varsigma_5,\zeta,\hat{\vartheta}),$$

where $\mathsf{F}_{\iota\iota}(\zeta,\hat{\vartheta})$ is given in (3.19).

**Step 3:** Determine the gains $\{M_\iota(\zeta,\hat{\vartheta})\in\mathbb{P}^{\nu_1\times\nu_3}(\zeta,\hat{\vartheta}),N_\jmath(\zeta,\hat{\vartheta})\in\mathbb{P}^{\nu_2\times\nu_1}(\zeta,\hat{\vartheta})\}$ and the radius $\delta$.

**Remark 3.1.** *Using (3.22), the Lyapunov-Krasovskii functional given in (3.3) can be equivalently reformulated as follows:*

$$\mathcal{F}(t)=\chi^TP^{-1}\chi+\int_t^{t+\varphi}e^{2\sigma(s-t-\varphi)}\chi^T(s-\varphi)\,P^{-1}XP^{-1}\chi(s-\varphi)\,ds, \tag{3.27}$$

*where P and X are defined as in (3.23). Thus, the matrices $P_1$, $P_2$, $X_1$, and $X_2$ are not assumed to exist independently. They are introduced through the Lyapunov-Krasovskii functional, and their existence is ensured by the feasibility of the resulting SOS conditions in Step 2 of Algorithm 3.1.*

**Remark 3.2.** *In [25], the exogenous term $\Delta(t,\vartheta(t))$ is assumed to satisfy a Lipschitz condition. In contrast, in our study, the exogenous is only assumed to be bounded, which is a natural and less restrictive assumption. Moreover, while the aforementioned work relies on a TA-SUF model and uses LMI as the main design tool, our approach uses a PF model exploits the SOS framework, which allows for a smaller number of fuzzy rules.*

**Remark 3.3.** *Although this work is the first to address the problem of designing an observer for the practical exponential stabilization of PFTD models and the first to adopt recent relaxed conditions for parameterized LMIs within the SOS framework, the framework could be further developed to address more intricate cases, thereby enhancing opportunities for comparative studies.*

- *The study in [26] addresses the same problem for delay-free PF models using the Hadamard fractional-order derivative. In contrast, our work considers the integer-order case with TD and seeks to achieve more relaxed conditions by using the recently developed parameterized LMI approach within the SOS framework. Extending the present results to the more general case involving the Hadamard fractional derivative remains an interesting direction for future research.*
- *The study in [27] investigates exponential stabilization of PF positive switched systems with TD, which represents a broader class due to the positivity and switching aspects. In contrast, our work considers a different, simpler class without positive switching, but it addresses bounded perturbations and partial-state measurements, which are not treated in [27]. Overall, extending the analysis of positive switched systems with TD to also incorporate bounded perturbations and partially unmeasurable states represents an interesting and valuable direction for future research, highlighting a complementary and promising generalization of these two approaches.*

**Discussion.** Although this work is, to the best of our knowledge, the first attempt to address the problem of practical exponential OB control for PF models with TD in a general framework and for TA-SUF models in particular, it still has certain limitations. Therefore, several aspects can be further improved and deserve future investigation.

- The proposed practical exponential OB control framework can be applied to engineering systems, such as mechanical systems, networked control systems, and industrial process control, where state estimation and stabilization under uncertainty are essential in future applications.
- The SVD-based decoupling used to convert bilinear PMIs into linear ones imposes additional structural constraints. Specifically, the output matrix $\Theta$ is assumed to be identical across all rules and of full column rank. Furthermore, the Lyapunov matrices are restricted to a diagonal block structure, which reduces the number of decision variables and thereby introduces conservatism. Addressing these limitations by developing alternative decoupling techniques such as the cone complementary linearization algorithm [28] constitutes a promising direction for future work.
- Reducing the conservatism of the proposed conditions constitutes an important direction for future work. In this regard, the use of improved Lyapunov-Krasovskii functionals incorporating double integral terms could be investigated. Moreover, adopting tighter bounding techniques for cross-terms in the derivation process, such as the exponential weighted integral inequality in [29], remains an open problem.
- Further to the practical exponential OB control of PFTD models investigated in this paper, interval stability, studied for other classes of systems in [30, 31], remains unexplored in this context.

## 4. Illustrative example

Let us consider the nonlinear TD system represented in the form of (2.1), where

$$\begin{aligned}
\tilde{q}_1(\vartheta_1) &= \begin{bmatrix} 0.7 + 0.3\vartheta_1 + 0.5\vartheta_1^2 & 0 \\ (1-\alpha)\,\epsilon(\vartheta_1) & -0.5 \end{bmatrix}, \ \tilde{q}_2(\vartheta_1) = \begin{bmatrix} 0 & 0 \\ \alpha\,\epsilon(\vartheta_1) & 0 \end{bmatrix}, \ \tilde{q}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
\Delta(t,\vartheta(t)) &= \begin{bmatrix} 0.5\cos(2\pi t) \\ 0.2\,e^{-0.7t}\sin(t) \end{bmatrix}, \ \Theta = \begin{bmatrix} 1 & 0 \end{bmatrix}, \ \varphi = 0.1,
\end{aligned}$$

in which $\alpha = 0.9$ and $\epsilon(\vartheta_1) = \frac{\sin(\vartheta_1)}{\vartheta_1}$.

Figure 1 presents the time trajectory of $\vartheta$ without control, revealing unstable behavior.

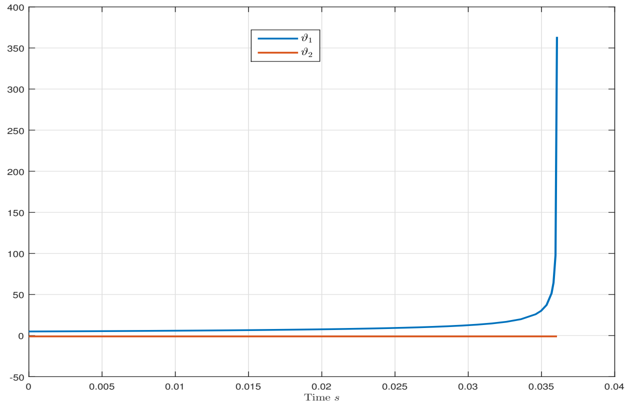


**Figure 1.** Time trajectory of $\vartheta$ without control.

Start with Step 1 of Algorithm 3.1. Applying the sector nonlinearity method to the nonpolynomial measurable nonlinearity $\epsilon(\vartheta_1)$, we obtain a PFTD model in the form of (2.6), where

$$
\begin{aligned}
& n = 2,\ \rho_1 = \frac{\epsilon(\vartheta_1) + 0.2172}{1.2172},\ \rho_2 = \frac{1 - \epsilon(\vartheta_1)}{1.2172},\ q_{11}(\zeta) = \begin{bmatrix} 0.7 + 0.3\zeta + 0.5\zeta^2 & 0 \\ -0.2172(1-\alpha) & -0.5 \end{bmatrix}, \\
& q_{12}(\zeta) = \begin{bmatrix} 0.7 + 0.3\zeta + 0.5\zeta^2 & 0 \\ 1-\alpha & -0.5 \end{bmatrix},\ q_{21} = \begin{bmatrix} 0 & 0 \\ -0.2172\alpha & 0 \end{bmatrix}, \\
& q_{22} = \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix},\ q_{31} = q_{32} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},
\end{aligned}
\tag{4.1}
$$

in which $\zeta$ is a measurable variable corresponding to the measured output $\eta = \vartheta_1$.

Applying Step 2 of Algorithm 3.1 with $\sigma = 0.16$ and $\{\varpi_k = 10^{-4}$ for $k \in \mathbb{I}^6$, $\varpi_{7_{\imath\jmath}}(\zeta, \hat{\vartheta}) = 10^{-4}$ for $(\imath, \jmath) \in \mathbb{I}^2\}$ yields a feasible solution.

The SOS conditions in this step are solved numerically using SOSTOOLS in combination with SeDuMi. First, the SOS constraints are formulated and converted into a semidefinite programming (SDP) problem using SOSTOOLS 4.00 [19]. The resulting SDP is then efficiently solved using the SeDuMi 1.02 solver [20]. The computational complexity of the proposed approach mainly depends on the size of the resulting semidefinite programming problem, which is determined by the system's order, polynomial degree, and number of decision variables.

Building on the feasible solution from Step 2, Step 3 yields $\delta = 8.626$ and the following gains:

$$
M_1(\zeta) = \begin{bmatrix} 4.6067\zeta^2 + 1.5092\zeta + 16.5205 \\ 0.2454 \times 10^{-3}\zeta^2 - 0.0805\zeta + 3.3477 \end{bmatrix}, \tag{4.2}
$$

$$
M_2(\zeta) = \begin{bmatrix} 4.7085\zeta^2 + 0.9740\zeta + 15.5912 \\ 0.0163 \times 10^{-3}\zeta^2 - 0.3539\zeta + 1.5887 \end{bmatrix}, \tag{4.3}
$$

$$
N_1(\zeta) = \begin{bmatrix} -4.2181\zeta^2 - 2.4106\zeta - 16.2191 & 0.8928\zeta^2 + 0.5569\zeta + 3.8361 \end{bmatrix}, \tag{4.4}
$$

$$N_2(\zeta) = \begin{bmatrix} -5.0132\zeta^2 - 1.6628\zeta - 16.3851 & 1.0634\zeta^2 + 0.3798\zeta + 3.7607 \end{bmatrix}. \tag{4.5}$$

We apply the observer-based controller to the system using the gains obtained in Step 3.

Figure 2 shows the time evolution of both the state $\vartheta$ and its estimate for initial conditions $\vartheta = \begin{bmatrix} 5 & -1 \end{bmatrix}$ and $\hat{\vartheta} = \begin{bmatrix} 0 & 0 \end{bmatrix}$. It demonstrates that the closed-loop system achieves PES. The control $r$ is shown in Figure 3.

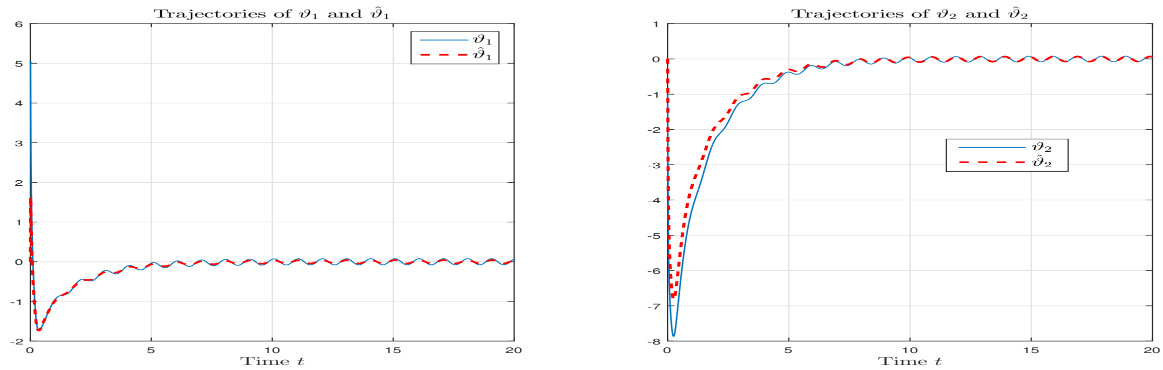


**Figure 2.** Time trajectory of $\vartheta$ with control.

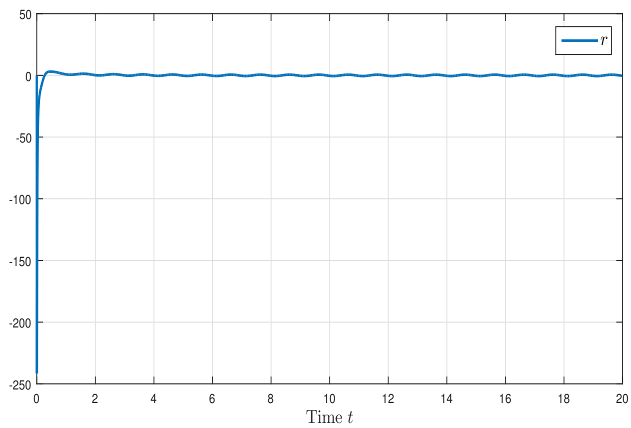


**Figure 3.** Time trajectory of control $r$.

Figure 4 shows the behaviors in the $\vartheta_1$–$\vartheta_2$ plane with control for different initial conditions. This figure highlights that the originally unstable uncontrolled system becomes PES under the SOS-designed controller.

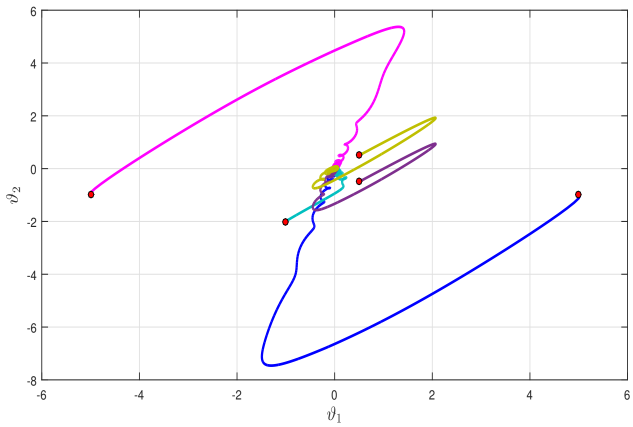


**Figure 4.** Behaviors in the $\vartheta_1$–$\vartheta_2$ plane with control for different initial conditions.

In order to compare the PF modeling approach with the TA-SUF approach, the considered nonlinear TD system is represented using a TA-SUF model. In this formulation, the polynomial term involved in $\tilde{q}_1(\vartheta_1)$ and $\tilde{q}_2(\vartheta_1)$ is treated as a premise variable. Accordingly, these matrices are expressed as follows:

$$\tilde{q}_1(\vartheta_1) = \begin{bmatrix} \epsilon_1(\vartheta_1) & 0 \\ (1-\alpha)\,\epsilon_2(\vartheta_1) & -0.5 \end{bmatrix},\ \tilde{q}_2(\vartheta_1) = \begin{bmatrix} 0 & 0 \\ \alpha\,\epsilon_2(\vartheta_1) & 0 \end{bmatrix},$$

where $\epsilon_1(\vartheta_1) = 0.7 + 0.3\vartheta_1 + 0.5\vartheta_1^2$ and $\epsilon_2(\vartheta_1) = \frac{\sin(\vartheta_1)}{\vartheta_1}$.

Introducing the two premise variables $\epsilon_1(\vartheta_1)$ and $\epsilon_2(\vartheta_1)$, and assuming $|\vartheta_1| < \vartheta_{1,max}$ with $\vartheta_{1,max} \in \mathbb{R}_{>0}$, the considered system can be represented in the form of a TA-SUF model. Since the TA-SUF model is a special case of the PF model, applying Step 1 of Algorithm 3.1 yields the model in (2.6) with constant matrices, where

$$\begin{aligned}
n &= 4,\ \rho_1 = \frac{\epsilon_1(\vartheta_1) - \epsilon_{1min}}{\epsilon_{1max} - \epsilon_{1min}} \cdot \frac{\epsilon_2(\vartheta_1) + 0.2172}{1.2172},\ \rho_2 = \frac{\epsilon_1(\vartheta_1) - \epsilon_{1min}}{\epsilon_{1max} - \epsilon_{1min}} \cdot \frac{1 - \epsilon_2(\vartheta_1)}{1.2172},\\
\rho_3 &= \frac{\epsilon_{1max} - \epsilon_1(\vartheta_1)}{\epsilon_{1max} - \epsilon_{1min}} \cdot \frac{\epsilon_2(\vartheta_1) + 0.2172}{1.2172},\ \rho_4 = \frac{\epsilon_{1max} - \epsilon_1(\vartheta_1)}{\epsilon_{1max} - \epsilon_{1min}} \cdot \frac{1 - \epsilon_2(\vartheta_1)}{1.2172},\\
q_{11} &= \begin{bmatrix} \epsilon_{1max} & 0 \\ -0.2172(1-\alpha) & -0.5 \end{bmatrix},\ q_{12} = \begin{bmatrix} \epsilon_{1max} & 0 \\ 1-\alpha & -0.5 \end{bmatrix},\\
q_{13} &= \begin{bmatrix} \epsilon_{1min} & 0 \\ -0.2172(1-\alpha) & -0.5 \end{bmatrix},\ q_{14} = \begin{bmatrix} \epsilon_{1min} & 0 \\ 1-\alpha & -0.5 \end{bmatrix},\\
q_{21} &= q_{23} = \begin{bmatrix} 0 & 0 \\ -0.2172\alpha & 0 \end{bmatrix},\ q_{22} = q_{24} = \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix},\ q_{31} = q_{32} = q_{33} = q_{34} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},
\end{aligned}$$

in which

$$\epsilon_{1min} = \min_{|\vartheta_1| < \vartheta_{1,max}} (\epsilon_1(\vartheta_1)),\ \epsilon_{1max} = \max_{|\vartheta_1| < \vartheta_{1,max}} (\epsilon_1(\vartheta_1)).$$

Hence, in order to derive a TA-SUF model, the polynomial terms must be treated as premise variables, which leads to an increase in the number of fuzzy rules. Moreover, determining the upper bounds of these premise variables requires us to assume a bounded modeling domain, expressed as $|\vartheta_1| < \vartheta_{1,max}$. Consequently, the resulting TA-SUF model is only semiglobal, even when a relatively large value of $\vartheta_{1,max}$ is selected. In contrast, the PF model provides a global representation, as it preserves the polynomial terms in their original form, without the need to impose any boundedness assumptions.

To the best of our knowledge, there are no existing results in the literature addressing the PES of PF models via OB control even for the special case of TA-SUF models. Therefore, to highlight the relationship with the LMI framework, we note that the proposed conditions in Theorem 3.2 reduce to LMI-based design conditions for TA-SUF models when all polynomial matrices are of degree zero. Solving these LMIs for the resulting TA-SUF model does not yield feasible solutions for large values of $\vartheta_{1,max}$ (e.g., $\vartheta_{1,max} > 544$), whereas the SOS-based formulation remains feasible for any $\vartheta_1$.

## 5. Conclusions

This paper has investigated the problem of PES for nonlinear TD systems. A class of PFTD models is used in place of the TA-SUF model to represent the dynamics of systems, aiming to minimize the number of fuzzy rules and enhance modeling flexibility. In addition, we have proposed an OB controller to ensure the PES of the considered class of PFTD models. By using a Lyapunov-Krasovskii functional, we establish sufficient bilinear PMI to guarantee the PES of the combined state estimation-error model. Subsequently, the bilinear PMI is converted into linear polynomial constraints through application of a SVD decoupling matrix technique. The efficacy of the proposed stability conditions is verified through a representative numerical example. In future work, we plan to develop an enhanced Lyapunov-Krasovskii functional with double integral term that explicitly incorporates additional system information, particularly regarding TD. This will be combined with refined mathematical tools, including Jensen-type inequalities, to obtain linear PMI. Furthermore, extending the framework from constant delays to time-varying delays, and incorporating parameter uncertainties into the OB control design, would enhance both the generality and practical applicability of the proposed results.

## Author contributions

S. Dhahri: Conceptualization, Methodology, Software, Formal analysis, Writing–original draft; E. B. Alaia: Investigation, Validation, Writing–review & editing; A. Alanazi: Resources, Funding acquisition, Visualization, Writing–review & editing; S. Almenwer: Data curation, Investigation, Resources, Writing–review & editing; H. Gassara: Supervision, Project administration, Validation, Writing–review & editing. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Acknowledgments

This work was funded by the Deanship of Graduate Studies and Scientific Research at Jouf University under grant No. (DGSSR-2025-02-01645).


## Conflict of interest

All authors declare no conflicts of interest in this paper.

## References


1. X. P. Xu, G. S. Zhai, Practical stability and stabilization of hybrid and switched systems, *IEEE Trans. Automat. Control*, **50** (2005), 1897–1903. https://doi.org/10.1109/TAC.2005.858680
2. A. BenAbdallah, I. Ellouze, M. A. Hammami, Practical stability of nonlinear time-varying cascade systems, *J. Dyn. Control Syst.*, **15** (2009), 45–62. https://doi.org/10.1007/s10883-008-9057-5
3. Y. Z. Tian, M. Q. Liang, Y. G. Sun, Practical exponential stability of switched homogeneous positive nonlinear systems with stable and unstable modes, *J. Franklin Inst.*, **360** (2023), 8398–8415. https://doi.org/10.1016/j.jfranklin.2023.06.031
4. Y. W. Huang, B. W. Wu, Y. E. Wang, L. L. Liu, Practical exponential stability of switched generalized homogeneous positive nonlinear systems with bounded disturbances, *Circuits Syst. Signal Process.*, **44** (2025), 3793–3809. https://doi.org/10.1007/s00034-024-02985-8
5. B. B. Hamed, I. Ellouze, M. A. Hammami, Practical uniform stability of nonlinear differential delay equations, *Mediterr. J. Math.*, **8** (2011), 603–616. https://doi.org/10.1007/s00009-010-0083-7
6. R. Villafuerte, S. Mondié, A. Poznyak, Practical stability of time-delay systems: LMI's approach, *Eur. J. Control*, **17** (2011), 127–138. https://doi.org/10.3166/ejc.17.127-138
7. M. Kharrat, H. Gassara, M. Rhaima, L. Mchiri, A. B. Makhlouf, Practical stability for conformable time-delay systems, *Discrete Dyn. Nat. Soc.*, **2023** (2023), 9375360. https://doi.org/10.1155/2023/9375360
8. T. Caraballo, F. Ezzine, M. A. Hammami, Practical stability of stochastic differential delay equations driven by G-Brownian motion with general decay rate, *Electron. J. Differ. Equ.*, **2024** (2024), 1–26. https://doi.org/10.58997/ejde.2024.70
9. T. S. Peng, H. B. Zeng, W. Wang, X. M. Zhang, X. G. Liu, General and less conservative criteria on stability and stabilization of T-S fuzzy systems with time-varying delay, *IEEE Trans. Fuzzy Syst.*, **31** (2023), 1531–1541. https://doi.org/10.1109/TFUZZ.2022.3204899
10. H. Zhang, J. Liu, S. Y. Xu, Z. Q. Zhang, Practical stabilization of networked Takagi-Sugeno fuzzy systems via improved Jensen inequalities, *IEEE Trans. Cybernet.*, **52** (2022), 4381–4390. https://doi.org/10.1109/TCYB.2020.3026375
11. K. Tanaka, H. Yoshida, H. Ohtake, H. O. Wang, A sum of squares approach to stability analysis of polynomial fuzzy systems, In: *2007 American Control Conference*, New York, 2007, 4071–4076. https://doi.org/10.1109/ACC.2007.4282579

12. K. Tanaka, H. Yoshida, H. Ohtake, H. O. Wang, A sum-of-squares approach to modeling and control of nonlinear dynamical systems with polynomial fuzzy systems, *IEEE Trans. Fuzzy Syst.*, **17** (2009), 911–922. https://doi.org/10.1109/TFUZZ.2008.924341
13. H. Gassara, A. El Hajjaji, M. Krid, M. Chaabane, Stability analysis and memory control design of polynomial fuzzy systems with time delay via polynomial Lyapunov-Krasovskii functional, *Int. J. Control Automat. Syst.*, **16** (2018), 2011–2020. https://doi.org/10.1007/s12555-017-0617-x
14. L. N. Fu, H. K. Lam, F. C. Liu, B. Xiao, Z. X. Zhong, Static output-feedback tracking control for positive polynomial fuzzy systems, *IEEE Trans. Fuzzy Syst.*, **30** (2022), 1722–1733. https://doi.org/10.1109/TFUZZ.2021.3065521
15. P. Selvaraj, R. Sakthivel, O. M. Kwon, R. Sakthivel, Event-triggered output feedback control design for polynomial fuzzy systems, *J. Franklin Inst.*, **361** (2024), 1078–1092. https://doi.org/10.1016/j.jfranklin.2023.12.048
16. P. Selvaraj, O. M. Kwon, S. H. Lee, R. Sakthivel, S. M. Lee, Event-triggered control design with varying gains for polynomial fuzzy systems against DoS attacks, *Math. Comput. Simul.*, **218** (2024), 1–14. https://doi.org/10.1016/j.matcom.2023.11.022
17. S. Dhahri, E. B. Alaia, H. Gassara, Observer-based finite-time boundedness of polynomial fuzzy models with time delay, *Int. J. Syst. Sci.*, **57** (2026), 1544–1555. https://doi.org/10.1080/00207721.2025.2532037
18. L. V. Hien, V. N. Phat, Exponential stability and stabilization of a class of uncertain linear time-delay systems, *J. Franklin Inst.*, **346** (2009), 611–625. https://doi.org/10.1016/j.jfranklin.2009.03.001
19. A. Papachristodoulou, J. Anderson, G. Valmorbida, S. Prajna, P. Seiler, P. A. Parrilo, et al., *SOSTOOLS: Sum of squares optimization toolbox for MATLAB User's guide, Version 4.00*, 2021, arXiv:1310.4716.
20. J. F. Sturm, Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones, *Optim. Methods Software*, **11** (1999), 625–653. https://doi.org/10.1080/10556789908805766
21. S. Prajna, A. Papachristodoulou, F. Wu, Nonlinear control synthesis by sum of squares optimization: a Lyapunov-based approach, In: *2004 5th Asian Control Conference (IEEE Cat. No.04EX904)*, Melbourne, VIC, Australia, 2004, 157–165.
22. D. W. Kim, D. Lee, Relaxed conditions for parameterized linear matrix inequality in the form of double fuzzy summation, *IEEE Trans. Fuzzy Syst.*, **32** (2024), 1608–1612. https://doi.org/10.1109/TFUZZ.2023.3315290
23. K. Tanaka, H. O. Wang, *Fuzzy control systems design and analysis*, New York: John Wiley & Sons, 2001. https://doi.org/10.1002/0471224596
24. A. Sala, C. Ariño, Polynomial fuzzy models for nonlinear control: a Taylor series approach, *IEEE Trans. Fuzzy Syst.*, **17** (2009), 1284–1295. https://doi.org/10.1109/TFUZZ.2009.2029235
25. O. Kahouli, H. Gassara, L. El Amraoui, M. Ayari, Observer-based exponential stabilization for time delay Takagi-Sugeno-Lipschitz models, *Mathematics*, **13** (2025), 3170. https://doi.org/10.3390/math13193170

26. H. Gassara, O. Naifar, M. Chaabane, A. B. Makhlouf, H. Arfaoui, M. Aldandani, Observer-based control for nonlinear Hadamard fractional-order systems via SOS approach, *Asian J. Control*, **27** (2025), 912–920. https://doi.org/10.1002/asjc.3497

27. X. M. Li, Y. N. Shan, H. K. Lam, Z. Y. Bao, J. D. Zhao, Exponential stabilization of polynomial fuzzy positive switched systems with time delay considering MDADT switching signal, *IEEE Trans. Fuzzy Syst.*, **32** (2024), 174–187. https://doi.org/10.1109/TFUZZ.2023.3289650

28. W. Wang, J. M. Liang, H. B. Zeng, X. M. Zhang, Novel looped functionals in designing output feedback controllers for aperiodic sampled-data control systems, *IEEE Trans. Automat. Sci. Eng.*, **22** (2025), 16397–16402. https://doi.org/10.1109/TASE.2025.3573304

29. N. Cheng, W. Wang, H. B. Zeng, X. Liu, X. M. Zhang, Novel exponential-weighted integral inequality for exponential stability analysis of time-varying delay systems, *Appl. Math. Lett.*, **172** (2026), 109730. https://doi.org/10.1016/j.aml.2025.109730

30. H. Geng, H. S. Zhang, S. F. Su, Asynchronously switched control with variable convergence rate for switched nonlinear systems: a persistent dwell-time scheme, *IEEE Trans. Fuzzy Syst.*, **32** (2024), 6695–6707. https://doi.org/10.1109/TFUZZ.2024.3459860

31. P. H. He, H. S. Zhang, S. F. Su, A sliding mode control method with variable convergence rate for nonlinear impulsive stochastic systems, *IEEE Trans. Cybernet.*, **55** (2025), 2213–2222. https://doi.org/10.1109/TCYB.2025.3551668

AIMS Press