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Research article

On independent and total variants of Roman and Italian domination in Mycielskian graphs

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Abstract: Let G be a graph and $\mu(G)$ its Mycielskian graph. Roman and Italian domination, together with their independent and total variants, have been extensively studied, including studies on graph operators and products. In this article, we investigated the behavior of these domination parameters in Mycielskian graphs and, in particular, we provided closed formulas for these invariants in $\mu(G)$ in terms of the corresponding values of G.

Keywords: Roman domination; Italian domination; Mycielskian graphs
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1. Introduction

Let G be a nontrivial connected graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. The Mycielskian of G, denoted by $\mu(G)$, is constructed by introducing n new vertices $u_1, \ldots, u_n$ together with an additional vertex w. Then, for each $i \in \{1, \ldots, n\}$, the edge wu_i is added to the graph. In addition, for each edge $v_iv_j \in E(G)$, we add the edges u_iv_j and v_iu_j to complete the construction of $\mu(G)$. Figure 1 illustrates a graph G together with its Mycielskian graph $\mu(G)$. This graph operator was introduced by J. Mycielski in [16] as a construction that increases the chromatic number while avoiding the creation of triangles. Beyond coloring, several works have examined how domination parameters behave under this operator (see, for instance, [7, 10–13, 15, 18]). Our goal is to advance this line of research through the study of the independent and total variants of classical Roman and Italian domination.

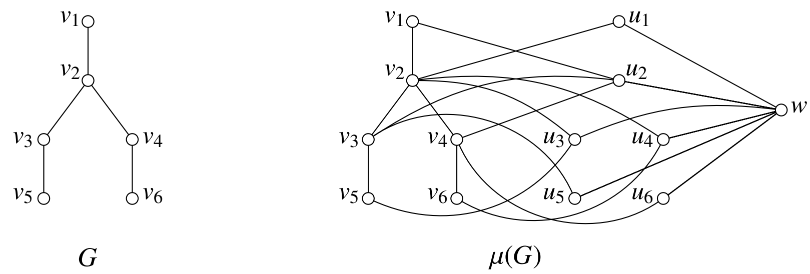

Figure 1. A graph G, and the corresponding Mycielskian graph $\mu(G)$.

For each vertex $v \in V(G)$, let $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$. The *minimum degree* and *maximum degree* of G are defined as $\delta(G) = \min\{|N_G(v)| : v \in V(G)\}$ and $\Delta(G) = \max\{|N_G(v)| : v \in V(G)\}$, respectively. We denote by C_n the cycle graph of order n. Let $D \subseteq V(G)$. The *boundary* of D is denoted by $B_G(D) = (\cup_{v\in D} N_G(v)) \setminus D$. As usual, the subgraph of G induced by D will be denoted by $G[D]$. We say that D is an *independent set* of G if $uv \notin E(G)$ for all $u, v \in D$. Furthermore, D is a *total dominating set* (TDS) of G if $N_G(v) \cap D \neq \emptyset$ for every $v \in V(G)$.

Let $f : V(G) \to \{0, 1, 2\}$ be a function defined on a graph G. The function f induces three subsets V_0, V_1 and V_2 of $V(G)$, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i \in \{0, 1, 2\}$. Accordingly, we denote such a function by $f(V_0, V_1, V_2)$. For any subset $X \subseteq V(G)$, we define $f(X) = \sum_{x\in X} f(x)$, and the *weight* of f is given by $\omega(f) = f(V(G)) = |V_1| + 2|V_2|$. We next present the formal definitions of the domination parameters considered in this article.

- A function $f(V_0, V_1, V_2)$ is called a *Roman dominating function* (RDF) on G whenever $N_G(v) \cap V_2 \neq \emptyset$ for all $v \in V_0$. An RDF is said to be an *independent Roman dominating function* (IRDF) if the set $V_1 \cup V_2$ is independent, and a *total Roman dominating function* (TRDF) if $V_1 \cup V_2$ is a TDS of G. The minimum weight among all RDFs, IRDFs, and TRDFs on G gives rise to the *Roman domination number*, *independent Roman domination number*, and *total Roman domination number*, denoted by $\gamma_R(G)$, $i_R(G)$, and $\gamma_{tR}(G)$, respectively. An RDF whose weight equals $\gamma_R(G)$ is referred to as a $\gamma_R(G)$-function. Throughout this paper, analogous terminology will be used for minimum-weight functions associated with the other domination parameters considered. The concepts of Roman domination and independent Roman domination were introduced by Cockayne et al. [8], while total Roman domination was later defined in [14].
- A function $f(V_0, V_1, V_2)$ is called an *Italian dominating function* (IDF) on G if every vertex $v \in V_0$ satisfies $f(N_G(v)) \geq 2$. An IDF is said to be an *independent Italian dominating function* (IIDF) whenever $V_1 \cup V_2$ is an independent set, and a *total Italian dominating function* (TIDF) whenever $V_1 \cup V_2$ is a TDS of G. The minimum weight among all IDFs, IIDFs, and TIDFs defines the *Italian domination number*, *independent Italian domination number*, and *total Italian domination number*, denoted by $\gamma_I(G)$, $i_I(G)$, and $\gamma_{tI}(G)$, respectively. These parameters were introduced in [6], [17], and [3], respectively.

In 2012, Kazemi [11] proved that for any graph G,

$$\gamma_R(\mu(G)) \in \{\gamma_R(G) + 1, \gamma_R(G) + 2\},$$

and that $\gamma_R(\mu(G)) = \gamma_R(G) + 1$ if, and only if, there exists a $\gamma_R(G)$-function $f(V_0, \emptyset, V_2)$ such that V_2 is a TDS of G. Moreover, in 2021, Varghese and Aparna Lakshmanan [18] showed that

$$\gamma_I(\mu(G)) \in \{\gamma_I(G) + 1, \gamma_I(G) + 2\}.$$

Our goal is to derive closed formulas for the independent and total variants of Roman and Italian domination numbers of the Mycielskian graphs $\mu(G)$ in terms of the corresponding parameters of G. In Section 2, we obtain closed formulas for $i_R(\mu(G))$ and $i_I(\mu(G))$. Section 3 is devoted to deriving closed formulas for $\gamma_{tR}(\mu(G))$ and $\gamma_{tI}(\mu(G))$. Finally, in Section 4 we characterize the graphs G for which $\gamma_I(\mu(G)) = \gamma_I(G) + 1$ and $\gamma_I(\mu(G)) = \gamma_I(G) + 2$, thereby filling an existing gap in the literature.

2. Closed formulas for $i_R(\mu(G))$ and $i_I(\mu(G))$

We first derive an explicit expression for $i_R(\mu(G))$ in terms of $i_R(G)$, where G is a nontrivial connected graph. To establish this result, we first require the following useful lemma.

Lemma 2.1. [1] *Let $f(V_0, V_1, V_2)$ be an RDF on a graph G. If V_2 is an independent set of G, then there exists an IRDF g on G such that $\omega(g) \leq \omega(f)$.*

Theorem 2.2. *Let G be a nontrivial connected graph. Then,*

$$i_R(\mu(G)) = i_R(G) + 2.$$

Proof. Let $U = \{u_1, \ldots, u_n\}$. Now, let g' be an $i_R(G)$-function, and define a function g on $\mu(G)$ as follows.

$$g(x) = \begin{cases} g'(x) & \text{if } x \in V(G), \\ 0 & \text{if } x \in U, \\ 2 & \text{if } x = w. \end{cases}$$

Observe that g is an IRDF on $\mu(G)$. Hence, $i_R(\mu(G)) \leq \omega(g) = \omega(g') + 2 = i_R(G) + 2$.

Finally, we proceed to prove that $i_R(\mu(G)) \geq i_R(G) + 2$. Let $f(V_0, V_1, V_2)$ be an $i_R(\mu(G))$-function. If $f(w) = 1$, then $f(u_i) = 0$ for every $i \in \{1, \ldots, n\}$, which leads to V_2 is not an independent set of $\mu(G)$, a contradiction. Hence, $f(w) \in \{0, 2\}$. Next, we analyze the following two complementary cases.

Case 1: $f(w) = 2$. In this case, it follows that $f(u_i) = 0$ for every $i \in \{1, \ldots, n\}$. Consequently, the restriction of f to $V(G)$ is an IRDF on G. Hence, $i_R(G) \leq f(V(G))$, which implies that $i_R(\mu(G)) = \omega(f) = f(V(G)) + f(N_{\mu(G)}[w]) \geq i_R(G) + 2$, as desired.

Case 2: $f(w) = 0$. In this case, we have that $U \cap V_2 \neq \emptyset$. Without loss of generality, assume that $U \cap V_2 = \{u_1, \ldots, u_k\}$. Next, we consider the following two complementary cases.

Subcase 2.1: $k = 1$. Observe that $v_1 \in V_1 \cup V_2$, since $N_G(v_1) \subseteq N_{\mu(G)}(u_1) \subseteq V_0$ and $U \cap V_2 = \{u_1\}$. If $U \cap V_1 = \emptyset$, then the restriction of f to $V(G)$ is an IRDF on G. Consequently, $i_R(\mu(G)) = \omega(f) = f(V(G)) + f(N_{\mu(G)}[w]) \geq i_R(G) + 2$, as desired. Henceforth, assume that $U \cap V_1 \neq \emptyset$. Now, let f' be the function defined on G as follows.

$$f'(v_i) = \begin{cases} \min\{f(v_i) + 1, 2\} & \text{if } i = 1, \\ f(v_i) & \text{if } i > 1. \end{cases}$$

Since $v_1 \in V_1 \cup V_2$, it is easy to check that f' is an IRDF on G, which implies that $i_R(G) \leq \omega(f')$. Consequently, $i_R(\mu(G)) = \omega(f) \geq \omega(f') + 1 + |U \cap V_1| \geq i_R(G) + 2$, as desired.

Subcase 2.2: $k \geq 2$. First, assume that $v_i \in V_1 \cup V_2$ for every subscript $i \in \{1, \ldots, k\}$. In this case, let f' be the function defined on G as follows.

$$f'(v_i) = \begin{cases} \min\{f(v_i) + 1, 2\} & \text{if } i \in \{1, \ldots, k\}, \\ f(v_i) & \text{if } i \in \{k+1, \ldots, n\}. \end{cases}$$

Observe that f' is an IRDF on G, which implies that $i_R(G) \leq \omega(f')$. Therefore, $i_R(\mu(G)) = \omega(f) \geq \omega(f') + k \geq i_R(G) + 2$, as desired. From now on, assume that $\{v_1, \ldots, v_k\} \cap V_0 \neq \emptyset$, and without loss of generality, assume that $v_1 \in V_0$. Since $u_1 \in V_2$, it follows that $N_G(v_1) \subseteq V_0$. Consequently, $N_{\mu(G)}(v_1) \cap V_2 \cap U \neq \emptyset$. Without loss of generality, assume that $N_{\mu(G)}(v_1) \cap V_2 \cap U = \{u_2, \ldots, u_r\}$, where clearly $r \leq k$. Moreover, observe that if $v_i \in V_1 \cup V_2$, then $u_i \in V_1 \cup V_2$. Let $W = \{v_i \in V(G) : f(u_i) > 0\}$ and $W^* = W \cap V_2$. It is easy to deduce that if $v_i \in W^*$, then $f(u_i) = 1$. Now, we define the function $f''(V_0'', V_1'', V_2'')$ on G as follows.

$$f''(v_i) = \begin{cases} 2 & \text{if } v_i = v_1, \\ 0 & \text{if } v_i \in \{v_2, \ldots, v_r\}, \\ 1 & \text{if } v_i \in W \setminus (W^* \cup \{v_1, \ldots, v_r\}), \\ f(v_i) & \text{if } v_i \in (V(G) \setminus W) \cup W^*. \end{cases}$$

Observe that $\omega(f'') + 2 \leq \omega(f) - f(u_2) + 2 = i_R(\mu(G))$. Let $v_i \in V_0''$. By the definition of f'', either $v_i \in \{v_2, \ldots, v_r\}$ or $v_i \notin W$. If $v_i \in \{v_2, \ldots, v_r\}$, then $N_G(v_i) \cap V_2'' \neq \emptyset$, since $v_1 \in N_G(v_i) \cap V_2''$. Now, assume that $v_i \notin W$. This implies that $f(u_i) = 0$, and as a consequence, $N_G(v_i) \cap V_2 \neq \emptyset$. Since $V_2 \cap V(G) \subseteq V_2''$, it follows that $N_G(v_i) \cap V_2'' \neq \emptyset$. Therefore, f'' is an RDF on G. Moreover, it easy to verify that $V_2'' = (V_2 \cap V(G)) \cup \{v_1\}$ is an independent set of G. By Lemma 2.1, there exists an IRDF g'' on G such that $\omega(g'') \leq \omega(f'')$. Consequently, $i_R(\mu(G)) \geq \omega(f'') + 2 \geq \omega(g'') + 2 \geq i_R(G) + 2$, as desired.

The above cases imply that $i_R(\mu(G)) = i_R(G) + 2$, which completes the proof. □

Finally, we establish an explicit expression for $i_I(\mu(G))$ in terms of $i_I(G)$, where G is a nontrivial connected graph. To prove this result, we first present the next proposition.

Proposition 2.3. *Let G be a connected graph of order $n \geq 2$. Then,*

(i) $i_I(G) \leq n - \Delta(G) + 1$.

(ii) *If G is connected, $\delta(G) \geq 2$ and $G \notin \{C_3, C_5\}$, then $i_I(G) \leq n - 2$.*

Proof. First, we proceed to prove (i). Since every IRDF on G is also an IIDF on G, it follows that $i_I(G) \leq i_R(G)$. Moreover, it was shown in [9] that $i_R(G) \leq n - \Delta(G) + 1$. By combining the above inequalities, it follows that $i_I(G) \leq n - \Delta(G) + 1$, which proves (i). Now, assume that (ii) holds, i.e., G is connected, $\delta(G) \geq 2$ and $G \notin \{C_3, C_5\}$. If $G \cong C_n$, then $i_I(G) \leq n - 2$. Otherwise, if $\Delta(G) \geq 3$, then by (i) we have that $i_I(G) \leq n - \Delta(G) + 1 \leq n - 2$, which completes the proof. □

Theorem 2.4. *Let G be a connected graph of order $n \geq 2$. Then,*

$$i_I(\mu(G)) = \begin{cases} n & \text{if } G \in \{C_3, C_5\}, \\ i_I(G) + 2 & \text{otherwise.} \end{cases}$$

Proof. If $G \in \{C_3, C_5\}$, then $i_I(\mu(G)) = |V(G)|$. Thus, we may assume for the rest of the proof that $G \notin \{C_3, C_5\}$. Let $U = \{u_1, \ldots, u_n\}$. Now, let h' be an $i_I(G)$-function, and define a function h on $\mu(G)$ as follows.

$$h(x) = \begin{cases} h'(x) & \text{if } x \in V(G), \\ 0 & \text{if } x \in U, \\ 2 & \text{if } x = w. \end{cases}$$

Observe that h is an IIDF on $\mu(G)$. Therefore, $i_I(\mu(G)) \leq \omega(h) = \omega(h') + 2 = i_I(G) + 2$.

Conversely, let $f(V_0, V_1, V_2)$ be an $i_I(\mu(G))$-function such that $|(V_1 \cup V_2) \cap V(G)|$ is maximum. Define $V_0^* = \{v_i \in V_0 \cap V(G) : f(N_G(v_i)) \leq 1\}$ and $U^* = \{u_i \in U : v_i \in V_0^*\}$. Suppose that $i_I(\mu(G)) = \omega(f) \leq i_I(G) + 1$. We next consider the three possible values that $f(w)$ can take.

Case 1: $f(w) = 1$. In this case, we have that $f(u_i) = 0$ for every $i \in \{1, \ldots, n\}$. This implies that $V_1 \cup V_2$ is not an independent set of $\mu(G)$, which is a contradiction.

Case 2: $f(w) = 2$. Again, we have that $f(u_i) = 0$ for every $i \in \{1, \ldots, n\}$. It is easy to verify that the restriction of f to $V(G)$ is an IIDF on G. Hence, $i_I(G) \leq f(V(G))$, and consequently, $i_I(\mu(G)) = \omega(f) = f(V(G)) + 2 \geq i_I(G) + 2$, which contradicts the assumption.

Case 3: $f(w) = 0$. By the definition of f, we have that $f(U) \geq 2$, which leads to $f(V(G)) \leq i_I(G) - 1$. Therefore, the restriction of f to $V(G)$ is not an IIDF on G. Since $(V_1 \cup V_2) \cap V(G)$ is an independent set of G, it follows that $V_0^* \neq \emptyset$, and hence $U^* \neq \emptyset$. Let $v_r \in V_0^*$ be a vertex for which $f(N_G(v_r))$ is maximum. We next distinguish the following two subcases.

Subcase 3.1: $f(N_G(v_r)) = 1$. Since $V_1 \cup V_2$ is an independent set of $\mu(G)$, we have that $f(u_r) = 0$. However, by the definition of f we obtain that $2 \leq f(N_{\mu(G)}(u_r)) = f(N_G(u_r)) + f(w) = f(N_G(v_r))$, a contradiction.

Subcase 3.2: $f(N_G(v_r)) = 0$. By the maximality of $f(N_G(v_r))$, every vertex $v_i \in V_0^*$ satisfies that $f(N_G(v_i)) = 0$. This condition, together with the fact that $f(w) = 0$, implies that $U^* \subseteq V_1 \cup V_2$. We now show that $\Delta(G[V_0^*]) \geq 2$. Observe that $B_G(V_0^*) \subseteq V_0$, and if $v_i \in B_G(V_0^*)$, then $u_i \in V_0$. If $\Delta(G[V_0^*]) = 0$, then there exists a vertex $v_j \in V_0^*$ such that $N_G(v_j) \subseteq B_G(V_0^*) \subseteq V_0$, which implies that $f(N_{\mu(G)}(v_j)) = 0$, a contradiction. Moreover, if $\Delta(G[V_0^*]) = 1$, then there exist two vertices $v_j, v_k \in V_0^*$ such that $v_jv_k \in E(G)$ and $B_G(\{v_j, v_k\}) \subseteq B_G(V_0^*) \subseteq V_0$. As a consequence, $u_j, u_k \in V_2$, which implies that the function $f'(V_0', V_1', V_2')$, defined by $f'(u_k) = 0$, $f'(v_j) = 2$ and $f'(x) = f(x)$ whenever $x \in V(\mu(G)) \setminus \{u_k, v_j\}$, is an $i_I(\mu(G))$-function with $|(V_1' \cup V_2') \cap V(G)| > |(V_1 \cup V_2) \cap V(G)|$, contradicting the choice of f. Therefore, $\Delta(G[V_0^*]) \geq 2$, as required. Let $g(W_0, W_1, W_2)$ be an $i_I(G[V_0^*])$-function. By Proposition 2.3-(i), it follows that $\omega(g) = i_I(G[V_0^*]) \leq |V_0^*| - \Delta(G[V_0^*]) + 1 \leq |V_0^*| - 1$. Define a function $g'(W_0', W_1', W_2')$ on G as follows.

$$g'(v_i) = \begin{cases} g(v_i) & \text{if } v_i \in V_0^*, \\ f(v_i) & \text{if } v_i \in V(G) \setminus V_0^*. \end{cases}$$

Since $B_G(V_0^*) \subseteq V_0$, it is easy to check that $W_1' \cup W_2' = (W_1 \cup W_2) \cup ((V_1 \cup V_2) \cap V(G))$ is an independent set of G. Moreover, every vertex $v_i \in W_0'$ satisfies that

- If $v_i \in W_0' \cap V_0^*$, then $g'(N_G(v_i)) = g(N_G(v_i)) \geq 2$.
- If $v_i \in W_0' \setminus V_0^*$, then $g'(N_G(v_i)) \geq f(N_G(v_i)) \geq 2$.

Hence, g' is an IIDF on G, which implies that

$$|V_0^*| + f(V(G)) + 1 \geq \omega(g) + f(V(G)) + 2 = \omega(g') + 2 \geq i_I(G) + 2. \tag{2.1}$$

If $B_G(V_0^*) \neq \emptyset$, then there exists a vertex $v_s \in (V_1 \cup V_2) \cap V(G)$, which leads to $u_s \in V_1 \cup V_2$. Hence, $f(U) \geq f(U^*) + f(u_s) \geq |V_0^*| + 1$, and by (2.1) we obtain that $i_I(\mu(G)) = \omega(f) = f(U) + f(V(G)) \geq i_I(G) + 2$, a contradiction. Since G is connected, we may henceforth assume that $V(G) = V_0^*$. Hence, $U = U^*$ and consequently, $f(U) \geq |V_0^*|$. If $f(U) \geq |V_0^*| + 1$, then by (2.1) we again obtain that $i_I(\mu(G)) = \omega(f) = f(U) + f(V(G)) \geq i_I(G) + 2$, a contradiction. Finally, if $f(U) = |V_0^*|$, then it is easy to check that $V_1 = U$ and $V_2 = \emptyset$, which implies that $\delta(G) \geq 2$ and $i_I(\mu(G)) = |V_1| = n$. Since G is connected and $G \notin \{C_3, C_5\}$, Proposition 2.3-(ii) implies that $n \geq i_I(G) + 2$, and hence $i_I(\mu(G)) = n \geq i_I(G) + 2$, which is again a contradiction.

Therefore, all possible values of $f(w)$ lead to a contradiction, implying that the assumption $i_I(\mu(G)) \leq i_I(G) + 1$ is false. Hence, $i_I(\mu(G)) \geq i_I(G) + 2$. Consequently, $i_I(\mu(G)) = i_I(G) + 2$, which completes the proof. □

3. Closed formulas for $\gamma_{tR}(\mu(G))$ and $\gamma_{tI}(\mu(G))$

We start this section by establishing a closed formula for $\gamma_{tR}(\mu(G))$ in terms of $\gamma_{tR}(G)$, where G is a connected graph of order at least three.

Theorem 3.1. *Let G be a connected graph of order $n \geq 3$. Then,*

$$\gamma_{tR}(\mu(G)) = \gamma_{tR}(G) + 2.$$

Proof. Since G is a connected graph with $n \geq 3$, there exists a $\gamma_{tR}(G)$-function g such that $g(v_i) = 2$ for some vertex $v_i \in V(G)$. By the definition of g, there exists a vertex $v_j \in N_G(v_i)$ such that $g(v_j) \geq 1$. Now, let $U = \{u_1, \ldots, u_n\}$, and we define a function g' on $\mu(G)$ as follows.

$$g'(x) = \begin{cases} 2 & \text{if } x = w, \\ g(v_j) & \text{if } x = u_j, \\ 0 & \text{if } x \in (U \setminus \{u_j\}) \cup \{v_j\}, \\ g(x) & \text{if } x \in V(G) \setminus \{v_j\}. \end{cases}$$

It is easy to check that g' is a TRDF on $\mu(G)$. Therefore, $\gamma_{tR}(\mu(G)) \leq \omega(g') = \omega(g) + 2 = \gamma_{tR}(G) + 2$.

Finally, we proceed to prove that $\gamma_{tR}(\mu(G)) \geq \gamma_{tR}(G) + 2$. For this purpose, let us consider a $\gamma_{tR}(\mu(G))$-function $f(V_0, V_1, V_2)$. Next, let us consider the following three complementary cases.

Case 1: $f(w) = 2$. In this case, it is straightforward to verify that the function f' defined on G by $f'(v_i) = \min\{f(v_i) + f(u_i), 2\}$ for every $i \in \{1, \ldots, n\}$, is a TRDF on G. Consequently, $\gamma_{tR}(G) + 2 \leq \omega(f') + f(w) \leq \omega(f) = \gamma_{tR}(\mu(G))$, as desired.

Case 2: $f(w) = 0$. In this case, and without loss of generality, assume that $f(u_1) = 2$. Let f' be the function defined on G as follows.

$$f'(v_i) = \begin{cases} f(v_i) & \text{if } i = 1, \\ \min\{f(v_i) + f(u_i), 2\} & \text{if } i > 1. \end{cases}$$

It is easy to check that f' is a TRDF on G. Hence, $\gamma_{tR}(G) + 2 \leq \omega(f') + f(u_1) \leq \omega(f) = \gamma_{tR}(\mu(G))$, as desired.

Case 3: $f(w) = 1$. In this case, and without loss of generality, assume that $f(u_1) + f(v_1) \geq f(u_i) + f(v_i)$ for every $i \in \{2, \ldots, n\}$. By the definition of f, it follows that $f(u_1) \in \{1, 2\}$. Now, we consider the following two complementary subcases.

Subcase 3.1: $f(u_1) = 2$. Let $f'(V_0', V_1', V_2')$ be the function defined on G as follows.

$$f'(v_i) = \begin{cases} \min\{f(v_i) + 1, 2\} & \text{if } i = 1, \\ \min\{f(v_i) + f(u_i), 2\} & \text{if } i > 1. \end{cases}$$

By the definition of f', every vertex $v_i \in V_0'$ satisfies that $u_i \in V_0$. Since $w \in V_1$, there exists a vertex $v_j \in N_G(v_i) \cap V_2$, which implies that $v_j \in N_G(v_i) \cap V_2'$. Moreover, it is easy to check that $V_1' \cup V_2'$ is a TDS of G. Consequently, f' is a TRDF on G. Hence, $\gamma_{tR}(G)+2 \leq \omega(f')+f(w)+f(u_1)-1 \leq \omega(f) = \gamma_{tR}(\mu(G))$, as desired.

Subcase 3.2: $f(u_1) = 1$. Let $f''(V_0'', V_1'', V_2'')$ be the function defined on G as follows.

$$f''(v_i) = \begin{cases} f(v_i) & \text{if } i = 1, \\ \min\{f(v_i) + f(u_i), 2\} & \text{if } i > 1. \end{cases}$$

By the definition of f'' and the fact that $U \cap V_2 = \emptyset$, it follows that every vertex $v_i \in V_0''$ satisfies that $N_G(v_i) \cap V_2'' \neq \emptyset$. Moreover, and by the assumption that $f(u_1) + f(v_1) \geq f(u_i) + f(v_i)$ for every $i \in \{2, \ldots, n\}$, it is easy to check that $V_1'' \cup V_2''$ is a TDS of G. Consequently, f'' is a TRDF on G. Hence, $\gamma_{tR}(G) + 2 \leq \omega(f'') + f(u_1) + f(w) \leq \omega(f) = \gamma_{tR}(\mu(G))$, as desired.

The above cases imply that $\gamma_{tR}(\mu(G)) \geq \gamma_{tR}(G) + 2$, as desired. Therefore, $\gamma_{tR}(\mu(G)) = \gamma_{tR}(G) + 2$, which completes the proof. □

We now derive a closed formula for the total Italian domination number of the Mycielskian graph of a nontrivial connected graph G, expressed in terms of the corresponding parameter of G.

Proposition 3.2. *For any connected graph G of order $n \geq 3$,*

$$\gamma_{tI}(\mu(G)) \in \{\gamma_{tI}(G) + 1, \gamma_{tI}(G) + 2\}.$$

Proof. Let g be a $\gamma_{tI}(G)$-function. Now, let $U = \{u_1, \ldots, u_n\}$, and we define a function g' on $\mu(G)$ as follows.

$$g'(x) = \begin{cases} 1 & \text{if } x \in \{w, u_1\}, \\ 0 & \text{if } x \in U \setminus \{u_1\}, \\ g(x) & \text{if } x \in V(G). \end{cases}$$

Since g is a TIDF on G, it is easy to check that g' is a TIDF on $\mu(G)$. Therefore, $\gamma_{tI}(\mu(G)) \leq \omega(g') = \omega(g) + 2 = \gamma_{tI}(G) + 2$.

Finally, we proceed to prove that $\gamma_{tI}(\mu(G)) \geq \gamma_{tI}(G) + 1$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{tI}(\mu(G))$-function. Since $f(N_{\mu(G)}(w)) \geq 1$, we have that $f(U) \geq 1$. Without loss of generality, assume that $u_1 \in V_1 \cup V_2$. Next, we consider the following two complementary cases.

Case 1: $f(w) \geq 1$. In this case, we consider the function $f'(V'_0, V'_1, V'_2)$ defined on G by $f'(v_i) = \min\{f(v_i) + f(u_i), 2\}$ for each $i \in \{1, \ldots, n\}$. By the definition of f', every vertex $v_i \in V'_0$ satisfies that $v_i \in V_0$. Hence, $f(N_{\mu(G)}(v_i)) \geq 2$, which implies that $f'(N_G(v_i)) \geq 2$. Moreover, every vertex $v_i \in V'_1 \cup V'_2$ satisfies that $f(N_{\mu(G)}(v_i)) \geq 1$, which implies that $f'(N_G(v_i)) \geq 1$. Hence, f' is a TIDF on G. Consequently, $\gamma_{tI}(G) + 1 \leq \omega(f') + f(w) \leq \omega(f) = \gamma_{tI}(\mu(G))$, as desired.

Case 2: $f(w) = 0$. Let $f''(V''_0, V''_1, V''_2)$ be the function defined on G as follows.

$$f''(v_i) = \begin{cases} \min\{f(v_i) + f(u_i) - 1, 2\} & \text{if } i = 1, \\ \min\{f(v_i) + f(u_i), 2\} & \text{if } i > 1. \end{cases}$$

By the definition of f'', every vertex $v_i \in V''_0$ satisfies that $\{u_i, v_i\} \cap V_0 \neq \emptyset$. Since $f(w) = 0$, it follows that $f(N_{\mu(G)}(u_i) \setminus \{w\}) \geq 2$, and hence $f''(N_G(v_i)) \geq 2$. Moreover, since $f(w) = 0$, every vertex $v_i \in V''_1 \cup V''_2$ satisfies that $f(N_{\mu(G)}(u_i) \setminus \{w\}) \geq 1$, which in turn implies that $f''(N_G(v_i)) \geq 1$. Hence, f'' is a TIDF on G with weight $\omega(f'') \leq f(V(G)) + f(U) - 1 = \omega(f) - 1$. Hence, $\gamma_{tI}(G) + 1 \leq \omega(f'') + 1 \leq \omega(f) = \gamma_{tI}(\mu(G))$, as desired.

The above cases imply that $\gamma_{tI}(\mu(G)) \geq \gamma_{tI}(G) + 1$, which completes the proof. □

By Proposition 3.2, we have that $\gamma_{tI}(\mu(G)) = \gamma_{tI}(G) + 1$ or $\gamma_{tI}(\mu(G)) = \gamma_{tI}(G) + 2$. We next characterize the graphs G for which each of these cases occurs. For any integer $k \geq 3$, let $\mathcal{G}_k$ denote the family of connected graphs G of order k for which there exists a $\gamma_{tI}(G)$-function $f(V_0, V_1, V_2)$ satisfying one of the conditions:

(C1) $V_2 \neq \emptyset$.

(C2) $V_2 = \emptyset$ and there exists a vertex $v \in V_1$ with $f(N_G(v)) \geq 2$ such that $V_1 \setminus \{v\}$ is a TDS of G.

Theorem 3.3. *Let G be a connected graph of order $n \geq 3$. Then,*

$$\gamma_{tI}(\mu(G)) = \begin{cases} \gamma_{tI}(G) + 1 & \text{if } G \in \mathcal{G}_n, \\ \gamma_{tI}(G) + 2 & \text{otherwise.} \end{cases}$$

Proof. By Proposition 3.2, it follows that $\gamma_{tI}(\mu(G)) \in \{\gamma_{tI}(G) + 1, \gamma_{tI}(G) + 2\}$. Hence, it suffices to show that $\gamma_{tI}(\mu(G)) = \gamma_{tI}(G) + 1$ if, and only if, $G \in \mathcal{G}_n$. First, we assume that $G \in \mathcal{G}_n$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{tI}(G)$-function satisfying either condition (C1) or (C2). Next, we analyze each of these two scenarios.

Case A1: Condition (C1) holds. Without loss of generality, assume that $v_1 \in V_2$. Now, let $U = \{u_1, \dots, u_n\}$, and we define a function f' on $\mu(G)$ as follows.

$$f'(x) = \begin{cases} 1 & \text{if } x \in \{w, u_1, v_1\}, \\ 0 & \text{if } x \in U \setminus \{u_1\}, \\ f(x) & \text{if } x \in V(G) \setminus \{v_1\}. \end{cases}$$

It is easy to check that f' is a TIDF on $\mu(G)$. Hence, $\gamma_{tI}(\mu(G)) \leq \omega(f') = \omega(f) + 1 = \gamma_{tI}(G) + 1$, and by Proposition 3.2, it follows that $\gamma_{tI}(\mu(G)) = \gamma_{tI}(G) + 1$.

Case A2: Condition (C2) holds. In this case, $V_2 = \emptyset$, and without loss of generality, assume that $v_1 \in V_1$ satisfies that $f(N_G(v_1)) \geq 2$ and $V_1 \setminus \{v_1\}$ is a TDS of G. Now, let $U = \{u_1, \dots, u_n\}$, and we define a function f'' on $\mu(G)$ as follows.

$$f''(x) = \begin{cases} 1 & \text{if } x \in \{w, u_1\}, \\ 0 & \text{if } x \in (U \setminus \{u_1\}) \cup \{v_1\}, \\ f(x) & \text{if } x \in V(G) \setminus \{v_1\}. \end{cases}$$

It is easy to see that f'' is a TIDF on $\mu(G)$. Consequently, $\gamma_{tI}(\mu(G)) \leq \omega(f'') = \omega(f) + 1 = \gamma_{tI}(G) + 1$. Furthermore, by Proposition 3.2, we obtain that $\gamma_{tI}(\mu(G)) = \gamma_{tI}(G) + 1$.

The above cases imply that if $G \in \mathcal{G}_n$, then $\gamma_{tI}(\mu(G)) = \gamma_{tI}(G) + 1$, as desired.

Conversely, assume that G satisfies that $\gamma_{tI}(\mu(G)) = \gamma_{tI}(G) + 1$. Let $g(W_0, W_1, W_2)$ be a $\gamma_{tI}(\mu(G))$-function. Since $g(N_{\mu(G)}(w)) \geq 1$, we have that $g(U) \geq 1$. Without loss of generality, assume that $(W_1 \cup W_2) \cap U = \{u_1, \dots, u_r\}$ and that $g(u_1) + g(v_1) \leq g(u_i) + g(v_i)$ for every $i \in \{1, \dots, r\}$. Now, we consider the function $g'(W_0', W_1', W_2')$ defined on G by $g'(v_i) = \min\{f(v_i) + f(u_i), 2\}$ for every $i \in \{1, \dots, n\}$. By the definition of g', every vertex $v_i \in W_0'$ satisfies that $v_i \in W_0$. Hence, $g'(N_G(v_i)) \geq g(N_{\mu(G)}(v_i)) \geq 2$. Moreover, every vertex $v_i \in W_1' \cup W_2'$ satisfies that $g'(N_G(v_i)) \geq g(N_{\mu(G)}(v_i)) \geq 1$. Hence, g' is a TIDF on G. As a consequence,

$$\gamma_{tI}(G) + g(w) \leq \omega(g') + g(w) \leq \omega(g) = \gamma_{tI}(\mu(G)) = \gamma_{tI}(G) + 1, \tag{3.1}$$

which implies that $g(w) \in \{0, 1\}$. Next, we analyze the following two complementary cases.

Case B1: $g(w) = 1$. In this case, all inequalities in (3.1) hold with equality. As a consequence, it follows that $\omega(g') = \gamma_{tI}(G)$ and, thus, g' is a $\gamma_{tI}(G)$-function. If $W_2' \neq \emptyset$, then condition (C1) holds and, consequently, $G \in \mathcal{G}_n$. Henceforth, we assume that $W_2' = \emptyset$. By the definition of g', it follows that $g(u_i) + g(v_i) \leq 1$ for every $i \in \{1, \dots, n\}$, which leads to $W_2 = \emptyset$. Hence, $g(u_1) = 1$ and $g(v_1) = 0$, and thus $v_1 \in W_1'$. In addition, we have that $g'(N_G(v_1)) \geq g(N_{\mu(G)}(v_1)) \geq 2$. We claim that $W_1' \setminus \{v_1\}$ is a TDS of G. By the definition of g', it follows that $N_G(v_i) \cap (W_1' \setminus \{v_1\}) \neq \emptyset$ for every vertex $v_i \in V(G) \setminus (N_G(v_1) \cap W_1')$. Now, let $v_i \in N_G(v_1) \cap W_1'$. Observe that $\{u_i, v_i\} \cap W_0 \neq \emptyset$. If $x \in \{u_i, v_i\} \cap W_0$, then $g(N_{\mu(G)}(x) \cap \{v_1, u_1, w\}) = 1$. Hence, $g(N_{\mu(G)}(x) \setminus \{v_1, u_1, w\}) \geq 1$, which implies that $N_G(v_i) \cap (W_1' \setminus \{v_1\}) \neq \emptyset$. Therefore, $W_1' \setminus \{v_1\}$ is a TDS of G, as desired. Thus, condition (C2) holds and, consequently, $G \in \mathcal{G}_n$.

Case B2: $g(w) = 0$. In this case, let $g''(W_0'', W_1'', W_2'')$ be the function defined on G as follows.

$$g''(v_i) = \begin{cases} \min\{g(v_i) + g(u_i) - 1, 2\} & \text{if } i = 1, \\ \min\{g(v_i) + g(u_i), 2\} & \text{if } i > 1. \end{cases}$$

By the definition of g'', every vertex $v_i \in W_0''$ satisfies that $u_i \in W_0$. Since $g(w) = 0$, it follows that $g''(N_G(v_i)) \geq g(N_{\mu(G)}(u_i) \setminus \{w\}) \geq 2$. Moreover, since $g(w) = 0$, every vertex $v_i \in W_1'' \cup W_2''$ satisfies that $g''(N_G(v_i)) \geq g(N_{\mu(G)}(u_i) \setminus \{w\}) \geq 1$. Hence, g'' is a TIDF on G, which implies that

$$\gamma_{tI}(G) \leq \omega(g'') \leq g(V(G)) + g(U) - 1 = \omega(g) - 1 = \gamma_{tI}(\mu(G)) - 1 = \gamma_{tI}(G).$$

Consequently, $\omega(g'') = \gamma_{tI}(G)$, and hence g'' is a $\gamma_{tI}(G)$-function. If $W_2'' \neq \emptyset$, then condition (C1) holds and, therefore, $G \in \mathcal{G}_n$. Henceforth, we assume that $W_2'' = \emptyset$. By the definition of g'' and the fact that $g(u_1) \geq 1$, it follows that $g(u_1) + g(v_1) \in \{1, 2\}$ and $g(u_i) + g(v_i) \leq 1$ for every $i \in \{2, \ldots, n\}$. Therefore, $(V(\mu(G)) \setminus \{u_1\}) \cap W_2 = \emptyset$. Next, we consider the following two complementary subcases.

Subcase B2.1: $g(u_1) + g(v_1) = 1$. In this subcase, $g(u_1) = 1$ and $g(v_1) = 0$. Since $g(w) = 0$, there exists a vertex $v_r \in N_{\mu(G)}(u_1)$ such that $g(v_r) \geq 1$. Since $W_2'' = \emptyset$, it follows that $g(v_r) = 1$ and $g(u_r) = 0$. Hence, $v_r \in W_1''$. In addition, we have that $g(N_{\mu(G)}(u_r) \setminus \{w, v_1\}) \geq 2$. This implies that $g''(N_G(v_r)) \geq 2$. By an argument analogous to that of Case B1, we conclude that $W_1'' \setminus \{v_r\}$ is a TDS of G. Therefore, condition (C2) holds and, consequently, $G \in \mathcal{G}_n$.

Subcase B2.2: $g(u_1) + g(v_1) = 2$. If $g(u_1) = g(v_1) = 1$, then since $g(w) = 0$, there exists a vertex u_r such that $g(u_r) = 1$. By the minimality of $g(u_1) + g(v_1)$, it follows that $g(v_r) = 1$, which implies that $g''(v_r) = 2$, a contradiction. Therefore, $g(u_1) = 2$ and $g(v_1) = 0$. Hence, $v_1 \in W_1''$. In addition, we have that $g''(N_G(v_1)) \geq g(N_{\mu(G)}(v_1)) \geq 2$. We claim that $W_1'' \setminus \{v_1\}$ is a TDS of G. By the definition of g'', it follows that $N_G(v_i) \cap (W_1'' \setminus \{v_1\}) \neq \emptyset$ for every vertex $v_i \in V(G) \setminus (N_G(v_1) \cap W_1'')$. Now, let $v_i \in N_G(v_1) \cap W_1''$. Since $g(N_{\mu(G)}(u_i) \cap \{v_1, w\}) \geq 1$, it follows that $N_G(v_i) \cap (W_1'' \setminus \{v_1\}) \neq \emptyset$. Therefore, $W_1'' \setminus \{v_1\}$ is a TDS of G, as desired. Thus, condition (C2) holds and, consequently, $G \in \mathcal{G}_n$.

The above cases imply that if $\gamma_{tI}(\mu(G)) = \gamma_{tI}(G) + 1$, then $G \in \mathcal{G}_n$, which completes the proof. □

4. Closed formula for $\gamma_I(\mu(G))$

As shown previously, it was established in [18] that the Italian domination number of a Mycielskian graph admits two possible values. Since the graphs attaining each of these values have not yet been characterized, it is natural to pursue such a characterization. In this direction, we provide a complete characterization in Theorem 4.1.

To this end, we first introduce a well-known domination parameter that is closely related to the Italian domination number of a graph. A function $f : V(G) \to \{0, 1, 2\}$ is called a $\{2\}$-dominating function ($\{2\}$DF) on G if $\sum_{x \in N_G[v]} f(x) \geq 2$ for every vertex $v \in V(G)$. The $\{2\}$-domination number of G, denoted by $\gamma_{\{2\}}(G)$, is the minimum weight $\omega(f) = \sum_{x \in V(G)} f(x)$ among all $\{2\}$DFs f on G. Observe that every $\{2\}$DF on G is also an IDF on G. Consequently, it follows that $\gamma_I(G) \leq \gamma_{\{2\}}(G)$. Recent results on this parameter can be found, for example, in [2, 4, 5].

Theorem 4.1. *For any graph G,*

$$\gamma_I(\mu(G)) = \begin{cases} \gamma_I(G) + 1 & \text{if } \gamma_I(G) = \gamma_{\{2\}}(G), \\ \gamma_I(G) + 2 & \text{otherwise.} \end{cases}$$

Proof. As shown in [18], we have that $\gamma_I(\mu(G)) \in \{\gamma_I(G) + 1, \gamma_I(G) + 2\}$. Hence, it suffices to show that $\gamma_I(\mu(G)) = \gamma_I(G) + 1$ if, and only if, $\gamma_I(G) = \gamma_{\{2\}}(G)$. First, we assume that $\gamma_I(G) = \gamma_{\{2\}}(G)$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{\{2\}}(G)$-function. Next, we analyze the following two complementary cases.

Case A1: $V_2 = \emptyset$. In this case, V_1 is a TDS of G and $|N_G(v_i) \cap V_1| \geq 2$ for every $v_i \in V_0$. Now, let $U = \{u_1, \ldots, u_n\}$ and let f' be the function defined on $\mu(G)$ as follows.

$$f'(x) = \begin{cases} 1 & \text{if } x = w, \\ 0 & \text{if } x \in U, \\ f(x) & \text{if } x \in V(G). \end{cases}$$

It is straightforward to verify that f' is an IDF on $\mu(G)$. Hence, $\gamma_I(G) + 1 \leq \gamma_I(\mu(G)) \leq \omega(f') = \omega(f) + 1 = \gamma_{\{2\}}(G) + 1 = \gamma_I(G) + 1$. As a consequence, $\gamma_I(\mu(G)) = \gamma_I(G) + 1$.

Case A2: $V_2 \neq \emptyset$. Let $U = \{u_1, \ldots, u_n\}$ and $U_2 = \{u_i \in U : v_i \in V_2\}$. Now, we define a function $f''(V_0'', V_1'', V_2'')$ on $\mu(G)$ as follows.

$$f''(x) = \begin{cases} 1 & \text{if } x \in V_1 \cup V_2 \cup U_2 \cup \{w\}, \\ 0 & \text{if } x \in V_0 \cup (U \setminus U_2). \end{cases}$$

Observe that $V_2'' = \emptyset$ and $V_0'' \subseteq V(G) \cup U$. If $v_i \in V_0''$, then it is straightforward to verify that $f''(N_{\mu(G)}(v_i)) \geq 2$. Moreover, if $u_i \in V_0''$, then by the definition of f'' we have that $f(v_i) \in \{0, 1\}$. In any case, $f(N_G(v_i)) \geq 1$, which implies that $f''(N_G(v_i)) \geq 1$. Consequently, $f''(N_{\mu(G)}(u_i)) = f''(N_G(v_i)) + f''(w) \geq 2$. Therefore, f'' is an IDF on $\mu(G)$. Hence, $\gamma_I(G) + 1 \leq \gamma_I(\mu(G)) \leq \omega(f'') = \omega(f) + 1 = \gamma_{\{2\}}(G) + 1 = \gamma_I(G) + 1$. As a consequence, $\gamma_I(\mu(G)) = \gamma_I(G) + 1$.

The above cases imply that if $\gamma_I(G) = \gamma_{\{2\}}(G)$, then $\gamma_I(\mu(G)) = \gamma_I(G) + 1$, as desired.

Conversely, assume that G satisfies $\gamma_I(\mu(G)) = \gamma_I(G) + 1$. Let $g(W_0, W_1, W_2)$ be a $\gamma_I(\mu(G))$-function. Next, we analyze the following complementary cases.

Case B1: $g(w) \geq 1$. In this case, we deduce that the function $g'(W_0', W_1', W_2')$, defined on G by $g'(v_i) = \min\{g(v_i) + g(u_i), 2\}$ for every $i \in \{1, \ldots, n\}$, is an IDF on G. Hence,

$$\gamma_I(G) + g(w) \leq \omega(g') + g(w) \leq \omega(g) = \gamma_I(\mu(G)) = \gamma_I(G) + 1, \tag{4.1}$$

which implies that $g(w) = 1$. As a consequence, all inequalities in (4.1) hold with equality. In particular, $\omega(g') = \gamma_I(G)$, which implies that g' is a $\gamma_I(G)$-function. We claim that g' is also a {2}DF on G. By definition, every vertex $v_i \in W_0' \cup W_2'$ satisfies that $g'(N_G[v_i]) \geq 2$. Now, let $v_i \in W_1'$. By definition of g', it follows that $\{v_i, u_i\} \cap W_0 \neq \emptyset$, which implies that $g'(N_G[v_i]) = g'(N_G(v_i)) + g'(v_i) \geq g(N_G(v_i)) + 1 \geq 2$. Therefore, g' is also a {2}DF on G, as desired. Consequently, $\gamma_I(G) \leq \gamma_{\{2\}}(G) \leq \omega(g') = \gamma_I(G)$, and hence $\gamma_{\{2\}}(G) = \gamma_I(G)$.

Case B2: $g(w) = 0$. Let $U = \{u_1, \ldots, u_n\}$. By definition of g, it follows that $g(U) = g(N_{\mu(G)}(w)) \geq 2$. Without loss of generality, assume that $(W_1 \cup W_2) \cap U = \{u_1, \ldots, u_r\}$ and that $g(u_1) \geq g(u_i)$ for every $i \in \{1, \ldots, r\}$. Now, let $g''(W_0'', W_1'', W_2'')$ be a function defined on G as follows.

$$g''(v_i) = \begin{cases} g(v_i) & \text{if } i = 1, \\ \min\{g(v_i) + g(u_i), 2\} & \text{if } i > 1. \end{cases}$$

Let $v_i \in W_0''$. If $i = 1$, then $g(v_1) = 0$, which implies that $g''(N_G[v_1]) = g''(N_G(v_1)) \geq g(N_{\mu(G)}(v_1)) \geq 2$. Moreover, if $i > 1$, then $g(u_i) = 0$, and consequently, $g''(N_G[v_i]) = g''(N_{\mu(G)}(u_i)) \geq 2$. Therefore, g'' is an IDF on G. Hence,

$$\gamma_I(G) + g(u_1) \leq \omega(g'') + g(u_1) \leq \omega(g) = \gamma_I(\mu(G)) = \gamma_I(G) + 1. \tag{4.2}$$

Since $g(u_1) \geq 1$, it follows that $g(u_1) = 1$, and by the maximality of $g(u_1)$, it follows that $r \geq 2$ and $g(u_i) = 1$ for every $i \in \{1, \ldots, r\}$. In addition, all inequalities in (4.2) hold with equality. In particular, $\omega(g'') = \gamma_I(G)$, which implies that g'' is a $\gamma_I(G)$-function.

Now, suppose that there exists two subscripts $k, l \in \{1, \ldots, r\}$ such that $f(v_k) = f(v_l) = 1$. Let g''' be the function defined on G as follows.

$$g'''(v_i) = \begin{cases} g(v_i) & \text{if } i \in \{l, k\}, \\ \min\{g(v_i) + g(u_i), 2\} & \text{if } i \in \{1, \ldots, n\} \setminus \{l, k\}. \end{cases}$$

It is easy to check that g''' is an IDF on G. Hence, $\gamma_I(G) + 2 \leq \omega(g''') + f(u_l) + f(u_k) \leq \omega(g) = \gamma_I(\mu(G))$, a contradiction. Therefore, without loss of generality, we may assume that $g(v_1) \in \{0, 2\}$. We claim that g'' is also a $\{2\}$DF on G. By definition, every vertex $v_i \in W_0'' \cup W_2''$ satisfies that $g''(N_G[v_i]) \geq 2$. Now, let $v_i \in W_1''$. By definition of g'', it follows that $\{v_i, u_i\} \cap W_0 \neq \emptyset$, which implies that $g''(N_G[v_i]) = g''(N_G(v_i)) + g''(v_i) \geq g(N_G(v_i)) + 1 \geq 2$. Therefore, g'' is also a $\{2\}$DF on G, as desired. Consequently, $\gamma_I(G) \leq \gamma_{\{2\}}(G) \leq \omega(g'') = \gamma_I(G)$, and hence $\gamma_{\{2\}}(G) = \gamma_I(G)$.

The above cases imply that if $\gamma_I(\mu(G)) = \gamma_I(G) + 1$, then $\gamma_I(G) = \gamma_{\{2\}}(G)$. Therefore, the proof is complete. □

5. Conclusions

In this paper, we derived closed formulas for the independent and total variants of Roman and Italian domination numbers of the Mycielskian graph $\mu(G)$ in terms of the corresponding parameters of G. In addition, we characterized the graphs G for which $\gamma_I(\mu(G)) = \gamma_I(G) + 1$ and $\gamma_I(\mu(G)) = \gamma_I(G) + 2$, thereby completing the study of the Italian domination number in Mycielskian graphs.

We believe that the results presented here open several avenues for further research. In particular, it would be interesting to investigate the behavior of other domination parameters on Mycielskian graphs, as well as on Splitting graphs, which are well-known subgraphs of Mycielskian graphs.

Author contributions

The results presented in this paper were obtained as a result of collective work sessions involving all authors. The process was organized and led by A.C.M. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there is no conflict of interest.

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