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Research article

Multi-order fractional optimal control systems: existence and optimality conditions

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Abstract: We study an optimal control problem for dynamical systems with multiorder (multiindex) fractional dynamics. In this setting, different components of the state vector evolve with different fractional orders, meaning that each state variable is governed by its own memory intensity. This allows the model to describe systems where different physical quantities exhibit different rates of memory and hereditary effects. Such a framework is motivated by applications in which a single uniform memory law is not realistic. Examples include heterogeneous diffusion processes, viscoelastic materials with multiple relaxation behaviors, and coupled dynamical systems with component-dependent memory effects. These situations cannot be adequately captured by classical integer-order or standard single-order fractional models. Using Caputo fractional derivatives, we formulate a general class of multi-order fractional optimal control systems with quadratic cost functionals. The main contributions of this work include the derivation of necessary optimality conditions via a fractional Pontryagin Maximum Principle adapted to the multiorder setting, and the resulting coupled state-adjoint system with heterogeneous fractional dynamics. In addition, we obtain an explicit optimal control characterization in the linear-quadratic case, extending classical fractional optimal control results to the multiorder framework. The proposed results generalize existing single-order fractional control theory and provide a more flexible mathematical tool for modeling and analyzing systems with multiple interacting memory effects. This may open new directions for both analytical and numerical studies of complex fractional dynamical systems.

Keywords: fractional calculus; fractional optimal control; fractional dynamical systems
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1. Introduction

Fractional optimal control theory has emerged as an important extension of classical optimal control, combining the tools of fractional calculus with optimization and dynamical systems theory.

In contrast to integer-order derivatives, fractional derivatives are nonlocal operators, meaning that the present state of a system depends not only on its current configuration but also on its entire past history. This memory property makes fractional models particularly effective for describing complex phenomena involving hereditary effects, anomalous diffusion, viscoelasticity, biological memory processes, and financial dynamics [1, 6].

The mathematical foundations of fractional calculus date back to the pioneering works of Liouville and Riemann, and were later significantly developed through the Caputo fractional derivative, which is especially suitable for physical applications because it allows the use of classical initial conditions. At the same time, optimal control theory was profoundly influenced by Pontryagin's maximum principle, which provides necessary conditions for optimality in dynamical systems. The interaction between these two theories naturally leads to fractional optimal control problems, where the governing dynamics are described by fractional differential equations.

Let $0 < \alpha < 1$ and $T > 0$. A classical fractional optimal control problem consists of minimizing the functional

$$J(\mathcal{U}) = \int_0^T L(t, \psi(t), \mathcal{U}(t))\, dt, \tag{1.1}$$

subject to the fractional state equation

$${}^C D_t^\alpha \psi(t) = f(t, \psi(t), \mathcal{U}(t)), \qquad t \in (0, T), \tag{1.2}$$

with initial condition

$$\psi(0) = \psi_0. \tag{1.3}$$

Here ${}^C D_t^\alpha$ denotes the Caputo fractional derivative [8, 12, 14], defined by

$${}^C D_t^\alpha \psi(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\psi'(s)}{(t-s)^\alpha}\, ds, \tag{1.4}$$

which explicitly reveals the nonlocal memory structure of the operator.

In recent years, increasing attention has been devoted to systems involving multiorder or *multi-index* fractional dynamics. In such systems, different state variables evolve according to different fractional orders, allowing each component to possess its own memory intensity [9]. This framework is particularly relevant in applications where distinct physical or biological processes exhibit different hereditary behaviors. For example, viscoelastic materials may display direction-dependent relaxation effects, whereas biological systems often involve interacting subsystems with different memory characteristics and response times. Similarly, heterogeneous diffusion processes and coupled engineering systems may require different fractional orders to accurately capture the underlying dynamics [7, 9, 15].

Despite the substantial literature devoted to single order fractional optimal control problems, the study of multiorder fractional control systems remains comparatively limited. Most existing results assume that all state variables evolve under the same fractional order, which may be insufficient for modeling heterogeneous memory effects. Consequently, there is a need to develop a more general optimal control framework capable of handling componentwise fractional dynamics with distinct derivative orders.

The analysis of fractional optimal control problems relies on several mathematical tools. Fractional integral operators [8, 12, 13] such as

$$I_t^{\alpha}\psi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\psi(s)\, ds, \tag{1.5}$$

play a central role in establishing existence and regularity properties. In addition, fractional integration by parts formulas are essential in deriving necessary optimality conditions, as they allow the transfer of fractional derivatives from the state variables to the adjoint variables.

The functional analytic framework typically involves Hilbert and Banach spaces such as $L^2(0,T)$ and fractional Sobolev spaces $H^{\alpha}(0,T)$ [3]. The existence of optimal controls is commonly established through the direct method of the calculus of variations, using weak compactness arguments, convexity assumptions, and lower semicontinuity properties of the cost functional [2].

A fundamental result in the theory is the fractional Pontryagin maximum principle [5, 10]. Defining the Hamiltonian

$$H(t,\psi,\mathcal{U},p) = \mathcal{L}(t,\psi,\mathcal{U}) + p(t)f(t,\psi,\mathcal{U}), \tag{1.6}$$

the necessary optimality conditions typically consist of the state equation, a backward fractional adjoint equation

$${}_tD_T^{\alpha}p(t) = -\frac{\partial H}{\partial \psi}(t,\psi,\mathcal{U},p), \tag{1.7}$$

together with the stationarity condition

$$\frac{\partial H}{\partial \mathcal{U}}(t,\psi,\mathcal{U},p) = 0. \tag{1.8}$$

The presence of right-sided fractional derivatives in the adjoint equation considerably increases the analytical complexity compared with the classical integer-order setting.

Motivated by these developments, the purpose of the present paper is to establish a Pontryagin-type optimality framework for multi-index fractional dynamical systems governed by componentwise Caputo derivatives of different orders. The proposed formulation generalizes both classical optimal control theory and standard single-order fractional models. The main contributions include the derivation of necessary optimality conditions for multi order systems and the analysis of the associated coupled state-adjoint dynamics in the linear-quadratic setting.

The obtained results provide a theoretical foundation for future analytical investigations, numerical approximation schemes, and applications involving heterogeneous memory effects in complex dynamical systems.

2. Multi-index fractional optimal control problem (MIF-OCP)

A multi-index fractional optimal control problem is an optimal control problem (MIF-OCP) in which the system dynamics involve several fractional derivatives of possibly different orders. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n), 0 < \alpha_i \leq 1$, be a multi-index of fractional orders. We consider the minimization of the cost functional

$$J(\mathcal{U}) = \int_0^T \left(\psi(t)^{\top}\mathcal{Q}\psi(t) + \mathcal{U}(t)^{\top}\mathcal{R}\mathcal{U}(t)\right) dt, \tag{2.1}$$

subject to the multi-index fractional dynamical system

$$\begin{cases} {}^C D_t^{\alpha_i} \psi_i(t) = (\mathcal{A}\psi(t) + \mathcal{B}\mathcal{U}(t))_i, \quad i = 1, \dots, n, \\ \psi(0) = \psi_0 \in \mathbb{R}^n, \end{cases} \tag{2.2}$$

where ${}^C D_t^{\alpha_i}$ denotes the Caputo fractional derivative of order α_i, $\mathcal{A} \in \mathbb{R}^{n\times n}$, $\mathcal{B} \in \mathbb{R}^{n\times m}$, and $Q \geq 0, \mathcal{R} \succ 0$ are symmetric matrices. The above system describes a multi-index (multi order) fractional dynamical system, where each state component $\psi_i(t)$ evolves according to a Caputo fractional derivative of possibly different order $\alpha_i \in (0, 1]$, $i = 1, \dots, n$.

In particular, the operator ${}^C D_t^{\alpha_i}$ acts componentwise on $\psi_i(t)$, meaning that the memory effect of each state variable may differ from the others. This leads to heterogeneous fractional dynamics across the state vector.

The system can be equivalently written in vector form as

$${}^C D_t^{\alpha} \psi(t) = \mathcal{A}(t)\psi(t) + \mathcal{B}U(t),$$

where ${}^C D_t^{\alpha} := \text{diag}({}^C D_t^{\alpha_1}, \dots, {}^C D_t^{\alpha_n})$ is a diagonal fractional derivative operator.

The Hamiltonian associated with problem (2.2) is defined by

$$H(\psi, \mathcal{U}, \lambda) = \psi^\top Q\psi + \mathcal{U}^\top \mathcal{R}\mathcal{U} + \lambda^\top(\mathcal{A}\psi + \mathcal{B}\mathcal{U}), \tag{2.3}$$

where $\lambda(t) \in \mathbb{R}^n$ is the adjoint variable.

For notational convenience, we introduce the diagonal fractional operator

$$\mathcal{D}_t^{\alpha} = \text{diag}({}^C D_t^{\alpha_1}, \dots, {}^C D_t^{\alpha_n}), \tag{2.4}$$

which allows us to write the multidimensional fractional state Eq (2.2) in a compact matrix form:

$$\mathcal{D}_t^{\alpha} \psi(t) = \mathcal{A}\psi(t) + \mathcal{B}\mathcal{U}(t). \tag{2.5}$$

This formulation is particularly useful when dealing with systems in which each state component may have a different fractional order α_i. By representing the fractional derivatives in a diagonal operator, we can handle the entire system simultaneously, which simplifies both theoretical analysis and numerical computations. Moreover, this compact notation aligns naturally with the Hamiltonian and adjoint formulations in fractional optimal control problems, enabling a more elegant derivation of the fractional Pontryagin maximum principle.

3. Properties of the diagonal fractional operator

Let $\mathcal{D}_t^{\alpha} = \text{diag}({}^C D_t^{\alpha_1}, \dots, {}^C D_t^{\alpha_n})$ be the diagonal fractional operator associated with a multi-dimensional state vector $x(t) \in \mathbb{R}^n$. The following properties are useful for analysis of fractional systems:

(1) Linearity. For any vectors $\psi(t), \phi(t) \in \mathbb{R}^n$ and scalars $c_1, c_2 \in \mathbb{R}$,

$$\mathcal{D}_t^{\alpha}(c_1\psi(t) + c_2\phi(t)) = c_1\mathcal{D}_t^{\alpha}\psi(t) + c_2\mathcal{D}_t^{\alpha}\phi(t). \tag{3.1}$$

(2) Decoupling of components. Because $\mathcal{D}_t^{\alpha}$ is diagonal, each state component is differentiated independently:

$$\mathcal{D}_t^{\alpha}\psi(t) = \begin{bmatrix} {}^{C}D_t^{\alpha_1}\psi_1(t) \\ \vdots \\ {}^{C}D_t^{\alpha_n}\psi_n(t) \end{bmatrix}. \tag{3.2}$$

This is particularly useful for systems with nonuniform fractional orders.

(3) Compatibility with matrix multiplication. For a constant matrix $A \in \mathbb{R}^{n\times n}$, the operator satisfies

$$\mathcal{D}_t^{\alpha}(\mathcal{A}\psi(t)) = \mathrm{diag}({}^{C}D_t^{\alpha_1}, \ldots, {}^{C}D_t^{\alpha_n})\mathcal{A}\psi(t), \tag{3.3}$$

allowing the state equation to be expressed compactly as

$$\mathcal{D}_t^{\alpha}\psi(t) = \mathcal{A}\psi(t) + \mathcal{B}\mathcal{U}(t). \tag{3.4}$$

(4) Facilitates fractional optimal control. In LQ problems, using $\mathcal{D}_t^{\alpha}$ allows the Hamiltonian, adjoint, and Riccati equations to be written compactly in matrix form, simplifying both analysis and numerical implementation.

(5) Time-shift invariance. For a time-shifted vector $x(t-\tau)$ with $\tau > 0$,

$$\mathcal{D}_t^{\alpha}\psi(t-\tau) = \begin{bmatrix} {}^{C}D_t^{\alpha_1}\psi_1(t-\tau) \\ \vdots \\ {}^{C}D_t^{\alpha_n}\psi_n(t-\tau) \end{bmatrix}, \tag{3.5}$$

which preserves the fractional memory effect of each component individually.

(6) Fractional integration property. For the Riemann-Liouville fractional integral operator $I_t^{\alpha_i}$,

$$\mathcal{D}_t^{\alpha} I_t^{\alpha}\psi(t) = \psi(t), \quad I_t^{\alpha} = \mathrm{diag}(I_t^{\alpha_1}, \ldots, I_t^{\alpha_n}). \tag{3.6}$$

This is the componentwise generalization of the standard Caputo derivative-integral relation.

4. Multi-index fractional Pontryagin maximum principle

The classical Pontryagin maximum principle provides necessary conditions for optimality in control systems governed by ordinary differential equations. In many applications, however, the system dynamics exhibit memory effects that are better modeled by fractional derivatives [4, 8, 15]. Moreover, in multidimensional systems, each component may have a different fractional order, giving rise to a *multi-index fractional control system*.

The following theorem extends the classical maximum principle to such multi-index fractional systems. It provides the necessary conditions for optimality in terms of a componentwise state equation, a right-sided fractional adjoint equation, and an explicit optimality condition expressed through the Hamiltonian. This result is fundamental for analyzing both theoretical properties and numerical schemes for linear-quadratic multi-index fractional control problems. In contrast to the standard Caputo fractional PMP, which is driven by a single fractional order and thus exhibits a uniform memory kernel across all state variables, the present PMP is formulated in a multi-index setting where different components evolve with different fractional orders. This leads to heterogeneous memory effects in the state system and a correspondingly more complex adjoint system, where the backward dynamics are coupled through multiple memory structures rather than a single fractional operator

Theorem 4.1. *Let $0 < \alpha_i < 1$, $i = 1, \dots, n$, and let $\mathcal{A} \in \mathbb{R}^{n\times n}$, $\mathcal{B} \in \mathbb{R}^{n\times m}$, $\mathcal{Q} \in \mathbb{R}^{n\times n}$ be symmetric positive semidefinite, and $\mathcal{R} \in \mathbb{R}^{m\times m}$ symmetric positive definite.*

Consider the multi-index fractional control system

$$ {}^{C}D_t^{\alpha_i}\psi_i(t) = (\mathcal{A}\psi(t) + \mathcal{B}\mathcal{U}(t))_i, \quad i = 1, \dots, n, \tag{4.1}$$

with initial condition $\psi(0) = \psi_0$, and the quadratic cost functional

$$J(\mathcal{U}) = \int_0^T \left(\psi(t)^\top \mathcal{Q}\psi(t) + \mathcal{U}(t)^\top \mathcal{R}\mathcal{U}(t)\right) dt. \tag{4.2}$$

If $\mathcal{U}^* \in L^2(0, T; \mathbb{R}^m)$ is an optimal control with corresponding state ψ^*, then there exists an adjoint function $\lambda \in L^2(0, T; \mathbb{R}^n)$ such that

(1) State system

$$ {}^{C}D_t^{\alpha_i}\psi_i^*(t) = (\mathcal{A}\psi^*(t) + \mathcal{B}\mathcal{U}^*(t))_i, \quad i = 1, \dots, n. \tag{4.3}$$

(2) Adjoint system

$$ {}_tD_T^{\alpha_i}\lambda_i(t) = -\frac{\partial H}{\partial \psi_i}(t, \psi^*, \mathcal{U}^*, \lambda) = -\left(2(\mathcal{Q}\psi^*(t))_i + (\mathcal{A}^\top\lambda(t))_i\right), \tag{4.4}$$

for $i = 1, \dots, n$, with terminal condition

$$\lambda(T) = 0, \tag{4.5}$$

where ${}_tD_T^{\alpha_i}$ denotes the right-sided Riemann-Liouville fractional derivative.

(3) Optimality condition

$$\frac{\partial H}{\partial \mathcal{U}}(t, \psi^*, \mathcal{U}^*, \lambda) = 2\mathcal{R}\mathcal{U}^*(t) + \mathcal{B}^\top\lambda(t) = 0, \tag{4.6}$$

which implies

$$\mathcal{U}^*(t) = -\tfrac{1}{2}\mathcal{R}^{-1}\mathcal{B}^\top\lambda(t). \tag{4.7}$$

Here the Hamiltonian is defined by

$$H(t, \psi, \mathcal{U}, \lambda) = \psi^\top \mathcal{Q}\psi + \mathcal{U}^\top\mathcal{R}\mathcal{U} + \lambda^\top(\mathcal{A}\psi + \mathcal{B}\mathcal{U}). \tag{4.8}$$

Proof. We divide the proof into several steps.

Step 1: Well-posedness of the state equation.

For any $\mathcal{U} \in L^2(0, T; \mathbb{R}^m)$, the multi-index Caputo system

$$ {}^{C}D_t^{\alpha_i}\psi_i(t) = (\mathcal{A}\psi(t) + \mathcal{B}\mathcal{U}(t))_i, \quad \psi(0) = \psi_0,$$

is equivalent to the Volterra integral equation

$$\psi_i(t) = \psi_{i0} + \frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i - 1}(\mathcal{A}\psi(s) + \mathcal{B}\mathcal{U}(s))_i\, ds.$$

Standard results for linear fractional differential equations ensure existence and uniqueness of a solution $\psi \in C([0, T]; \mathbb{R}^n)$.

Step 2: Existence of an optimal control.

The control-to-state map $\mathcal{U} \mapsto \psi$ is linear and continuous. Since Q is positive semidefinite and $\mathcal{R}$ is positive definite, the functional

$$J(u) = \int_0^T (\psi^\top Q\psi + \mathcal{U}^\top \mathcal{R}\mathcal{U})\, dt$$

is strictly convex and coercive on $L^2(0, T; \mathbb{R}^m)$. Hence there exists a unique optimal control $\mathcal{U}^*$.

Step 3: First variation of the state.

Let $\mathcal{U}^\varepsilon = \mathcal{U}^* + \varepsilon\mathcal{V}$ with $v \in L^2(0, T; \mathbb{R}^m)$. Let ψ^ε be the corresponding state and define

$$z(t) = \frac{d}{d\varepsilon}\psi^\varepsilon(t)\Big|_{\varepsilon=0}.$$

Differentiating the state equation yields the linearized system

$${}^C D_t^{\alpha_i} z_i(t) = (\mathcal{A}z(t) + \mathcal{B}v(t))_i, \quad z(0) = 0.$$

Step 4: First variation of the cost functional.

Differentiating $J(u^\varepsilon)$ gives

$$\delta J(\mathcal{U}^*; \mathcal{V}) = 2\int_0^T (z^{*\top} Qz + \mathcal{U}^{*\top} R\mathcal{V})\, dt.$$

Since $\mathcal{U}^*$ is optimal, we must have

$$\delta J(\mathcal{U}^*; \mathcal{V}) = 0 \quad \text{for all } \mathcal{V}.$$

Step 5: Fractional integration by parts.

Multiply the linearized equation by an adjoint function $\lambda(t)$ and integrate:

$$\int_0^T \lambda^\top ({}^C D_t^\alpha z - Az - Bv)\, dt = 0.$$

Using the fractional integration-by-parts formula [8], for each i,

$$\int_0^T \lambda_i\, {}^C D_t^{\alpha_i} z_i\, dt = \int_0^T z_i\, {}_t D_T^{\alpha_i} \lambda_i\, dt + [z_i(t) I_{t,T}^{1-\alpha_i} \lambda_i(t)]_0^T.$$

Since $z(0) = 0$, the boundary term reduces to $z(T)\lambda(T)$. Imposing the terminal condition

$$\lambda(T) = 0$$

eliminates the boundary contribution.

Thus,

$$\int_0^T z^\top ({}_t D_T^\alpha \lambda - \mathcal{A}^\top \lambda)\, dt - \int_0^T \lambda^\top \mathcal{B}\mathcal{V}\, dt = 0.$$

Step 6: Choice of adjoint equation.

Choose λ satisfying

$${}_tD_T^{\alpha_i}\lambda_i = -2(Q\psi^*)_i - (\mathcal{A}^\top\lambda)_i, \quad \lambda(T) = 0.$$

Then we obtain

$$2\int_0^T \psi^{*\top} Qz\, dt = \int_0^T \lambda^\top \mathcal{B}\mathcal{V}\, dt.$$

Step 7: Optimality condition.

Substituting into the first variation gives

$$\delta J(\mathcal{U}^*; \mathcal{V}) = \int_0^T (\mathcal{B}^\top\lambda + 2\mathcal{R}\mathcal{U}^*)^\top \mathcal{V}\, dt.$$

Since $\mathcal{V}$ is arbitrary in L^2, the fundamental lemma of the calculus of variations implies

$$\mathcal{B}^\top\lambda(t) + 2\mathcal{R}\mathcal{U}^*(t) = 0 \quad \text{a.e. on } (0, T).$$

Because $\mathcal{R}$ is invertible,

$$\mathcal{U}^*(t) = -\tfrac{1}{2}\mathcal{R}^{-1}\mathcal{B}^\top\lambda(t).$$

The state equation, adjoint equation, and optimality condition are therefore satisfied, completing the proof. □

Remark 4.1. *Theorem 4.1 extends [11, Theorem 1] to the multi-index fractional case. In particular, the scalar case reduces to the result established in [11, Theorem 1].*

Example 4.1. (Fractional linear quadratic control problem)

We illustrate Theorem 4.1 with a one-dimensional fractional linear-quadratic problem.

Problem formulation. Let $0 < \alpha < 1$ and $T > 0$. Minimize the cost functional

$$J(\mathcal{U}) = \int_0^T \left(\psi(t)^2 + \mathcal{U}(t)^2\right) dt, \tag{4.9}$$

subject to the fractional state equation

$${}^CD_t^\alpha\psi(t) = \gamma\psi(t) + \beta\mathcal{U}(t), \quad t \in (0, T), \tag{4.10}$$

with initial condition

$$\psi(0) = \psi_0, \tag{4.11}$$

where $a, b \in \mathbb{R}$.

This corresponds to the matrices

$$\mathcal{A} = \gamma, \quad \mathcal{B} = \beta, \quad Q = 1, \quad \mathcal{R} = 1.$$

Step 1: The Hamiltonian is defined by

$$H(\psi, \mathcal{U}, \lambda) = \psi^2 + \mathcal{U}^2 + \lambda(\gamma\psi + \beta\mathcal{U}). \tag{4.12}$$

Step 2: State equation.

From the theorem, the optimal state ψ^* satisfies

$$ {}^C D_t^{\alpha} \psi^*(t) = \gamma\psi^*(t) + \beta\mathcal{U}^*(t). \tag{4.13} $$

Step 3: Adjoint equation.

The adjoint equation is

$$ {}^C D_{T-t}^{\alpha} \lambda(t) = -\frac{\partial H}{\partial \psi} = -(2\psi^*(t) + \gamma\lambda(t)), \tag{4.14} $$

with terminal condition

$$ \lambda(T) = 0. \tag{4.15} $$

Step 4: Optimality condition.

The stationarity condition gives

$$ \frac{\partial H}{\partial \mathcal{U}} = 2\mathcal{U}^*(t) + \beta\lambda(t) = 0, \tag{4.16} $$

hence, the optimal control is

$$ \mathcal{U}^*(t) = -\frac{\beta}{2}\lambda(t). \tag{4.17} $$

Step 5: Coupled optimality system.

Substituting $\mathcal{U}^*$ into the state equation, we obtain the coupled fractional system:

$$ {}^C D_t^{\alpha} \psi^*(t) = \gamma\psi^*(t) - \frac{\beta^2}{2}\lambda(t), \tag{4.18} $$

$$ {}^C D_{T-t}^{\alpha} \lambda(t) = -2\psi^*(t) - \gamma\lambda(t), \tag{4.19} $$

with boundary conditions

$$ \psi^*(0) = \psi_0, \qquad \lambda(T) = 0. $$

Remark 4.2. *This is a two-point boundary value problem involving:*

(i) A forward Caputo fractional equation for the state,
(ii) A backward Caputo fractional equation for the adjoint.

Even in this simple linear-quadratic case, the presence of fractional derivatives makes the system nonlocal in time. Analytical solutions can be expressed in terms of Mittag -Leffler functions, whereas numerical solutions typically require predictor-corrector schemes.

Example 4.2. (Multi-index fractional linear-quadratic control problem)

We consider a two dimensional fractional optimal control problem with different fractional orders $\alpha_1, \alpha_2 \in (0, 1)$.

Problem formulation.

Minimize the cost functional

$$ J(\mathcal{U}) = \int_0^T \left(\psi_2(t)^2 + \mathcal{U}(t)^2\right) dt, \tag{4.20} $$

subject to the multi-index fractional state system

$$\begin{cases} {}^{C}D_t^{\alpha_1}\psi_1(t) = \gamma_{11}\psi_1(t) + \gamma_{12}\psi_2(t) + \beta_1\mathcal{U}(t), \\ {}^{C}D_t^{\alpha_2}\psi_2(t) = \gamma_{21}\psi_1(t) + \gamma_{22}\psi_2(t) + \beta_2\mathcal{U}(t), \end{cases} \tag{4.21}$$

with initial conditions

$$\psi_1(0) = \psi_{10}, \qquad \psi_2(0) = \psi_{20}.$$

In compact diagonal form, introduce

$$\mathcal{D}_t^{\alpha} = \mathrm{diag}\big({}^{C}D_t^{\alpha_1}, {}^{C}D_t^{\alpha_2}\big),$$

so that

$$\mathcal{D}_t^{\alpha}\psi(t) = \mathcal{A}\psi(t) + \mathcal{B}\mathcal{U}(t), \tag{4.22}$$

where

$$\mathcal{A} = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}.$$

Step 1: The Hamiltonian is

$$H(\psi, \mathcal{U}, \lambda) = \psi^{\top}\psi + \mathcal{U}^2 + \lambda^{\top}(\mathcal{A}\psi + \mathcal{B}\mathcal{U}), \tag{4.23}$$

where $\lambda = (\lambda_1, \lambda_2)^{\top}$.
Step 2: State equations.

The optimal state ψ^* satisfies

$${}^{C}D_t^{\alpha_i}\psi_i^*(t) = (\mathcal{A}\psi^*(t) + \mathcal{B}\mathcal{U}^*(t))_i, \qquad i = 1, 2. \tag{4.24}$$

Step 3: Adjoint equations.

The adjoint system is

$${}^{C}D_{T-t}^{\alpha_i}\lambda_i(t) = -\left(2\psi_i^*(t) + (\mathcal{A}^{\top}\lambda(t))_i\right), \qquad i = 1, 2, \tag{4.25}$$

with terminal conditions

$$\lambda_1(T) = 0, \qquad \lambda_2(T) = 0.$$

In matrix form,

$$\mathcal{D}_{T-t}^{\alpha}\lambda(t) = -(2\psi^*(t) + \mathcal{A}^{\top}\lambda(t)). \tag{4.26}$$

Step 4: Optimality condition.

The stationarity condition gives

$$\frac{\partial H}{\partial \mathcal{U}} = 2\mathcal{U}^*(t) + \mathcal{B}^{\top}\lambda(t) = 0, \tag{4.27}$$

hence

$$\mathcal{U}^*(t) = -\frac{1}{2}\mathcal{B}^{\top}\lambda(t). \tag{4.28}$$

Step 5: Coupled multi-index optimality system.

The complete fractional optimality system is

$$\begin{cases} \mathcal{D}_t^{\alpha}\psi^*(t) = \mathcal{A}\psi^*(t) - \frac{1}{2}\mathcal{B}\mathcal{B}^\top\lambda(t), \\ \mathcal{D}_{T-t}^{\alpha}\lambda(t) = -(2\psi^*(t) + \mathcal{A}^\top\lambda(t)), \\ \psi^*(0) = \psi_0, \qquad \lambda(T) = 0. \end{cases} \tag{4.29}$$

Remark 4.3. *This example clearly shows the multi-index structure:*

(i) Each state component has a different fractional order α_i.
(ii) The adjoint system involves backward fractional derivatives of the same multiorder.
(iii) The system forms a coupled two-point boundary value problem.

When $\alpha_1 = \alpha_2 = 1$, the classical linear-quadratic optimal control system is recovered. For $0 < \alpha_i < 1$, the solution generally involves matrix-valued Mittag-Leffler functions.

Example 4.3. (n-dimensional multi-index fractional linear-quadratic problem)

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ with $0 < \alpha_i < 1$ for $i = 1, \ldots, n$. We consider the n-dimensional fractional optimal control problem.

Cost functional.

$$J(\mathcal{U}) = \int_0^T (\psi(t)^\top Q\psi(t) + \mathcal{U}(t)^\top\mathcal{R}\mathcal{U}(t))\, dt, \tag{4.30}$$

where $Q \in \mathbb{R}^{n\times n}$ is symmetric positive semidefinite and $\mathcal{R} \in \mathbb{R}^{m\times m}$ is symmetric positive definite.

Multi-index fractional state equation.

$${}^C D_t^{\alpha_i}\psi_i(t) = (\mathcal{A}\psi(t) + \mathcal{B}\mathcal{U}(t))_i, \quad i = 1, \ldots, n, \tag{4.31}$$

with initial condition

$$\psi(0) = \psi_0 \in \mathbb{R}^n. \tag{4.32}$$

Introduce the diagonal fractional operator

$$\mathcal{D}_t^{\alpha} = \operatorname{diag}({}^C D_t^{\alpha_1}, \ldots, {}^C D_t^{\alpha_n}),$$

so that the state system can be written compactly as

$$\mathcal{D}_t^{\alpha}\psi(t) = \mathcal{A}\psi(t) + \mathcal{B}\mathcal{U}(t). \tag{4.33}$$

Hamiltonian.

The Hamiltonian function is defined by

$$H(\psi, \mathcal{U}, \lambda) = \psi^\top Q\psi + \mathcal{U}^\top\mathcal{R}\mathcal{U} + \lambda^\top(\mathcal{A}\psi + \mathcal{B}\mathcal{U}), \tag{4.34}$$

where $\lambda(t) \in \mathbb{R}^n$ is the adjoint variable.

State equations.

The optimal state ψ^* satisfies

$$\mathcal{D}_t^{\alpha}\psi^*(t) = \mathcal{A}\psi^*(t) + \mathcal{B}\mathcal{U}^*(t). \tag{4.35}$$

Adjoint equations.

For each $i = 1, \dots, n$, the adjoint satisfies

$$ {}^{C}D^{\alpha_i}_{T-t}\lambda_i(t) = -\frac{\partial H}{\partial \psi_i} = -\left((2Q\psi^*(t))_i + (\mathcal{A}^\top \lambda(t))_i\right), \tag{4.36} $$

with terminal condition

$$ \lambda(T) = 0. \tag{4.37} $$

It can be expressed in compact matrix form as

$$ \mathcal{D}^{\alpha}_{T-t}\lambda(t) = -(2Q\psi^*(t) + \mathcal{A}^\top \lambda(t)). \tag{4.38} $$

Optimality condition.

The stationarity condition gives

$$ \frac{\partial H}{\partial \mathcal{U}} = 2\mathcal{R}\mathcal{U}^*(t) + \mathcal{B}^\top \lambda(t) = 0, \tag{4.39} $$

hence, the optimal control is

$$ \mathcal{U}^*(t) = -\frac{1}{2}\mathcal{R}^{-1}\mathcal{B}^\top \lambda(t). \tag{4.40} $$

Complete multi-index optimality system

The coupled fractional two-point boundary value problem becomes

$$ \begin{cases} \mathcal{D}^{\alpha}_{t}\psi^*(t) = \mathcal{A}\psi^*(t) - \frac{1}{2}\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top \lambda(t), \\ \mathcal{D}^{\alpha}_{T-t}\lambda(t) = -(2Q\psi^*(t) + \mathcal{A}^\top \lambda(t)), \\ \psi^*(0) = \psi_0, \qquad \lambda(T) = 0. \end{cases} \tag{4.41} $$

Remark 4.4. *(i) Each state component evolves with its own fractional order α_i.*

(ii) The adjoint system involves backward fractional derivatives of the same multi-order structure.

(iii) When $\alpha_i = 1$ for all i, the classical n-dimensional linear-quadratic optimal control system is recovered.

(iv) For $0 < \alpha_i < 1$, the solution generally involves matrix Mittag-Leffler functions and does not reduce to a classical Riccati differential equation.

The following corollary provides an explicit representation of the optimal control derived from the multi-index fractional Pontryagin maximum principle and the associated optimality conditions.

Corollary 4.1. *Under the assumptions of the fractional Pontryagin maximum principle, if R is positive definite, then the optimal control is uniquely given by*

$$ \mathcal{U}^*(t) = -\tfrac{1}{2}\mathcal{R}^{-1}\mathcal{B}^\top \lambda(t). \tag{4.42} $$

Proof. Because $\mathcal{R}$ is positive definite, it is invertible. From the optimality condition

$$ 2\mathcal{R}\mathcal{U}^*(t) + \mathcal{B}^\top \lambda(t) = 0, $$

we directly obtain

$$ \mathcal{U}^*(t) = -\tfrac{1}{2}\mathcal{R}^{-1}\mathcal{B}^\top \lambda(t). $$

□

Corollary 4.2 (Fractional two-point boundary value problem)**.** *The optimal pair* (ψ^*, λ) *satisfies the coupled fractional system*

$$\begin{cases} {}^C D_t^\alpha \psi^*(t) = \mathcal{A}\psi^*(t) + \mathcal{B}\mathcal{U}^*(t), \\ {}^C D_{T-t}^\alpha \lambda(t) = -2Q\psi^*(t) - \mathcal{A}^\top \lambda(t), \end{cases}$$

with boundary conditions

$$\psi^*(0) = \psi_0, \qquad \lambda(T) = 0.$$

Proof. From the fractional Pontryagin maximum principle, if u^* is optimal, and x^* is the associated state trajectory, then there exists an adjoint function $\lambda(t)$ such that

(1) State equation.

$${}^C D_t^\alpha \psi^*(t) = \mathcal{A}\psi^*(t) + \mathcal{B}\mathcal{U}^*(t),$$

with the prescribed initial condition

$$\psi^*(0) = \psi_0.$$

(2) Adjoint equation.

$${}^C D_{T-t}^\alpha \lambda(t) = -2Q\psi^*(t) - \mathcal{A}^\top \lambda(t),$$

together with the transversality (terminal) condition

$$\lambda(T) = 0.$$

Thus, the optimal pair (ψ^*, λ) satisfies a coupled system of fractional differential equations: The state equation evolves forward in time with initial condition at $t = 0$, whereas the adjoint equation evolves backward in time with terminal condition at $t = T$.

Therefore, the optimality system constitutes a fractional two-point boundary value problem with boundary conditions

$$\psi^*(0) = \psi_0, \qquad \lambda(T) = 0.$$

This completes the proof. □

Corollary 4.3 (Fractional Hamiltonian system)**.** *The optimality system can be written compactly as a fractional Hamiltonian system*

$$\begin{cases} {}^C D_t^\alpha \psi^*(t) = \dfrac{\partial H}{\partial \lambda}, \\ {}^C D_{T-t}^\alpha \lambda(t) = -\dfrac{\partial H}{\partial \psi}, \end{cases}$$

with

$$\frac{\partial H}{\partial \mathcal{U}} = 0.$$

Proof. Recall that the Hamiltonian associated with the fractional optimal control problem is

$$H(\psi, U, \lambda) = \psi^\top Q\psi + \mathcal{U}^\top \mathcal{R}\mathcal{U} + \lambda^\top(\mathcal{A}\psi + \mathcal{B}\mathcal{U}).$$

From the fractional Pontryagin maximum principle, the optimal triple $(\psi^*, \mathcal{U}^*, \lambda)$ satisfies

(i) State equation.

$${}^C D_t^\alpha \psi^*(t) = \mathcal{A}\psi^*(t) + \mathcal{B}\mathcal{U}^*(t).$$

Observe that

$$\frac{\partial H}{\partial \lambda} = \mathcal{A}\psi + \mathcal{B}\mathcal{U}.$$

Evaluating at the optimal triple gives

$$^{C}D_t^{\alpha}\psi^*(t) = \frac{\partial H}{\partial \lambda}(\psi^*, \mathcal{U}^*, \lambda).$$

(ii) Adjoint equation.

$$^{C}D_{T-t}^{\alpha}\lambda(t) = -2\mathcal{Q}\psi^*(t) - \mathcal{A}^\top \lambda(t).$$

On the other hand,

$$\frac{\partial H}{\partial \psi} = 2\mathcal{Q}\psi + \mathcal{A}^\top \lambda.$$

Hence,

$$^{C}D_{T-t}^{\alpha}\lambda(t) = -\frac{\partial H}{\partial \psi}(\psi^*, \mathcal{U}^*, \lambda).$$

(iii) Optimality condition.

$$2\mathcal{R}\mathcal{U}^*(t) + \mathcal{B}^\top \lambda(t) = 0.$$

However

$$\frac{\partial H}{\partial \mathcal{U}} = 2Ru + B^\top \lambda,$$

therefore,

$$\frac{\partial H}{\partial \mathcal{U}}(\psi^*, \mathcal{U}^*, \lambda) = 0.$$

Combining (i)–(iii), we obtain the compact fractional Hamiltonian system

$$\begin{cases} ^{C}D_t^{\alpha}\psi^*(t) = \dfrac{\partial H}{\partial \lambda}, \\ ^{C}D_{T-t}^{\alpha}\lambda(t) = -\dfrac{\partial H}{\partial \psi}, \end{cases} \qquad \text{with} \quad \frac{\partial H}{\partial \mathcal{U}} = 0.$$

This completes the proof. □

Corollary 4.4 (Reduction to the classical case)**.** *If $\alpha_i = 1$ for all $i = 1, \ldots, n$, then the system reduces to the classical Pontryagin maximum principle:*

$$\dot{x}^*(t) = Ax^*(t) + Bu^*(t),$$

$$\dot{\lambda}(t) = -2Qx^*(t) - A^\top \lambda(t),$$

with $\lambda(T) = 0$.

Proof. Recall that for $0 < \alpha_i \leq 1$, the optimality system is given by

$$^{C}D_t^{\alpha_i}\psi_i^*(t) = (\mathcal{A}\psi^*(t) + \mathcal{B}\mathcal{U}^*(t))_i,$$

$$^{C}D_{T-t}^{\alpha_i}\lambda_i(t) = -2(\mathcal{Q}\psi^*(t))_i - (\mathcal{A}^\top \lambda(t))_i,$$

with terminal condition $\lambda_i(T) = 0$.

If $\alpha_i = 1$ for all i, then the Caputo fractional derivative of order one coincides with the classical first derivative, that is,

$$^C D_t^1 \psi(t) = \frac{d}{dt}\psi(t) = \dot{\psi}(t),$$

and similarly,

$$^C D_{T-t}^1 \lambda(t) = \frac{d}{dt}\lambda(t) = \dot{\lambda}(t).$$

Therefore, the state equation becomes

$$\dot{\psi}^*(t) = \mathcal{A}\psi^*(t) + \mathcal{B}\mathcal{U}^*(t).$$

Likewise, the adjoint equation reduces to

$$\dot{\lambda}(t) = -2\mathcal{Q}\psi^*(t) - \mathcal{A}^\top \lambda(t),$$

with terminal condition

$$\lambda(T) = 0.$$

Moreover, the optimality condition

$$2\mathcal{R}\mathcal{U}^*(t) + \mathcal{B}^\top \lambda(t) = 0$$

remains unchanged.

Hence, when $\alpha_i = 1$, the fractional optimality system reduces exactly to the classical Pontryagin maximum principle for linear-quadratic control problems.

This completes the proof. □

The following corollary discusses the associated fractional Riccati equation arising from the multi-Index fractional Pontryagin maximum principle and its role in the characterization of optimal controls.

Corollary 4.5 (Fractional Riccati equation)**.** *Let $\psi^*(t) \in \mathbb{R}^n$ and $\lambda(t) \in \mathbb{R}^n$ be the optimal state and adjoint variables. Assume that there exists a sufficiently smooth matrix-valued function*

$$P : [0, T] \to \mathbb{R}^{n\times n}, \qquad P \in C^1([0, T]; \mathbb{R}^{n\times n}),$$

such that the adjoint variable admits the representation

$$\lambda(t) = P(t)\psi^*(t). \tag{4.43}$$

Then $P(t)$ formally satisfies the fractional Riccati-type equation

$$^C D_{T-t}^\alpha P(t) = -2\mathcal{Q} - \mathcal{A}^\top P(t) - P(t)\mathcal{A} + \tfrac{1}{2}P(t)\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top P(t). \tag{4.44}$$

Proof. From the fractional Pontryagin maximum principle, the adjoint equation is

$$^C D_{T-t}^\alpha \lambda(t) = -2\mathcal{Q}\psi^*(t) - \mathcal{A}^\top \lambda(t), \tag{4.45}$$

and the optimal control is

$$\mathcal{U}^*(t) = -\frac{1}{2}\mathcal{R}^{-1}\mathcal{B}^\top\lambda(t). \tag{4.46}$$

Assume the ansatz $\lambda(t) = P(t)\psi^*(t)$. Then

$${}^C D^\alpha_{T-t}\lambda(t) = {}^C D^\alpha_{T-t}(P(t)\psi^*(t)). \tag{4.47}$$

Using the fractional Leibniz-type expansion (formal identity) [12, 14]

$${}^C D^\alpha(fg)(t) = \sum_{k=0}^{\infty}\binom{\alpha}{k}\left(D^k f(t)\right)\left({}^C D^{\alpha-k}g(t)\right), \tag{4.48}$$

we obtain

$${}^C D^\alpha_{T-t}\lambda(t) = \sum_{k=0}^{\infty}\binom{\alpha}{k}(D^k P(t))({}^C D^{\alpha-k}_{T-t}\psi^*(t)). \tag{4.49}$$

We separate the leading term $k = 0$:

$${}^C D^\alpha_{T-t}\lambda(t) = ({}^C D^\alpha_{T-t}P(t))\psi^*(t) + \sum_{k=1}^{\infty}\binom{\alpha}{k}(D^k P(t))({}^C D^{\alpha-k}_{T-t}\psi^*(t)). \tag{4.50}$$

Formal truncation assumption: We assume that higher-order memory interaction terms ($k \geq 1$) are negligible in the Riccati reduction framework, which is standard in PMP-type derivations.

Hence,

$${}^C D^\alpha_{T-t}\lambda(t) \approx ({}^C D^\alpha_{T-t}P(t))\psi^*(t) + P(t)\,{}^C D^\alpha_{T-t}\psi^*(t). \tag{4.51}$$

Proceeding as in the classical fractional PMP derivation, and using

$${}^C D^\alpha_{T-t}\psi^*(t) = -(\mathcal{A}\psi^*(t) + \mathcal{B}\mathcal{U}^*(t)),$$

we obtain

$${}^C D^\alpha_{T-t}\lambda(t) = ({}^C D^\alpha_{T-t}P(t))\psi^*(t) - P(t)\mathcal{A}\psi^*(t) + \tfrac{1}{2}P(t)\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top P(t)\psi^*(t). \tag{4.52}$$

Comparing with the adjoint equation and factoring $\psi^*(t)$ yields

$${}^C D^\alpha_{T-t}P(t) = -2\mathcal{Q} - \mathcal{A}^\top P(t) - P(t)\mathcal{A} + \tfrac{1}{2}P(t)\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top P(t). \tag{4.53}$$

This completes the proof. □

Remark 4.5. *The Riccati-type equation in Corollary 4.4 is derived formally using the ansatz $\lambda(t) = P(t)\psi^*(t)$ together with a fractional Leibniz-type rule. because fractional derivatives generally do not satisfy the classical product rule and inherently involve memory effects, higher-order terms in the resulting expansion are neglected. Accordingly, the obtained Riccati equation should be interpreted as a heuristic reduction of the Pontryagin system, which is valid only under additional regularity and structural assumptions.*

5. Pontryagin maximum principle for multi-index fractional optimal control problems

The following theorem establishes a multi-index fractional Pontryagin maximum principle for the considered optimal control problem. It provides necessary optimality conditions in terms of the Hamiltonian system, the adjoint equations, and the corresponding stationarity and transversality conditions.

Theorem 5.1 (Multi-index fractional Pontryagin maximum principle)**.** *Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ with $0 < \alpha_i < 1$ for $i = 1, \ldots, n$. Consider the fractional optimal control problem*

$$\min_{\mathcal{U} \in \mathcal{U}_{ad}} J(\mathcal{U}) = \int_0^T \mathcal{L}(t, \psi(t), \mathcal{U}(t))\, dt, \tag{5.1}$$

subject to the multi-index fractional state system

$${}^C D_t^{\alpha_i} \psi(t) = f_i(t, \psi(t), \mathcal{U}(t)), \quad i = 1, \ldots, n, \tag{5.2}$$

with initial condition

$$\psi(0) = \psi_0 \in \mathbb{R}^n. \tag{5.3}$$

Assume that:

(1) $f = (f_1, \ldots, f_n)$ is continuously differentiable in (x, u),
(2) $\mathcal{L}$ is continuously differentiable in $(\psi, \mathcal{U})$,
(3) $\mathcal{U}_{ad} \subset L^2(0, T; \mathbb{R}^m)$ is convex.

If $\mathcal{U}^*$ is an optimal control with corresponding state x^*, then there exists an adjoint function $\lambda(t) = (\lambda_1(t), \ldots, \lambda_n(t))^\top$ such that the following conditions hold:

(1) State equations.

$${}^C D_t^{\alpha_i} \psi_i^*(t) = f_i(t, \psi^*(t), \mathcal{U}^*(t)), \quad i = 1, \ldots, n. \tag{5.4}$$

(2) Adjoint equations.

$${}^C D_{T-t}^{\alpha_i} \lambda_i(t) = -\frac{\partial H}{\partial \psi_i}(t, \psi^*(t), \mathcal{U}^*(t), \lambda(t)), \quad i = 1, \ldots, n, \tag{5.5}$$

with terminal conditions

$$\lambda_i(T) = 0, \quad i = 1, \ldots, n. \tag{5.6}$$

(3) Optimality condition.

$$\frac{\partial H}{\partial \mathcal{U}}(t, \psi^*(t), \mathcal{U}^*(t), \lambda(t)) = 0 \quad a.e.\ t \in (0, T), \tag{5.7}$$

where the Hamiltonian is defined by

$$H(t, \psi, \mathcal{U}, \lambda) = \mathcal{L}(t, \psi, \mathcal{U}) + \sum_{i=1}^n \lambda_i f_i(t, \psi, \mathcal{U}). \tag{5.8}$$

Proof. Let $\mathcal{U}^*$ be an optimal control and let ψ^* be the corresponding state trajectory. For any $v \in L^2(0, T; \mathbb{R}^m)$, consider the perturbation

$$\mathcal{U}^\varepsilon(t) = \mathcal{U}^*(t) + \varepsilon v(t),$$

where ε is sufficiently small such that $\mathcal{U}^\varepsilon \in \mathcal{U}_{ad}$.

Let ψ^ε be the solution of the perturbed system

$${}^C D_t^{\alpha_i} \psi_i^\varepsilon(t) = f_i(t, \psi^\varepsilon(t), \mathcal{U}^\varepsilon(t)), \quad i = 1, \ldots, n,$$

with $\psi^\varepsilon(0) = \psi_0$.

Step 1: Linearized state system.

Define the variation

$$z_i(t) = \frac{d}{d\varepsilon} \psi_i^\varepsilon(t) \Big|_{\varepsilon=0}.$$

Using the differentiability of f_i, we obtain the linearized system

$${}^C D_t^{\alpha_i} z_i(t) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial \psi_j}(t, \psi^*, \mathcal{U}^*) z_j(t) + \sum_{k=1}^{m} \frac{\partial f_i}{\partial \mathcal{U}_k}(t, \psi^*, \mathcal{U}^*) v_k(t),$$

with

$$z_i(0) = 0, \quad i = 1, \ldots, n.$$

Step 2: First variation of the cost.

Optimality of $\mathcal{U}^*$ implies

$$\frac{d}{d\varepsilon} J(\mathcal{U}^\varepsilon) \Big|_{\varepsilon=0} = 0.$$

Hence,

$$\delta J = \int_0^T \left[\sum_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial \psi_i}(t, \psi^*, \mathcal{U}^*) z_i(t) + \sum_{k=1}^{m} \frac{\partial \mathcal{L}}{\partial \mathcal{U}_k}(t, \psi^*, \mathcal{U}^*) v_k(t) \right] dt.$$

Step 3: Adjoint system.

We introduce the adjoint variables $\lambda_i(t)$ satisfying

$${}^C D_{T-t}^{\alpha_i} \lambda_i(t) = -\frac{\partial H}{\partial \psi_i}(t, \psi^*(t), \mathcal{U}^*(t), \lambda(t)), \quad i = 1, \ldots, n,$$

with terminal condition

$$\lambda_i(T) = 0.$$

The Hamiltonian is

$$H(t, \psi, \mathcal{U}, \lambda) = \mathcal{L}(t, \psi, \mathcal{U}) + \sum_{i=1}^{n} \lambda_i f_i(t, \psi, \mathcal{U}).$$

Step 4: Fractional integration by parts (key step).

For each component i, we use the Caputo fractional integration -by- parts formula [14]:

$$\int_0^T \lambda_i(t)\, {}^C D_t^{\alpha_i} z_i(t)\, dt = \Big[z_i(t)\, I_{T-t}^{1-\alpha_i} \lambda_i(t) \Big]_0^T + \int_0^T z_i(t)\, {}^C D_{T-t}^{\alpha_i} \lambda_i(t)\, dt.$$

We now analyze the boundary terms:

(i) Because $z_i(0) = 0$, the contribution at $t = 0$ vanishes.

(ii) At $t = T$, the adjoint is defined with terminal condition $\lambda_i(T) = 0$, which implies the right-sided fractional integral term vanishes at the terminal point in the variational sense.

Hence,

$$\left[z_i(t)\, I_{T-t}^{1-\alpha_i} \lambda_i(t)\right]_0^T = 0.$$

Therefore,

$$\int_0^T \lambda_i(t)\, {}^C D_t^{\alpha_i} z_i(t)\, dt = \int_0^T z_i(t)\, {}^C D_{T-t}^{\alpha_i} \lambda_i(t)\, dt.$$

Step 5: Substitution and cancellation.

Multiply the linearized state equation by $\lambda_i(t)$, integrate over $(0, T)$, and sum over i:

$$\sum_{i=1}^{n} \int_0^T \lambda_i(t)\, {}^C D_t^{\alpha_i} z_i(t)\, dt = \sum_{i=1}^{n} \int_0^T \lambda_i(t) \left(\sum_{j=1}^{n} \frac{\partial f_i}{\partial \psi_j} z_j + \sum_{k=1}^{m} \frac{\partial f_i}{\partial \mathcal{U}_k} v_k \right) dt.$$

Using fractional integration by parts, the left-hand side becomes

$$\sum_{i=1}^{n} \int_0^T z_i(t)\, {}^C D_{T-t}^{\alpha_i} \lambda_i(t)\, dt.$$

Substituting the adjoint equation

$${}^C D_{T-t}^{\alpha_i} \lambda_i(t) = -\frac{\partial H}{\partial \psi_i},$$

we obtain exact cancellation of all terms involving $z_i(t)$.

Thus,

$$\delta J = \int_0^T \left(\frac{\partial \mathcal{L}}{\partial \mathcal{U}} + \lambda^\top \frac{\partial f}{\partial \mathcal{U}} \right) v(t)\, dt.$$

Step 6: Optimality condition.

Since $v(t)$ is arbitrary and $\mathcal{U}_{ad}$ is convex, it follows that

$$\frac{\partial \mathcal{L}}{\partial \mathcal{U}} + \lambda^\top \frac{\partial f}{\partial \mathcal{U}} = 0 \quad \text{a.e. on } (0, T),$$

that is,

$$\nabla_{\mathcal{U}} H(t, \psi^*, \mathcal{U}^*, \lambda) = 0.$$

Conclusion. We have derived the necessary optimality conditions consisting of the state system, the adjoint system, and the stationarity condition. This completes the proof.

□

Consider a two-dimensional system with nonlinear dynamics:

State system.

$$\begin{cases} {}^C D_t^{\alpha_1} \psi_1(t) = \sin(\psi_2(t)) + \mathcal{U}(t), \\ {}^C D_t^{\alpha_2} \psi_2(t) = \psi_1(t)^2 - \psi_2(t) + \mathcal{U}(t), \end{cases} \tag{5.9}$$

with initial conditions

$$x_1(0) = 1, \quad x_2(0) = 0,$$

where $0 < \alpha_1, \alpha_2 < 1$.

Cost functional (general nonlinear form).

$$J(\mathcal{U}) = \int_0^T (\psi_1(t)^2 + e^{\psi_2(t)} + \mathcal{U}(t)^4)\, dt. \tag{5.10}$$

Notice that this is non-quadratic, nonlinear in both the state and control.

Step 1: The Hamiltonian is

$$H(t, \psi, \mathcal{U}, \lambda) = \psi_1^2 + e^{\psi_2} + \mathcal{U}^4 + \lambda_1(\sin(\psi_2) + \mathcal{U}) + \lambda_2(\psi_1^2 - \psi_2 + \mathcal{U}),$$

with adjoint variables $\lambda_1(t), \lambda_2(t)$.

Step 2: Adjoint equations.

From the multi-index PMP

$${}^C D_{T-t}^{\alpha_1} \lambda_1(t) = -\frac{\partial H}{\partial \psi_1} = -(2\psi_1 + 2\lambda_2 \psi_1), \tag{5.11}$$

$${}^C D_{T-t}^{\alpha_2} \lambda_2(t) = -\frac{\partial H}{\partial \psi_2} = -(e^{x\psi 2} + \lambda_1 \cos(\psi_2) - \lambda_2). \tag{5.12}$$

Terminal conditions:

$$\lambda_1(T) = 0, \quad \lambda_2(T) = 0.$$

Step 3: Optimality condition.

$$\frac{\partial H}{\partial \mathcal{U}} = 4\mathcal{U}^3 + \lambda_1 + \lambda_2 = 0,$$

so the optimal control satisfies the nonlinear algebraic equation

$$\mathcal{U}^*(t) = -\left(\frac{\lambda_1(t) + \lambda_2(t)}{4}\right)^{1/3}.$$

Step 4: Coupled fractional system.

The optimality system consists of

$$\begin{cases} {}^C D_t^{\alpha_1} \psi_1 = \sin(\psi_2) + \mathcal{U}^*(t), \\ {}^C D_t^{\alpha_2} \psi_2 = \psi_1^2 - \psi_2 + \mathcal{U}^*(t), \\ {}^C D_{T-t}^{\alpha_1} \lambda_1 = -(2\psi_1 + 2\lambda_2 x_1), \\ {}^C D_{T-t}^{\alpha_2} \lambda_2 = -(e^{\psi_2} + \lambda_1 \cos(\psi_2) - \lambda_2), \end{cases}$$

with boundary conditions:

$$\psi_1(0) = 1, \quad \psi_2(0) = 0, \quad \lambda_1(T) = 0, \quad \lambda_2(T) = 0.$$

Remark 5.1. *(i) The control law is nonlinear in the adjoint variables due to u^4.*
(ii) The system is fully nonlinear and multi-index fractional.

(iii) The existence of optimal control follows from convexity in u (here u^4) and Lipschitz continuity in the state.

(iv) This illustrates that the PMP applies beyond linear-quadratic problems: it gives necessary conditions even for nonlinear multi-index systems.

Conclusion. The multi-index fractional Pontryagin maximum principle establishes necessary optimality conditions for linear-quadratic control systems governed by componentwise Caputo fractional dynamics. The result generalizes the classical maximum principle by incorporating heterogeneous memory effects through distinct fractional orders. The derived state, adjoint, and optimality conditions provide a theoretical foundation for further analytical investigations. Some algorithms for the numerical solution of the proposed system, as well as related numerical applications, will be addressed in future work on this topic.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interest regarding the publication of this manuscript.

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