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Research article

Self-adjoint extensions of quantum graphs with eigenparameter-dependent vertex conditions: An operator-theoretic and symplectic geometry approach

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Abstract: We studied differential operators on compact metric graphs subject to vertex conditions that depend linearly on the spectral parameter. Such eigenparameter-dependent conditions arise naturally in several models of mathematical physics but complicate the standard operator-theoretic formulation of the problem. To address this difficulty, we introduced an extended Hilbert space framework that converts the original boundary value problem into an equivalent eigenvalue problem for an operator acting on a larger space. Within this setting, we established a characterization of self-adjoint realizations in terms of algebraic conditions on the matrices defining the vertex relations. The analysis was further interpreted using Hermitian symplectic geometry, which provides a natural description of admissible boundary conditions. As a consequence of the self-adjointness results, we showed that the associated operator possesses a compact resolvent and therefore has a purely discrete spectrum consisting of real eigenvalues with finite multiplicities. An illustrative example on a star graph was included to demonstrate the applicability of the framework and the resulting spectral properties. We also established connections to time-dependent problems: through Fourier analysis, eigenparameter-dependent conditions become first-order dynamic conditions for the heat equation, second-order conditions for the wave equation, and quantum dot models for the Schrödinger equation. Our framework unifies operator-theoretic and symplectic geometric approaches, providing verifiable criteria for self-adjointness applicable to arbitrary compact metric graphs. The results extend existing approaches for interval and simple graph models to general compact metric graphs and provide a systematic method for treating eigenparameter-dependent vertex conditions in the spectral theory of quantum graphs.

Keywords: quantum graphs; eigenparameter-dependent vertex conditions; self-adjoint operators; symplectic geometry; extension theory; dynamic boundary conditions; spectral analysis
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1. Introduction

Differential operators on metric graphs, commonly referred to as quantum graphs, provide a natural mathematical framework for the analysis of wave propagation and quantum dynamics on network-like structures. Over the past decades, quantum graph models have attracted considerable attention due to their applications in mathematical physics, spectral theory, and engineering systems such as quantum wires, optical waveguides, and transport processes in complex networks [1]. From a mathematical point of view, these models combine techniques from functional analysis, spectral theory, and the theory of differential equations on domains with nontrivial topology [2].

A fundamental aspect in the analysis of quantum graph operators is the formulation of appropriate vertex conditions ensuring that the associated differential operator defines a self-adjoint operator in the underlying Hilbert space. Self-adjointness is essential for the development of a consistent spectral theory and guarantees important physical properties such as real spectra and unitary time evolution. A well-established theory shows that self-adjoint realizations of second-order differential operators on metric graphs can be characterized through algebraic relations between boundary values of functions and their derivatives at the vertices. This matrix-based description [3] of vertex conditions has become a standard tool in the spectral analysis of quantum graphs and has been extensively developed in the literature [1].

In most classical studies, the vertex conditions are assumed to be independent of the spectral parameter. However, in a variety of applications, one encounters boundary relations that involve the eigenvalue parameter explicitly. Such eigenparameter-dependent conditions arise, for instance, in models of thin structures shrinking to graphs [4,5], photonic crystals [6], and networks with internal vertex structures [7,8]. The appearance of the spectral parameter in the boundary relations introduces significant analytical difficulties, since the resulting boundary value problem can no longer be directly associated with a standard self-adjoint operator acting in the natural Hilbert space of the system.

Eigenparameter-dependent boundary value problems have been investigated in several contexts, particularly for differential equations posed on intervals or for special graph configurations such as star graphs, see [9] and references therein. In these settings, various techniques have been developed to handle the dependence on the spectral parameter, including the introduction of auxiliary variables or the use of extended state spaces. Nevertheless, most available results remain restricted to relatively simple geometries, and a systematic operator-theoretic treatment for general compact metric graphs has received comparatively little attention.

In [9], we considered two boundary value problems on a compact star graph where some vertex conditions depend on the spectral parameter. In the first problem, we imposed Neumann-Kirchhoff conditions at the central vertex and eigenparameter-dependent conditions at the boundary vertices. On the contrary, in the second problem, we considered an eigenparameter-dependent condition at the central vertex, whereas we applied separated conditions at the boundary vertices. By generalizing the operator-theoretic approach in [10], we first introduced suitable operators within extended Hilbert spaces and proved the self-adjointness of the related operators. However, we were not able to characterize all self-adjoint realizations; instead, we proved only the self-adjointness of the two operators. Moreover, these two operators were defined via specific vertex conditions (e.g., Neumann-Kirchhoff or separated). In this paper, we aim to expand on our previous findings [9] in two significant ways. First, we aim to address the eigenparameter-dependent vertex conditions on a general compact metric graph, thereby extending the implications of our results to a wider context. Second, rather than enforcing specific vertex conditions, such as Neumann-Kirchhoff or separated conditions, we will permit the vertex conditions to take the most general form.

Despite the substantial progress in the spectral theory of quantum graphs, the case where vertex conditions depend explicitly on the spectral parameter remains comparatively underdeveloped, particularly for general compact graph topologies. Existing studies have largely focused on interval problems or on special graph structures where the analysis can be carried out more directly. For general metric graphs, however, the presence of the spectral parameter in the vertex relations introduces additional technical complications because the associated boundary value problem does not correspond in a straightforward way to a self-adjoint operator in the underlying Hilbert space. Consequently, the classical extension theory commonly used in the analysis of quantum graph operators cannot be applied directly in this setting.

The present work addresses this difficulty by developing an operator-theoretic framework for differential operators on compact metric graphs subject to vertex conditions that depend linearly on the spectral parameter. The main idea is to embed the original boundary value problem into an extended Hilbert space in which the eigenparameter appearing in the vertex relations can be incorporated into additional components of the state vector. Within this formulation the problem can be recast as a standard eigenvalue problem for a suitably defined operator acting on the enlarged space. This approach makes it possible to apply methods from the theory of self-adjoint extensions and provides a natural framework for analyzing the spectral properties of the system.

Within this extended Hilbert space setting, using the symplectic geometry approach, we obtain an explicit algebraic characterization of self-adjoint realizations in terms of the matrices defining the vertex relations. The resulting conditions generalize the classical matrix characterization of self-adjoint vertex conditions for quantum graph operators to the case where the boundary relations depend on the spectral parameter. The analysis can also be interpreted geometrically in terms of Hermitian symplectic structures associated with the boundary data of the graph.

An additional consequence of the developed framework is that, for compact metric graphs, the resulting operator possesses a compact resolvent. As a result, the spectrum consists entirely of real eigenvalues of finite multiplicity that accumulate only at infinity. Thus, the spectral problem associated with eigenparameter-dependent vertex conditions exhibits a discrete spectral structure analogous to that of classical quantum graph models.

The novel contributions of this paper can be summarized as follows:

(1) **Generalization to arbitrary compact metric graphs:** While previous studies were confined to intervals or star graphs (see [9] and references therein), we consider a general compact metric graph. This significantly extends the applicability of the results to more complex network structures encountered in real-world applications.

(2) **Characterization of self-adjointness via symplectic geometry:** We provide a classification of all self-adjoint realizations for operators with eigenparameter-dependent vertex conditions. By employing a Hermitian symplectic framework, we show that self-adjointness is equivalent to the condition that each vertex subspace M_v is Lagrangian in $\mathbb{C}^{2d_v}$. This generalizes the classical results of Kostrykin and Schrader [3] to the eigenparameter-dependent setting.

(3) **Explicit verifiable conditions in terms of vertex matrices:** Theorem 3.6 translates the abstract geometric condition into a concrete algebraic criterion: at every vertex $v \in V_1$, the matrix

$C_v = (A_v \ \ B_v)$ must have maximal rank and $A_v B_v^*$ must be self-adjoint. These conditions are straightforward to verify in practice, making the theory accessible for applications.

(4) **Compactness of the resolvent:** We prove that the resolvent of a self-adjoint realization on a compact metric graph is compact (Theorem 3.7).

(5) **Detailed spectral analysis for an example:** We illustrate the theoretical results through a concrete example of a star graph with three edges. We derive the secular determinant and compute eigenvalues numerically.

(6) **Connection to time-dependent problems:** We establish a direct link between eigenparameter-dependent vertex conditions and dynamic boundary conditions for the heat, wave, and Schrödinger equations.

(7) **Unified operator-theoretic and geometric approach:** We bridge two powerful methodologies—the extended Hilbert space technique [10] and symplectic extension theory [11]—to provide a comprehensive framework for eigenparameter-dependent problems. This synthesis not only clarifies the underlying structure but also opens new avenues for studying more general boundary conditions (e.g., nonlinear or time-dependent).

The paper is organized as follows. Section 2 reviews basic definitions and tools, including quantum graphs and symplectic geometry. Section 3 presents the main results, including the construction of the extended Hilbert space and operator, the characterization of self-adjointness, and the compactness of the resolvent for self-adjoint realizations. Section 4 provides a detailed example and spectral analysis of it. Section 5 establishes the connection to time-dependent problems. Finally, Section 6 concludes the paper with a summary and outlook for future research.

2. Basic definitions, notations, and tools

2.1. Basic notions on quantum graphs

We summarise the basic definitions and properties related to quantum graphs. The material presented here is taken from [1].

Definition 2.1. *Consider a graph* $\Gamma = (V, E)$*, where* V *and* E *denote the sets of vertices and edges, respectively. If each edge* $e \in E$ *is assigned an interval* $[0, l_e]$*, then* Γ *is called a metric graph. A metric graph which has a finite number of edges with finite lengths is called a compact metric graph.*

This study is concerned solely with compact metric graphs; thus, we assume all metric graphs are compact from here onward. Moreover, we assume that there are no isolated vertices, loops, or multiple edges, for simplicity and without loss of generality. Note that one can always add a vertex of degree 2 with Neumann-Kirchhoff conditions for a second-order operator without changing the spectral properties and, thus, breaking any loop or multiple edge.

Metric graphs allow us to represent each point on an edge e by a local coordinate x_e. Consequently, when defining a function f on Γ, we consider it as a vector $f = (f_e)_{e \in E}$ of functions such that each component f_e is defined along the edge e, namely, $f_e : [0, l_e] \to \mathbb{C}$.

Since we are concerned with second-order differential operators, the functions f defined on the metric graph Γ belong to the Sobolev space $H^2(\Gamma)$. This ensures that both the first and second derivatives of f are well-defined in the L^2-sense, which is essential for our analysis of second-order operators.

The spaces $L^2(\Gamma)$ and $H^2(\Gamma)$ are defined as the direct sums:

$$L^2(\Gamma) := \bigoplus_{e\in E} L^2(0,l_e), \quad H^2(\Gamma) := \bigoplus_{e\in E} H^2(0,l_e),$$

where for each edge $e \in E$,

$$\begin{aligned} L^2(0,l_e) &:= \left\{ f_e : [0,l_e] \to \mathbb{C} \,\middle|\, f_e \text{ is measurable and } \int_0^{l_e} |f_e|^2\, dx < \infty \right\}, \\ H^2(0,l_e) &:= \left\{ f_e \in L^2(0,l_e) \,\middle|\, f_e', f_e'' \in L^2(0,l_e) \right\}. \end{aligned}$$

Definition 2.2. *A quantum graph is defined as a metric graph equipped with a differential operator acting on functions defined on the graph, accompanied by appropriate vertex conditions. Usually, the differential operator is taken as the Schrödinger operator.*

We consider the Laplacian

$$\mathcal{H} : f = (f_e)_{e\in E} \to (\mathcal{H}_e f_e)_{e\in E} = \left(-\frac{d^2}{dx_e^2} f_e(x_e) \right)_{e\in E},$$

which is acting on the compact metric graph $\Gamma = (V, E)$.

We now investigate the types of boundary conditions that yield a self-adjoint operator $\mathcal{H}$. In this work, we focus specifically on local vertex conditions, those that concern the function values and derivatives at a single vertex v.

We denote the degree of a vertex v by d_v and the set of edges containing v by E_v. Vertex conditions at v can be described in terms of a pair of $d_v \times d_v$ matrices, A_v and B_v. The associated matching rule is given by

$$A_v F(v) + B_v F'(v) = 0, \tag{2.1}$$

where $F(v)$ represents the vector of function values at v along incident edges, while $F'(v)$ represents the vector of their outgoing derivatives evaluated at v. Namely, $F(v) = (f_e(v))_{e\in E_v}$ and $F'(v) = (f_e'(v))_{e\in E_v}$ are d_v-dimensional column vectors. Here, the outgoing derivative at a vertex v is defined as

$$f_e'(v) = \begin{cases} f_e'(0), & \text{if } v \text{ is the initial vertex of } e, \\ -f_e'(l_e), & \text{if } v \text{ is the terminal vertex of } e. \end{cases} \tag{2.2}$$

Note that, in this paper, we assume all the derivatives at the vertices are outgoing derivatives (see (2.2)).

Theorem 2.1. *[3] Let D be the operator acting as $-\frac{d^2}{dx_e^2}$ on each edge e of a compact metric graph Γ such that the domain consists of functions $f \in H^2(\Gamma)$ satisfying the condition (2.1) for all v. Then, D is self-adjoint if and only if at every vertex v, we have*

(1) The rank of the matrix $(A_v \ B_v)$ is maximal;
(2) $A_v B_v^* = B_v A_v^*$.

2.2. Symplectic geometry and extension theory

We summarize the symplectic geometry approach for self-adjoint extensions of symmetric operators, see [1,2,11].

Let X be an n-dimensional complex vector space. A sesqui-linear 2-form on X is a function $\langle .,. \rangle : X \times X \to \mathbb{C}$ that is linear on the first and anti-linear (conjugate-linear) on the second argument. If for every $u \in X \setminus \{0\}$, there exists $v \in X$ such that $\langle u, v\rangle \neq 0$, then the form $\langle .,. \rangle$ is non-degenerate. If all $u, v \in X$ satisfy

$$\langle u, v\rangle = -\overline{\langle v, u\rangle},$$

then the form $\langle .,. \rangle$ is Hermitian symplectic. X is called a Hermitian symplectic space if it is equipped with a non-degenerate Hermitian symplectic sesqui-linear 2-form.

Let $B = \{e_1,\ e_2, \cdots, e_n\}$ be a basis of X. Then, one can associate the Hermitian symplectic form with an $n \times n$ matrix ω, where

$$\omega_{ij} = \langle e_i, e_j\rangle. \tag{2.3}$$

This allows us to represent

$$\langle u, v\rangle = (u, \omega v),$$

where the inner product on the right-hand side is the standard inner product associated to basis B.

A Hermitian symplectic space is called canonical if there exists a basis such that the matrix ω defined by (2.3) equal

$$\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix},$$

where I_m denotes the $m \times m$ identity matrix.

Let X be a Hermitian symplectic space and Y be a linear subspace. Then, $u, v \in X$ are skew-orthogonal if $\langle u, v\rangle = 0$. $Y^\perp$ denotes the space of all vectors skew-orthogonal to all elements in Y. Y is called isotropic if $Y \subset Y^\perp$ and Lagrangian if $Y = Y^\perp$.

Lemma 2.1 (Lemma 5 in [11])**.** *An even-dimensional Hermitian symplectic space is canonical iff it contains a Lagrangian subspace.*

Let X be a Hermitian symplectic space with even dimension n. Then, the dimension of any Lagrangian subspace in X is $m = n/2$ (see [11]).

3. Self-adjoint realizations and their characterization

Let $\Gamma = (V, E)$ be a compact metric graph where

- V is a finite set of vertices;
- E is the set of edges;
- Each edge $e \in E$ is identified with the interval $[0, l_e]$, where $0 < l_e < \infty$.

Let us consider the Hamiltonian

$$\ell : (f_e)_{e\in E} \to \left(-\frac{d^2}{dx_e^2} f_e(x_e)\right)_{e\in E} \tag{3.1}$$

on $\Gamma = (V, E)$.

We decompose the vertex set V into two nonempty and disjoint subsets:

$$V = V_1 \cup V_2,$$

where $|V_2| = n$.

We consider the boundary value problem (BVP) generated by

$$\ell f = \lambda f, \tag{3.2}$$

where $f = (f_e)_{e\in E}$ and $\lambda \in \mathbb{C}$ is the spectral parameter. For vertices $v \in V_1$, we impose the vertex condition

$$A_v F(v) + B_v F'(v) = 0, \tag{3.3}$$

where $A_v,\ B_v$ are $d_v \times d_v$ matrices. For vertices $v_i \in V_2$, $i = 1, 2, \dots, n$, we assume that

$$f \text{ is continuous at } v_i, \text{ i.e., } f(v_i) := f_e(v_i) = f_k(v_i),\ \forall e, k \in E_{v_i}, \tag{3.4}$$

and impose the eigenparameter-dependent conditions

$$\theta_{i1} f(v_i) - \theta_{i2} \sum_{e\in E_{v_i}} f_e'(v_i) = \lambda \left(\phi_{i1} f(v_i) - \phi_{i2} \sum_{e\in E_{v_i}} f_e'(v_i) \right), \tag{3.5}$$

where $\theta_{i1}, \theta_{i2}, \phi_{i1}, \phi_{i2}$ are real numbers.

Let us assume

$$\gamma_i := \phi_{i1}\theta_{i2} - \theta_{i1}\phi_{i2} > 0, \quad i = 1, 2, \dots, n. \tag{3.6}$$

To examine this BVP involving eigenparameter-dependent conditions, we introduce an operator-theoretic formulation, thereby reducing it to an eigenvalue problem of an operator in an extended Hilbert space. To this end, we define a new Hilbert space $H_1 := L^2(\Gamma) \oplus \mathbb{C}^n$ with the inner product

$$(F, G) := (f, g)_{L^2(\Gamma)} + \sum_{i=1}^{n} \frac{1}{\gamma_i} a_i \overline{b_i}, \quad F = (f, a_1, \dots, a_n), G = (g, b_1, \dots, b_n) \in H_1. \tag{3.7}$$

Note that the positivity of γ_i is required for the positive definiteness of the inner product (3.7).

We shall use the following notations:

$$R_i(f) := \theta_{i1} f(v_i) - \theta_{i2} \sum_{e\in E_{v_i}} f_e'(v_i), \quad i = 1, 2, \dots, n,$$

$$R_i'(f) := \phi_{i1} f(v_i) - \phi_{i2} \sum_{e\in E_{v_i}} f_e'(v_i), \quad i = 1, 2, \dots, n.$$

Our aim is to formulate the above BVP as an eigenvalue problem of an operator A defined in the new Hilbert space H_1. To this end, let us introduce the appropriate domain $Dom(A)$ of this operator. We define $Dom(A)$ as the subspace of H_1 consisting of vectors $F = (f, a_1, a_2, \dots, a_n) \in H_1$ such that

- $f \in H^2(\Gamma) = \bigoplus_{e\in E} H^2(0, l_e)$;
- f satisfies (3.3) and (3.4);
- $a_i = R_i'(f), \quad i = 1, 2, \dots, n.$

Now we can define the operator $A : Dom(A) \subset H_1 \to H_1$ with the formula

$$AF := A(f, a_1, a_2, \dots, a_n) = (\ell f, R_1(f), R_2(f), \dots, R_n(f)).$$

It is clear that λ is an eigenvalue of the BVP iff λ is an eigenvalue of A. This enables us to get rid of the eigenparameter in vertex condition (3.5) by extending the Hilbert space $L^2(\Gamma)$.

3.1. Minimal and maximal operators

To place the boundary value problems (3.2)–(3.5) within a rigorous extension theory framework, we first define two reference operators acting on the extended Hilbert space $H_1 = L^2(\Gamma) \oplus \mathbb{C}^n$.

Maximal operator. Let $A_{\max}$ be the operator defined on the domain

$$\mathrm{Dom}(A_{\max}) = \{G = (g, b_1, \dots, b_n) \in H_1 : g \in H^2(\Gamma),\ g \text{ is continuous at every } v_i \in V_2\}$$

by

$$A_{\max} G = (\ell g,\ R_1(g),\ \dots,\ R_n(g)),$$

where $\ell g = -g''$ componentwise, and the functionals R_i are as defined above. No vertex conditions are imposed at vertices in V_1 nor at V_2 apart from the continuity requirement. The operator $A_{\max}$ is closed and densely defined; it is the maximal operator associated with the differential expression.

Minimal operator. The minimal operator $A_{\min}$ is defined as the restriction of $A_{\max}$ to the domain

$$\begin{aligned}&\mathrm{Dom}(A_{\min})\\ &= \{F = (f, a_1, \dots, a_n) \in \mathrm{Dom}(A_{\max}) : f(v) = 0,\ f'(v) = 0,\ \forall v \in V_1,\ a_i = R_i'(f) = 0,\ \forall i = 1, \dots, n\}.\end{aligned}$$

Equivalently, $\mathrm{Dom}(A_{\min})$ consists of those vectors in $\mathrm{Dom}(A_{\max})$ whose boundary traces at all vertices vanish and whose extra components a_i are zero. A standard integration by parts shows that $A_{\min}$ is symmetric and closed. Moreover, one verifies that $A_{\min}$ is densely defined (its domain contains smooth functions with compact support in the interior of each edge) and that

$$A_{\min}^* = A_{\max}.$$

Relation to the operator A**.** The operator A defined above is a restriction of $A_{\max}$:

$$A \subset A_{\max}, \qquad \mathrm{Dom}(A) = \{F = (f, a_1, \dots, a_n) \in \mathrm{Dom}(A_{\max}) : (3.3) \text{ holds at } V_1,\ a_i = R_i'(f)\}.$$

Furthermore, $A_{\min} \subset A$ because any function satisfying the vanishing conditions trivially satisfies (3.3) and $R_i'(f) = 0$. Hence A is an extension of $A_{\min}$. The theory of self-adjoint extensions of symmetric operators (e.g., via boundary triples or symplectic geometry) can therefore be applied.

Theorem 3.1. *A is densely defined in $H_1 = L^2(\Gamma) \oplus \mathbb{C}^n$.*

Proof. We must show that $\mathrm{Dom}(A)$ is dense in H_1. The proof proceeds in three steps.

Step 1: A dense subspace of smooth functions vanishing at all vertices. Let $\mathcal{D} \subset C^\infty(\Gamma)$ be the set of all smooth functions that vanish in an open neighborhood of every vertex of Γ. For any $\phi \in \mathcal{D}$, we have $\phi(v) = 0$ and $\phi'(v) = 0$ for all vertices $v \in V$. Consequently:

- For $v \in V_1$, the vertex condition (3.3) is satisfied trivially because $F(v) = 0$ and $F'(v) = 0$.
- For $v_i \in V_2$, the continuity condition (3.4) holds (all values are zero).
- $R_i'(\phi) = \phi_{i1}\phi(v_i) - \phi_{i2}\partial\phi(v_i) = 0$ for every i.

Hence $(\phi, 0, \dots, 0) \in \mathrm{Dom}(A)$ for every $\phi \in \mathcal{D}$. Since $\mathcal{D}$ is dense in $L^2(\Gamma)$ (smooth functions with compact support in the interior of edges are dense), the set $\{(\phi, 0) : \phi \in \mathcal{D}\}$ is dense in the subspace $L^2(\Gamma) \oplus \{0\}^n \subset H_1$.

Step 2: Construction of functions ψ_i that isolate the i-th extra component. Fix $i \in \{1, \dots, n\}$ (corresponding to vertex $v_i \in V_2$). We construct a function $\psi_i \in H^2(\Gamma)$ with the following properties:

(1) $\mathrm{supp}(\psi_i)$ is contained in a small neighborhood of v_i that contains no other vertex.
(2) ψ_i is continuous at v_i: for all edges $e, e' \in E_{v_i}$, $\psi_{i,e}(v_i) = \psi_{i,e'}(v_i)$.
(3) $R_i'(\psi_i) = 1$ and for all $j \neq i$, $R_j'(\psi_i) = 0$ (the latter holds automatically because the support avoids v_j).
(4) The L^2-norm $\|\psi_i\|_{L^2(\Gamma)}$ can be made arbitrarily small by shrinking the support.

Explicit construction. Let $r > 0$ be smaller than half the length of every edge incident to v_i and smaller than the distance from v_i to any other vertex. On each incident edge $e \in E_{v_i}$, parameterize x_e so that $x_e = 0$ at v_i (the outgoing derivative convention will be respected). Define a smooth bump function $\eta : [0, \infty) \to [0, 1]$ such that $\eta(s) = 1$ for $s \leq r/2$, $\eta(s) = 0$ for $s \geq r$, and η is decreasing and in C^∞. On edge e, set

$$\psi_{i,e}(x_e) = c_i \cdot \eta(x_e),$$

where $c_i \in \mathbb{C}$ is a constant to be determined. Extend ψ_i by zero on all other edges. Clearly $\mathrm{supp}(\psi_i)$ lies within distance r of v_i and contains no other vertex. The function is continuous at v_i because all incident edges have the same value c_i. Its outgoing derivative at v_i is

$$\partial\psi_i(v_i) = \sum_{e \in E_{v_i}} \psi_{i,e}'(0) = \sum_{e \in E_{v_i}} c_i \cdot \eta'(0) = c_i \cdot d_{v_i} \cdot \eta'(0).$$

Since $\eta'(0) = 0$ (a flat bump), we actually get $\partial\psi_i(v_i) = 0$ if we choose η with $\eta'(0) = 0$. That simplifies the computation. However, we can also choose η with a non-zero derivative at 0; the important point is that both $\psi_i(v_i) = c_i$ and $\partial\psi_i(v_i)$ are proportional to c_i. In fact, we can directly prescribe the desired value of $R_i'(\psi_i) = \phi_{i1}\psi_i(v_i) - \phi_{i2}\partial\psi_i(v_i)$. Because this is a linear functional in c_i, we can solve for c_i to make $R_i'(\psi_i) = 1$, provided the coefficient is non-zero. The coefficient is non-zero exactly because $\gamma_i = \phi_{i1}\theta_{i2} - \theta_{i1}\phi_{i2} > 0$ implies ϕ_{i1} and ϕ_{i2} are not both zero, and the linear functional R_i' is non-trivial on the one-dimensional space of constant multiples of our bump. More concretely: choose a bump with $\eta'(0) = 0$ (e.g., $\eta(s) = 1$ for $s \leq r/2$ with a smooth cut-off). Then $\partial\psi_i(v_i) = 0$, so $R_i'(\psi_i) = \phi_{i1}c_i$. Setting $c_i = 1/\phi_{i1}$ works if $\phi_{i1} \neq 0$. If $\phi_{i1} = 0$, then $\phi_{i2} \neq 0$ (since $\gamma_i > 0$), and we can instead choose a bump with $\eta(0) = 0$ and $\eta'(0) \neq 0$ (e.g., a linear ramp) to make $\psi_i(v_i) = 0$ and $\partial\psi_i(v_i) = c_i'$. Then, $R_i'(\psi_i) = -\phi_{i2}c_i'$, and we set $c_i' = -1/\phi_{i2}$. Thus in all cases, we can achieve $R_i'(\psi_i) = 1$. Moreover, the L^2-norm satisfies $\|\psi_i\|_{L^2}^2 \leq |c_i|^2 \sum_{e \in E_{v_i}} r \cdot \max|\eta|^2$, which tends to 0 as $r \to 0$. By taking r sufficiently small, we can make $\|\psi_i\|_{L^2(\Gamma)}$ as small as desired.

Step 3: Approximation of an arbitrary vector. Let $F = (f, a_1, \dots, a_n) \in H_1$ be arbitrary. Choose a sequence $(\phi_k)_{k \geq 1} \subset \mathcal{D}$ such that $\phi_k \to f$ in $L^2(\Gamma)$ (this is possible because $\mathcal{D}$ is dense). For each i, select ψ_i as constructed in Step 2 with support so small that

$$\|\psi_i\|_{L^2(\Gamma)} \leq \frac{\varepsilon}{n \max(1, |a_i|)} \quad \text{and} \quad \mathrm{supp}(\psi_i) \cap \mathrm{supp}(\psi_j) = \varnothing \text{ for } i \neq j.$$

Also, the supports avoid all vertices $v \in V_1$ (so that ψ_i vanishes on V_1). This is possible because each $v_i \in V_2$ is isolated and we can take r small enough.

Define

$$f_k = \phi_k + \sum_{i=1}^{n} a_i \psi_i.$$

Then:

- For $v \in V_1$: Since ϕ_k and all ψ_i vanish in a neighborhood of v, we have $f_k(v) = 0$ and $f_k'(v) = 0$. Hence the vertex condition (3.3) is satisfied.
- For $v_j \in V_2$: Continuity of f_k at v_j follows because ϕ_k vanishes near v_j (as $\phi_k \in \mathcal{D}$) and each ψ_i is continuous at v_j by construction. Moreover, for $i \neq j$, ψ_i vanishes near v_j, and ψ_j is continuous by construction.
- For each i, $R_i'(f_k) = R_i'(\phi_k) + \sum_{j=1}^{n} a_j R_i'(\psi_j) = 0 + a_i \cdot 1 = a_i$ because $R_i'(\phi_k) = 0$ (since ϕ_k vanishes at all vertices) and $R_i'(\psi_j) = \delta_{ij}$.

Therefore $F_k := (f_k, a_1, \ldots, a_n) = (f_k, R_1'(f_k), \ldots, R_n'(f_k)) \in \mathrm{Dom}(A)$.

Now estimate the difference:

$$\|F_k - F\|_{H_1}^2 = \|f_k - f\|_{L^2(\Gamma)}^2 \leq \left(\|\phi_k - f\|_{L^2(\Gamma)} + \sum_{i=1}^{n} |a_i| \|\psi_i\|_{L^2(\Gamma)} \right)^2 .$$

Given $\varepsilon > 0$, choose k large enough so that $\|\phi_k - f\|_{L^2(\Gamma)} < \varepsilon/2$, and choose ψ_i (by taking their supports sufficiently small) such that $\sum_{i=1}^{n} |a_i| \|\psi_i\|_{L^2(\Gamma)} < \varepsilon/2$. Then $\|F_k - F\|_{H_1} < \varepsilon$. Hence F is a limit of vectors in $\mathrm{Dom}(A)$. Since F was arbitrary, $\mathrm{Dom}(A)$ is dense in H_1. □

Theorem 3.2. *A is closed.*

Proof. Let $(F_m)_{m \geq 1} \subset \mathrm{Dom}(A)$ be a sequence such that

$$F_m \to F \quad \text{in } H_1, \qquad AF_m \to G \quad \text{in } H_1,$$

with $F = (f, a_1, \ldots, a_n)$ and $G = (g_0, g_1, \ldots, g_n)$. Write

$$F_m = (f_m, a_{1,m}, \ldots, a_{n,m}), \quad a_{i,m} = R_i'(f_m),$$

$$AF_m = (\ell f_m,\ R_1(f_m), \ldots, R_n(f_m)).$$

Since $f_m \to f$ in $L^2(\Gamma)$ and $\ell f_m \to g_0$ in $L^2(\Gamma)$, and $\ell f_m = -f_m''$, we have $f_m'' \to -g_0$ in $L^2(\Gamma)$. On each edge e (a compact interval), the operator $\frac{d^2}{dx^2}$ with domain $H^2(0, l_e)$ is closed as an operator from L^2 to L^2. Hence the limit function f belongs to $H^2(\Gamma)$ and satisfies $f'' = -g_0$ componentwise, i.e., $\ell f = g_0$.

For every vertex $v_i \in V_2$, the trace maps $f \mapsto f(v_i)$ and $f \mapsto \partial f(v_i) := \sum_{e \in E_{v_i}} f_e'(v_i)$ are continuous from $H^2(\Gamma)$ to $\mathbb{C}$. Therefore

$$f_m(v_i) \to f(v_i), \qquad \partial f_m(v_i) \to \partial f(v_i).$$

Consequently,

$$a_{i,m} = R_i'(f_m) = \phi_{i1} f_m(v_i) - \phi_{i2} \partial f_m(v_i) \ \longrightarrow\ \phi_{i1} f(v_i) - \phi_{i2} \partial f(v_i) = R_i'(f).$$

By assumption, $a_{i,m} \to a_i$ in $\mathbb{C}$. Hence $a_i = R'_i(f)$; in particular, the required relation for membership in Dom(A) holds.

For vertices $v \in V_1$, we have, for each m, $A_v F_m(v) + B_v F'_m(v) = 0$. The continuity of the trace $H^2(\Gamma) \ni f \mapsto (F(v), F'(v)) \in \mathbb{C}^{2d_v}$ implies $F_m(v) \to F(v)$ and $F'_m(v) \to F'(v)$. Hence the limit satisfies $A_v F(v) + B_v F'(v) = 0$. For $v_i \in V_2$, continuity of f at v_i follows from the fact that each f_m is continuous at v_i and the trace is continuous. Thus f fulfills all vertex conditions (3.3) and (3.4).

The functionals R_i are also continuous from $H^2(\Gamma)$ to $\mathbb{C}$ because they involve only point evaluations of f and its outgoing derivatives. Hence $R_i(f_m) \to R_i(f)$. By assumption, the $(i+1)$-th component of AF_m converges: $R_i(f_m) \to g_i$. Therefore $g_i = R_i(f)$ for each i.

We have shown that $F = (f, R'_1(f), \ldots, R'_n(f)) \in \mathrm{Dom}(A)$ and that

$$AF = (\ell f, R_1(f), \ldots, R_n(f)) = (g_0, g_1, \ldots, g_n) = G.$$

Hence A is closed. □

Now, we shall determine the conditions on the matrices in (3.3), which give rise to a self-adjoint operator A.

For $F = (f, a_1, a_2, \ldots, a_n), G = (g, b_1, b_2, \ldots, b_n) \in Dom(A)$, we have

$$(F, AG) - (AF, G) = \sum_{e\in E} \int_0^{l_e} \left[f_e(-\overline{g''_e}) - (-f''_e)\overline{g_e} \right] dx_e + \sum_{i=1}^{n} \frac{1}{\gamma_i} \left(R'_i(f)\overline{R_i(g)} - R_i(f)\overline{R'_i(g)} \right).$$

Integrating by parts, we obtain

$$(F, AG) - (AF, G) = \sum_{e\in E} \left[f'_e(x_e)\overline{g_e(x_e)} - f_e(x_e)\overline{g'_e(x_e)} \right] \Big|_0^{l_e} + \sum_{i=1}^{n} \frac{1}{\gamma_i} \left(R'_i(f)\overline{R_i(g)} - R_i(f)\overline{R'_i(g)} \right).$$

Moreover, using the vectors $F(v)$ and $F'(v)$, we can write

$$(F, AG) - (AF, G) = \sum_{v\in V} (F(v), G'(v)) - (F'(v), G(v)) + \sum_{i=1}^{n} \frac{1}{\gamma_i} \left(R'_i(f)\overline{R_i(g)} - R_i(f)\overline{R'_i(g)} \right),$$

where on the right-hand side of the equation, the inner product is the standard one in $\mathbb{C}^{d_v}$. The decomposition $V = V_1 \cup V_2$ allows us to represent the last equation

$$\begin{aligned}(F, AG) - (AF, G) = &\sum_{v\in V_1} (F(v), G'(v)) - (F'(v), G(v)) + \sum_{i=1}^{n} \left(f(v_i) \sum_{e\in E_{v_i}} \overline{g'_e(v_i)} - \overline{g(v_i)} \sum_{e\in E_{v_i}} f'_e(v_i) \right) \\ &+ \sum_{i=1}^{n} \frac{1}{\gamma_i} \left(R'_i(f)\overline{R_i(g)} - R_i(f)\overline{R'_i(g)} \right). \qquad (3.8)\end{aligned}$$

Let $i \in \{1, 2, \ldots, n\}$ and let us denote

$$\partial f(v_i) := \sum_{e\in E_{v_i}} f'_e(v_i), \quad \partial g(v_i) := \sum_{e\in E_{v_i}} g'_e(v_i).$$

Then, we have

$$\begin{aligned}R_i'(f)\overline{R_i(g)} - R_i(f)\overline{R_i'(g)} &= [\phi_{i1}f(v_i) - \phi_{i2}\partial f(v_i)][\theta_{i1}\overline{g(v_i)} - \theta_{i2}\overline{\partial g(v_i)}]\\ &\quad - [\theta_{i1}f(v_i) - \theta_{i2}\partial f(v_i)][\phi_{i1}\overline{g(v_i)} - \phi_{i2}\overline{\partial g(v_i)}].\end{aligned}$$

Expanding the products, we see that

$$R_i'(f)\overline{R_i(g)} - R_i(f)\overline{R_i'(g)} = f(v_i)\overline{\partial g(v_i)}[\theta_{i1}\phi_{i2} - \phi_{i1}\theta_{i2}] + \partial f(v_i)\overline{g(v_i)}[\phi_{i1}\theta_{i2} - \theta_{i1}\phi_{i2}].$$

This equation, indeed, motivates the definition of γ_i. Now, using the definition of γ_i, the last equation simplifies to

$$R_i'(f)\overline{R_i(g)} - R_i(f)\overline{R_i'(g)} = -\gamma_i f(v_i)\overline{\partial g(v_i)} + \gamma_i\partial f(v_i)\overline{g(v_i)}.$$

Therefore, we obtain that

$$f(v_i)\overline{\partial g(v_i)} - \partial f(v_i)\overline{g(v_i)} + \frac{1}{\gamma_i}\left(R_i'(f)\overline{R_i(g)} - R_i(f)\overline{R_i'(g)}\right) = 0. \tag{3.9}$$

Consequently, Eq (3.8) reduces to

$$(F, AG) - (AF, G) = \sum_{v\in V_1}(F(v), G'(v)) - (F'(v), G(v)). \tag{3.10}$$

Using the right-hand side of (3.10), we can use a symplectic geometry approach to characterize all matrices $\{A_v, B_v : v \in V_1\}$ that give rise to a self-adjoint operator A.

We assume $v \in V_1$ hereafter. First, we decompose every column vector $x \in \mathbb{C}^{2d_v}$ into $x = \begin{pmatrix}x_1\\ x_2\end{pmatrix}$, where x_1, x_2 are d_v-dimensional column vectors. Next, let us define a sesqui-linear form on $\mathbb{C}^{2d_v}$:

$$\langle x, y\rangle_v := (x_1, y_2) - (x_2, y_1), \quad x, y \in \mathbb{C}^{2d_v}, \tag{3.11}$$

where the inner product is the standard inner product in $\mathbb{C}^{d_v}$. It is easy to show that (3.11) defines a Hermitian symplectic form on $\mathbb{C}^{2d_v}$.

The vertex condition (3.3) can be written in the form

$$C_vF_v = 0, \quad F_v := \begin{pmatrix}F(v)\\ F'(v)\end{pmatrix}, \quad C_v := (A_v\ \ B_v),$$

and they induce a linear subspace of $\mathbb{C}^{2d_v}$ for each v. Namely, we define

$$M_v := \{F_v = \begin{pmatrix}F(v)\\ F'(v)\end{pmatrix} \in \mathbb{C}^{2d_v} :\ f_e \in H^2(0, l_e),\ \forall e \in E_v,\ C_vF_v = 0\}.$$

We can easily extend the Hermitian symplectic form (3.11) to $T := \bigoplus_{v\in V_1}\mathbb{C}^{2d_v}$. Namely, for $X = \bigoplus_{v\in V_1}X^{(v)},\ Y = \bigoplus_{v\in V_1}Y^{(v)} \in T$, we define

$$\langle X, Y\rangle := \sum_{v\in V_1}\langle X^{(v)}, Y^{(v)}\rangle_v. \tag{3.12}$$

Now we can obtain a linear subspace $M := \bigoplus_{v\in V_1} M_v$ of T by collecting individual subspaces M_v. Moreover, there is a one-to-one correspondence between

$$\tilde{F} = \bigoplus_{v\in V_1} F_v = \bigoplus_{v\in V_1} \begin{pmatrix} F(v) \\ F'(v) \end{pmatrix} \in M$$

and

$$F = (f, R_1'(f), R_2'(f), \dots, R_n'(f)) \in Dom(A).$$

Hereafter, we shall identify the vectors

$$\tilde{F} = \bigoplus_{v\in V_1} F_v = \bigoplus_{v\in V_1} \begin{pmatrix} F(v) \\ F'(v) \end{pmatrix} \in M$$

and

$$F = (f, R_1'(f), R_2'(f), \dots, R_n'(f)) \in Dom(A).$$

Let $F = (f, a_1, a_2, \dots, a_n), G = (g, b_1, b_2, \dots, b_n) \in Dom(A)$. Then, using (3.10) and (3.12), one obtains the key relation

$$(F, AG) - (AF, G) = 0 \Leftrightarrow \langle \tilde{F}, \tilde{G} \rangle = 0. \tag{3.13}$$

The spaces introduced above form the following coherent picture:

- For each vertex $v \in V_1$, the boundary values F_v live in $\mathbb{C}^{2d_v}$, which we equip with the symplectic form $\langle \cdot, \cdot \rangle_v$ from (3.11).
- The vertex condition (3.3) selects a linear subspace $M_v \subset \mathbb{C}^{2d_v}$ (the set of admissible boundary data at v).
- Collecting all vertices, the global trace space is $T = \bigoplus_{v\in V_1} \mathbb{C}^{2d_v}$ with the symplectic form $\langle \cdot, \cdot \rangle$ from (3.12), and the admissible global traces form $M = \bigoplus_v M_v \subset T$.
- For any $F \in \mathrm{Dom}(A)$, its boundary values give an element $\tilde{F} \in M$; conversely, every $\tilde{F} \in M$ corresponds to some $F \in \mathrm{Dom}(A)$.
- The integration-by-parts formula (3.10) then becomes $(F, AG) - (AF, G) = \langle \tilde{F}, \tilde{G} \rangle$ for all $F, G \in \mathrm{Dom}(A)$. Hence the symplectic orthogonality relations in T directly control the symmetry and self-adjointness of A.

This geometric viewpoint is made precise in the following theorems, where the isotropy of M corresponds to symmetry (Theorem 3.3) and the Lagrangianity of M (i.e., $M = M^{\perp}$) corresponds to self-adjointness (Theorem 3.4). The algebraic criteria in Theorem 3.6 follow immediately from the characterization of Lagrangian subspaces in $\mathbb{C}^{2d_v}$.

Theorem 3.3. *The following are equivalent:*

(i) A is symmetric.
(ii) M is an isotropic subspace of T.
(iii) M_v is an isotropic subspace of $\mathbb{C}^{2d_v}$ for every $v \in V_1$.

Proof. The equivalence of (ii) with (iii) follows directly from the fact that M is a direct sum of M_v for $v \in V_1$. Now let us prove the equivalence of (i) with (ii). Suppose that A is symmetric. Then, we have

$$(F, AG) - (AF, G) = 0, \quad \forall F = (f, a_1, a_2, \dots, a_n), G = (g, b_1, b_2, \dots, b_n) \in Dom(A).$$

Next, the equivalence (3.13) yields

$$\langle \tilde{F}, \tilde{G} \rangle = 0, \quad \forall \tilde{F}, \tilde{G} \in M.$$

This implies $M \subset M^{\perp}$, i.e., M is isotropic.

Conversely, let M be an isotropic subspace of T. Then, we have $M \subset M^{\perp}$ and this implies

$$\langle \tilde{F}, \tilde{G} \rangle = 0, \quad \forall \tilde{F}, \tilde{G} \in M.$$

Finally, the equivalence (3.13) gives

$$(F, AG) - (AF, G) = 0, \quad \forall F = (f, a_1, a_2, \ldots, a_n), G = (g, b_1, b_2, \ldots, b_n) \in Dom(A).$$

As a result, A is symmetric. ◻

Theorem 3.4. *The following are equivalent:*

(i) A is self-adjoint.
(ii) M is a Lagrangian subspace of T.
(iii) M_v is a Lagrangian subspace of $\mathbb{C}^{2d_v}$ for every $v \in V_1$.

Proof. **(ii) ⇔ (iii).** The symplectic form on T is the direct sum of the forms on each $\mathbb{C}^{2d_v}$. Hence for a direct sum subspace $M = \bigoplus_v M_v$,

$$M^{\perp} = \bigoplus_v M_v^{\perp}.$$

Therefore $M = M^{\perp}$ if and only if $M_v = M_v^{\perp}$ for every $v \in V_1$, i.e., each M_v is Lagrangian. Thus (ii) and (iii) are equivalent.

(i) ⇔ (ii). Recall from (3.13) that for every $F, G \in \mathrm{Dom}(A)$, we have

$$(F, AG) - (AF, G) = \sum_{v \in V_1} \big((F(v), G'(v)) - (F'(v), G(v))\big) = \langle \tilde{F}, \tilde{G} \rangle, \tag{3.14}$$

where $\tilde{F} = \bigoplus_{v \in V_1} \begin{pmatrix} F(v) \\ F'(v) \end{pmatrix} \in M$ and $\tilde{G}$ can be found in a similar way.

Now take any $F \in \mathrm{Dom}(A)$ and any $G \in \mathrm{Dom}(A_{\max})$. Using integration by parts on each edge and the definition of the inner product in H_1, we obtain

$$\begin{aligned}(AF, G) = (f, \ell g)_{L^2(\Gamma)} &+ \sum_{i=1}^{n} \frac{1}{\gamma_i} R_i(f) \overline{b_i} - \sum_{v \in V_1} \big((F(v), G'(v)) - (F'(v), G(v))\big) \\ &- \sum_{i=1}^{n} \Big(f(v_i) \overline{\partial g(v_i)} - \partial f(v_i) \overline{g(v_i)} \Big).\end{aligned} \tag{3.15}$$

The last sum over i is precisely the contribution from the vertices $v_i \in V_2$.

Using (3.9), (3.15) reduces to

$$(AF, G) = (f, \ell g)_{L^2(\Gamma)} - \sum_{v \in V_1} \big((F(v), G'(v)) - (F'(v), G(v))\big) + \sum_{i=1}^{n} \frac{1}{\gamma_i} R_i(f) \overline{b_i}$$

$$+\sum_{i=1}^{n}\frac{1}{\gamma_i}\Big(R_i'(f)\overline{R_i(g)}-R_i(f)\overline{R_i'(g)}\Big).$$

Combining the terms containing $R_i(f)$, we obtain

$$\begin{aligned}(AF,G)=(f,\ell g)_{L^2(\Gamma)}-\sum_{v\in V_1}\big((F(v),G'(v))-(F'(v),G(v))\big)\\+\sum_{i=1}^{n}\frac{1}{\gamma_i}R_i(f)(\overline{b_i}-\overline{R_i'(g)})+\sum_{i=1}^{n}\frac{1}{\gamma_i}R_i'(f)\,\overline{R_i(g)}.\end{aligned}\tag{3.16}$$

On the other hand, for any $H=(h_0,h_1,\dots,h_n)\in H_1$,

$$(F,H)=(f,h_0)_{L^2(\Gamma)}+\sum_{i=1}^{n}\frac{1}{\gamma_i}R_i'(f)\overline{h_i}.\tag{3.17}$$

Now $G\in\mathrm{Dom}(A^*)$ iff there exists $H\in H_1$ such that $(AF,G)=(F,H)$ for all $F\in\mathrm{Dom}(A)$. From (3.16) and (3.17), we see that a natural candidate is $H=A_{\max}G$, i.e., $h_0=\ell g$ and $h_i=R_i(g)$. Then

$$(F,A_{\max}G)=(f,\ell g)_{L^2(\Gamma)}+\sum_{i=1}^{n}\frac{1}{\gamma_i}R_i'(f)\,\overline{R_i(g)}.$$

Subtracting (3.16) from this yields

$$(F,A_{\max}G)-(AF,G)=\sum_{v\in V_1}\big((F(v),G'(v))-(F'(v),G(v))\big)-\sum_{i=1}^{n}\frac{1}{\gamma_i}R_i(f)(\overline{b_i}-\overline{R_i'(g)}).\tag{3.18}$$

To deduce that the coefficients in (3.18) must vanish individually for all $F\in\mathrm{Dom}(A)$, we construct explicit test functions that isolate each term.

Let $v_0\in V_1$ have degree d_{v_0}. Choose a coordinate system such that each incident edge e is parametrized by $x_e\in[0,l_e]$ with $x_e=0$ at v_0. For any vectors $\alpha,\beta\in\mathbb{C}^{d_{v_0}}$, we can construct a function $f\in H^2(\Gamma)$ with the following properties:

- The support of f is contained in a small neighborhood of v_0 that contains no other vertex.
- On each edge e incident to v_0, f is a smooth bump function satisfying $f_e(0)=\alpha_e$ and $f_e'(0)=\beta_e$, and f_e vanishes identically on the rest of Γ.
- For all other vertices $v\neq v_0$, $f(v)=0$ and $f'(v)=0$.
- For every $v_i\in V_2$, $f(v_i)=0$ and $\partial f(v_i)=0$ (since the support avoids V_2).

Such a function exists by taking, for instance, cubic polynomials matching the prescribed boundary values at 0 and decaying to zero before reaching the other endpoint.

For a fixed vertex $v_0\in V_1$ and fixed indices $p,q\in\{1,\dots,d_{v_0}\}$, choose α and β such that

$$(F(v_0),G'(v_0))-(F'(v_0),G(v_0))=\alpha_p\,\overline{G_q'(v_0)}-\beta_p\,\overline{G_q(v_0)}$$

is the only possibly non-zero term, while for all other components, the contribution vanishes. This is achieved by taking α with a single non-zero entry in position p, and β with a single non-zero entry in position q (or zero, depending on which term we want to test). Because the symplectic form is non-degenerate, by varying these choices, we can isolate each coordinate of $G(v_0)$ and $G'(v_0)$.

The function f constructed above does not necessarily satisfy the vertex condition $A_vF(v) + B_vF'(v) = 0$ at v_0 (or at other vertices in V_1). However, we can modify f by adding a correction term that:

- belongs to $\ker C_v$ (i.e., satisfies the homogeneous condition) at all $v \in V_1$,
- does not change the prescribed boundary values $(F(v), F'(v))$ at v_0 (e.g., by adding a function that vanishes to second order at v_0),
- and vanishes on all other vertices.

This is possible because the space of functions satisfying the homogeneous condition and having given boundary data is non-empty whenever the data are compatible with the condition. Since (3.3) imposes exactly d_{v_0} independent linear constraints, we can always adjust a function by a term that does not affect the boundary values we wish to test. Consequently, for any prescribed complex numbers $\delta_1, \dots, \delta_{d_{v_0}}$ and $\varepsilon_1, \dots, \varepsilon_{d_{v_0}}$, we can find an $F \in \mathrm{Dom}(A)$ whose boundary trace at v_0 equals (δ, ε) and whose traces at all other vertices vanish. Hence the first sum in (3.18) can be made to equal any chosen linear functional of G and G' at v_0.

Now fix an index $i \in \{1, \dots, n\}$. We construct a function $f \in \mathrm{Dom}(A)$ such that:

- f is supported in a small neighborhood of the vertex $v_i \in V_2$ containing no other vertex.
- f is continuous at v_i (by construction, e.g., using a radially symmetric bump on the star of edges incident to v_i).
- The values $f(v_i)$ and $\partial f(v_i)$ are chosen so that

  $$R_i(f) = \theta_{i1} f(v_i) - \theta_{i2} \partial f(v_i) = 1,$$

  and for all $j \neq i$, $R_j(f) = 0$ (automatically because the support avoids v_j).
- Moreover, we arrange that for all $v \in V_1$, $f(v) = 0$ and $f'(v) = 0$ (by shrinking the support away from V_1).
- Finally, we ensure that f satisfies the vertex condition (3.3) at V_1 trivially (since all traces vanish) and the continuity condition (3.4) at V_2 (by the radial construction).

The existence of such an f follows from the same bump-function argument used in the proof of dense definiteness (Theorem 3.1), where we already constructed functions ψ_i with $R'_i(\psi_i) = 1$. For R_i instead of R'_i, we simply replace ϕ by θ in the linear combination. Because $\gamma_i > 0$, the two linear functionals R_i and R'_i are linearly independent, so we can independently prescribe $R_i(f)$ and $R'_i(f)$. In particular, we can achieve $R_i(f) = 1$ while keeping $R'_i(f) = 0$ (or any other value). For the purpose of isolating the term involving $R_i(f)\,(\overline{b_i} - \overline{R'_i(g)})$ in (3.18), we only need $R_i(f) \neq 0$ and $R_j(f) = 0$ for $j \neq i$, together with vanishing boundary traces at V_1.

With the above constructions, we can now argue. For a fixed G in the candidate adjoint domain, we require that the left-hand side of (3.18) is zero for all $F \in \mathrm{Dom}(A)$ (by the definition of the adjoint). Using the test functions described:

(1) Choose F supported near a single vertex $v_0 \in V_1$ with $R_i(f) = 0$ for all i. Then the second sum vanishes, and we obtain

$$(F(v_0), G'(v_0)) - (F'(v_0), G(v_0)) = 0$$

for all choices of F_{v_0} that lie in M_{v_0}. Because the symplectic form is non-degenerate on $\mathbb{C}^{2d_{v_0}}$, this forces G_{v_0} to be skew-orthogonal to every vector in M_{v_0}; i.e., $G_{v_0} \in M_{v_0}^{\perp}$. Repeating for each $v \in V_1$ gives $\tilde{G} \in M^{\perp}$.

(2) Choose F supported near $v_i \in V_2$ with $R_i(f) = 1$, $R_j(f) = 0$ for $j \neq i$, and vanishing boundary traces at V_1. Then (3.18) reduces to

$$-\frac{1}{\gamma_i}(\overline{b_i} - \overline{R_i'(g)}) = 0,$$

hence $\overline{b_i} = \overline{R_i'(g)}$, i.e., $b_i = R_i'(g)$. This holds for each $i = 1, \dots, n$.

We conclude that $G \in \mathrm{Dom}(A)$ satisfies $b_i = R_i'(g)$ and $\tilde{G} \in M^{\perp}$. Thus we have shown that

$$\mathrm{Dom}(A^*) = \{G \in \mathrm{Dom}(A_{\max}) : b_i = R_i'(g),\ \tilde{G} \in M^{\perp}\}.$$

Conversely, if $G \in \mathrm{Dom}(A^*)$, then there exists some H; comparing with $A_{\max}G$ and using the independence of the variations forces the same conditions, $b_i = R_i'(g)$ and $\tilde{G} \in M^{\perp}$, and H must equal $A_{\max}G$. Therefore

$$\mathrm{Dom}(A^*) = \{G \in \mathrm{Dom}(A_{\max}) : b_i = R_i'(g) \text{ for all } i,\ \tilde{G} \in M^{\perp}\}.$$

Recall that $\mathrm{Dom}(A)$ consists exactly of those vectors in $\mathrm{Dom}(A_{\max})$ satisfying $b_i = R_i'(g)$ and $\tilde{G} \in M$ (by the definition of M). Thus

$$\mathrm{Dom}(A) = \{G \in \mathrm{Dom}(A_{\max}) : b_i = R_i'(g),\ \tilde{G} \in M\}.$$

From Theorem 3.3, we know that A is symmetric iff M is isotropic. In particular, if A is self-adjoint, then it is symmetric, so M is isotropic, which implies $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(A^*)$. Self-adjointness requires $\mathrm{Dom}(A) = \mathrm{Dom}(A^*)$. Comparing the descriptions of the two domains, this equality holds precisely when $M = M^{\perp}$, i.e., when M is Lagrangian. Conversely, if M is Lagrangian, then M is isotropic (so A is symmetric) and $M = M^{\perp}$ gives $\mathrm{Dom}(A) = \mathrm{Dom}(A^*)$; hence A is self-adjoint.

Thus A is self-adjoint if and only if M is a Lagrangian subspace of T. This establishes (i) ⇔ (ii). □

Next, we prove that self-adjoint realizations exist.

Theorem 3.5. *The vertex condition matrices* $\{C_v :\ v \in V_1\}$ *that makes the operator A self-adjoint exist.*

Proof. Using Theorem 3.4, the proof reduces to showing that Lagrangian subspaces of $\mathbb{C}^{2d_v}$ exist for every $v \in V_1$. To this end, we shall use the symplectic geometry approach [11]. First, let us denote the standard basis of $\mathbb{C}^{2d_v}$ by $B = \{e_i\}_{i=1}^{2d_v}$. Then, it is straightforward to check that the matrix defined by (2.3) of the symplectic form (3.11) associated to this basis equals to

$$\begin{pmatrix} 0 & I_{d_v} \\ -I_{d_v} & 0 \end{pmatrix}.$$

Therefore, the Hermitian symplectic space $\mathbb{C}^{2d_v}$ with the symplectic form (3.11) is canonical. Lemma 2.1 implies that Lagrangian subspaces exist and this completes the proof. □

Corollary 3.1. *If A is self-adjoint, then* rank $C_v = d_v$ *for every* $v \in V_1$.

Proof. The dimension of each Lagrangian subspace of $\mathbb{C}^{2d_v}$ should be d_v (see Section 2.2). If A is self-adjoint, then M_v is a Lagrangian subspace of $\mathbb{C}^{2d_v}$ for every $v \in V_1$, by Theorem 3.4. Therefore, rank $C_v = d_v$ for every $v \in V_1$. □

Theorem 3.6. *A is self-adjoint iff the matrix C_v has the maximal rank and $A_v B_v^*$ is self-adjoint for every $v \in V_1$.*

Proof. Suppose that C_v has the maximal rank, i.e., rank $C_v = d_v$. Then, one can easily show that (see Lemma 2.2 in [3]) M_v is a Lagrangian subspace of $\mathbb{C}^{2d_v}$ iff $A_v B_v^*$ is self-adjoint. This completes the proof using Theorem 3.4 and Corollary 3.1. □

3.2. *Compact resolvent and discrete spectrum*

Before proving compactness of the resolvent, we establish a key equivalence between the graph norm of A and the H^2-norm of the function component.

Let A be a self-adjoint operator as defined in Section 3. For $U = (u, a_1, \ldots, a_n) \in \mathrm{Dom}(A)$, recall that $a_i = R_i'(u)$. Define the graph norm

$$\|U\|_A^2 = \|U\|_{H_1}^2 + \|AU\|_{H_1}^2,$$

and the H^2-norm of the function component

$$\|u\|_{H^2(\Gamma)}^2 = \sum_{e \in E} (\|u_e\|_{L^2(0,l_e)}^2 + \|u_e'\|_{L^2(0,l_e)}^2 + \|u_e''\|_{L^2(0,l_e)}^2).$$

Lemma 3.1. *There exists a constant $C > 0$, independent of u, such that for every $U = (u, R_1'(u), \ldots, R_n'(u)) \in \mathrm{Dom}(A)$,*

$$\|u\|_{H^2(\Gamma)} \le C\|U\|_A.$$

Proof. On each edge $e \cong [0, l_e]$, the function u_e satisfies $-u_e'' = (\pi_1(AU))_e$ in $L^2(0, l_e)$, where π_1 denotes projection onto the first component $L^2(\Gamma)$. By standard Sobolev regularity for one-dimensional second-order ODEs, for any $u_e \in H^2(0, l_e)$, we have the estimate

$$\|u_e\|_{H^2(0,l_e)} \le C_e(\|u_e\|_{L^2(0,l_e)} + \|u_e''\|_{L^2(0,l_e)}).$$

(Indeed, on an interval, the H^2-norm is equivalent to $\|u_e\|_{L^2} + \|u_e''\|_{L^2}$ because the intermediate derivative u_e' can be bounded via the interpolation inequality $\|u_e'\|_{L^2}^2 \le \|u_e\|_{L^2}\|u_e''\|_{L^2}$ together with the fact that u_e and u_e' are continuous up to the boundary.) Summing over edges gives

$$\|u\|_{H^2(\Gamma)}^2 \le C_1(\|u\|_{L^2(\Gamma)}^2 + \|u''\|_{L^2(\Gamma)}^2).$$

But $\|u''\|_{L^2(\Gamma)} = \|\pi_1(AU)\|_{L^2(\Gamma)} \le \|AU\|_{H_1}$. Moreover, $\|u\|_{L^2(\Gamma)} \le \|U\|_{H_1}$. Hence

$$\|u\|_{H^2(\Gamma)}^2 \le C_1(\|U\|_{H_1}^2 + \|AU\|_{H_1}^2) = C_1\|U\|_A^2.$$

Taking $C = \sqrt{C_1}$ completes the proof. □

Remark 3.1. *The lemma shows that the graph norm on* Dom(A) *is stronger than the* H^2*-norm of the function component. Conversely, it is obvious that* $\|U\|_A \leq C_2\|u\|_{H^2(\Gamma)}$ *because the extra components* $R_i'(u)$ *are continuous linear functionals on* $H^2(\Gamma)$ *and* AU *involves* u'' *and* $R_i(u)$*, which are also continuous on* $H^2(\Gamma)$*. Thus the graph norm is equivalent to the* H^2*-norm on the function component together with the finite-dimensional part. Consequently,* Dom(A) *equipped with the graph norm is continuously embedded in* $H^2(\Gamma) \oplus \mathbb{C}^n$ *and the embedding* Dom(A) $\hookrightarrow H_1$ *is compact because the embedding* $H^2(\Gamma) \hookrightarrow L^2(\Gamma)$ *is compact (due to Rellich–Kondrachov theorem on each bounded interval) and the extra components are finite-dimensional.*

Theorem 3.7 (Compact resolvent and discrete spectrum)**.** *Let* A *be the operator defined in Section 3 on the extended Hilbert space* $H_1 = L^2(\Gamma) \oplus \mathbb{C}^n$*. If* A *is self-adjoint, then its resolvent* $(A - \lambda I)^{-1}$ *is compact for every* $\lambda \in \rho(A)$*. Consequently, the spectrum of* A *consists only of isolated real eigenvalues of finite multiplicity that accumulate only at infinity; i.e.,*

$$\sigma(A) = \{\lambda_1, \lambda_2, \dots\}, \qquad |\lambda_n| \to \infty,$$

and the corresponding eigenfunctions form a complete orthonormal basis of H_1.

Proof. Since A is self-adjoint, its resolvent is bounded. For any $\lambda \in \rho(A)$, the resolvent $R(\lambda) = (A - \lambda I)^{-1}$ maps H_1 onto Dom(A) (with the graph norm). The closed graph theorem guarantees that $R(\lambda) : H_1 \to$ Dom(A) is an isomorphism when Dom(A) is equipped with the graph norm.

Consider the embedding

$$J : \mathrm{Dom}(A) \hookrightarrow H_1, \qquad J(U) = U.$$

Lemma 3.1 shows that the graph norm dominates the H^2-norm of the function component, and the latter is compactly embedded in $L^2(\Gamma)$ (Rellich–Kondrachov). Moreover, the extra components $a_i = R_i'(u)$ are continuous linear functionals on $H^2(\Gamma)$; weak convergence in H^2 implies convergence of these numbers. Hence J is a compact operator (it maps bounded sequences in Dom(A) to sequences in H_1 that have a convergent subsequence).

The resolvent can be written as the composition

$$R(\lambda) : H_1 \xrightarrow{\text{isomorphism}} \mathrm{Dom}(A) \xrightarrow{J} H_1.$$

The composition of a bounded isomorphism and a compact operator is compact. Therefore $R(\lambda)$ is compact.

For a self-adjoint operator with a compact resolvent, standard spectral theory implies that the spectrum is purely discrete: it consists of isolated eigenvalues of finite multiplicity with no finite accumulation point, and the eigenfunctions form a complete orthonormal basis. □

4. An example and physical interpretation

In this section, we illustrate the theoretical results obtained in Section 3 with a concrete example. We consider a simple yet nontrivial finite compact metric graph and explicitly verify the self-adjointness conditions derived in Theorem 3.6.

4.1. A simple star graph with three edges

Let Γ be a star graph consisting of a central vertex v_0 and three boundary vertices v_1, v_2, v_3 (see Figure 1). Each edge e_i is identified with the interval $[0, l_i]$ for $i = 1, 2, 3$, where the endpoint 0 corresponds to the central vertex v_0 and l_i corresponds to the boundary vertex v_i. We set $l_1 = 1$, $l_2 = 2$, and $l_3 = 3$ for simplicity.

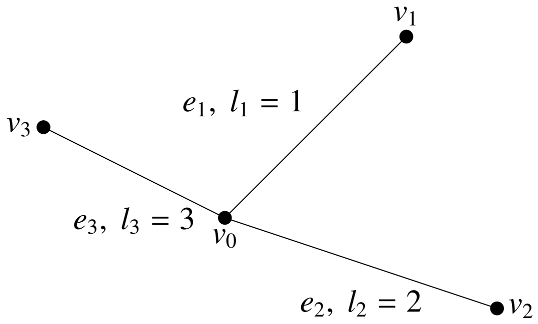

Figure 1. A star graph with three edges and a central vertex v_0.

We decompose the vertex set as $V = V_1 \cup V_2$ with:

- $V_1 = \{v_0\}$ (central vertex): standard vertex conditions of type (3.3);
- $V_2 = \{v_1, v_2, v_3\}$ (boundary vertices): eigenparameter-dependent conditions of type (3.5).

4.2. Vertex conditions

At the central vertex, we impose the following matrices:

$$A_{v_0} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_{v_0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

These matrices correspond to the following vertex conditions:

$$f_1(v_0) = f_2(v_0) \quad \text{(continuity between edges 1 and 2)}, \tag{4.1}$$

$$f_2(v_0) = f_3(v_0) \quad \text{(continuity between edges 2 and 3)}, \tag{4.2}$$

$$f_1'(v_0) + f_2'(v_0) + f_3'(v_0) = 0 \quad \text{(Kirchhoff condition)}. \tag{4.3}$$

The combined matrix $C_{v_0} = (A_{v_0}\ B_{v_0})$ is:

$$C_{v_0} = \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right).$$

Clearly, $\operatorname{rank}(C_{v_0}) = 3 = d_{v_0}$, so the maximal rank condition is satisfied. Moreover,

$$A_{v_0} B_{v_0}^* = 0,$$

which is trivially self-adjoint.

At each boundary vertex v_i (for $i = 1, 2, 3$), we impose eigenparameter-dependent condition (3.5) with coefficients in Table 1.

Table 1. Coefficients for vertex conditions at boundary vertices.

Vertex	θ_{i1}	θ_{i2}	ϕ_{i1}	ϕ_{i2}	$\gamma_i = \phi_{i1}\theta_{i2} - \theta_{i1}\phi_{i2}$
v_1	1	0	0	-1	1
v_2	1	0	0	-2	2
v_3	1	1	1	0	1

The vertex conditions at v_i become:

$$\frac{f_1'(l_1)}{f_1(l_1)} = -\frac{1}{\lambda}, \tag{4.4}$$

$$\frac{f_2'(l_2)}{f_2(l_2)} = -\frac{1}{2\lambda}, \tag{4.5}$$

$$\frac{f_3'(l_3)}{f_3(l_3)} = \lambda - 1. \tag{4.6}$$

One can easily verify that the functions appearing on the right-hand sides of Eqs (4.4)–(4.6) are rational Herglotz–Nevanlinna functions. Recall that a function $Q : \mathbb{C}^+ \to \mathbb{C}$ is called a Herglotz–Nevanlinna function if it is analytic and satisfies

$$\operatorname{Im} Q(z) \geq 0, \qquad z \in \mathbb{C}^+.$$

For more details, see [12].

4.3. Extended Hilbert space and operator A

Following Section 3, we construct the extended Hilbert space $H_1 = L^2(\Gamma) \oplus \mathbb{C}^3$ with the following inner product:

$$(F, G) = (f, g)_{L^2(\Gamma)} + \sum_{i=1}^{3} \frac{1}{\gamma_i} a_i \overline{b_i},$$

where $\gamma_1 = 1, \gamma_2 = 2, \gamma_3 = 1$.

For any function $f \in H^2(\Gamma)$ satisfying the vertex conditions (4.1)–(4.3) and (4.4)–(4.6), we define:

$$R_i'(f) = \phi_{i1} f(v_i) - \phi_{i2} f'(v_i), \quad R_i(f) = \theta_{i1} f(v_i) - \theta_{i2} f'(v_i).$$

Specifically:

$$R_1'(f) = -f_1'(l_1), \quad R_1(f) = f_1(l_1), \tag{4.7}$$

$$R_2'(f) = -2f_2'(l_2), \quad R_2(f) = f_2(l_2), \tag{4.8}$$

$$R_3'(f) = f_3(l_3), \quad R_3(f) = f_3(l_3) + f_3'(l_3). \tag{4.9}$$

The operator A is defined on vectors $F = (f, R_1'(f), R_2'(f), R_3'(f)) \in H_1$ by:

$$AF = (\ell f, R_1(f), R_2(f), R_3(f)).$$

4.4. *Verification of self-adjointness*

By Theorem 3.6, A is self-adjoint iff for every $v \in V_1$ (here, only v_0), C_v has maximal rank and $A_v B_v^*$ is self-adjoint. We have already verified that C_{v_0} has maximal rank and $A_{v_0} B_{v_0}^*$ is self-adjoint. Therefore, A is self-adjoint.

4.5. *Physical interpretation*

The conditions $\gamma_i > 0$ ensure that the inner product in the extended space is positive definite, which is essential for a probabilistic interpretation in quantum mechanics. Moreover, the eigenparameter-dependent conditions (4.4)–(4.6) can be used to model physical situations where the boundary has an internal structure that interacts with the incident wave, with the interaction strength depending on the energy λ.

4.6. *Spectral analysis of the example graph*

We now analyze the spectral properties of the operator A constructed in Section 4.3. The eigenvalues λ of the boundary value problem satisfy Eq (3.2) on each edge, together with the vertex conditions (4.1)–(4.3) at the central vertex and (4.4)–(4.6) at the boundary vertices.

4.6.1. General solution on each edge

On each edge e_i, the differential equation $-f_i'' = \lambda f_i$ has the general solution:

$$f_i(x) = \alpha_i \cos(kx) + \beta_i \sin(kx), \quad i = 1, 2, 3,$$

where $k = \sqrt{\lambda}$ denotes the principal square root (taking $\sqrt{\lambda} = i\kappa$ for $\lambda < 0$ if necessary). The derivatives are:

$$f_i'(x) = k\left(-\alpha_i \sin(kx) + \beta_i \cos(kx)\right).$$

4.6.2. Boundary conditions at $v_i \in V_2$

At each boundary vertex v_i, the local coordinate is $x = l_i$. Applying conditions (4.4)–(4.6):

$$\alpha_1 \left[\cos(k) - k^3 \sin(k)\right] + \beta_1 \left[\sin(k) + k^3 \cos(k)\right] = 0, \tag{4.10}$$

$$\alpha_2 \left[\cos(2k) - 2k^3 \sin(2k)\right] + \beta_2 \left[\sin(2k) + 2k^3 \cos(2k)\right] = 0, \tag{4.11}$$

$$\alpha_3 \left[(k^2 - 1)\cos(3k) + k \sin(3k)\right] + \beta_3 \left[(k^2 - 1)\sin(3k) - k\cos(3k)\right] = 0. \tag{4.12}$$

4.6.3. Central vertex conditions at v_0

At the central vertex v_0 (local coordinate $x = 0$), continuity conditions (4.1) and (4.2) give:

$$f_1(0) = f_2(0) \quad \Rightarrow \quad \alpha_1 = \alpha_2, \tag{4.13}$$

$$f_2(0) = f_3(0) \quad \Rightarrow \quad \alpha_2 = \alpha_3. \tag{4.14}$$

Thus, $\alpha_1 = \alpha_2 = \alpha_3 \equiv \alpha$.

Kirchhoff condition (4.3) at v_0 gives:

$$f_1'(0) + f_2'(0) + f_3'(0) = 0.$$

Since $f_i'(0) = k\beta_i$, we obtain:

$$\beta_1 + \beta_2 + \beta_3 = 0. \tag{4.15}$$

4.6.4. The secular equation

Let us define

$$\begin{aligned}
a_1(k) &= \cos(k) - k^3 \sin(k),\\
b_1(k) &= \sin(k) + k^3 \cos(k),\\
a_2(k) &= \cos(2k) - 2k^3 \sin(2k),\\
b_2(k) &= \sin(2k) + 2k^3 \cos(2k),\\
a_3(k) &= (k^2 - 1)\cos(3k) + k \sin(3k),\\
b_3(k) &= (k^2 - 1)\sin(3k) - k \cos(3k).
\end{aligned}$$

We now have a homogeneous linear system:

$$\begin{aligned}
a_1(k)\alpha + b_1(k)\beta_1 &= 0,\\
a_2(k)\alpha + b_2(k)\beta_2 &= 0,\\
a_3(k)\alpha + b_3(k)\beta_3 &= 0,\\
\beta_1 + \beta_2 + \beta_3 &= 0,
\end{aligned}$$

with the unknowns $\alpha, \beta_1, \beta_2, \beta_3$. We can write this system in the matrix form

$$M(k)X = 0,$$

where $X = (\alpha, \beta_1, \beta_2, \beta_3)^T$ and the coefficient matrix is

$$M(k) = \begin{pmatrix} a_1(k) & b_1(k) & 0 & 0 \\ a_2(k) & 0 & b_2(k) & 0 \\ a_3(k) & 0 & 0 & b_3(k) \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

For nontrivial solutions, the determinant of the coefficient matrix must vanish, yielding the secular equation:

$$F(k) := \det M(k) = 0.$$

It easily follows that

$$F(k) = a_1(k)b_2(k)b_3(k) + a_2(k)b_1(k)b_3(k) + a_3(k)b_1(k)b_2(k).$$

4.6.5. Numerical computation of eigenvalues

We numerically solve $F(k) = 0$ for $k \in \mathbb{R}$. Figure 2 shows the graph of the secular function F with respect to k, where the zeros correspond to the roots of the spectral parameters λ. The associated eigenvalues are given by $\lambda_n = k_n^2$. Table 2 lists the first ten computed eigenvalues and their multiplicities.

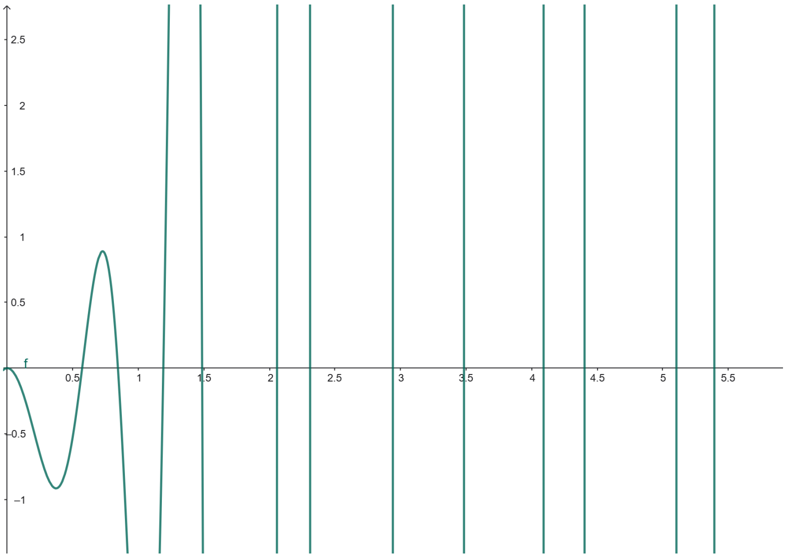

Figure 2. Plot of the secular function $F(k)$.

Table 2. First ten eigenvalues of the star graph with eigenparameter-dependent vertex conditions.

n	k_n	λ_n	Multiplicity
1	0.571428	0.326530	1
2	0.844449	0.713094	1
3	1.190725	1.417827	1
4	1.486648	2.210123	1
5	2.058119	4.235855	1
6	2.310470	5.338272	1
7	2.941629	8.653185	1
8	3.484322	12.140500	1
9	4.091481	16.740222	1
10	4.403939	19.394678	1

4.6.6. Special case: $\lambda = 0$

For $\lambda = 0$, the differential equation becomes $-f_i'' = 0$, so $f_i(x) = \alpha_i + \beta_i x$. The vertex conditions become:

- At v_1: $f_1(1) = 0 \Rightarrow \alpha_1 + \beta_1 = 0$;
- At v_2: $f_2(2) = 0 \Rightarrow \alpha_2 + 2\beta_2 = 0$;

- At v_3: $-f_3(3) = f_3'(3) \Rightarrow -\alpha_3 - 3\beta_3 = \beta_3 \Rightarrow \alpha_3 + 4\beta_3 = 0$;
- At v_0: Continuity gives $\alpha_1 = \alpha_2 = \alpha_3 \equiv \alpha$, and the Kirchhoff condition gives $\beta_1 + \beta_2 + \beta_3 = 0$.

Solving yields $\alpha = 0$ and $\beta_1 = \beta_2 = \beta_3 = 0$, so only the trivial solution exists. Thus, $\lambda = 0$ is not an eigenvalue.

4.6.7. Spectral properties and physical interpretation

As expected for a finite compact graph, the spectrum is purely discrete with $\lambda_n \to \infty$ as $n \to \infty$. The eigenvalues depend continuously on the parameters θ_{ij}, ϕ_{ij} and the edge lengths l_i. This allows for "spectral engineering" by tuning these parameters.

5. Relation to time-dependent problems (formal discussion)

This section provides a purely formal connection between eigenparameter-dependent vertex conditions and dynamic boundary conditions. A rigorous treatment of the resulting time-dependent problems, including well-posedness, semigroup generation, and stability, is beyond the scope of this paper and is left for future research. The discussion below is intended to illustrate how the stationary problem arises naturally from time-dependent evolution equations via separation of variables, and to suggest possible physical interpretations.

Eigenparameter-dependent vertex conditions appear naturally when one applies separation of variables to time-dependent evolution equations on a metric graph. Consider the heat, wave, or Schrödinger equations:

$$\partial_t u = \partial_x^2 u, \qquad \partial_t^2 u = \partial_x^2 u, \qquad i\partial_t \psi = -\partial_x^2 \psi.$$

Using the standard separation of variables ansatz:

$$u(x,t) = f(x)e^{-\lambda t}, \qquad u(x,t) = f(x)e^{i\sqrt{\lambda}t}, \qquad \psi(x,t) = f(x)e^{-i\lambda t},$$

each of these problems reduces formally to the stationary equation

$$-f'' = \lambda f.$$

Consequently, a vertex condition that depends on the spectral parameter,

$$\theta_1 f(v) - \theta_2 \sum_{e\in E_v} f_e'(v) = \lambda \left(\phi_1 f(v) - \phi_2 \sum_{e\in E_v} f_e'(v) \right),$$

suggests, at the formal level, a dynamic boundary condition involving time derivatives. For instance, for the heat equation, the above condition becomes

$$\theta_1 u(v,t) - \theta_2 \sum_{e\in E_v} \partial_x u_e(v,t) + \phi_1 \partial_t u(v,t) - \phi_2 \sum_{e\in E_v} \partial_t \partial_x u_e(v,t) = 0,$$

which is a first-order dynamic condition. For the wave equation, one obtains second-order dynamic conditions involving $\partial_t^2 u$, while for the Schrödinger equation, the vertex condition couples the boundary values with the time derivative of the wave function.

We emphasise that these are formal calculations. A complete analysis of the corresponding initial-boundary value problems (e.g., existence, uniqueness, and asymptotic behavior of solutions) requires a separate investigation, which we leave for future work. The present discussion merely highlights a conceptual link between eigenparameter-dependent vertex conditions and dynamic boundary conditions, thereby motivating the spectral analysis carried out in this paper.

6. Conclusions

This paper provided a classification of self-adjoint realizations of the Laplacian with eigenparameter-dependent vertex conditions on finite compact metric graphs. By constructing an extended Hilbert space $H_1 = L^2(\Gamma) \oplus \mathbb{C}^n$, we transformed the original boundary value problem into a standard eigenvalue problem for an operator A, effectively eliminating the spectral parameter from the vertex conditions.

Using Hermitian symplectic geometry, we proved that self-adjointness of A is equivalent to the condition that at every vertex $v \in V_1$, the matrices satisfy $\operatorname{rank}(A_v B_v) = d_v$ and $A_v B_v^*$ is self-adjoint—a direct generalization of the classical Kostrykin–Schrader criterion. We further established that self-adjoint realizations have a compact resolvent, yielding purely discrete real spectra.

The connection to time-dependent problems was demonstrated: eigenparameter-dependent conditions transform via Fourier analysis into dynamic boundary conditions for heat, wave, and Schrödinger equations. A detailed star graph example illustrated the practical verification of our conditions and numerical spectral computation.

This work unifies operator-theoretic and symplectic approaches, providing accessible algebraic criteria applicable to arbitrary finite compact metric graphs. Future directions include nonlinear vertex conditions, infinite graphs, higher-order operators, and inverse problems.

Author contributions

Zineb Zellak: Conceptualization, software, validation, writing – original draft, writing – review and editing; Gökhan Mutlu: Conceptualization, project administration, supervision, writing – original draft, writing – review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Data availability statement

We used Maple and MATLAB for the computations to obtain Table 2. We share the codes via the link: https://github.com/Zineb-Zellak/First-ten-eigenvalues-of-the-star-graph-with-eigenparameter-dependent-vertex-conditions.

Conflict of interest

The authors declare no conflicts of interest.

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